**Nonlinear Functional Analysis and Applications**
**Vol. 30, No. 2 (2025), pp. 513-528**
**ISSN: 1229-1595(print), 2466-0973(online)**

https://doi.org/10.22771/nfaa.2025.30.02.13
http://nfaa.kyungnam.ac.kr/journal-nfaa

# GLOBAL ANALYSIS OF A TIME-FRACTIONAL ORDER IN SPATIO-TEMPORAL BASIC VIRUS DYNAMIC MODELING

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**Abstract.** The main objective of this study is to investigate and analyze a spatio-temporal model of viral infection including a fractional derivative order. This model represents the dynamics of infection through partial differential equations integrating spatial diffusion to depict the spread of viruses. We assume in our model, the diffusion of the free viruses. First, we establish the existence, uniqueness and limits of solutions. The infection-free equilibrium points and the endemic equilibrium point are given in terms of the basic reproduction number. We conclude then that the overall stability of each equilibrium is mainly determined by this number. After validating our theoretical results by numerical simulations, we also made a numerical comparison between two schemes: one using a normal derivative and the other using a fractional derivative. It has been observed that the order of fractional derivatives has no impact on the stability of equilibria, but only on the speed of convergence towards stable states.

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[0]Received September 20, 2024. Revised January 5, 2025. Accepted January 7, 2025.
[0]2020 Mathematics Subject Classification: 34A12, 34D23, 26A33, 92B05.
[0]Keywords: Global stability, fractional derivative, reactiondiffusion systems, viral infection, numerical simulation, Lyapunov function.
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## 1. Introduction

Infectious diseases pose a significant challenge to global public health. In line with data from the World Health Organization, infectious diseases result in an annual toll of over 17 million fatalities [11, 17], prompting researchers in the mathematical and biological sciences to develop strategies for controlling and mitigating these conditions. Through various mathematical modeling techniques, epidemiologists have gained a thorough understanding of epidemic dynamics.

In this context, mathematicians have proposed, analyzed and controlled several infectious diseases such as Hepatitis B Virus (HBV) [1, 18], Hepatitis C Virus (HCV) [22, 24] and Coronavirus Disease 2019 (Covid-19) [12, 19]. This mathematical analytical approach proves to be cost-effective and yields faster results compared to traditional experimental methods, which are often time-consuming and expensive. This advantage encourages many mathematicians and biologists to design robust epidemiological models capable of providing an accurate representation of reality, with significant implications for society and the economy. Understanding the spread of viral infections is crucial for developing effective prevention, control, and treatment strategies. In this perspective, mathematical models play a decisive role by providing tools to describe and analyze the dynamics of these diseases.

The mathematical modeling of infectious diseases typically initiates with basic models employing ordinary differential equations (ODEs) and subsequently advances to models involving partial differential equations (PDEs). In recent years, researchers have introduced models that describe the propagation of infectious diseases using equations featuring fractional derivatives, incorporating the effect of memory.

Korobeinikov's [15] seminal model offers a concise yet comprehensive depiction of the dynamics underlying the transmission and propagation of viral infections. This model serves as a valuable foundation for understanding and analyzing the mechanisms influencing the spread of infectious diseases.

$$\begin{cases} \dfrac{dX(t)}{dt} = \Lambda - \beta X(t)V(t) - \mu X(t), \\ \dfrac{dZ(t)}{dt} = \beta X(t)V(t) - (b+c)Z(t), \\ \dfrac{dY(t)}{dt} = cZ(t) - aY(t), \\ \dfrac{dV(t)}{dt} = kY(t) - uV(t). \end{cases} \tag{1.1}$$

In this model, we have $X(t)$ the uninfected cell population (the susceptible cells), $Z(t)$ the exposed cells, $Y(t)$ the infected cells, $V(t)$ the free virus particles. Cells are assumed to reproduce at a constant $\lambda$ rate, and all newly produced cells are uninfected (susceptiblee cells). The average lifetimes of susceptible cells, exposed cells, infected cells and free virus are $\mu$, $b$, $a$ and $u$ respectively. Free virus is produced from cells infected at the rate $kY$ and infects susceptiblee cells at the rate $\beta XV$, and $c$ is the average time of the latent state. Naturally, the system is only defined for non-negative $X, Z, Y$ and $V$, all coefficients are assumed to be positive.

In recent years, fractional calculus has garnered significant attention from researchers. Classical models have demonstrated lower accuracy in predicting the temporal dynamics of diseases, whereas models incorporating non-integer order derivatives offer improved information retention and allocation for large-scale analysis [2, 3, 8, 9, 20].

A promising approach in the study of infectious diseases is the use of spatio-temporal models. These models integrate both the geographical aspect, by taking into account the spatial spread of infections and the temporal aspect, by examining the evolution of the infection over time. The study investigates and analyzes the dynamical stability of a spatial-temporal $SIR$ model (where $S$, $I$, and $R$ represent the susceptible, infected, and recovered populations, respectively), incorporating a fractional order derivative and a saturated incidence function, as discussed in [6]. Moreover, Bounkaicha et al. [7] extend the investigation by introducing an additional compartment of exposed individuals ($E$) and studying the stability of equilibria in a spatial-temporal $SEIR$ model. Also incorporating a fractional order derivative, the local stability of a spatio-temporal model $SIR$ with loss of immunity is studied in [5].

