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ADDITIONAL STABILITY RESULTS FOR QUARTIC LIE $*$-DERIVATIONS

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Abstract. In this article, we introduce additional stability results of quartic Lie $*$-derivations by using direct method and alternative fixed point method.

1. Introduction

In 1940, Ulam [17] raised the question concerning the stability of group homomorphisms. Let G be a group and let $G^{'}$ be a metric group with the metric $d(\cdot,\ \cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $f : G \to G^{'}$ satisfies the inequality

$$d(f(xy), f(x)f(y)) < \delta$$

for all $x, y \in G$, then there exists a homomorphism $F : G \to G^{'}$ with $d(f(x), F(x)) < \varepsilon$ for all $x \in G$? The case of approximately additive mappings was solved by Hyers [6] under the assumption that X and Y are Banach spaces. Hyers method used in [6], which is often called the direct method, has been applied for studying the stability of various functional equations. Most popular technique of proving the stability of functional equations except for direct method is the fixed point method [2, 3, 13].

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In 1999, Rassias [14] has studied the Hyers–Ulam stability problem of the quartic functional equation

$$f(x+2y)+f(x-2y)+6f(x)=4f(x+y)+4f(x-y)+24f(y), \tag{1.1}$$

of which the general solution is called the quartic mapping.

Let X and Y be vector spaces. Now, we introduce some basic concepts concerning 4-additive symmetric mappings [8]. A mapping $A_4 : X^4 \to Y$ is called 4-additive if it is additive in each variable. A mapping A_4 is said to be symmetric if $A_4(x_1, x_2, x_3, x_4) = A_4(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)})$ for every permutation $\{\sigma(1), \sigma(2), \sigma(3), \sigma(4)\}$ of $\{1,2,3,4\}$. If A_4 is 4-additive symmetric mapping, then $A^4(x) := A_4(x, x, x, x)$ will denote the diagonal of the 4-additive symmetric mapping A_4. Then it follows $A^4(qx) = q^4A^4(x)$ for all $x \in X$ and all $q \in Q$. In general, we refer that the generalized concepts of n-additive symmetric mappings are found in [16] and [18]. In 2003, Chung and Sahoo [4] obtained the general solution of the equation (1.1) by using the properties of the form $A(x, x, x, x)$, where the function $A : \mathbb{R}^4 \to \mathbb{R}$ is symmetric and additive in each variable. Recently, Kang and Koh [8] have established general solution of the following functional equation

$$\begin{aligned} f(ma+b)-f(a-mb)+\frac{1}{2}m(m^2+1)f(a-b)+(m^4-1)f(b) \quad (1.2)\\ =\frac{1}{2}m(m^2+1)f(a+b)+(m^4-1)f(a), \end{aligned}$$

for all vectors a, b in a complex normed $*$-algebra, where $m(m \neq 0, \pm 1)$ is a fixed integer, and then they have investigated the generalized Hyers–Ulam stability of the equation associated with approximate quartic Lie $*$-derivations. First of all, it is known from [8] that a mapping $f : X \to Y$ with $f(0) = 0$ is a solution of the equation (1.2) if and only if f is of the form $f(x) = A^4(x)$ for all $x \in X$, and thus it is quartic.

In this paper, we introduce to investigate additional stability results and refined stability theorems of the quartic functional equation (1.2) associated with quartic Lie $*$-derivations by using direct method and alternative fixed point method.

2. Stability for Approximate Quartic Lie $*$-Derivations

In this section, we will research the Hyers–Ulam stability of the quartic Lie $*$-derivation by using directed method and a fixed point method. Let A be a complex normed $*$-algebra and M a Banach A-bimodule with linear involution $*$. At the same time we denote $\|\cdot\|$ as norms on a normed $*$-algebra A and a Banach A-bimodule M. A mapping $f : A \to M$ is called quartic homogeneous if it satisfies $f(\mu a) = \mu^4 f(a)$ for all $a \in A$ and $\mu \in \mathbb{C}$. A quartic homogeneous mapping $f : A \to M$ is called a quartic derivation if

$$f(xy) = f(x)y^4 + x^4 f(y)$$

for all $x, y \in A$. A quartic homogeneous mapping f is called a quartic Lie derivation if

$$f([x, y]) = [f(x), y^4] + [x^4, f(y)]$$

for all $x, y \in A$, where $[x, y] := xy - yx$. In addition, a quartic Lie derivation f is called a quartic Lie $*$-derivation if it satisfies $f(x^*) = f(x)^*$ for all $x \in A$.

