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FIXED POINT RESULTS IN C*-ALGEBRA-VALUED GENERALIZED METRIC SPACES WITH APPLICATIONS

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Abstract. In this paper, we investigate new fixed point theorems within the framework of C*-algebra-valued generalized metric spaces. We extend the concepts of the Hardy-Rogers contraction principle as well as those of Ćirić-Jachymski-Matkowski in these spaces. Our results improve and extend some recent works available in the literature. Several non-trivial examples are given to illustrate our results. As applications, we discuss the existence and the uniqueness of solution of an integral equation and of an operator equation.

1. Introduction

The fixed point theory plays a crucial role in the development of methods to solve problems in mathematics and science. Specifically, this theory has been widely employed to prove the existence and uniqueness of solutions to various functional equations.

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Since the pioneering establishment of the famous Banach contraction principle, fixed point theory and its applications have developed rapidly in the past one hundred years and the Banach contraction principle has been the subject of numerous generalizations and extensions. Among these generalizations, the Hardy-Rogers contraction [16], introduced in 1973, is noteworthy. The Hardy-Rogers contraction is denoted by the following theorem. In a complete metric space (X, d), if $T : X \to X$ is a mapping such that

$$d(Tx, Ty) \leq u_1 d(x, y) + u_2 d(x, Tx) + u_3 d(y, Ty) + u_4 d(y, Tx) + u_5 d(x, Ty)$$

for all $x, y \in X$, with u_i are positive constants, such that $\sum_{i=1}^{5} u_i < 1$, then T has a unique fixed point x^* in X.

Another interesting generalization of the Banach contraction principle was proposed in 1969 by Meir and Keeler [23]. This generalized version initiated many studies in this field and led to significant contributions to metric fixed point theory. It also facilitate the way for further generalizations, such as Matkowski contractions [27], Wardowski contractions [34], and Ćirić-Jachymski-Matkowski contractions [11, 15], which state that, in a complete metric space (X, d), if $T : X \to X$ is a mapping that satisfies the following conditions

(1) For all $\varepsilon > 0$, there exists a number $\delta > 0$ satisfying $d(x, y) < \varepsilon + \delta \Longrightarrow d(Tx, Ty) < \varepsilon \quad$ for all $x, y \in X$
(2) $x \neq y \Longrightarrow d(Tx, Ty) < d(x, y)$ for all $x, y \in X$,

then T has a unique fixed point x^* in X.

In 2007, Huang and Zhang [18] introduced the innovative concept of conical metric spaces, which represents an extension of metric spaces where the underlying space of the metric is replaced by a Banach space E.

In 2013, Liu and Xu [24] developed the notion of conical metric spaces over Banach algebras, thereby substituting the Banach space E with a Banach algebra A as the underlying structure. Liu and Xu demonstrated several fixed point theorems for generalized Lipschitz mappings using the concept of spectral radius while imposing natural and weaker constraints.

Building on previous developments, Ma et al. [25] replaced the Banach algebra $\mathbb{A}$ with a C*-algebra and introduced the concept of a C*-algebra-valued metric in 2014. This advancement led to the demonstration of several fixed point theorems within this new framework. For more details, see [9, 26, 28, 32, 31].

In 2015, Jleli and Samet introduced an innovative concept of generalized metric spaces, known as JS-metric spaces [19], which encompass various types of topological spaces. These spaces include not only standard metric spaces but also b-metric spaces [12], extensions of Kamran's b-metric spaces [22], dislocated metric spaces [17], Branciari's rectangular spaces [3], and Nakano's modular metric spaces [30]. In this framework, Jleli and Samet developed generalized versions of metric fixed-point theorems. More recently, in 2019, Chaira et al. [5, 6, 7] extended some well-known fixed point theorems, such as those for Banach, Chatterjea, and Kannan contractions, by applying them to the generalized metric space of Samet-Jleli equipped with a digraph.