To better understand the dynamics of propagation and the underlying mechanisms, in this study, we propose a spatio-temporal model of viral infection using a fractional derivative.

$$\begin{cases} {}_0^C D_t^\alpha X(x,t) = \Lambda - \beta X(x,t)V(x,t) - \mu X(x,t), \\ {}_0^C D_t^\alpha Z(x,t) = \beta X(x,t)V(x,t) - (b+c)Z(x,t), \\ {}_0^C D_t^\alpha Y(x,t) = cZ(x,t) - aY(x,t), \\ {}_0^C D_t^\alpha V(x,t) = d\Delta V(x,t) + kY(x,t) - uV(x,t), \end{cases} \tag{1.2}$$

where, $\Delta = \frac{\partial^2}{\partial x^2}$ is the Laplace operator, ${}_0^C D_t^\alpha$ is the time fractional derivative of order $\alpha$ in the sense of Caputo ($0 < \alpha \leq 1$), and $d$ represent the diffusion coefficients for the free virus.

$$X(x,0) = X_0 \geq 0;\ Z(x,0) = Z_0 \geq 0;\ Y(x,0) = Y_0 \geq 0;\ V(x,0) = V_0 \geq 0, \tag{1.3}$$

and zero-flux boundary conditions

$$\frac{\partial V(x,t)}{\partial \eta} = 0, \qquad \forall\ x \in \partial\Omega, \tag{1.4}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, and $\frac{\partial .}{\partial \eta}$ is the normal derivative.

This document is organized coherently into several sections. First, in the next section, the necessary definitions and lemmas are presented to lay the foundations of the study. Then, in Section 3, the existence result is exposed and the equilibria are clearly exposed. The global stability of the equilibria is retained in section 4. Section 5 is devoted to various numerical simulations, thus making it possible to illustrate and verify the theoretical results. Finally, in the last section, a conclusion is formulated, summarizing the main findings and possibly opening avenues for future research. This structure ensures smooth reading and facilitates understanding of the study carried out.

## 2. Preliminaries

In this section, we will present some definitions and basic result. We first present Mittag-Leffler function [13].

**Definition 2.1.** The Mittag-Leffler function, $E_\alpha(\mathcal{Z})$, is defined as

$$E_\alpha(\mathcal{Z}) = \sum_{k=0}^{\infty} \frac{\mathcal{Z}^k}{\Gamma(k\alpha+1)}, \quad \alpha > 0, \quad \mathcal{Z} \in \mathbb{C}, \tag{2.1}$$

where

$$\Gamma(z) = \int_0^{+\infty} e^{-\tau}\tau^{z-1} d\tau, \tag{2.2}$$

is the Gamma function.

**Definition 2.2.** (Riemann-Liouville fractional integral [10, 14]) Let $\mathcal{F}$ be a function such that $\mathcal{F} \in L^1(\mathbb{R}^+)$, the Riemann-Liouville fractional integral with $\alpha > 0$ of $\mathcal{F}$ is defined as

$$I^\alpha \mathcal{F}(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \mathcal{F}(\tau)\ d\tau. \tag{2.3}$$

**Definition 2.3.** (The Caputo fractional derivative [14]) Let $\alpha > 0$, let $n \in \mathbb{N}$ satisfy $n-1 < \alpha \leq n$. The Caputo fractional derivative of order $\alpha$ applied to the function $\mathcal{F} \in C^n([0,+\infty), \mathbb{R})$ is given by

$${}_0^C D_t^\alpha \mathcal{F}(t) = I^{n-\alpha} D^n \mathcal{F}(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{\mathcal{F}^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}}\ d\tau, \tag{2.4}$$

where $D = \frac{d}{dt}$. In particular, if $0 < \alpha < 1$, we have $n = 1$, then

$$ {}_0^C D_t^\alpha \mathcal{F}(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\mathcal{F}'(\tau)}{(t-\tau)^\alpha}\, d\tau. \tag{2.5}$$

LaSalle's principle of invariance is a widely used tool to study the asymptotic behavior of solutions of differential equations [4, 16].

**Theorem 2.4.** (LaSalle's principle of invariance) *Let $\mathcal{X}^*$ be an equilibrium point of a Cauchy problem, the equilibrium $\mathcal{X}^*$ is asymptotically stable if there exists a continuous function $\mathbf{F}$ defined in a neighborhood $\mathcal{U} \subset \mathbb{R}^n$ of $\mathcal{X}^*$ with values in $\mathbb{R}$, differentiable on $\mathcal{U} \setminus \mathcal{X}^*$ such that*

(1) $\mathbf{F}(\mathcal{X}^*) = 0$ *and* $\mathbf{F}(\mathcal{X}(t)) > 0$ *for all* $\mathcal{X}(t) \in \mathcal{U} \setminus \{\mathcal{X}^*\}$.
(2) ${}_0^C D_t^\alpha \mathbf{F}(\mathcal{X}(t)) \leq 0$ *for all* $\mathcal{X} \in \mathcal{U} \setminus \{\mathcal{X}^*\}$.
(3) *The set* $S = \{\mathcal{X}(t) \in \mathcal{U} / {}_0^C D_t^\alpha \mathbf{F}(\mathcal{X}(t)) = 0\}$ *does not contain any trajectory of the system other than* $\mathcal{X}(t) = \mathcal{X}^*$.