Recently, the related properties of various derivations in different algebraic structures have been investigated by many authors [1, 9, 10, 11].

Throughout the paper, let n_0 be a positive integer and

$$\mathbb{T}^1_{\frac{1}{n_0}} := \{exp(i\theta) : 0 \leq \theta \leq \frac{2\pi}{n_0}\}.$$

For a given mapping $f : A \to M$ we denote the following abbreviations

$$\begin{aligned} \Delta_\mu f(a, b) \quad &:= \quad f(m\mu a + \mu b) - f(\mu a - m\mu b) + \frac{\mu^4}{2} m(m^2 + 1) f(a - b) \\ &\quad\quad +\mu^4(m^4 - 1) f(b) - \frac{\mu^4}{2} m(m^2 + 1) f(a + b) \\ &\quad\quad -\mu^4(m^4 - 1) f(a), \\ QDf(a, b) \quad &:= \quad f([a, b]) - [f(a), b^4] - [a^4, f(b)] \end{aligned}$$

for all $a, b \in A$, $\mu \in \mathbb{T}^1_{\frac{1}{n_0}}$, where $m \in \mathbb{Z}(m \neq 0, \pm 1)$ is fixed.

The following theorem is an alternative stability result of the quartic functional equation (1.2) associated with quartic Lie $*$-derivations by using direct method, which is similarly verified as in the proof of [8, Theorem 3.2].

Theorem 2.1. *Suppose that there exist a mapping $f : A \to M$ with $f(0) = 0$ and two functions $\phi_1 : A^3 \to [0, \infty)$, $\phi_2 : A^2 \to [0, \infty)$ such that*

$$\|\Delta_\mu f(a, b) + f(c^*) - f(c)^*\| \leq \phi_1(a, b, c), \tag{2.1}$$

$$\|QDf(a, b)\| \leq \phi_2(a, b), \tag{2.2}$$

in which

$$\Phi(a, b, c) := \sum_{j=0}^{\infty} \frac{\phi_1(m^j a, m^j b, m^j c)}{m^{4j}} < \infty, \tag{2.3}$$

$$\lim_{k\to\infty} \frac{\phi_2(m^k a, m^k b)}{m^{8k}} = 0$$

for all $\mu \in \mathbb{T}^1_{\frac{1}{n_0}}$ and all $a, b, c \in A$. Furthermore, if for each fixed $a \in A$ the mapping $r \mapsto f(ra)$ from $\mathbb{R}$ to M is continuous, then there exists a unique *quartic Lie $*$-derivation $L : A \to M$ such that*

$$\|f(a) - L(a)\| \leq \frac{1}{m^4}\Phi(a,0,0) \tag{2.4}$$

for all $a \in A$.

On the other hand, we introduce to investigate another stability result of the quartic functional equation (1.2) associated with quartic Lie $*$-derivations by using direct method, which is an additional main stability theorem.

Theorem 2.2. *Suppose that there exist a mapping $f : A \to M$ and two functions $\phi_1 : A^3 \to [0,\infty)$, $\phi_2 : A^2 \to [0,\infty)$ such that*

$$\begin{aligned} \|\Delta_\mu f(a,b) + f(c^*) - f(c)^*\| &\leq \phi_1(a,b,c), & (2.5)\\ \|QDf(a,b)\| &\leq \phi_2(a,b), & (2.6)\end{aligned}$$

in which

$$\Phi_1(a,b,c) := \sum_{j=1}^{\infty} m^{4j}\phi_1\Big(\frac{a}{m^j}, \frac{b}{m^j}, \frac{c}{m^j}\Big) < \infty, \tag{2.7}$$

$$\lim_{k\to\infty} m^{8k}\phi_2(\frac{a}{m^k}, \frac{b}{m^k}) = 0$$

for all $\mu \in \mathbb{T}^1_{\frac{1}{n_0}}$ and all $a,b,c \in A$. In addition, if for each fixed $a \in A$ the mapping $r \mapsto f(ra)$ from $\mathbb{R}$ to M is continuous, then there exists a unique quartic Lie $*$-derivation $L : A \to M$ such that

$$\|f(a) - L(a)\| \leq \frac{1}{m^4}\Phi_1(a,0,0) \tag{2.8}$$

for all $a \in A$.

