Nonlinear Functional Analysis and Applications
Vol. 25, No. 1 (2020), pp. 1-12
ISSN: 1229-1595(print), 2466-0973(online)

https://doi.org/10.22771/nfaa.2020.25.01.01
http://nfaa.kyungnam.ac.kr/journal-nfaa

PROXIMITY POINTS FOR CYCLIC 2-CONVEX CONTRACTION MAPPINGS

M. S. Khan[1], M. Menaka[2], Geno Kadwin Jacob[3] and M. Marudai[4]

[1]College of Science, Department of Mathematics, Sultan Qaboos University
PoBox 36, PCode 123, Muscat, Sultanate of Oman
e-mail: `mohammad@squ.edu.om`

[2]Department of Mathematics, Bharathidasan University
Tiruchirappalli-620 024, Tamil Nadu, India
e-mail: `menakamurugesan1987@gmail.com`

[3]Department of Mathematics, Loyola College
Chennai 600 034, Tamil Nadu, India
e-mail: `genomaths@gmail.com`

[4]Department of Mathematics, Bharathidasan University
Tiruchirappalli-620 024, Tamil Nadu, India
e-mail: `mmarudai@yahoo.co.in`

Abstract. In this paper, the existence of proximity point for cyclic 2-convex contraction mappings, weakly cyclic 2-convex contraction mappings and M-weakly cyclic 2-convex contraction mappings are proved in the metric space setting. Our result is an natural generalization to result discussed in Istraescu [6].

1. Introduction

Let X be any set and $T : X \to X$ be a contraction. In 1922, Banach proved the following fixed point theorem of contraction mappings. It is assumed that X should be a complete metric space with metric d and $T : X \to X$ is required to be a contraction, that is, there must exist $L \in [0, 1)$ such that

[0]Received September 4, 2018. Revised October 5, 2019.
[0]2010 Mathematics Subject Classification: 54E70, 47H25.
[0]Keywords: Proximity point, cyclic 2-convex contraction mappings, weakly cyclic 2-convex contraction mappings and M-weakly cyclic 2-convex contraction mappings.
[0]Corresponding author: M. S. Khan(`mohammad@squ.edu.om`).

$d(f(x), f(y)) \leq Ld(x, y)$ for all $x, y \in X$. Then T has a unique fixed point in X. Thereafter many authors generalized this theorem. After several generalizations to contraction mapping, in 1982. Istraescu [6] introduced convex contraction mapping of order 2 as in the following definition and proved fixed point theorems for such mappings and a related class of mappings satisfying a convexity condition with respect to diameters of bounded sets.

2. Preliminaries

Definition 2.1. A continuous mapping $f : X \to X$ is said to be convex contraction mapping of order 2 if there exists the constants $a, b \in [0, 1)$ such that the following conditions hold:

(1) $a + b < 1$,
(2) $d(f^2(x), f^2(y)) \leq a\, d(f(x), f(y)) + b\, d(x, y)$ *for all* $x, y \in X$.

In 2003, Kirk et al. [5] introduced the concept of cyclic map on $\cup_{i=i}^{m} A_i$ as follows:

Definition 2.2. ([5]) Let A_i, $i = 1, 2, ...m$ be nonempty closed subsets of a metric space X. A map $T : \cup_{i=i}^{m} A_i \to \cup_{i=1}^{m} A_i$ is a cyclic map if T satisfies:

$$T(A_i) \subset A_{i+1} \text{ for } 1 \leq i \leq m-1 \text{ and } T(A_m) \subset A_1.$$

Let A, B are nonempty subsets of a set X and T be a cyclic map on $A \cup B$. For each $x \in X$, define

$$d(x, A) = \inf_{y \in A} d(x, y)$$

and

$$d(A, B) = \inf_{x \in A} d(x, B).$$

A point $x \in A$ is said to be proximity point of T if it satisfies $d(x, T(x)) = d(A, B)$. Such results are discussed by Kirk et al. [5]. Recently, to prove the existence of proximity point for cyclic decreasing contraction, Chen [2] introduced the following:

Definition 2.3. ([2]) If $\lim_{k\to\infty} T^{n_k}(x)$ exists for some $x \in A \cup B$ and some subsequence $\{n_i\}_{i=1}^{\infty}$ of $\mathbb{N}$, and

$$d(T(\lim_{i\to\infty} T^{n_i}(x)), \lim_{i\to\infty} T^{n_i}(x)) \leq \lim_{n\to\infty} d(T^{n+1}(x), T^n(x)) \tag{2.1}$$

then T is said to satisfy the cyclic limiting contraction.

Definition 2.4. A subset A of a metric space X is said to be boundedly compact if each bounded sequence in A has a convergent subsequence.

Using these concepts, we now prove the existence of proximity point for cyclic 2-convex contraction. We introduce new concepts called weakly cyclic 2-convex contraction mapping and M-weakly cyclic 2-convex contraction mapping and obtain the existence of proximity point for these concepts. These are generalization of Istraescu [6].

3. Main Results

3.1. **Proximity Point for Cyclic 2-Convex Contraction Mappings.** In this section, we introduce the following definition which generalizes cyclic contraction mappings of Kirk et al. [5] and Istraescu [6]. We obtain proximity point for cyclic 2-convex contraction mappings.

Definition 3.1. Let A and B be nonempty subsets of a metric space (X, d) and $T : A \cup B \to A \cup B$ a continuous mapping. If T is cyclic and for any $x \in A \cup B$, there exists a nonnegative constants a, b with $a + b < 1$ such that

$$d(T^2(x), T^2(y)) \leq a\, d(T(x), T(y)) + b\, d(x, y) + (1 - a - b)d(A, B), \qquad (3.1)$$

then T is said to be cyclic 2-convex contraction.

Theorem 3.2. *Let A, B be two nonempty closed subsets of a complete metric space (X, d) and T be a cyclic 2-convex contraction on $A \cup B$. Then for any $x_0 \in A \cup B$ the sequence $d(T^n(x_0), T^{n+1}(x_0))$ converges to $d(A, B)$.*

Proof. Let $x_0 \in A \cup B$ be arbitrary. Define $x_n = T^n(x_0)$ and let $k = \max\{d(x_2, x_1),\, d(x_1, x_0)\}$. Since T is cyclic 2-convex contraction on $A \cup B$,

$$\begin{aligned} d(x_3, x_2) &\leq a\, d(x_2, x_1) + b\, d(x_1, x_0) + (1 - a - b)d(A, B) \\ &\leq (a + b)k + d(A, B), \end{aligned}$$

$$\begin{aligned} d(x_4, x_3) &\leq a\, d(x_3, x_2) + b\, d(x_2, x_1) + (1 - a - b)d(A, B) \\ &\leq a(a + b)k + bk + d(A, B) \\ &\leq (a + b)k + d(A, B), \end{aligned}$$

$$\begin{aligned} d(x_5, x_4) &\leq a\, d(x_4, x_3) + b\, d(x_3, x_2) + (1 - a - b)d(A, B) \\ &\leq a\big[(a + b)k + d(A, B)\big] + b\big[(a + b)k + d(A, B)\big] \\ &\quad + (1 - a - b)d(A, B) \\ &\leq a(a + b)k + b(a + b)k + d(A, B) \\ &\leq (a + b)^2 k + d(A, B) \end{aligned}$$

and

$$\begin{aligned} d(x_6, x_5) &\leq a\, d(x_5, x_4) + b\, d(x_4, x_3) + (1-a-b)d(A,B) \\ &\leq a\big[(a+b)^2 k + d(A,B)\big] + b\big[(a+b)k + d(A,B)\big] \\ &\quad + (1-a-b)d(A,B) \\ &\leq a(a+b)^2 k + b(a+b)k + d(A,B) \\ &\leq (a+b)^2 k + d(A,B). \end{aligned}$$