**Lemma 2.5.** *Let $\Psi$ be a positive function defined by $\Psi(y) = y - (\ln(y)+1), y > 0$, and $y(t) \in \mathbb{R}+^*$ a continuous differentiable function, for all $\alpha \in (0, 1]$ and $t \geq t_0$*

$$ y^* {}_{t_0}^C D_t^\alpha \left[\Psi\left(\frac{y(t)}{y^*}\right)\right] \leq \left(\frac{y(t)-y^*}{y(t)}\right) {}_{t_0}^C D_t^\alpha y(t),\ y^* \in \mathbb{R}_+^*. \tag{2.6}$$

## 3. The equilibria and existence solution

3.1. **The basic reproduction number and the equilibria.** In order to determine the basic reproduction number, we will utilize the approach described by Van Den Driessche and Watmough in [23].

To determine $\mathcal{R}_0$, we are just need the class $Z$, $Y$ and $V$.
In what follows, we denote $Z(x,t) = Z_{x,t}$, $Y(x,t) = Y_{x,t}$ and $V(x,t) = V_{x,t}$,

$$ \begin{cases} {}_0^C D_t^\alpha Z_{x,t} = \beta X_{x,t} V_{x,t} - (b+c) Z_{x,t}, \\ {}_0^C D_t^\alpha Y_{x,t} = c Z_{x,t} - a Y_{x,t}, \\ {}_0^C D_t^\alpha V_{x,t} = d\Delta V_{x,t} + k Y_{x,t} - u V_{x,t}. \end{cases} \tag{3.1}$$

We have $\mathcal{R}_0 = \rho(\mathbb{F}\mathbf{V}^{-1})$ ($\rho$ is the spectral radius) with

$$ \mathbb{F} = \begin{pmatrix} 0 & 0 & \frac{\beta\Lambda}{\mu} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.2}$$

and

$$\mathbb{V} = \begin{pmatrix} b+c & 0 & 0 \\ -c & a & 0 \\ 0 & -k & u \end{pmatrix}. \tag{3.3}$$

By performing simple calculations, the basic reproduction number of the proposed model is given by

$$\mathcal{R}_0 = \rho(\mathbb{F}\mathbb{V}^{-1}) = \frac{\beta\Lambda ck}{au\mu(b+c)}. \tag{3.4}$$

The equilibria of the model (1.2) are the solutions of the following system

$$\begin{cases} \Lambda - \beta XV - \mu X = 0, \\ \beta XV - (b+c)Z = 0, \\ cZ - aY = 0, \\ kY - uV = 0. \end{cases} \tag{3.5}$$

The model (1.2) has two equilibria, are given by the disease-free equilibrium

$$E_f = (X_f, 0, 0, 0) = \left(\frac{\Lambda}{\mu}, 0, 0, 0\right), \tag{3.6}$$

and the endemic equilibrium

$$E^* = (X^*, Z^*, Y^*, V^*) \tag{3.7}$$

with

$$X^* = \frac{\Lambda}{\mu\mathcal{R}_0}, \quad Z^* = \frac{au}{kc}V^*, \quad Y^* = \frac{u}{k}V^* \quad \text{and} \quad V^* = \frac{\mu}{\beta}(\mathcal{R}_0 - 1). \tag{3.8}$$

**Remark 3.1.** From the components of the endemic equilibrium, it is clear that $\mathcal{E}^*$ exists when $\mathcal{R}_0 > 1$.

### 3.2. **Existence, non-negative and boundedness of solutions.**

3.2.1. **Existence, non-negative of the solutions.** Let $\mathbb{X} = C(\bar{\Omega}, \mathbb{R})$ and $J = (J_1, J_2, J_3, J_4)$, $\lambda = (0, 0, 0, d)$ and $A$ the linear diffusion operator

$$A : D(A) \subset \mathbb{X}^4 \to \mathbb{X}^4$$
$$AJ = \lambda\Delta J = (0, 0, 0, d\Delta J_4), \quad \forall\ J \in D(A)$$

with

$$D(A) = \left\{ J \in \mathbb{X}^4 : \Delta J \in \mathbb{X}^4, \ \ \frac{\partial J}{\partial \eta} = 0_{\mathbb{R}^4} \ \text{ for } \ x \in \partial\Omega \right\}. \tag{3.9}$$

The function $f$ is defined by $f : \mathbb{X}^4 \times [0, T] \mapsto \mathbb{X}^4$ such that

$$f_{x,t} = f(J_{x,t}) = (f_1(J_{x,t}), f_2(J_{x,t}), f_3(J_{x,t}), f_4(J_{x,t}))$$

with

$$\begin{cases} f_1(J) = \Lambda - \mu J_1 - \beta J_1 J_4, \\ f_2(J) = \beta J_1 J_4 - (b+c) J_2, \\ f_3(J) = c J_2 - a J_3, \\ f_4(J) = k J_3 - u J_4. \end{cases} \tag{3.10}$$

Hence, we can rewrite the model (1.2) in the following form:

$$\begin{cases} {}_0^C D_t^\alpha J = AJ + f(J), \\ J(0) = J_0, \end{cases} \tag{3.11}$$

with $J = (X, Z, Y, V)$ and $J_0 = (X_0, Z_0, Y_0, V_0)$. We have the following proposition.