Proof. At first, we note $f(0) = 0$ by letting $a,b,c := 0$ because of $\phi_1(0,0,0) = 0$. After putting $b = c := 0$ and $\mu = 1$ in the inequality (2.5), we see

$$\|f(a) - m^4 f(\frac{a}{m})\| \leq \phi_1(\frac{a}{m},0,0) \tag{2.9}$$

for all $a \in A$. By induction, it follows that for any positive integers n the following inequality

$$\|f(a) - m^{4n} f(\frac{a}{m^n})\| \leq \frac{1}{m^4}\sum_{j=1}^{n} m^{4j}\phi_1(\frac{a}{m^j},0,0) \tag{2.10}$$

holds for all $a \in A$, and hence one deduces

$$\|m^{4t} f(\frac{a}{m^t}) - m^{4k} f(\frac{a}{m^k})\| \leq \frac{1}{m^4}\sum_{j=k+1}^{t} m^{4j}\phi_1(\frac{a}{m^j},0,0) \tag{2.11}$$

for all integers $t > k \geq 0$ and all $a \in A$, which tends to zero as $k \to \infty$. Hence, $\{m^{4k} f(\frac{a}{m^k})\}_{k=0}^{\infty}$ is a Cauchy sequence in the complete space M, and so we can define a mapping $L : A \to M$ as

$$L(a) = \lim_{k\to\infty} m^{4k} f(\frac{a}{m^k}) \tag{2.12}$$

for all $a \in A$. Then, we claim the mapping L is quartic. In fact, we figure out that

$$\begin{aligned} \|\Delta_\mu L(a,b)\| &= \lim_{k\to\infty} m^{4k} \|\Delta_\mu f(\frac{a}{m^k}, \frac{b}{m^k})\| \\ &\leq \lim_{k\to\infty} m^{4k} \phi_1(\frac{a}{m^k}, \frac{b}{m^k}, 0) = 0 \end{aligned} \tag{2.13}$$

for all $a, b \in A$ and $\mu \in \mathbb{T}^1_{\frac{1}{n_0}}$. On taking $\mu = 1$ in the inequality (2.13), one should conclude that the mapping L is quartic. In addition, taking $n \to \infty$ in the inequality (2.10), we notices that the quartic mapping L satisfies the approximation (2.8) near f.

Now, we prove L is quartic homogeneous. It follows from the inequality (2.13) that $\Delta_\mu L(a, 0) = 0$, which yields

$$L(\mu a) = \mu^4 L(a)$$

for all $\mu \in \mathbb{T}^1_{\frac{1}{n_0}}$ and $a \in A$, and so, in turn, we lead to

$$L(\nu a) = \nu^4 L(a)$$

for all $\nu = \mathbb{T}^1$ and all $a \in A$. Under the assumption the mapping $r \mapsto f(ra)$ from $\mathbb{R}$ to M is continuous, one establishes

$$L(ra) = r^4 L(a) \tag{2.14}$$

for all $r \in \mathbb{R}$ and all $a \in A$ by the same reasoning as in the paper [7, 5]. Thus, for any $\omega \in \mathbb{C}(\omega \neq 0)$ it follows that

$$\begin{aligned} L(\omega a) &= L\Big(|\omega| \frac{\omega}{|\omega|} a\Big) = |\omega|^4 \Big(\frac{|\omega|}{\omega}\Big)^4 L(\frac{\omega}{|\omega|} a) \\ &= |\omega|^4 \Big(\frac{\omega}{|\omega|}\Big)^4 L(a) = \omega^4 L(a) \end{aligned}$$

for all $a \in A$, from which we conclude that L is quartic homogeneous.