By the induction principle, let us assume that the following hold.

$$d(x_{2m-1}, x_{2m-2}) \leq (a+b)^{m-1} k + d(A,B)$$

and

$$d(x_{2m}, x_{2m-1}) \leq (a+b)^{m-1} k + d(A,B).$$

Therefore,

$$\begin{aligned} d(x_{2m+1}, x_{2m}) &\leq a\, d(x_{2m}, x_{2m-1}) + b\, d(x_{2m-1}, x_{2m-2}) + (1-a-b)d(A,B) \\ &\leq a\big[(a+b)^{m-1} k + d(A,B)\big] + b\big[(a+b)^{m-1} k + d(A,B)\big] \\ &\quad + (1-a-b)d(A,B) \\ &\leq a(a+b)^{m-1} k + b(a+b)^{m-1} k + d(A,B) \\ &= (a+b)^m k + d(A,B) \end{aligned}$$

and

$$\begin{aligned} d(x_{2m+2}, x_{2m+1}) &\leq a\, d(x_{2m+1}, x_{2m}) + b\, d(x_{2m}, x_{2m-1}) + (1-a-b)d(A,B) \\ &\leq a\big[(a+b)^m k + d(A,B)\big] + b\big[(a+b)^{m-1} k + d(A,B)\big] \\ &\quad + (1-a-b)d(A,B) \\ &\leq a(a+b)^{m-1} k + b(a+b)^{m-1} k + d(A,B) \\ &= (a+b)^m k + d(A,B) \\ &\to d(A,B) \text{ (as } m \to \infty). \end{aligned}$$

Hence

$$\lim_{m\to\infty} d(x_{2m+1}, x_{2m}) \leq d(A,B).$$

But

$$\lim_{m\to\infty} d(x_{2m+1}, x_{2m}) \geq d(A,B).$$

Let $n = 2m$. Then $d(x_{n+1}, x_n) \to d(A,B)$. This completes the proof. □

Theorem 3.3. *Let (X, d) be a complete metric space, A and B be nonempty closed subsets of X. Let $T : A \cup B \to A \cup B$ be a cyclic 2-convex contraction. If for some $x_0 \in A \cup B$ and subsequence $\{n_i\}_{i=1}^{\infty}$ on $\mathbb{N}$,*

$$p = \lim_{i\to\infty} T^{n_i}(x_0),$$

then p is a proximity point of T.

Proof. Suppose $p = \lim_{i\to\infty} T^{n_i}(x_0)$. Since T is continuous and since d is jointly continuous, we have

$$d(p, T(p)) = \lim_{n\to\infty} d(T^n(x_0), T^{n+1}(x_0)).$$

Since T is a cyclic 2-convex contraction, by Theorem 3.2

$$\lim_{n\to\infty} d(T^n(x_0), T^{n+1}(x_0)) = d(A, B)$$

and hence it follows that

$$d(p, T(p)) = d(A, B). \tag{3.2}$$

Hence p is a proximity point of T. □

Lemma 3.4. *Let A and B be two nonempty closed subsets of a complete metric space X and let $T : A \cup B \to A \cup B$ be a cyclic 2-convex contraction. Then for $x_0 \in A \cup B$, and $x_n = T^n x_0$ the sequence $\{x_{2n}\}$ is bounded.*

Proof. Without loss of generality, let $x_0 \in A$. Suppose that $\{x_{2n}\}$ is not bounded. Then there exists positive integer N_0 such that

$$d(T^3x_0, T^{2N_0+2}) > M \text{ and } \ d(T^3x_0, T^{2N_0}) \le M.$$

where

$$M > \max\left\{\frac{(a+b)^2 d(Tx_0, T^3x_0)}{1-(a+b)^2} + \frac{abd(x_0, T^2x_0)}{1-(a+b)^2} + d(A, B), d(T^2x_0, T^3x_0)\right\}.$$