**Proposition 3.2.** *Let $0 < \alpha \leq 1$, for all $J_0 \in D(A)$, problem (3.11) has a unique positive solution $J \in C\left([0,T]; X^4\right)$ with*

$$J(t) = \int_0^\infty \Phi_\alpha(\theta) Q\left(t^\alpha \theta\right) J_0 d\theta + F(t) \tag{3.12}$$

*and*

$$F(t) = \alpha \int_0^t \int_0^\infty \theta (t-\tau)^{\alpha-1} \Phi_\alpha(\theta) Q\left((t-\tau)^\alpha \theta\right) f(\tau) d\theta d\tau, \tag{3.13}$$

*where $\Phi_\alpha(\theta)$ is a probability density function defined on $(0, \infty)$.*

3.2.2. **Boundedness of the solutions.** Now, we will prove the boundedness of the solutions. Let

$$N(t) = \int_\Omega \left[X_{x,t} + Z_{x,t} + Y_{x,t} + V_{x,t}\right] dx, \tag{3.14}$$

so

$$\begin{aligned} {}_0^C D_t^\alpha N(t) &\leq \int_\Omega (\Lambda - m(X_{x,t} + Z_{x,t} + Y_{x,t} + V_{x,t}) dx, \\ &\leq \Lambda |\Omega| - m \int_\Omega (X_{x,t} + Z_{x,t} + Y_{x,t} + V_{x,t} dx \end{aligned} \tag{3.15}$$

with $m = \min(m, b, a-b, u)$. So we have

$${}_0^C D_t^\alpha N(t) \leq \Lambda |\Omega| - m N(t). \tag{3.16}$$

Using the Laplace transform, we get

$$N(t) \leq N(0) E_\alpha\left(-m t^\alpha\right) + \frac{\Lambda}{\mu}\left(1 - E_\alpha\left(-m t^\alpha\right)\right), \tag{3.17}$$

because of $0 \leq E_\alpha\left(-\mu t^\alpha\right) \leq 1$, we conclude that $N(t) \leq N(0) + \frac{\Lambda}{\mu}$ hence the results.

## 4. Global stability of the equilibria

In this section, we will study the global stability of the equilibria $E_f$ and $E^*$ as a function of the basic reproduction number $\mathcal{R}_0$.

**Theorem 4.1.** *If $\mathcal{R}_0 \leq 1$ the diseas-free equilibrium $E_f$ is globally asymptotically stable.*

*Proof.*

$$U(X_{x,t}, Z_{x,t}, Y_{x,t}, V_{x,t}) = \int_\Omega \left[ X_f \Psi\left(\frac{X_{x,t}}{X_f}\right) + \frac{b+c}{c} Y_{x,t} + Z_{x,t} + \frac{a(b+c)}{ck} V_{x,t} \right] dx. \tag{4.1}$$

We apply the time fractional derivative to the function $U$, according to Lemma 2.5, we have

$$\begin{aligned} {}_0^C D_t^\alpha U &= \int_\Omega \left[ X_f {}_0^C D_t^\alpha \Psi\left(\frac{X_{x,t}}{X_f}\right) + \frac{b+c}{c} {}_0^C D_t^\alpha Y_{x,t} + {}_0^C D_t^\alpha Z_{x,t} + \frac{a(b+c)}{ck} {}_0^C D_t^\alpha V_{x,t} \right] dx \\ &\leq \int_\Omega \left[ \left(1 - \frac{X_f}{X_{x,t}}\right)(\Lambda - \beta X_{x,t} V_{x,t} - \mu X_{x,t}) + \frac{b+c}{c}(cZ_{x,t} + aY_{x,t}) \right. \\ &\quad + \frac{a(b+c)}{c}(cZ_{x,t} + aY_{x,t}) + (\beta X_{x,t} V_{x,t} - (b+c) Z_{x,t}) \\ &\quad \left. + \frac{a(b+c)}{ck}(d\Delta V_{x,t} + kY_{x,t} - uV_{x,t}) \right] dx. \end{aligned} \tag{4.2}$$

We have $\int_\Omega \Delta V_{x,t}(x,t)dx = 0$ and $\mu x_f = \Lambda$, by simple calculations on

$${}_0^C D_t^\alpha U \leq \lambda \left(2 - \frac{X_{x,t}}{X_f} - \frac{X_f}{X_{x,t}}\right) + \frac{\lambda\beta}{\mu} \frac{1}{R_0} (R_0 - 1) V_{x,t}. \tag{4.3}$$