To prove that L is a quartic Lie $*$-derivation, replacing a by $\frac{a}{m^k}$ and b by $\frac{b}{m^k}$ in the inequality (2.6), one obtains

$$\begin{aligned} \|QDL(a,b)\| &= \lim_{k\to\infty} \|m^{8k} QDf(\frac{a}{m^k}, \frac{b}{m^k})\| \\ &\leq \lim_{k\to\infty} m^{8k} \phi_2(\frac{a}{m^k}, \frac{b}{m^k}) = 0 \end{aligned}$$

for all $a, b \in A$, which means that L is a quartic Lie derivation. At this time, by replacing $a = b := 0$ and $c := \frac{c}{m^k}$ in the inequality (2.5), we have

$$\|m^{4k} f(\frac{c^*}{m^k}) - m^{4k} f(\frac{c}{m^k})^*\| \leq m^{4k}\phi_1(0, 0, \frac{c}{m^k}),$$

which yields

$$L(c^*) = L(c)^*$$

for all $c \in A$. Therefore, L is a quartic Lie $*$-derivation satisfying the approximation (2.8).

Finally, we will show that the quartic Lie $*$-derivation satisfying the inequality (2.8) is unique. Thus, we assume $L' : A \to M$ is another quartic Lie $*$-derivation satisfying the approximation (2.8). Then

$$\begin{aligned} \|L(a) - L'(a)\| &= m^{4k}\|L(\frac{a}{m^k}) - L'(\frac{a}{m^k})\| \\ &\leq m^{4k}\Big(\|L(\frac{a}{m^k}) - f(\frac{a}{m^k})\| + \|f(\frac{a}{m^k}) - L'(\frac{a}{m^k})\|\Big) \\ &\leq m^{4k}\frac{2}{m^4}\sum_{j=1}^{\infty} m^{4j}\phi_1(\frac{a}{m^{j+k}}, 0, 0) \\ &= \frac{2}{m^4}\sum_{j=k+1}^{\infty} m^{4j}\phi_1(\frac{a}{m^j}, 0, 0), \end{aligned}$$

which tends to zero as $k \to \infty$. Hence the uniqueness of L was proved. $\square$

Corollary 2.3. *Let $\theta_i (i = 1, 2)$, r_j be positive real numbers with $r_j > 4 (j = 1, \cdots, 5)$. Suppose that a mapping $f : A \to M$ satisfies the followings*

$$\begin{aligned} \|\Delta_\mu f(a, b) + f(c^*) - f(c)^*\| &\leq \theta_1(\|a\|^{r_1} + \|b\|^{r_2} + \|c\|^{r_3}), \\ \|QDf(a, b)\| &\leq \theta_2(\|a\|^{2r_4} + \|b\|^{2r_5}) \end{aligned}$$

for all $\mu \in \mathbb{T}^1_{\frac{1}{n_0}}$ and all $a, b, c \in A$. Then there exists a unique quartic Lie $*$-derivation $L : A \to M$ such that

$$\|f(a) - L(a)\| \leq \frac{\theta_1}{|m|^{r_1} - m^4}\|a\|^{r_1}$$

for all $a \in A$.

Proof. On taking $\phi_1(a, b, c) = \theta_1(\|a\|^{r_1} + \|b\|^{r_2} + \|c\|^{r_3})$ and $\phi_2(a, b) = \theta_2(\|a\|^{2r_4} + \|b\|^{2r_5})$ in Theorem 2.2 for all $a, b, c \in A$, we obtain the desired results. $\square$

Now, we recall the following theorem which is related to the alternative of fixed point theory [12, 15].

Theorem 2.4. *(The alternative of fixed point [12], [15]). Suppose that we are given a complete generalized metric space (Ω, d) and a strictly contractive mapping $T : \Omega \to \Omega$ with Lipschitz constant l. Then for each given $x \in \Omega$, either*

$$d(T^n x, T^{n+1} x) = \infty \text{ for all } \quad n \geq 0$$

or there exists a natural number n_0 such that

(1) $d(T^n x, T^{n+1} x) < \infty$ *for all* $n \geq n_0$ *;*
(2) *the sequence* $(T^n x)$ *is convergent to a fixed point* y^* *of* T *;*
(3) y^* *is the unique fixed point of* T *in the set* $\triangle = \{y \in \Omega | d(T^{n_0} x, y) < \infty\}$;
(4) $d(y, y^*) \leq \frac{1}{1-l} d(y, Ty)$ *for all* $y \in \triangle$.

The following theorem is an alternative stability result of the quartic functional equation (1.2) associated with quartic Lie $*$-derivations by using direct method, which is similarly verified as in the proof of [8, Theorem 3.7].

