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ON Γ-RECURRENT C_0-SEMIGROUPS AND THEIR PROPERTIES

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Abstract. This paper introduces the concept of Γ-recurrent C_0-semigroups and provides various illustrative examples. We demonstrate that Γ-recurrence is preserved under inverse. Surprisingly, we establish the existence of Γ-recurrent C_0-semigroups on finite-dimensional Banach spaces. Additionally, we explore the notion of Γ-recurrent vectors and show that if a C_0-semigroup possesses a Γ-recurrent vector, it necessarily has many such vectors. Finally, we prove that when Γ is closed under multiplication, the Γ-recurrence of a C_0-semigroup is equivalent to having a dense set of Γ-recurrent vectors.

1. Introduction

Presume X is a Banach space. Assume T is an operator on X. In the theory of dynamical systems, there are various types of operators. An operator T on X is recurrent if, for any nonempty open set U of X, there exists $n \in \mathbb{N}$ such that $T^n(U) \cap U \neq \phi$ [13, Definition 1.1]. Authors widely investigate the properties of recurrent operators in [13]. Furthermore, [15] determines conditions for the recurrence of composition operators. One can also see [17] and [10].

An operator T on X is supercyclic if for any nonempty open sets U and V of X, there exist $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}$ such that $\lambda T^n(U) \cap V \neq \phi$ [14]. T is a power-bounded operator if, there exists $K > 0$ such that for any $n \in \mathbb{N}$,

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$\|T^n\| \leq K$. A recurrent operator, that is, power bounded is not supercyclic [13, Proposition 3.2].

Super-recurrent operators are defined in [5]. An operator T on X is super-recurrent if for any nonempty open set U of X, there exists $n \in \mathbb{N}$ and $\lambda \in \mathbb{C}$ such that $\lambda T^n(U) \cap U \neq \phi$. Presume $x \in X$ is a nonzero vector. If a strictly increasing sequence (k_n) of positive integers exists, and $(\alpha_{k_n}) \subseteq \mathbb{C}$ such that $\alpha_{k_n} T^{k_n} x \to x$, when n tends to infinity, then x is named a suer-recurrent vector for T [5]. It is proved in [5] that an operator T on X is super-recurrent if and only if it has a dense set of super-recurrent vectors in X. Proerties of Γ-suercyclic operators are also investigated in [7].

Let us assume that Γ is a nonempty subset of complex numbers. An operator T on X is Γ-supercyclic if $\{\lambda T^n x : \lambda \in \Gamma, n \in \mathbb{N}_0\}$ is dense in X for some $x \in X$ [12]. There are some conditions in [12] that under them Γ-supercyclicity and supercyclicity are equivalent. Proerties of Γ-suercyclic operators are also investigated in [7]. Moreover, Γ-supercyclicity has been studied for a particular case of Γ. As example, when $\Gamma = \{1\}$, and $\Gamma = \mathbb{D}$, where $\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$, one can see respectively, [3] and [4].

C_0-semigroup is another structure that is of interest in dynamical systems theory. A family of operators $(T_t)_{t\geq 0}$ on X is a C_0-semigroup [14] if

(1) $T_0 = I$,
(2) $T_{p+q} = T_p T_q$ for any $p, q \geq 0$,
(3) $\lim_{p\to q} T_p x = T_q x$ for any $x \in X$.

The concept of recurrent is defined in [16] for C_0-semigroups. A C_0-semigroup $(T_t)_{t\geq 0}$ on X is recurrent if, for any nonempty open set U of X, there exists $t_0 > 0$ such that $T_{t_0}(U) \cap U \neq \phi$. Recurrent C_0-semigroups exist in both finite and infinite-dimensional spaces [16]. Authors in [9] extended recurrent to a set of operators and investigated the recurrence of their direct sum.