Now

$$\begin{aligned}
M &< d(T^3x_0, T^{2N_0+2}x_0)\\
&\le a\, d(T^2x_0, T^{2N_0+1x_0}) + bd(Tx_0, T^{2N_0}x_0) + (1-(a+b))d(A, B)\\
&\le a\{ad(Tx_0, T^{2N_0}x_0) + bd(x_0, T^{2N_0-1}x_0) + (1-(a+b))d(A, B)\}\\
&\quad + bd(Tx_0, T^{2N_0}x_0) + (1-(a+b))d(A, B)\\
&= (a^2+b)d(Tx_0, T^{2N_0}x_0) + abd(x_0, T^{2N_0-1}x_0) + (1+a)(1-(a+b))d(A, B)\\
&\le (a+b)^2 d(Tx_0, T^{2N_0}x_0) + abd(x_0, T^{2N_0-1}x_0) + (1-(a+b)^2)d(A, B).
\end{aligned}$$

Hence

$$\begin{aligned}\frac{M - abd(x_0, T^{2N_0-1}x_0) - d(A,B)}{(a+b)^2} + d(A,B) &< d(Tx_0, T^{2N_0}x_0)\\ &\leq d(Tx_0, T^3x_0)\\ &\quad + d(T^3x_0, T^{2N_0}x_0)\\ &\leq d(Tx_0, T^3x_0) + M.\end{aligned}$$

Therefore

$$M - abd(x_0, T^{2N_0-1}x_0) - d(A,B) + (a+b)^2 d(A,B) \leq (a+b)^2(d(Tx_0, T^3x_0) + M)$$

and

$$M < \frac{(a+b)^2 d(Tx_0, T^3x_0)}{1-(a+b)^2} + \frac{abd(x_0, T^2x_0)}{1-(a+b)^2} + d(A,B).$$

This is a contradiction. Hence $\{x_{2n}\}$ is bounded. □

Theorem 3.5. *Let (X,d) be a complete metric space and A, B be nonempty closed subsets of (X,d). Suppose $T : A \cup B \to A \cup B$ is a cyclic 2-convex contraction map. If A or B is boundedly compact then there exists $p_0 \in A \cup B$ which is a proximity point of T.*

Proof. Without loss of generality, let $x_0 \in A$ and A is boundedly compact. By Lemma 3.4 $\{x_{2n}\}$ is bounded in A and hence $\{x_{2n}\}$ has a convergent subsequence say $\{x_{2n_k}\}$. Thus there exists $p_0 \in A$ such that $x_{2n_k} \to p_0$ as $k \to \infty$. Therefore, by Theorem 3.3, p_0 is a best proximity point of T. □

Corollary 3.6. *Let (X,d) be a complete metric space and A, B be nonempty subsets of (X,d) such that $A \cap B \neq \emptyset$. Suppose $T : A \cup B \to A \cup B$ is a cyclic 2-convex contraction map. Then $p = \lim_{n\to\infty} T^n x$, is a fixed point of T.*

NOTE: The Theorem still holds when $A = B$.

3.2. **Proximity Point for Weakly Cyclic 2-Convex Contraction Mappings.** In this section, we introduce the following definition which generalizes cyclic contraction mappings of Kirk et al. [5] and Vasile I. Istraescu [6]. We obtain proximity point for weakly cyclic 2-convex contraction mappings.

Definition 3.7. Let A and B be nonempty subsets of a metric space (X,d) and $T : A \cup B \to A \cup B$ a mapping. If T is cyclic and for any $x \in A \cup B$, there exists a nonnegative constants a, b with $a + b < 1$ such that

$$d(T^2(x), T^2(y)) \leq a\, d(T(x), T(y)) + b\, d(x,y) + (1-a-b)d(A,B), \tag{3.3}$$

then T is said to be weakly cyclic 2-convex contraction.

Note that a continuous weakly cyclic 2-convex contraction is a cyclic 2-convex contraction.