It is obvious that $R_0 \leq 1$ guarantees ${}_0^C D_t^\alpha U \leq 0$ for all $X_{x,t}, V_{x,t} > 0$.

$$\left\{ (X_{x,t}, Z_{x,t}, Y_{x,t}, V_{x,t}) \in \mathbb{R}_+^2 : {}_0^C D_t^\alpha U(t) = 0 \right\} = \{E_f\}. \tag{4.4}$$

According to the principle of LaSalle invariance, $E_f$ is globally asymptotically stable. □

**Theorem 4.2.** *The endemic equilibrium $E^*$ is globally asymptotically stable.*

*Proof.* To prove this theorem, we propose the positive Lyapunov function defined by

$$\begin{aligned} \mathcal{V}_{x,t}(X, Y_{x,t}, V_{x,t}) = \int_\Omega \left[ X^* \Psi\left(\frac{X_{x,t}}{X^*}\right) + Z^* \Psi\left(\frac{Z_{x,t}}{Z^*}\right) + \frac{b+c}{c} Y^* \Psi\left(\frac{Y_{x,t}}{Y^*}\right) \right. \\ \left. + \frac{a(b+c)}{ck} V^* \Psi\left(\frac{V_{x,t}}{V^*}\right) \right] dx. \end{aligned} \tag{4.5}$$

Using (1.2), we get

$$\begin{aligned} {}_0^C D_t^\alpha \mathcal{V}_{x,t} = \mu X^* \left(2 - \frac{X_{x,t}}{X^*} - \frac{X^*}{X_{x,t}}\right) \\ + (b+c)Z^* \left(4 - \frac{X^*}{X_{x,t}} - \frac{X_{x,t}V_{x,t}Z^*}{X^*V^*Z_{x,t}} - \frac{Z_{x,t}Y^*}{Z^*Y_{x,t}} - \frac{Y_{x,t}V^*}{Y^*V_{x,t}}\right). \end{aligned} \tag{4.6}$$

Since we always have the arithmetic mean is greater than or equal to the geometric mean, then

$$2 - \frac{X_{x,t}}{X^*} - \frac{X^*}{X_{x,t}} \leq 0 \quad \text{and} \quad 4 - \frac{X^*}{X_{x,t}} - \frac{X_{x,t}V_{x,t}Z^*}{X^*V^*Z_{x,t}} - \frac{Z_{x,t}Y^*}{Z^*Y_{x,t}} - \frac{Y_{x,t}V^*}{Y^*V_{x,t}} \leq 0. \tag{4.7}$$

It is easy to see that if $X^*, Z^* \geq 0$, then

$${}_0^C D_t^\alpha \mathcal{V}_{x,t} \leq 0$$

for all $X_{x,t}$, $Y_{x,t}$, $Z_{x,t}$, $V_{x,t} > 0$.

$$\left\{(S,I) \in \mathbb{R}_+^2 : D^\alpha \mathcal{V}_{x,t}(t) = 0\right\} = \{E^*\}. \tag{4.8}$$

According to the principle of LaSalle invariance, the equilibrium $E^*$ is globally asymptotically stable. □

## 5. Numerical simulation

In this section, we provide numerical simulations for the proposed epidemic model (1.2). The purpose of these simulations is to validate the theoretical findings discussed in the preceding section. To approximate the diffusion expression, the finite difference method with the Euler scheme was employed. For the Caputo order fractional derivative, the Euler fractional method described in [21] was utilized. The initial conditions were set as constant values. The program was implemented using MATLAB.

### 5.1. **Stability analysis of equilibria.**

In this subsection, we turn our attention to the numerical simulation focusing on the stability of equilibria.

Specifically, Figure 1 illustrates the stability of the disease-free equilibrium $E_f$ for given parameter values: $\alpha = 0.8$, $\Lambda = 0.6$, $\beta = 0.001$, $\mu = 0.02$, $b = 0.001$, $c = 0.001$, $a = 0.01$, $u = 0.01$, $k = 0.001$ and $d = 0.20$. Under these conditions, the basic reproduction number is less than unity, that is, $\mathcal{R}_0 = 0.15 \leq 1$, indicating the prediction of disease extinction.

The simulation results demonstrate convergence towards the disease-free equilibrium $E_f = (30, 0, 0, 0)$. Consequently, in accordance with Theorem 4.1, $E_f$ is proven to be globally asymptotically stable. This implies that the SIR dynamics exhibit convergence towards $E_f$.