A C_0-semigroup $(T_t)_{t\geq 0}$ on X is supercyclic if there exists $x \in X$ such that $\{\lambda T_t x : \lambda \in \mathbb{C}, t \geq 0\}$ is dense in X [14]. A C_0-semigroup $(T_t)_{t\geq 0}$ on X is Γ-supercyclic if $x \in X$ exists such that $\overline{\{\gamma T_t x : \gamma \in \Gamma, t \geq 0\}} = X$ [1]. By definition, Γ-supercyclic C_0-semigroups are supercyclic. Furthermore, $(T_t)_{t\geq 0}$ is Γ-supercyclic if and only if for any nonempty open sets U and V of X, $\gamma \in \Gamma$ and $t_0 > 0$ exist such that $\gamma T_{t_0}(U) \cap V \neq \phi$ [1, Proposition 3.6].

In this paper, we extend this idea to C_0-semigroups and define Γ-recurrent C_0-semigroups.

Section 2 defines Γ-recurrent C_0-semigroups. It proves some equivalent conditions for Γ-recurrent. Furthermore, it proves that if an invertible C_0-semigroup is Γ-recurrent, its inverse is Γ-recurrent and vice versa. Section

3 introduces Γ-recurrent vectors and investigates their properties. It demonstrates that if Γ is closed under multiplication, then having a dense set of Γ-recurrent vectors is equivalent to the Γ-recurrent of the C_0-semigroup. Section 4, presents various examples of Γ-recurrent C_0-semigroups. These examples show that Γ-recurrent operators can appear on finite-dimensional Banach spaces.

2. Definitions and some results

First, we present the definition of the Γ-recurrent C_0-semigroups in this section as follows.

Definition 2.1. A C_0-semigroup $(T_t)_{t\geq 0}$ on X is called Γ-recurrent, if for any nonempty open set U of X, $\lambda \in \Gamma$, where $\lambda \neq 0$, and $t_0 > 0$ exist so that

$$\lambda T_{t_0}(U) \cap U \neq \phi.$$

The following lemma is concluded if in the Definition 2.1, $1 \in \Gamma$.

Lemma 2.2. *A recurrent C_0-semigroup is Γ-recurrent if $1 \in \Gamma$.*

Without any condition on Γ, the next lemma shows that Γ-supercyclic C_0-semigroups are Γ-recurrent.

Lemma 2.3. *If $(T_t)_{t\geq 0}$ is an Γ-supercyclic C_0-semigroup, then it is Γ-recurrent.*

The following proposition states an equivalent condition for Γ-recurrent.

Proposition 2.4. *A C_0-semigroup $(T_t)_{t\geq 0}$ on X is Γ-recurrent if and only if there exists $0 \neq \alpha \in \Gamma$ and $t > 0$ such that $T_t^{-1}(U) \cap \alpha U \neq \phi$.*

Proof. The assertion is deduced from this fact that for $0 \neq \alpha \in \Gamma$ and $t > 0$, $\alpha T_t(U) \cap U \neq \phi$ if and only if $\alpha U \cap T_t^{-1}(U) \neq \phi$. □

An important question is that does Γ-recurrent preserve under inverse? The next theorem answers to this question. Note that we say Γ is closed under inverse if when $\alpha \in \Gamma$ is invertible, then $\alpha^{-1} \in \Gamma$.

Theorem 2.5. *Assume $(T_t)_{t\geq 0}$ is an invertible C_0-semigroup on X. If Γ is closed under inverse, then $(T_t)_{t\geq 0}$ is Γ-recurrent if and only if $(T_t^{-1})_{t\geq 0}$ is Γ-recurrent.*

Proof. Let $(T_t)_{t\geq 0}$ be an invertible and Γ-recurrent C_0-semigroup. Suppose U is a nonempty open subset of X. So, there exist $0 \neq \alpha \in \Gamma$ and $t_0 > 0$ such that $\alpha T_{t_0}(U) \cap U \neq \phi$. Hence, $U \cap \alpha^{-1} T_{t_0}^{-1}(U) \neq \phi$. Now, $\alpha^{-1} \in \Gamma$ because Γ is closed under inverse. So, $(T_t^{-1})_{t\geq 0}$ is Γ-recurrent. The converse can be proved similarly. □

Theorem 2.5 states a sufficient condition for Γ-recurrent of the inverse of an invertible C_0-semigroup. Does the assertion remain true when Γ is not closed under inverse? Does an example of an invertible Γ-recurrent C_0-semigroup $(T_t)_{t\geq 0}$ exist such that Γ is not closed under inverse and $(T_t^{-1})_{t\geq 0}$ is Γ-recurrent?