In the continuity of the ideas mentioned above, Chaira et al. [4] are recently introduced, in 2024, a new concept of generalized metric spaces called a C*-algebra-valued generalized metric space. This innovative concept encompasses various topological spaces, including JS-metric spaces, C*-algebra-valued metric spaces as considered by Ma et al., and C*-algebra-valued b-metric spaces as studied by Kamran et al., as well. Within this framework, we establish the extension of the Hardy-Rogers and Ćirić-Jachymski-Matkowski contractions. We apply our results to solve an integral equation and an operator equation.

2. Preliminaries

In this section, we will first review some notable concepts and definitions of C*-algebras spaces that will be utilized in the subsequent sections.

Let $\mathbb{A}$ be a unital algebra (over the field $\mathbb{C}$) with the unit element $1_{\mathbb{A}}$ and the zero element $0_{\mathbb{A}}$. A conjugate linear map $* : \mathbb{A} \to \mathbb{A}$ is an involution on $\mathbb{A}$ if $(uv)^* = v^*u^*$ and $(u^*)^* = u$ for all u, v in $\mathbb{A}$. The pair $(\mathbb{A}, *)$ is called a $*$-algebra.

Now let recall that a Banach $*$-algebra is a $*$-algebra $\mathbb{A}$ endowed with a complete sub-multiplicative norm. A C*-algebra is a Banach $*$-algebra $(\mathbb{A}, *)$ such that $\|uu^*\| = \|u\|^2$ for all u of $\mathbb{A}$.

We denote by $\mathbb{A}_h$ the set of all elements u of $\mathbb{A}$ satisfying $u^* = u$. An element u in $\mathbb{A}$ is said to be positive if $u \in \mathbb{A}_h$ and $\sigma(u) \subseteq \mathbb{R}^+$, where $\sigma(u)$ is the spectrum of u, and we write in this case $0_{\mathbb{A}} \preceq u$. By using the positive elements, we define a partial order on $\mathbb{A}_h$ as follows

$$u \preceq v \text{ if and only if } 0_{\mathbb{A}} \preceq v - u.$$

Finally, we denote by $\mathbb{A}^+$ the set of all positive elements of $\mathbb{A}$.

In the following, we recall some useful results.

Lemma 2.1. ([13, 29]) *Let $\mathbb{A}$ be a unitary C*-algebra with a unit $1_{\mathbb{A}}$.*

(i) *For any $u \in \mathbb{A}^+$, we have*

$$u \preceq 1_{\mathbb{A}} \Longleftrightarrow \|u\| \leqslant 1.$$

(ii) *If $0_{\mathbb{A}} \preceq u \preceq v$, then $\|u\| \leqslant \|v\|$.*

(iii) *Suppose that $u, v \in \mathbb{A}^+$ such that $uv = vu$. Then $0_{\mathbb{A}} \preceq uv$.*

Lemma 2.2. ([25]) *Let $\mathbb{A}$ be a unital C*-algebra with unit $1_{\mathbb{A}}$. For all $u, v \in \mathbb{A}$ and $w \in \mathbb{A}'_{+} := \mathbb{A}^{+} \cap \mathbb{A}'$, if $u \preceq v$, then $wu \preceq wv$, where*

$$\mathbb{A}' = \{u \in \mathbb{A} : uv = vu, \ \forall\, v \in \mathbb{A}\}.$$

Theorem 2.3. ([13]) *If u is an arbitrary element of a C*-algebra $\mathbb{A}$, then u^*u is positive.*

Lemma 2.4. ([2]) *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of non-negative real numbers and let $(a_n)_{n\in\mathbb{N}}$ be a real sequence in $[0,1]$ such that $\sum_{n=0}^{\infty} a_n = \infty$. If for a given $\varepsilon > 0$ there exists a positive integer n_0 such that*

$$x_{n+1} \leq (1 - a_n)x_n + \varepsilon a_n$$

for all $n \geq n_0$, then we have $0 \leq \limsup_{n\to\infty} x_n \leq \varepsilon$.