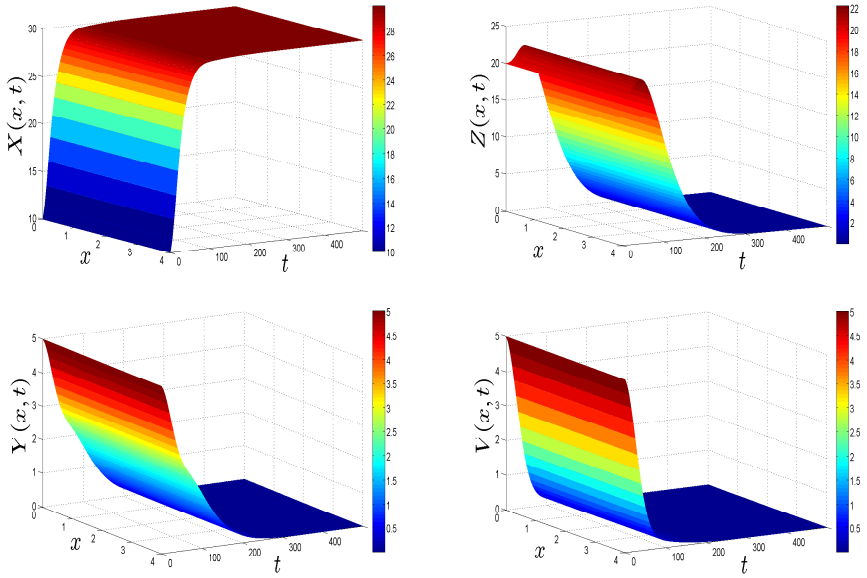


FIGURE 1. The stability of the disease-free equilibrium $E_f$.

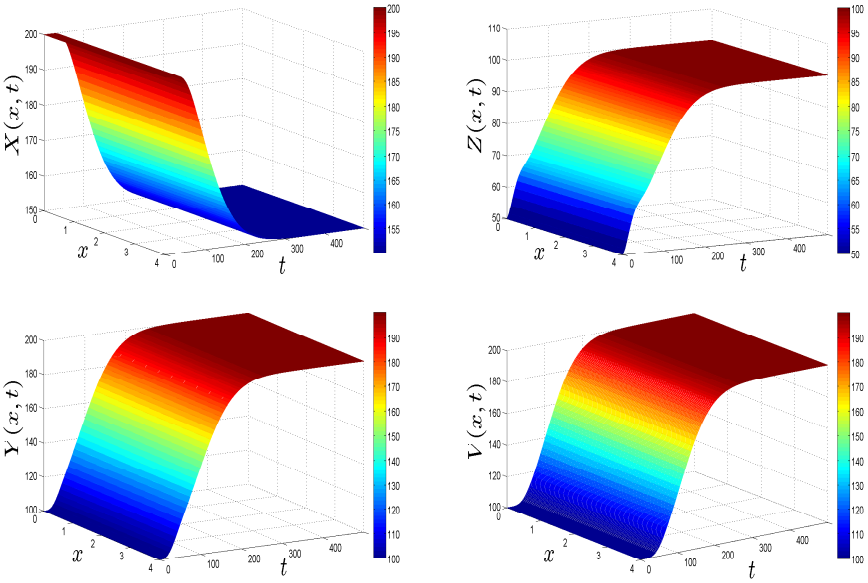


FIGURE 2. The stability of the endemic equilibrium $E^*$.

Furthermore, in Figure 2, the stability of the endemic equilibrium $E^*$ is depicted for specified values of $\alpha = 0.7$, $\Lambda = 6$, $\beta = 0.0001$, $\mu = 0.02$, $b = 0.01$, $c = 0.02$, $a = 0.01$, $u = 0.01$, $k = 0.01$, and $d = 0.20$. Under these parameter settings, the basic reproduction number is $\mathcal{R}_0 = 2 > 1$, indicating a scenario where the infection is expected to persist.

The simulation results illustrate convergence towards the endemic equilibrium $\mathcal{E}^* = (150, 200, 200, 100)$. In accordance with Theorem 4.2, $\mathcal{E}^*$ is established as globally asymptotically stable, signifying the persistence of the infection.

Effect of the Fractional Derivative Order-5.2. Effect of the Fractional Derivative Order.

5.2. **Effect of the fractional derivative order.** Figures 3 and 4 shows the effect of the order of the fractional derivative. It is clear that the order of the drift has no effect on the stability of the equilibrium points, but it does have an effect on the speed of the convergence. Indeed, the order of the largest derivative converges more rapidly in relation to the smallest values.