Proposition 2.6. *Assume $(T_t)_{t\geq 0}$ is a C_0-semigroup on X, and Γ is closed under multiplication. Then, if $(T_t)_{t\geq 0}$ is Γ-recurrent, there exists an infinite set of positive real numbers t so that $\alpha U \cap T_t^{-1}(U) \neq \phi$, where $\alpha \in \Gamma$ and is not unique.*

Proof. Presume $U \subseteq X$ is nonempty and open. So, $\alpha \in \Gamma$ and there exists $t_0 > 0$ such that $\alpha T_{t_0}(U) \cap U \neq \phi$ or equivalently $\alpha U \cap T_{t_0}^{-1}(U) \neq \phi$. Consider $V := \alpha U \cap T_{t_0}^{-1}(U)$. Hence, V is nonempty and open. Therefore, there exist $\lambda \in \Gamma$ and $s_0 > 0$ such that $\lambda V \cap T_{s_0}^{-1}(V) \neq \phi$. Hence,

$$\lambda(\alpha U \cap T_{t_0}^{-1}(U)) \cap T_{s_0}^{-1}(\alpha U \cap T_{t_0}^{-1}(U)) \neq \phi.$$

Consequently,

$$(\lambda\alpha)U \cap T_{s_0+t_0}^{-1}(U) \neq \phi.$$

If consider $t_1 := s_0 + t_0$, then $t_1 > t_0$ and there exists $\mu \in \Gamma$ such that $\mu U \cap T_{t_1}^{-1}(U) \neq \phi$. □

In the following, we define the notion of Γ-recurrent operators.

Definition 2.7. An operator T on a Banach space X is called Γ-recurrent, if for any nonempty open set U of X, there exists $\lambda \in \Gamma$, where $\lambda \neq 0$, and $n \in \mathbb{N}$ such that $\lambda T^n(U) \cap U \neq \phi$.

We prove that once a C_0-semigroup has an Γ-recurrent operator, it is an Γ-recurrent C_0-semigroup.

Theorem 2.8. *If a C_0-semigroup $(T_t)_{t\geq 0}$ contains an Γ-recurrent operator, then it is a Γ-recurrent C_0-semigroup.*

Proof. Assume T_s is an Γ-recurrent operator for some $s > 0$. Let U be a nonempty open subset of X. Hence, there exist $0 \neq \lambda \in \Gamma$ and $n \in \mathbb{N}$ such that $\lambda T_s^n(U) \cap U \neq \phi$. By properties of a C_0-semigroup, $T_s^n = T_{sn}$. Therefore,

$$\lambda T_{sn}(U) \cap U \neq \phi.$$

This means $(T_t)_{t\geq 0}$ is Γ-recurrent. □

Now it is natural to ask does the converse of Theorem 2.8 hold?

This section ends by stating two equivalent conditions for Γ-recurrent.

Theorem 2.9. *Assume $(T_t)_{t\geq 0}$ is a C_0-semigroup on X. Then the following are equivalent.*

(1) *$(T_t)_{t\geq 0}$ is Γ-recurrent.*
(2) *There exists a sequence (k_n) of positive integers which for any $x \in X$, and there exists $(t_{k_n}) \subseteq \Gamma$ and $(x_{k_n}) \subseteq X$ such that*
$$x_{k_n} \to x \quad \text{and} \quad \lambda_{k_n} T_{t_{k_n}}(x_{k_n}) \to x.$$
(3) *For any W-neighborhood of zero and any $x \in X$, there exist $z \in X$, $\lambda \in \Gamma$ and $t > 0$ such that*
$$z - x \in W \quad \text{and} \quad \lambda T_t z - x \in W.$$

Proof. (1) ⇒ (2). Suppose $x \in X$. Then $U_n = B(x, \frac{1}{n})$ for all positive integer n is an open set. So, there exist $k_n \in \mathbb{N}$ and $\lambda_{k_n} \in \Gamma$ such that $(\lambda_{k_n} U_n) \cap U_n \neq \phi$.