Theorem 2.5. *Suppose that there exist a mapping $f : A \to M$ with $f(0) = 0$ and two functions $\phi_1 : A^3 \to [0, \infty)$, $\phi_2 : A^2 \to [0, \infty)$ such that*

$$\begin{aligned} \|\Delta_\mu f(a, b) + f(c^*) - f(c)^*\| &\leq \phi_1(a, b, c), \\ \|QDf(a, b)\| &\leq \phi_2(a, b), \end{aligned}$$

in which there are constants $l_i (i = 1, 2) \in (0, 1)$ satisfying

$$\begin{aligned} \phi_1(ma, mb, mc) &\leq m^4 l_1 \phi_1(a, b, c), \\ \phi_2(ma, mb) &\leq m^8 l_2 \phi_2(a, b) \end{aligned}$$

for all $\mu \in \mathbb{T}^1_{\frac{1}{n_0}}$ and all $a, b, c \in A$. In addition, if for each fixed $a \in A$ the mapping $r \mapsto f(ra)$ from $\mathbb{R}$ to M is continuous, then there exists a unique quartic Lie $*$-derivation $L : A \to M$ such that

$$\|f(a) - L(a)\| \leq \frac{1}{m^4(1 - l_1)} \phi_1(a, 0, 0)$$

for all $a \in A$.

Using the fixed point method, we investigate another stability result of the quartic functional equation (1.2) associated with quartic Lie $*$-derivations.

Theorem 2.6. *Suppose that there exist a mapping $f : A \to M$ and two functions $\phi_1 : A^3 \to [0, \infty)$, $\phi_2 : A^2 \to [0, \infty)$ such that*

$$\|\Delta_\mu f(a, b) + f(c^*) - f(c)^*\| \leq \phi_1(a, b, c), \tag{2.15}$$

$$\|QDf(x, y)\| \leq \phi_2(x, y) \tag{2.16}$$

in which there are constants $l_i (i = 1, 2) \in (0, 1)$ satisfying

$$\begin{aligned} \phi_1(\frac{a}{m}, \frac{b}{m}, \frac{c}{m}) &\leq \frac{l_1}{m^4}\phi_1(a, b, c), \\ \phi_2(\frac{a}{m}, \frac{b}{m}) &\leq \frac{l_2}{m^8}\phi_2(a, b) \end{aligned} \tag{2.17}$$

for all $\mu \in \mathbb{T}^1_{\frac{1}{n_0}}$ and all $a, b, c \in A$. In addition, if for each fixed $a \in A$ the mapping $r \mapsto f(ra)$ from $\mathbb{R}$ to M is continuous, then there exists a unique quartic Lie $*$-derivation $L : A \to M$ such that

$$\|f(a) - L(a)\| \leq \frac{l_1}{m^4(1 - l_1)}\phi_1(a, 0, 0) \tag{2.18}$$

for all $a \in A$.

Proof. Above all, we note $f(0) = 0$ because of $\phi_1(0, 0, 0) = 0$ by (2.17). Now, consider the following function space

$$\Omega := \{g : A \to M, g(0) = 0\},$$

which is equipped with the generalized metric d on Ω as follows:

$$d(g, h) := \inf\{\lambda \in (0, \infty) : \|g(a) - h(a)\| \leq \lambda\phi_1(a, 0, 0) \ \forall a \in A\}.$$

Then, it is not difficult to prove that (Ω, d) is a complete generalized metric space. Additionally, we consider a mapping $T : \Omega \to \Omega$ defined as

$$T(g)(a) = m^4 g(\frac{a}{m}) \tag{2.19}$$

for all $a \in A$. Then, it follows that for any λ with $d(g, h) \leq \lambda$, where $g, h \in \Omega$,

$$\|g(a) - h(a)\| \leq \lambda\phi_1(a, 0, 0),$$

from which we get

$$\begin{aligned} \|T(g)(a) - T(h)(a)\| &= m^4\|g(\frac{a}{m}) - h(\frac{a}{m})\| \\ &\leq m^4\lambda\phi_1(\frac{a}{m}, 0, 0) \\ &\leq l_1\lambda\phi_1(a, 0, 0), \end{aligned}$$

which implies $d(Tg, Th) \leq l_1\lambda$. Since λ is arbitrary with $d(g, h) \leq \lambda$, one concludes that