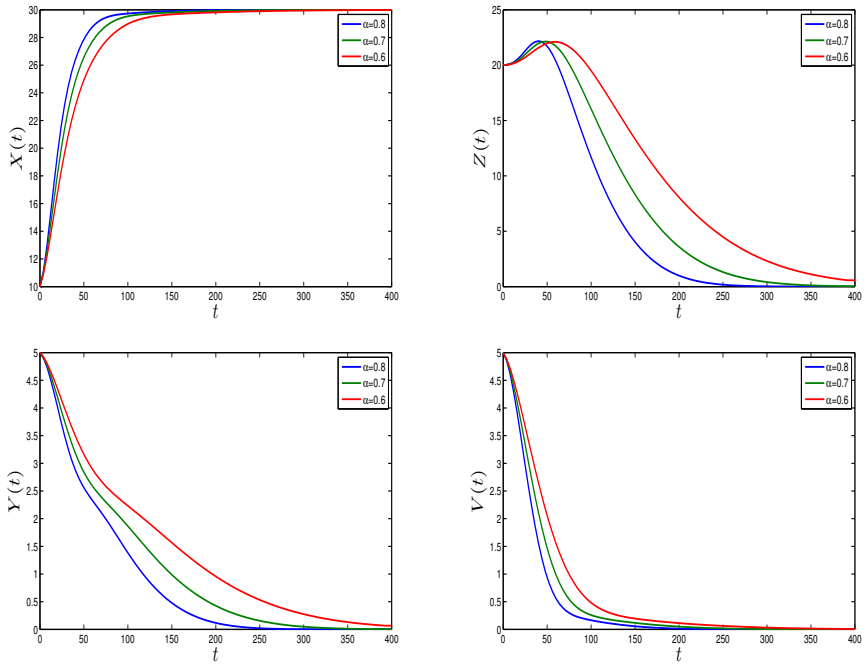


FIGURE 3. Effect of the fractional derivative on the stability of the disease-free equilibrium $E_f$.

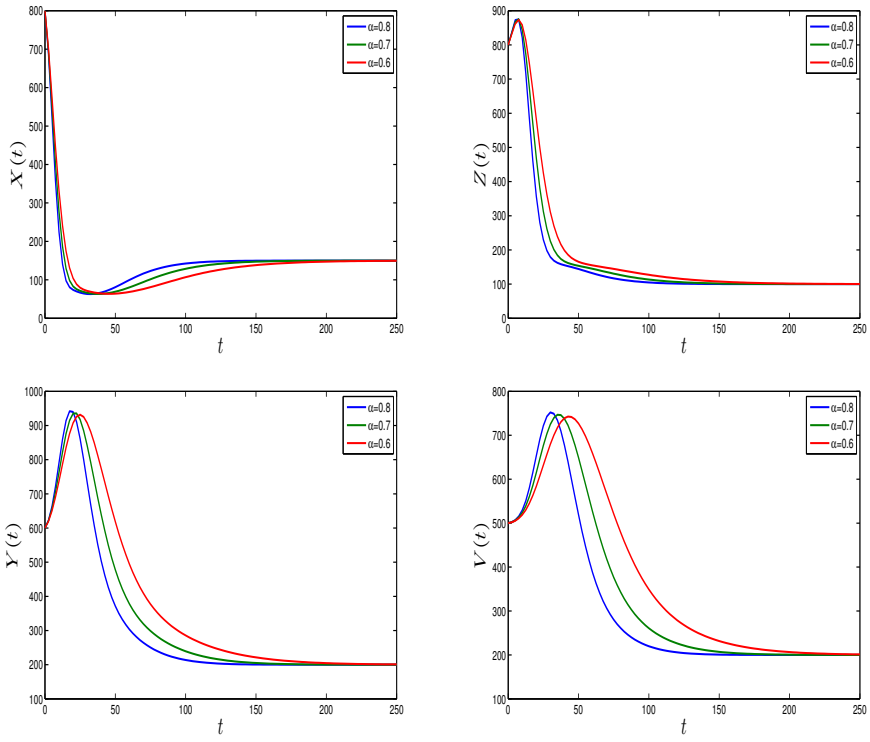


FIGURE 4. Effect of the fractional derivative on the stability of the endemic equilibrium $E^*$.

5.3. **Numerical comparison of the noninteger derivative with the ordinary derivative.** In this subsection, we will compare the numerical simulations employing both the ordinary differential equation in model (1.1) and the fractional differential derivative in model (1.2). To elucidate, equation (5.1) presents the schematic representation of the standard derivative, while Equation (5.2) delineates the schematic representation of the fractional derivative.

The numerical solution of system (1) at the discretized point $(x_i, t_{m+1})$ is given by

$$\begin{cases} X_{i,m+1} = X_{i,m} + (\Lambda - \beta X_{i,m} V_{i,m} - \mu X_{i,m})ht, \\ Z_{i,m+1} = Z_{i,m} + (\beta X_{i,m} V_{i,m} - (b+c) Z_{i,m})ht, \\ Y_{i,m+1} = Y_{i,m} + (cZ_{i,m} + aY_{i,m} - \nu R_{i,m})ht, \\ V_{i,m+1} = V_{i,m} + (kY_{i,m} - uV_{i,m})ht. \end{cases} \tag{5.1}$$

The numerical solution of system (2) at the discretized point $(x_i, t_{m+1})$ is given by