Let $x_{k_n} \in U_n$ such that $\lambda_{k_n} x_{k_n} \in U_n$. Therefore,
$$\|x_{k_n} - x\| < \frac{1}{n} \quad \text{and} \quad \|\lambda_{k_n} x_{k_n} - x\| < \frac{1}{n}.$$
Hence, $x_{k_n} \to x$ and $\lambda_{k_n} x_{k_n} \to x$.

(2) ⇒ (3). Let W be a neighborhood of zero and let $x \in X$. So, there exists $n \in \mathbb{N}$ such that $B(0, \frac{1}{n}) \subseteq W$. By hypothesis, $x_{k_n} \to x$ and $\lambda_{k_n} T_{t_{k_n}}(x_{k_n}) \to x$. Hence, $m \in \mathbb{N}$ can be found such that $\|x_{k_m} - x\| < \frac{1}{n}$ and $\|\lambda_{k_m} T_{t_{k_m}}(x_{k_m}) - x\| < \frac{1}{n}$. Therefore, $x_{k_m} - x \in W$ and $\lambda_{k_m} T_{t_{k_m}}(x_{k_m}) - x \in W$.

(3) ⇒ (1). Let U be a nonempty open subset of X and let $x \in U$. There exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq U$. Assume $W_n = B(0, \frac{1}{n})$. By hypothesis, for any $n \in \mathbb{N}$, there exist $k_n \in \mathbb{N}$, $0 \neq \lambda_{k_n} \in \Gamma$, and $z_n \in X$ such that
$$\|\lambda_{k_n} T_{t_{k_n}}(z_n) - x\| < \frac{1}{n} \quad \text{and} \quad \|z_n - x\| < \frac{1}{n}.$$
Suppose $n \in \mathbb{N}$ is such that $\frac{1}{n} < \varepsilon$. Hence, $z_n \in U$ and $\lambda_{k_n} T_{t_{k_n}}(z_n) \in U$. So, $\lambda_{k_n} T_{t_{k_n}}(U) \cap U \neq \phi$. □

3. Γ-recurrent vectors

This section begins with the definition of the Γ-recurrent vectors.

Definition 3.1. A vector $x \in X$ is a Γ-recurrent vector for $(T_t)_{t\geq 0}$, if a strictly increasing sequence (t_k) of positive real numbers and a sequence $(\lambda_{t_k}) \subseteq \Gamma$ exist such that $\lambda_{t_k} \neq 0$ and $\lambda_{t_k} T_{t_k} x \to x$. We signify the set of Γ-recurrent vectors of $(T_t)_{t\geq 0}$ by $\Gamma Rec(T_t)_{t\geq 0}$.

In accordance with Definition 3.1, the next theorem approves an equivalent condition for Γ-recurrent.

Theorem 3.2. *Assume $(T_t)_{t\geq 0}$ is a C_0-semigroup on X. If $\overline{\Gamma Rec(T_t)_{t\geq 0}} = X$, then $(T_t)_{t\geq 0}$ is Γ-recurrent. The converse is true when Γ is closed under multiplication.*

Proof. Suppose $\overline{\Gamma Rec(T_t)_{t\geq 0}} = X$. Assume U is a nonempty open subset of X. By hypothesis, there exists $x \in U \cap \Gamma Rec(T_t)_{t\geq 0}$. Therefore, $x \in U$ and $x \in \Gamma Rec(T_t)_{t\geq 0}$. Hence, there exists a strictly increasing sequence (t_k) of positive real numbers and a sequence $(\lambda_{t_k}) \subseteq \Gamma$ such that $\lambda_{t_k} \neq 0$ and

$$\lambda_{t_k} T_{t_k} x \to x. \tag{3.1}$$

Remember that U is open and $x \in U$. So, there exists $\varepsilon > 0$ such that

$$B(x, \varepsilon) \subseteq U. \tag{3.2}$$

By (3.1), there exists t_{k_0} such that

$$\|\lambda_{t_{k_0}} T_{t_{k_0}} x - x\| < \varepsilon.$$

Therefore, $\lambda_{t_{k_0}} T_{t_{k_0}} x \in B(x, \varepsilon)$. Then, $\lambda_{t_{k_0}} T_{t_{k_0}} x \in U$ by (3.2). Hence,

$$\lambda_{t_{k_0}} T_{t_{k_0}}(U) \cap U \neq \phi.$$

This means $(T_t)_{t\geq 0}$ is Γ-recurrent.