$$d(Tg, Th) \leq l_1 d(g, h)$$

for all $g, h \in \Omega$. This means that T is a strictly contractive self-mapping on Ω with Lipschitz constant l_1. On taking $\mu = 1$ and $b = c := 0$ in the inequality (2.15), one obtains

$$\|f(a) - m^4 f(\frac{a}{m})\| \leq \phi_1(\frac{a}{m}, 0, 0) \leq \frac{l_1}{m^4}\phi_1(a, 0, 0)$$

for all $a \in A$, and so $d(f, Tf) \leq \frac{l_1}{m^4}$. Now, applying Theorem 2.4 to the function space (Ω, d), we know that there exists a fixed point L of T in Ω such that

$$L(a) = m^4 L(\frac{a}{m}), \text{ and } L(a) = \lim_{k\to\infty} m^{4k} f(\frac{a}{m^k}), \tag{2.20}$$
$$d(f, L) \leq \frac{1}{1-l_1} d(f, Tf) \leq \frac{l_1}{m^4(1-l_1)}$$

for all $a \in A$, from which we conclude that the mapping L satisfies the approximate inequality (2.18) near f, and it is unique up to $d(f, L) < \infty$.

On the other hand, replacing a by $\frac{a}{m^k}$, b by $\frac{b}{m^k}$ and $c = 0$ in the inequality (2.15), one has

$$m^{4k}\|\triangle_\mu f(\frac{a}{m^k}, \frac{b}{m^k})\| \leq m^{4k}\phi_1(\frac{a}{m^k}, \frac{b}{m^k}, 0) \leq l_1^k \phi_1(a, b, 0),$$

which tends to zero as $k \to \infty$, and so, we get $\triangle_\mu L(a, b) = 0$ for all $a, b \in A$ and all $\mu \in \mathbb{T}^1_{\frac{1}{n_0}}$. Thus, the mapping L is a quartic homogeneous by the same argument as in the proof of Theorem 2.2.

To prove that L is a quartic Lie $*$-derivation, replacing a by $\frac{a}{m^k}$ and b by $\frac{b}{m^k}$ in the inequality (2.16), one obtains

$$\begin{aligned}\|QDL(a, b)\| &= \lim_{k\to\infty} \|m^{8k} QDf(\frac{a}{m^k}, \frac{b}{m^k})\| \\ &\leq \lim_{k\to\infty} l_2^k \phi_2(a, b) = 0\end{aligned}$$

for all $a, b \in A$, which means that L is a quartic Lie derivation. At this time, by replacing $a = b := 0$ and $c := \frac{c}{m^k}$ in the inequality (2.5), we have

$$\begin{aligned}\|L(c^*) - L(c)^*\| &= \lim_{k\to\infty} \|m^{4k} f(\frac{c^*}{m^k}) - m^{4k} f(\frac{c}{m^k})^*\| \\ &\leq \lim_{k\to\infty} m^{4k}\phi_1(0, 0, \frac{c}{m^k}) \\ &\leq \lim_{k\to\infty} l_1^k \phi_1(0, 0, c) = 0\end{aligned}$$

for all $c \in A$. Therefore, the mapping L is a quartic Lie $*$-derivation satisfying the approximation (2.18). $\square$

Corollary 2.7. *Let $\theta_i (i = 1, 2)$, r be positive real numbers with $r > 4$. Suppose that a mapping $f : A \to M$ satisfies*

$$\begin{aligned} \|\Delta_\mu f(a, b) + f(c^*) - f(c)^*\| &\leq \theta_1(\|a\|^r + \|b\|^r + \|c\|^r), \\ \|QDf(a, b)\| &\leq \theta_2(\|a\|^{2r} + \|b\|^{2r}) \end{aligned}$$

for all $\mu \in \mathbb{T}^1_{\frac{1}{n_0}}$ and all $a, b, c \in A$. Then there exists a unique quartic Lie $*$-derivation $L : A \to M$ satisfying

$$\|f(a) - L(a)\| \leq \frac{\theta_1}{|m|^r - m^4}\|a\|^r$$

for all $a \in A$.