The concept of C*-algebra-valued generalized metric space was introduced by Chaira et al. in [4, 8].

Definition 2.5. ([4]) Let X be a nonempty set, $\mathbb{A}$ be a C*-algebra and $\mathcal{D} : X \times X \to \mathbb{A}$ be a given mapping. Let $x \in X$ and $(x_n)_{n\in\mathbb{N}}$ be sequence in X. We say that $\{x_n\}_{n\in\mathbb{N}}$ is $\mathcal{D}$-convergent to x with respect to $\mathbb{A}$ and we write $\lim_{n\to\infty} \|\mathcal{D}(x_n, x)\| = 0$, if for given $\varepsilon > 0$, there exists a positive integer N such that $\|\mathcal{D}(x_n, x)\| < \varepsilon$ for all $n > N$.

For every $x \in X$, let us define the set

$$\mathcal{C}(\mathcal{D}, X, x) = \{(x_n)_{n\in\mathbb{N}} \subset X : \lim_{n\to+\infty} \|\mathcal{D}(x_n, x)\| = 0\}.$$

Definition 2.6. ([4]) Let X be a nonempty set. Suppose that the mapping $\mathcal{D} : X \times X \to \mathbb{A}$ is defined and satisfies the following properties.

- (D_1) $0_{\mathbb{A}} \preceq \mathcal{D}(x, y)$ for all x and y in X;
- (D_2) $\mathcal{D}(x, y) = 0_{\mathbb{A}} \Longrightarrow x = y$;
- (D_3) $\mathcal{D}(x, y) = \mathcal{D}(y, x)$ for all x and y in X;
- (D_4) There exists $c \in \mathbb{A}_+$ with $c \neq 0_{\mathbb{A}}$ such that if $(x, y) \in X \times X$, $(x_n)_{n\in\mathbb{N}} \in \mathcal{C}(\mathcal{D}, X, x)$ and $\limsup_{n\to\infty} \|\mathcal{D}(x_n, y)\| < \infty$, then

$$\mathcal{D}(x, y) \preceq \big(\limsup_{n\to+\infty} \|\mathcal{D}(x_n, y)\|\big)c.$$

In this case, $\mathcal{D}$ is said to be a C*-algebra-valued generalized metric on X and $(X, \mathbb{A}, \mathcal{D})$ is said to be a C*-algebra-valued generalized metric space.

Let recall also, the concept of $\mathcal{D}$-convergent and $\mathcal{D}$-Cauchy sequence with respect to $\mathbb{A}$ as well as the $\mathcal{D}$-completeness.

Definition 2.7. ([4]) Let $(X, \mathbb{A}, \mathcal{D})$ be a C*-algebra-valued generalized metric space.

(i) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is said to be $\mathcal{D}$-convergent to x with respect to $\mathbb{A}$ if $(x_n) \in \mathcal{C}(\mathcal{D}, X, x)$.

(ii) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is said to be a $\mathcal{D}$-Cauchy sequence with respect to $\mathbb{A}$ if for given $\varepsilon > 0$, there exists a positive integer N such that $\|\mathcal{D}(x_n, x_m)\| < \varepsilon$ for all $n, m \geq N$.

(iii) We say that $(X, \mathbb{A}, \mathcal{D})$ is a $\mathcal{D}$-complete C*-algebra-valued generalized metric space if every $\mathcal{D}$-Cauchy sequence is $\mathcal{D}$-convergent with respect to $\mathbb{A}$ in X.