Now, suppose $(T_t)_{t\geq 0}$ is a Γ-recurrent C_0-semigroup and assume Γ is closed under multiplication. Let $x_0 \in X$. Consider $U_0 = B(x_0, \varepsilon)$. By Γ-recurrency of $(T_t)_{t\geq 0}$, there exist $\lambda_0 \in \Gamma$ and $t_0 > 0$ such that $(\lambda_0 U_0) \cap T_{t_0}^{-1}(U_0) \neq \phi$ or equivalently $U_0 \cap {\lambda_0}^{-1} T_{t_0}^{-1}(U_0) \neq \phi$.

Suppose $x_1 \in U_0 \cap ({\lambda_0}^{-1} T_{t_0}^{-1}(U_0))$. Consider $U_1 = B(x_1, \varepsilon_1)$ such that $\varepsilon_1 < \frac{1}{2}$ and

$$U_1 = B(x_1, \varepsilon_1) \subseteq U_0 \cap ({\lambda_0}^{-1} T_{t_0}^{-1}(U_0)). \tag{3.3}$$

Another by Γ-recurrency of $(T_t)_{t\geq 0}$ there exist $\lambda_1 \in \Gamma$ and $t_1 > t_0$ such that

$$U_1 \cap {\lambda_1}^{-1} T_{t_1}^{-1}(U_1) \neq \phi.$$

Now, suppose $x_2 \in U_1 \cap {\lambda_1}^{-1} T_{t_1}^{-1}(U_1)$. Consider $U_2 = B(x_2, \varepsilon_2)$ such that $\varepsilon_2 < \frac{1}{2^2}$ and

$$U_2 = B(x_2, \varepsilon_2) \subseteq U_1 \cap ({\lambda_1}^{-1} T_{t_1}^{-1}(U_1)). \tag{3.4}$$

Inductively, we can consider $U_n = B(x_n, \varepsilon_n)$ such that $\varepsilon_n < \frac{1}{2^n}$ and

$$U_n = B(x_n, \varepsilon_n) \subseteq U_{n-1} \cap ({\lambda_{n-1}}^{-1} T_{t_{n-1}}^{-1}(U_{n-1})). \tag{3.5}$$

Hence, by (3.5),

$$U_n \subseteq U_{n-1} \quad \text{and} \quad U_n \subseteq {\lambda_{n-1}}^{-1} T_{t_{n-1}}^{-1}(U_{n-1}).$$

Consequently,

$$U_n \subseteq U_{n-1} \quad \text{and} \quad \lambda_{n-1}T_{t_{n-1}}U_n \subseteq U_{n-1}. \tag{3.6}$$

By Cantor theorem and (3.6), there exists $z \in X$ such that $\cap_{n=1}^{\infty} U_n = \{z\}$. Hence,

$$\lambda_{n-1}T_{t_{n-1}}z \ \rightarrow \ z.$$

This means that z is a Γ-recurrent vector. □

In the following, some statements are proved, which show that once a C_0-semigroup has a Γ-recurrent vector, it has, except zero, an invariant subspace of them.

Theorem 3.3. *If x is a Γ-recurrent vector for $(T_t)_{t\geq 0}$, then λx is a Γ-recurrent vector for it for any $\lambda \in \mathbb{C}$ with $\lambda \neq 0$.*

Proof. Suppose x is a Γ-recurrent vector for $(T_t)_{t\geq 0}$. Then, a strictly increasing sequence (t_k) and $(\lambda_{t_k}) \subseteq \Gamma$ exist so that $\lambda_{t_k} \neq 0$ and

$$\lambda_{t_k}T_{t_k}xp \ \rightarrow \ x. \tag{3.7}$$

Now, let $\lambda \in \mathbb{C}$ and $\lambda \neq 0$. Hence, $\lambda_{t_k}T_{t_k}(\lambda x) = \lambda\lambda_{t_k}T_{t_k}x$. By (3.7),

$$\lambda\lambda_{t_k}T_{t_k}x \ \rightarrow \ \lambda x.$$

So, λx is a Γ-recurrent vector for $(T_t)_{t\geq 0}$. □

Theorem 3.4. *If x is a Γ-recurrent vector for $(T_t)_{t\geq 0}$, then $T_s^m x$ is a Γ-recurrent vector for it for any $s > 0$ and $m \in \mathbb{N}$.*