Proof. The proof follows from Theorem 2.6 by taking $\phi_1(a, b, c) = \theta_1(\|a\|^r + \|b\|^r + \|c\|^r)$ and $\phi_2(a, b) = \theta_2(\|a\|^{2r} + \|b\|^{2r})$ for all $a, b, c \in A$. □

Remark 2.8. Under the same conditions (2.17) of Theorem 2.6, we observe that

$$\begin{aligned} \Phi_1(a, b, c) &= \sum_{j=1}^{\infty} m^{4j}\phi_1\Big(\frac{a}{m^j}, \frac{b}{m^j}, \frac{c}{m^j}\Big) \\ &\leq \sum_{j=1}^{\infty} l_1^j \phi_1(a, b, c) \\ &= \frac{l_1}{1 - l_1}\phi_1(a, b, c) < \infty, \end{aligned}$$

and

$$\lim_{k\to\infty} m^{8k}\phi_2(\frac{a}{m^k}, \frac{b}{m^k}) \leq \lim_{k\to\infty} l_2^k \phi_2(a, b) = 0$$

for all $a, b, c \in A$. Thus, applying Theorem 2.2 to Theorem 2.6, we also get the desired stability result (2.18).

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References

[1] H. Azadi Kenary and A. Ghaffaripour, *Almost Jordan quartic homomorphisms and Jordan quartic derivations on fuzzy Banach algebras,* Nonlinear Funct. Anal. Appl., **21**, (3) (2016), 531–543.

[2] J.A. Baker, *The stability of certain functional equations,* Proc. Amer. Math. Soc. **112** (1991), 729–732.

[3] K. Ciepliński, *Applications of fixed point theorems to the Hyers–Ulam stability of functional equations-a survey,* Ann. Funct. Anal., **3** (2012), 151–164.

[4] J.K. Chung and P.K. Sahoo, *On the general solution of a quartic functional equation,* Bull. Korean Math. Soc., **40**(4) (2003), 565–576.

[5] S. Czerwik, *On the stability of the quadratic mapping in normed spaces,* Abh. Math. Sem. Univ. Hamburg, **62** (1992), 59–64.

[6] D.H. Hyers, *On the stability of the linear functional equation,* Proc. Nat. Acad. Sci. U.S.A. **27** (1941), 222–224.

[7] K.-W. Jun and H.-M. Kim, *Solution of Ulam stability problem for approximately bi-quadratic mappings and functional inequalities,* Contemporary Geometry and Topology and Related Topics, Cluj-Napoca, August 19-25, 2007, pages 16.

[8] D.-S. Kang and H.-J. Koh, *A fixed point approach to the stability of quartic Lie ∗-derivations,* Korean J. Math., **24**(4) (2016), 587–600.

[9] B. Khosravi, M.B. Moghimi and A. Najati, *Generalized Hyers–Ulam stability of a functional equation of Hosszú type,* Nonlinear Funct. Anal. Appl., **23**(1) (2018), 157–166.

[10] K. Kim, *A note on generalized ∗-derivations of prime ∗-rings,* J. Chungcheong Math. Soc., **30**(2) (2017), 251–258.

[11] K. Kim and Y. Lee, *Symmetric bi-(f,g)-derivations in lattices,* J. Chungcheong Math. Soc., **29**(3) (2016), 491–502.

[12] B. Margolis and J.B. Diaz, *A Fixed point theorem of the alternative for contrac-tions on a generalized complete metric space,* Bull. Amer. Math. Soc., **126**, 74 (1968), 305–309.

[13] V. Radu, *The fixed point alternative and the stability of functional equations,* Fixed Point Theory **4** (2003), 91–96.

[14] J.M. Rassias, *Solution of the Ulam stability problem for quartic mappings,* Glasnik Matematicki Series III, **34** no. 2 (1999) 243–252.

[15] I.A. Rus, *Principles and Applications of Fixed Point Theory,* Ed. Dacia, Cluj-Napoca, **1979** (in Romanian).

[16] P.K. Sahoo, *A generalized cubic functional equation, Acta Math. Sinica (English Series),* **21**(5) (2005), 1159–1166.

[17] S.M. Ulam, *Problems in Modern Mathematics,* Chap. VI, Wiley, New York, (1960).

[18] T.Z. Xu, J.M. Rassias and W.X. Xu, *A generalized mixed quadratic-quartic functional equation,* Bull. Malays. Math. Sci. Soc., **35**(3) (2012), 633–649.