As shown in [4], the limit of $\{x_n\}_{n\in\mathbb{N}}$ of $\mathcal{D}$-convergent with respect to $\mathbb{A}$ sequence is unique, that is, for all $(x, y) \in X^2$, we have

$$\mathcal{C}(\mathcal{D}, X, x) \cap \mathcal{C}(\mathcal{D}, X, y) \neq \emptyset \Longrightarrow x = y.$$

Note that, in C*-algebra-valued generalized metric space, a $\mathcal{D}$-convergent with respect to $\mathbb{A}$ sequence may not be a $\mathcal{D}$-Cauchy sequence with respect to $\mathbb{A}$, as shown by the following example.

Example 2.8. Let $X = \mathbb{R}^+$ and $\mathbb{A}$ be the set of continuous linear operators on a Hilbert space H, equipped with the usual norm

$$\|v\| = \sup_{x\in H\setminus\{0\}} \frac{\|v(x)\|_H}{\|x\|_H}.$$

Then $\mathbb{A}$ becomes a C*-algebra.

Consider the mapping $\mathcal{D} : X \times X \to \mathbb{A}$ defined by

$$\mathcal{D}(x, y) = \begin{cases} \big(x + y\big).u^*u, & \text{if } x = 0 \text{ or } y = 0, \\ \\ \left(5 + \dfrac{x + y}{4}\right).u^*u, & \text{otherwise,} \end{cases}$$

where $u \in \mathbb{A}$ with $u \neq 0_{\mathbb{A}}$.

Now, in this structure, let us consider the sequence $\{x_n\}_{n\geqslant 1}$, where $x_n = \dfrac{1}{n}$ for all $n \geqslant 1$. Then, we have $\lim\limits_{n\to+\infty} \|\mathcal{D}(x_n, 0)\| = 0$, implies that the sequence $\{x_n\}_{n\geq 1}$ is $\mathcal{D}$-convergent to 0 with respect to $\mathbb{A}$. But, $\lim\limits_{n,m\to+\infty} \|\mathcal{D}(x_n, x_m)\| = 5\|u\|^2$. This shows that $\{x_n\}_{n\geq 1}$ is a $\mathcal{D}$-convergent with respect to $\mathbb{A}$ sequence but not a $\mathcal{D}$-Cauchy sequence.

3. Main results

In this section, we introduce the extension of generalized Hardy-Rogers contraction and we establish the existence of fixed points for this type of mapping. Our results are supported by some examples. As application, we also use this results to resolve integral equations.

Definition 3.1. Let $(X, \mathbb{A}, \mathcal{D})$ be a C*-algebra-valued generalized metric space. We say that a mapping $T : X \to X$ is a C*-algebra-valued generalized Hardy-Rogers contractive mapping on X, if there exists $u_i \in \mathbb{A}^+$ for $i \in \{1, 2, 3, 4, 5\}$ such that $\sum_{i=1}^{5} \|u_i\| < 1$ such that for all $x, y \in X$,

$$\begin{aligned} \mathcal{D}(Tx, Ty) \preceq{}& u_1\mathcal{D}(x, y) + u_2\mathcal{D}(x, Tx) + u_3\mathcal{D}(y, Ty) \\ &+ u_4\mathcal{D}(y, Tx) + u_5\mathcal{D}(x, Ty). \end{aligned} \tag{3.1}$$

Remark 3.2. Note that in the case where $\mathbb{A} = \mathbb{R}$, the relation (3.1) is reduced to a standard contraction according to Hardy-Rogers [16].

For an element x_0 in a C*-algebra-valued generalized metric space $(X, \mathbb{A}, \mathcal{D})$ and for an arbitrary mapping $T : X \to X$, we denote by the following

$$\delta(\mathcal{D}, T, x_0) := \sup \Big\{ \|\mathcal{D}(T^i x_0, T^j x_0)\| : i, j \in \mathbb{N} \Big\}.$$

Theorem 3.3. *Let $(X, \mathbb{A}, \mathcal{D})$ be a $\mathcal{D}$-complete C*-algebra-valued generalized metric space with constant c and $T : X \to X$ be a C*-algebra-valued generalized Hardy-Rogers contractive mapping on X such that $\|u_2\|\|c\| + \|u_4\| < 1$.*
If there exists an x_0 in X such that $\delta(\mathcal{D}, T, x_0) < \infty$, then T has a unique fixed point x^* in X.