Theorem 3.8. *Let A and B be two nonempty closed subsets of a complete metric space (X, d) and T be a weakly cyclic 2-convex contraction on $A \cup B$. Then for any $x_0 \in A \cup B$ the sequence $d(T^n(x_0), T^{n+1}(x_0))$ converges to $d(A, B)$.*

Proof. Let $x_0 \in A \cup B$ be arbitrary. Define $x_n = T^n(x_0)$ and let $k = \max\{d(x_2, x_1),\, d(x_1, x_0)\}$. Since T is weakly cyclic 2-convex contraction on $A \cup B$,

$$\begin{aligned} d(x_3, x_2) &\leq a\, d(x_2, x_1) + b\, d(x_1, x_0) + (1 - a - b) d(A, B) \\ &\leq (a + b)k + d(A, B), \end{aligned}$$

$$\begin{aligned} d(x_4, x_3) &\leq a\, d(x_3, x_2) + b\, d(x_2, x_1) + (1 - a - b) d(A, B) \\ &\leq a(a + b)k + bk + d(A, B) \\ &\leq (a + b)k + d(A, B), \end{aligned}$$

$$\begin{aligned} d(x_5, x_4) &\leq a\, d(x_4, x_3) + b\, d(x_3, x_2) + (1 - a - b) d(A, B) \\ &\leq a\big[(a + b)k + d(A, B)\big] + b\big[(a + b)k + d(A, B)\big] \\ &\quad + (1 - a - b) d(A, B) \\ &\leq a(a + b)k + b(a + b)k + d(A, B) \\ &\leq (a + b)^2 k + d(A, B) \end{aligned}$$

and

$$\begin{aligned} d(x_6, x_5) &\leq a\, d(x_5, x_4) + b\, d(x_4, x_3) + (1 - a - b) d(A, B) \\ &\leq a\big[(a + b)^2 k + d(A, B)\big] + b\big[(a + b)k + d(A, B)\big] \\ &\quad + (1 - a - b) d(A, B) \\ &\leq a(a + b)^2 k + b(a + b)k + d(A, B) \\ &\leq (a + b)^2 k + d(A, B). \end{aligned}$$

By the induction principle, lets us assume that the following holds.

$$d(x_{2m-1}, x_{2m-2}) \leq (a + b)^{m-1} k + d(A, B)$$

and

$$d(x_{2m}, x_{2m-1}) \leq (a + b)^{m-1} k + d(A, B).$$

Therefore,

$$\begin{aligned} d(x_{2m+1}, x_{2m}) &\leq a\, d(x_{2m}, x_{2m-1}) + b\, d(x_{2m-1}, x_{2m-2}) + (1-a-b)d(A,B) \\ &\leq a\big[(a+b)^{m-1}k + d(A,B)\big] + b\big[(a+b)^{m-1}k + d(A,B)\big] \\ &\quad + (1-a-b)d(A,B) \\ &\leq a(a+b)^{m-1}k + b(a+b)^{m-1}k + d(A,B) \\ &= (a+b)^m k + d(A,B) \end{aligned}$$

and

$$\begin{aligned} d(x_{2m+2}, x_{2m+1}) &\leq a\, d(x_{2m+1}, x_{2m}) + b\, d(x_{2m}, x_{2m-1}) + (1-a-b)d(A,B) \\ &\leq a\big[(a+b)^{m}k + d(A,B)\big] + b\big[(a+b)^{m-1}k + d(A,B)\big] \\ &\quad + (1-a-b)d(A,B) \\ &\leq a(a+b)^{m-1}k + b(a+b)^{m-1}k + d(A,B) \\ &= (a+b)^m k + d(A,B). \end{aligned}$$

Since $a+b<1$, as $m \to \infty$

$$d(x_{2m+1}, x_{2m}) \to d(A,B).$$

This completes the proof. □

Theorem 3.9. *Let (X,d) be a metric space, A and B be nonempty closed subsets of X. Let $T : A\cup B \to A\cup B$ satisfies*

(1) *cyclic 2-convex contraction and*
(2) *cyclic limiting contraction.*

If for some $x_0 \in A\cup B$ and subsequence $\{n_i\}_{i=1}^{\infty}$ on $\mathbb{N}$, $p = \lim_{i\to\infty} T^{n_i}(x_0)$, then p is a proximity point of T.