Proof. Suppose x is a Γ-recurrent vector for $(T_t)_{t\geq 0}$. Then, there exists a strictly increasing sequence (t_k) and $(\lambda_{t_k}) \subseteq \Gamma$ such that $\lambda_{t_k} \neq 0$ and

$$\lambda_{t_k}T_{t_k}x \ \rightarrow \ x. \tag{3.8}$$

Assume $m \in \mathbb{N}$ and $s > 0$. Then $\lambda_{t_k}T_{t_k}(T_s^m x) = T_s^m(\lambda_{t_k}T_{t_k}x)$ by definition of a C_0-semigroup. Also, from (3.8) and continuity of T_s^m it is concluded that

$$T_s^m(\lambda_{t_k}T_{t_k}x) \ \rightarrow \ T_s^m(x).$$

Hence, $T_s^m x$ is an Γ- recurrent vector for $(T_t)_{t\geq 0}$. □

Theorem 3.3 and Theorem 3.4 give the following corollaries.

Corollary 3.5. *If x is an Γ-recurrent vector for $(T_t)_{t\geq 0}$, then $p(T_s)x$ is a Γ-recurrent vector for $(T_t)_{t\geq 0}$ for any $s > 0$, where $p(T)$ is a nonzero polynomial.*

Corollary 3.6. *While $(T_t)_{t\geq 0}$ has an Γ-recurrent vector, it has an invariant subspace of them except zero.*

4. Some examples

This section presents some examples of Γ-recurrent C_0-semigroups. Some of them, lead to new results. The first example is offered by using Theorem 3.2 as follows.

Example 4.1. Consider C_0-semigroup $(T_t)_{t\geq 0}$ on X that is defined with $T_t = e^{2\pi ti}I$. Hence, for any $x \in \mathbb{C}$,

$$T_t x = e^{2\pi ti}I(x) = (\cos 2\pi t + i \sin 2\pi t)x.$$

Suppose $t_k := k$. Then,

$$T_{t_k} x = (\cos 2\pi k + i \sin 2\pi k)x = x.$$

Hence, $T_{t_k} x \to x$. Suppose $\Gamma \subseteq \mathbb{C}$ is a set such that $1 \in \Gamma$. So, any $x \in \mathbb{C}$ with $x \neq 0$ is a Γ-recurrent vector for $(T_t)_{t\geq 0}$. Therefore, $(T_t)_{t\geq 0}$ is Γ-recurrent.

In the following, we construct a non-supercyclic Γ-recurrent C_0-semigroup.

Example 4.2. Assume $(T_t)_{t\geq 0}$ is a C_0-semigroup on X so that $T_{t_0} = I$ for some $t_o > 0$. Then, as proved in [18, Lemma 2.4], $(T_t)_{t\geq 0}$ is not supercyclic. Suppose $\Gamma \subseteq \mathbb{C}$ is a set such that $1 \in \Gamma$. Then $(T_t)_{t\geq 0}$ is Γ-recurrent since for any nonempty open set U of X, $T_{t_0}(U) \cap U \neq \phi$.

The following theorem, helps us to construct examples of Γ-recurrent C_0-semigroups.

Theorem 4.3. *If $(T_t \oplus S_t)_{t\geq 0}$ is a Γ-recurrent C_0-semigroup on $X \oplus Y$, then $(T_t)_{t\geq 0}$ is Γ-recurrent on X, and $(S_t)_{t\geq 0}$ is Γ-recurrent on Y.*

Proof. Let $U \oplus V$ be a nonempty subset of $X \oplus Y$. Then there exist $0 \neq \lambda \in \Gamma$ and $t_0 > 0$ such that

$$\lambda(T_{t_0} \oplus S_{t_0})(U \oplus V) \cap (U \oplus V) \neq \phi.$$

Hence, $\lambda T_{t_0} U \cap U \neq \phi$ and $\lambda S_{t_0} V \cap V \neq \phi$. Therefore, $(T_t)_{t\geq 0}$ and $(S_t)_{t\geq 0}$ are Γ-recurrent. □