Proof. Let $n \in \mathbb{N}$ $(n \geqslant 1)$. Since T is a C*-algebra-valued generalized Hardy-Rogers contractive mapping on X, then for all $i, j \in \mathbb{N}$, we have

$$\begin{aligned} \mathcal{D}(T^{n+i}x_0, T^{n+j}x_0) &= \mathcal{D}(TT^{n-1+i}x_0, TT^{n-1+j}x_0) \\ &\preceq u_1\mathcal{D}(T^{n-1+i}x_0, T^{n-1+j}x_0) \\ &\quad + u_2\mathcal{D}(T^{n-1+i}x_0, T^{n+i}x_0) \\ &\quad + u_3\mathcal{D}(T^{n-1+j}x_0, T^{n+j}x_0) \\ &\quad + u_4\mathcal{D}(T^{n-1+j}x_0, T^{n+i}x_0) \\ &\quad + u_5\mathcal{D}(T^{n-1+i}x_0, T^{n+j}x_0). \end{aligned}$$

Hence,

$$\begin{aligned}\mathcal{D}(T^{n+i}x_0, T^{n+j}x_0) &\preceq \Big(\|u_1\|\|\mathcal{D}(T^{n-1+i}x_0, T^{n-1+j}x_0)\| \\ &\quad + \|u_2\|\|\mathcal{D}(T^{n-1+i}x_0, T^{n+i}x_0)\| \\ &\quad + \|u_3\|\|\mathcal{D}(T^{n-1+j}x_0, T^{n+j}x_0)\| \\ &\quad + \|u_4\|\|\mathcal{D}(T^{n-1+j}x_0, T^{n+i}x_0)\| \\ &\quad + \|u_5\|\|\mathcal{D}(T^{n-1+i}x_0, T^{n+j}x_0)\|\Big) \cdot 1_{\mathbb{A}} \\ &\preceq \big(\sum_{i=1}^{5}\|u_i\|\big)\delta(\mathcal{D}, T, T^{n-1}x_0) \cdot 1_{\mathbb{A}} \\ &\preceq \alpha\delta(\mathcal{D}, T, T^{n-1}x_0) \cdot 1_{\mathbb{A}},\end{aligned}$$

where $\alpha = \sum_{i=1}^{5}\|u_i\| < 1$. By Lemma 2.1, we obtain

$$\|\mathcal{D}(T^{n+i}x_0, T^{n+j}x_0)\| \preceq \alpha\delta(\mathcal{D}, T, T^{n-1}x_0) \text{ for all } i, j \in \mathbb{N}.$$

So,

$$\delta(\mathcal{D}, T, T^{n}x_0) \preceq \alpha\delta(\mathcal{D}, T, T^{n-1}x_0) \text{ for all } n \in \mathbb{N}^*.$$

Using induction reasoning, we get

$$\delta(\mathcal{D}, T, T^{n}x_0) \preceq \alpha^n\delta(\mathcal{D}, T, x_0).$$

So, for all $(i, j) \in \mathbb{N} \times \mathbb{N}$, we have

$$\begin{aligned}\mathcal{D}(T^{n+i}x_0, T^{n+j}x_0) &\preceq \alpha\delta(\mathcal{D}, T, T^{n-1}x_0) \cdot 1_{\mathbb{A}} \\ &\preceq \alpha^{n-1}\delta(\mathcal{D}, T, x_0) \cdot 1_{\mathbb{A}}.\end{aligned}$$