Proof. Since T satisfies cyclic limiting contraction, we have

$$d(p, T(p)) \leq \lim_{n\to\infty} d(T^n(x_0), T^{n+1}(x_0)).$$

Since T satisfies a cyclic 2-convex contraction, $\lim_{n\to\infty} d(T^n(x_0), T^{n+1}(x_0)) = d(A,B)$ by Theorem 3.8 and hence it follows that

$$d(p, T(p)) \leq d(A,B). \tag{3.4}$$

Moreover, since $p \in A\cup B$

$$d(p, T(p)) \geq d(A,B). \tag{3.5}$$

By equations (3.4) and (3.5), we have p is a proximity point of T. □

Theorem 3.10. *Let (X, d) be a complete metric space and A, B be nonempty closed subsets of (X, d). Suppose $T : A \cup B \to A \cup B$ is a weakly cyclic 2-convex contraction map and cyclic limiting contraction. If A or B is boundedly compact then there exists $p_0 \in A \cup B$ which is a proximity point of T.*

Proof. Without loss of generality, let $x_0 \in A$ and A is boundedly compact. By Lemma 3.4 $\{x_{2n}\}$ is bounded in A and hence $\{x_{2n}\}$ has a convergent subsequence say $\{x_{2n_k}\}$. Thus there exists $p_0 \in A$ such that $x_{2n_k} \to p_0$ as $k \to \infty$. Therefore, by Theorem 3.9, p_0 is a best proximity point of T. □

Corollary 3.11. *Let (X, d) be a complete metric space and A, B be nonempty closed subsets of (X, d) such that $A \cap B \neq \emptyset$. Suppose $T : A \cup B \to A \cup B$ is a cyclic 2-convex contraction map. Then $p = \lim_{n\to\infty} T^n x$, is a fixed point of T.*

NOTE: The Theorem still holds when $A = B$.

3.3. **Proximity Point for M-Weakly Cyclic 2-Convex Contraction Mappings.** In this section, we introduce the following definition which generalizes cyclic contraction mappings of Kirk et al. [5] and Istraescu [6]. We obtain proximity point for M-Weakly cyclic 2-convex contraction mappings.

Definition 3.12. Let A and B be nonempty subsets of a metric space (X, d) and $T : A \cup B \to A \cup B$ be a continuous mapping. T is said to be M-weakly cyclic 2-convex contraction if T is cyclic and for any $x, y \in A \cup B$, there exists a nonnegative constants a, b, c with $2a + b + 2c < 1$ such that

$$\begin{aligned} d(T^2(x), T^2(y)) \;\leq\; & a\,[d(x, T(x)) + d(y, T(y))] + b\,d(x, y) \\ & + c[d(x, T(y)) + d(y, T(x))] \\ & + (1 - (2a + b + 2c))d(A, B). \end{aligned} \tag{3.6}$$

Theorem 3.13. *Let A, B be two nonempty closed subsets of a complete metric space (X, d) and T be a M-weakly cyclic 2-convex contraction on $A \cup B$. Then for any $x_0 \in A \cup B$ the sequence $d(T^n(x_0), T^{n+1}(x_0))$ converges to $d(A, B)$.*