Example 4.4. Let X be a Banach space on real line $\mathbb{R}$. Suppose $(T_t)_{t\geq 0}$ is a hypercyclic C_0-semigroup on X. Assume $A_t \in L(\mathbb{R}^2)$, is an operator with the matrix $\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$ for $t \geq 0$. Then $(A_t \oplus T_t)_{t\geq 0}$ is supercyclic on $\mathbb{R}^2 \oplus X$ [18, Example B]. Hence, $(A_t \oplus T_t)_{t\geq 0}$ is Γ-recurrent on $\mathbb{R}^2 \oplus X$, where $\Gamma = \mathbb{C}$. Especially, $(A_t)_{t\geq 0}$ is a Γ-recurrent C_0-semigroup on $\mathbb{R}^2$ by Theorem 4.3.

Example 4.4 presents a Γ-recurrent C_0-semigroup on a finite-dimensional space. So, the following result can be stated.

Corollary 4.5. *Γ-recurrent C_0-semigroups exist on finite-dimensional spaces.*

References

[1] A. Abbar, *Γ-supercyclicity for strongly continuous semigroups*, Complex Anal. Oper. Theory, **13**(8) (2019), 3923–3942.

[2] A. Abbar, *Γ-supercyclicity for bilateral shift operators and translation semigroups*, Bulletin of L.N. Gumilyov ENU. Mathematics. Computer Science. Mechanics series, **129**(4) (2019), 73–79.

[3] M. Amouch and O. Benchiheb, *On linear dynamics of sets of operators*, Turkish J. Math., **43**(1) (2019), 402–411.

[4] M. Amouch and O. Benchiheb, *Diskcyclicity of sets of operators and applications*, Acta Math. Sin. English Series, **36** (2020), 1203–1220.

[5] M. Amouch and O. Benchiheb, *On a class of super-recurrent operators*, Filomat, **36**(11) (2022), 3701–3708.

[6] M. Ansari, *Supermixing and hypermixing operators*, J. Math. Anal.Appl., **498** (2021), 124952.

[7] N. Bamerni, *Some Properties of Gamma-supercyclic operators*, Acad. J. Nawroz Univ., **12**(4) (2023), 302–305.

[8] F. Bayart, U. B. Darji and B. Pires, *Topological transitivity and mixing of composition operators*, J. Math. Anal.Appl., **465**(1) (2018), 125–139.

[9] O. Benchiheb and M. Amouch, *On Recurrent Sets of Operators*, Bol. Soc. Paran. Mat., **42** (2024), 1–9.

[10] O. Benchiheb and M. Amouch, *Subspace-super recurrence of operators*, Filomat, **38**(9) (2024), 3093–3103.

[11] J. Bes, A. Peris and Y. Puig, *Strong transitivity properties for operators*, J. Diff. Equ., **266**(2-3) (2019), 1313–1337.

[12] S. Charpentier, R. Ernst and Q. Menet, *Γ-supercyclicity*, J. Funct. Anal., **270**(12) (2016), 4443–4465.

[13] G. Costakis, A. Manoussos and I. Parissis, *Recurrent linear operators*, Complex Anal. Oper. Theory., **8**(8) (2014), 1601–1643.

[14] K.G. Grosse-Erdmann and A. Peris Manguillot, *Linear Chaos*, Springer-Verlag, London, UK, 2011.

[15] N. Karim, O. Benchiheb and M. Amouch, *Recurrence of multiples of composition operators on weighted Dirichlet spaces*, Adv. Oper. Theory., **7**(2) (2022), 23.

[16] M. Moosapoor, *On the Recurrent C_0-Semigroups, Their Existence, and Some Criteria*, J. Math., **2021**(1) (2021), 6756908.

[17] M. Moosapoor, *On subspace-recurrent operators*, Tamkang J. Math., **53**(4) (2022), 363–371.

[18] S. Shkarin, *On supercyclicity of operators from a supercyclic semigroup*, J. Math. Anal. Appl., **382** (2011), 516–522.