Since the sequence $\left\{\alpha^{n-1}\delta(\mathcal{D}, T, x_0) \cdot 1_{\mathbb{A}}\right\}_{n\in\mathbb{N}}$ is $\mathcal{D}$-convergent to $0_{\mathbb{A}}$ with respect to $\mathbb{A}$, it follows that $\left\{\mathcal{D}(T^{n+i}x_0, T^{n+j}x_0)\right\}_{n\in\mathbb{N}}$ is $\mathcal{D}$-convergent to $0_{\mathbb{A}}$ with respect to $\mathbb{A}$. Therefore, $\{T^n x_0\}_{n\in\mathbb{N}}$ is a $\mathcal{D}$-Cauchy sequence with respect to $\mathbb{A}$ in X. Since $(X, \mathbb{A}, \mathcal{D})$ is $\mathcal{D}$-complete, the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ convergent to a $x^* \in X$ with respect to $\mathbb{A}$.

Now, let us show that x^* is a fixed point of T, that is, $Tx^* = x^*$. Indeed,

$$\begin{aligned}\|\mathcal{D}(Tx^*, T^{n+1}x_0)\| &\leqslant \|u_1\|\|\mathcal{D}(x^*, T^n x_0)\| + \|u_2\|\|\mathcal{D}(x^*, Tx^*)\| \\ &\quad + \|u_3\|\|\mathcal{D}(T^n x_0, T^{n+1}x_0)\| \\ &\quad + \|u_4\|\|\mathcal{D}(T^n x_0, Tx^*)\| \\ &\quad + \|u_5\|\|\mathcal{D}(x^*, T^{n+1}x_0)\|.\end{aligned}$$

By setting

$$\begin{cases} \lambda_n = \|\mathcal{D}(Tx^*, T^n x_0)\|, \\ \gamma_n = \|u_1\|\|\mathcal{D}(x^*, T^n x_0)\| + \|u_3\|\|\mathcal{D}(T^n x_0, T^{n+1} x_0)\| \\ \qquad +\|u_5\|\|\mathcal{D}(x^*, T^{n+1} x_0)\| \\ \text{and} \\ \beta = \|u_2\|\|\mathcal{D}(x^*, Tx^*)\|, \end{cases}$$

we obtain $\lambda_{n+1} \leqslant \|u_4\|\lambda_n + \gamma_n + \beta$.

Since $\lim_{n\to\infty} \gamma_n = 0$, we have for all $\varepsilon > \dfrac{\beta}{1-\|u_4\|} \geqslant 0$, there exists an integer N_ε such that $\gamma_n \leqslant \varepsilon(1-\|u_4\|) - \beta$ for all integer $n \geq N_\varepsilon$. Then, we get

$$\lambda_{n+1} \leqslant \|u_4\|\lambda_n + \varepsilon(1-\|u_4\|) \text{ for all } n \geq N_\varepsilon.$$

By using Lemma 2.4, we get $0 \leqslant \limsup_{n\to\infty} \lambda_n \leq \varepsilon$ for all $\varepsilon > \dfrac{\beta}{1-\|u_4\|}$. Then,

$$0 \leqslant \limsup_{n\to\infty} \lambda_n \leqslant \frac{\beta}{1-\|u_4\|}.$$

Since $\{T^{n+1}x_0\}$ is $\mathcal{D}$-convergent to x^* with respect to $\mathbb{A}$, we have

$$\begin{aligned} \mathcal{D}(Tx^*, x^*) &\preceq \big(\limsup_{n\to\infty} \|\mathcal{D}(Tx^*, T^{n+1}x_0)\|\big) \cdot c \\ &\preceq \big(\limsup_{n\to\infty} \lambda_n\big) \cdot c \\ &\preceq \frac{\beta}{1-\|u_4\|} \cdot c. \end{aligned}$$

By Lemma 2.1, we get

$$\begin{aligned} \|\mathcal{D}(Tx^*, x^*)\| &\leqslant \frac{\beta\|c\|}{1-\|u_4\|} \\ &\leqslant \frac{\|c\|u_2\|}{1-\|u_4\|}\|\mathcal{D}(Tx^*, x^*)\|. \end{aligned}$$