$$\begin{cases} X_{i,m+1} = X_{0,m} \\ \qquad + \dfrac{h_t^\alpha}{\Gamma(\alpha+1)}\Big[\sum\limits_{j=0}^{n}\left((n-j+1)^\alpha - (n-j)^\alpha\right)\left(\Lambda - \beta X_{i,m}V_{i,m} - \mu X_{i,m}\right)\Big], \\ Z_{i,m+1} = Z_{0,m} \\ \qquad + \dfrac{h_t^\alpha}{\Gamma(\alpha+1)}\Big[\sum\limits_{j=0}^{n}\left((n-j+1)^\alpha - (n-j)^\alpha\right)\left(\beta X_{i,m}V_{i,m} - (b+c)Z_{i,m}\right)\Big], \\ Y_{i,m+1} = Y_{0,m} \\ \qquad + \dfrac{h_t^\alpha}{\Gamma(\alpha+1)}\Big[\sum\limits_{j=0}^{n}\left((n-j+1)^\alpha - (n-j)^\alpha\right)\left(cZ_{i,m} + aY_{i,m} - \nu R_{i,m}\right)\Big], \\ V_{i,m+1} = V_{0,m} + \dfrac{h_t^\alpha}{\Gamma(\alpha+1)}\Big[\sum\limits_{j=0}^{n}((n-j+1)^\alpha \\ \qquad - (n-j)^\alpha)\Big(d\dfrac{V_{i+1,m} - 2V_{i,m} + V_{i-1,m}}{h_x^2} + kY_{i,m} - uV_{i,m}\Big)\Big]. \end{cases} \tag{5.2}$$

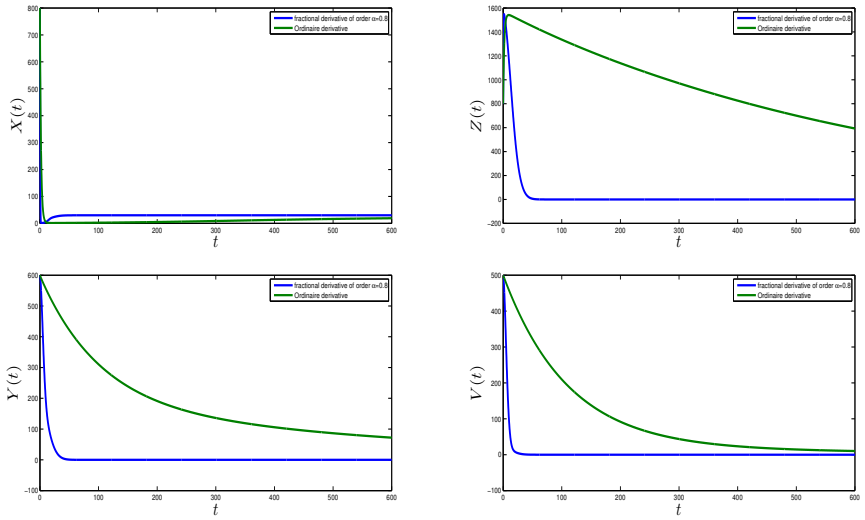


FIGURE 5. Comparison of Numerical Simulations: fractional derivative and standard derivative for the disease-free equilibrium $E_f$.

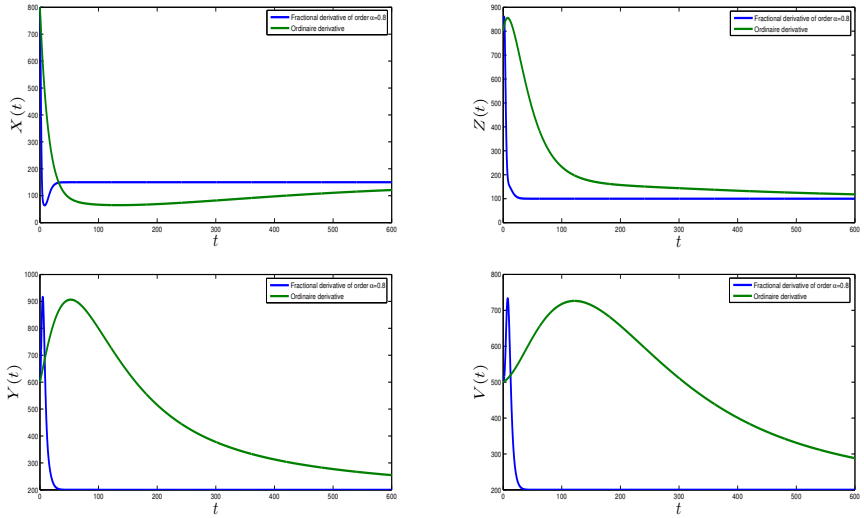


FIGURE 6. Comparison of Numerical Simulations: fractional derivative and standard derivative of the endemic equilibrium $E^*$.

Figures 5 and 6 illustrate the difference between the numerical simulation of the fractional derivative and that of the ordinary derivative.

## 6. Conclusion

This study focuses on a spatio-temporal model of viral infection, which is described by differential equations incorporating diffusion and a fractional derivative $\alpha$. Our analysis begins by establishing the existence, uniqueness, positivity, and unboundedness of the solution for this dynamic model. We also computed the basic reproduction number $\mathcal{R}$ and identified the two equilibrium points of the model based on this parameter. Subsequently, we proved the global stability of the disease-free equilibrium $E_f$ when $\mathcal{R} < 1$, indicating the absence of long-term infection in this scenario. However, when $\mathcal{R} > 1$, the endemic equilibrium $E^*$ was found to be locally asymptotically stable, suggesting the persistence of the disease over an extended period of infection. To validate our theoretical findings, we conducted numerical simulations, confirming the robustness of the results.

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