Proof. Let $x_0 \in A \cup B$ be arbitrary. Define $x_n = T^n(x_0)$ and let $k = \max\{d(x_2, x_1),\, d(x_1, x_0)\}$. Since T is M-weakly cyclic 2-convex contraction on $A \cup B$,

$$\begin{aligned} d(x_3, x_2) &\leq a[d(x_0, x_1) + d(x_1, x_2)] + b\,d(x_1, x_0) + c[d(x_0, x_2) + d(x_1, x_1)] \\ &\quad + (1 - (2a + b + 2c))d(A, B) \\ &\leq (a + c)d(x_1, x_2) + (a + b + c)d(x_0, x_1) + (1 - (2a + b + 2c))d(A, B) \\ &\leq (2a + b + 2c)k + d(A, B), \end{aligned}$$

$$\begin{aligned}
d(x_4, x_3) &\leq (a+c)\, d(x_3, x_2) + (a+b+c) d(x_2, x_1) \\
&\quad + (1 - (2a+b+2c)) d(A, B) \\
&\leq (a+c)[(2a+b+2c)k + d(A, B)] + (a+b+c)k \\
&\quad + (1 - (2a+b+2c)) d(A, B) \\
&\leq (a+c)k + (a+c) d(A, B) + (a+b+c)k \\
&\quad + (1 - (2a+b+2c)) d(A, B) \\
&\leq (2a+b+2c)k + d(A, B),
\end{aligned}$$

$$\begin{aligned}
d(x_5, x_4) &\leq (a+c) d(x_4, x_3) + (a+b+c)\, d(x_3, x_2) \\
&\quad + (1 - (2a+b+2c)) d(A, B) \\
&\leq (a+c)\big[(2a+b+2c)k + d(A, B)\big] \\
&\quad + (a+b+c)\big[(2a+b+2c)k + d(A, B)\big] \\
&\quad + (1 - (2a+b+2c)) d(A, B) \\
&= (2a+b+2c)^2 k + d(A, B)
\end{aligned}$$

and

$$\begin{aligned}
d(x_6, x_5) &\leq (a+c) d(x_5, x_4) + (a+b+c) d(x_4, x_3) \\
&\quad + (1 - (2a+b+2c)) d(A, B) \\
&\leq (a+c)\big[(2a+b+2c)^2 k + d(A, B)\big] \\
&\quad + (a+b+c)\big[(2a+b+2c)k + d(A, B)\big] \\
&\quad + (2a+b+2c)k + d(A, B) \\
&\leq (a+c)\big[(2a+b+2c)k + d(A, B)\big] \\
&\quad + (a+b+c)\big[(2a+b+2c)k + d(A, B)\big] \\
&\quad + (2a+b+2c)k + d(A, B) \\
&= (2a+b+2c)^2 k + d(A, B).
\end{aligned}$$

By the induction principle, lets us assume that the following hold.

$$d(x_{2m-1}, x_{2m-2}) \leq (2a+b+2c)^{m-1} k + d(A, B)$$

and

$$d(x_{2m}, x_{2m-1}) \leq (2a+b+2c)^{m-1} k + d(A, B).$$

Therefore,

$$\begin{aligned}
d(x_{2m+1}, x_{2m}) &\leq (a+c)d(x_{2m}, x_{2m-1}) + (a+b+c)d(x_{2m-1}, x_{2m-2}) \\
&\quad + (1-(2a+b+2c))d(A,B) \\
&\leq (a+c)\big[(2a+b+2c)^{m-1}k + d(A,B)\big] \\
&\quad + (a+b+c)\big[(2a+b+2c)^{m-1}k + d(A,B)\big] \\
&\quad + (1-(2a+b+2c))d(A,B) \\
&= (a+c)(2a+b+2c)^{m-1}k + (a+b+c)(2a+b+2c)^{m-1}k \\
&\quad + d(A,B) \\
&= (2a+b+2c)^m k + d(A,B)
\end{aligned}$$

and

$$\begin{aligned}
d(x_{2m+2}, x_{2m+1}) &\leq (a+c)d(x_{2m+1}, x_{2m}) + (a+b+c)d(x_{2m}, x_{2m-1}) \\
&\quad + (1-(2a+b+2c))d(A,B) \\
&\leq (a+c)\big[(2a+b+2c)^{m}k + d(A,B)\big] \\
&\quad + (a+b+c)\big[(2a+b+2c)^{m-1}k + d(A,B)\big] \\
&\quad + (1-(2a+b+2c))d(A,B) \\
&\leq (a+c)(2a+b+2c)^{m-1}k + (a+b+c)(2a+b+2c)^{m-1}k \\
&\quad + d(A,B) \\
&= (2a+b+2c)^m k + d(A,B).
\end{aligned}$$

Since $2a+b+2c<1$,

$$\lim_{m\to\infty} d(x_{2m+1}, x_{2m}) \leq d(A,B).$$

But

$$\lim_{n\to\infty} d(x_{2m+1}, x_{2m}) \geq d(A,B).$$

Let $n=2m$. Then $\lim\limits_{n\to\infty} d(x_{n+1}, x_n) = d(A,B)$. □

Theorem 3.14. *Let (X,d) be a metric space, A, B be nonempty closed subsets of X. Let $T : A\cup B \to A\cup B$ be a M-weakly cyclic 2-convex contraction. If for some $x_0 \in A\cup B$ and subsequence $\{n_i\}_{i=1}^{\infty}$ on $\mathbb{N}$, $p = \lim\limits_{i\to\infty} T^{n_i}(x_0)$, then p is a proximity point of T.*

Proof. Since T is continuous and d is jointly continuous, we have

$$d(p, T(p)) = \lim_{n\to\infty} d(T^n(x_0), T^{n+1}(x_0)).$$

Since T is M-weakly cyclic 2-convex contraction,

$$\lim_{n\to\infty} d(T^n(x_0), T^{n+1}(x_0)) = d(A,B)$$

by Theorem 3.13 and hence it follows that

$$d(p, T(p)) = d(A, B). \tag{3.7}$$

Thus p is a proximity point of T. □

Theorem 3.15. *Let (X, d) be a complete metric space and A, B be nonempty closed subsets of (X, d). Suppose $T : A \cup B \to A \cup B$ is a M-weakly cyclic 2-convex contraction map. If A or B is boundedly compact then there exists $p_0 \in A \cup B$ which is a proximity point of T.*

Proof. Without loss of generality, let $x_0 \in A$ and A is boundedly compact. By Lemma 3.4 $\{x_{2n}\}$ is bounded in A and hence $\{x_{2n}\}$ has a convergent subsequence say $\{x_{2n_k}\}$. Thus there exists $p_0 \in A$ $x_{2n_k} \to p_0$ as $k \to \infty$. Therefore, by Theorem 3.14, p_0 is a best proximity point of T. □

Corollary 3.16. *Let (X, d) be a complete metric space and A, B be nonempty subsets of (X, d) such that $A \cap B \neq \emptyset$. Suppose $T : A \cup B \to A \cup B$ is a cyclic 2-convex contraction map. Then $p = \lim_{n\to\infty} T^n x$, is a fixed point of T.*

NOTE: The Theorem still holds when $A = B$.

References

[1] A. Antony Eldred and P. Veeramani, *Existence and convergence of best proximity points,* J, Math. Anal. Appl., **323**(2) (2006), 79-89.

[2] Y.C. Chen, *Existence Results of Best Proximity Points for Cyclic Contractions in Metric Space,* Inter. J. Math. Anal., **11**(4) (2017), 163-172.

[3] G.K. Jacob, A.H. Ansari, C. Park and N. Annamalai, *Common fixed point results for weakly compatible mappings using $C-$class functions*, J. Comput. Anal. Appl., **25**(1) (2018), 184-194.

[4] G.K. Jacob, M. Marudai, *Fixed point and best proximity point results for generalized cyclic coupled mappings*, Thai J. Math., **14**(2) (2016), 431-441.

[5] W.A. Kirk, P.S. Srinavasan and P. Veeramani, *Fixed points for mapping satisfying cyclical contractive conditions*, Fixed Point Theory, **4**(1) (2003), 79-89.

[6] V.I. Istraescu, *Some fixed point theorems for convex contraction mappings and mappings with convex diminishing diameters-I*, Ann. Mat. Pura Appl., **130** (1982), 89-104.