Thus,

$$\big(1 - \frac{\|c\|\|u_2\|}{1-\|u_4\|}\big)\|\mathcal{D}(Tx^*, x^*)\| \leqslant 0.$$

Since $\dfrac{\|c\|\|u_2\|}{1-\|u_4\|} < 1$, $\|\mathcal{D}(Tx^*, x^*)\| = 0$, which implies that $\mathcal{D}(Tx^*, x^*) = 0_{\mathbb{A}}$. Therefore, $Tx^* = x^*$.

Now, suppose that the mapping T has two fixed points x^* and y^* in X. Since T is C*-algebra-valued Hardy-Rogers contractive mapping on X, we have

$$\begin{aligned}\mathcal{D}(x^*, x^*) &= \mathcal{D}(Tx^*, Tx^*)\\ &\preceq u_1\mathcal{D}(x^*, x^*) + u_2\mathcal{D}(x^*, Tx^*) + u_3\mathcal{D}(x^*, Tx^*)\\ &\quad + u_4\mathcal{D}(x^*, Tx^*) + u_5\mathcal{D}(x^*, Tx^*).\\ &= u_1\mathcal{D}(x^*, x^*) + u_2\mathcal{D}(x^*, x^*) + u_3\mathcal{D}(x^*, x^*)\\ &\quad + u_4\mathcal{D}(x^*, x^*) + u_5\mathcal{D}(x^*, x^*).\end{aligned}$$

Then $(1-\alpha)\|\mathcal{D}(x^*, x^*)\| \leqslant 0$, which gives $\|\mathcal{D}(x^*, x^*)\| = 0$, therefore $\mathcal{D}(x^*, x^*) = 0_{\mathbb{A}}$. Similarly, we get also $\mathcal{D}(y^*, y^*) = 0_{\mathbb{A}}$.

On the other hand, we have

$$\begin{aligned}\mathcal{D}(x^*, y^*) &= \mathcal{D}(Tx^*, Ty^*)\\ &\preceq u_1\mathcal{D}(x^*, y^*) + u_2\mathcal{D}(x^*, Tx^*)\\ &\quad + u_3\mathcal{D}(y^*, Ty^*) + u_4\mathcal{D}(y^*, Tx^*) + u_5\mathcal{D}(x^*, Ty^*)\\ &= u_1\mathcal{D}(x^*, y^*) + u_2\mathcal{D}(x^*, x^*) + u_3\mathcal{D}(y^*, y^*)\\ &\quad + u_4\mathcal{D}(y^*, x^*) + u_5\mathcal{D}(x^*, y^*).\\ &= u_1\mathcal{D}(x^*, y^*) + u_4\mathcal{D}(x^*, y^*) + u_5\mathcal{D}(x^*, y^*).\end{aligned}$$

Then, we have

$$\|\mathcal{D}(x^*, y^*)\| \leqslant \|u_1\|\|\mathcal{D}(x^*, y^*)\| + \|u_4\|\|\mathcal{D}(x^*, y^*)\| + \|u_5\|\|\mathcal{D}(x^*, y^*)\|,$$

this implies that

$$\Big(1 - (\|u_1\| + \|u_4\| + \|u_5\|)\Big)\|\mathcal{D}(x^*, y^*)\| \leqslant 0.$$

Since $\|u_1\| + \|u_4\| + \|u_5\| < 1$, so $\|\mathcal{D}(x^*, y^*)\| = 0$. Therefore, $\mathcal{D}(x^*, y^*) = 0_{\mathbb{A}}$ and $x^* = y^*$, which show the uniqueness of the fixed point of T. □

Example 3.4. Let $X = \mathbb{R}^+$, the set of positive real numbers, and let $\mathbb{A} = \mathcal{M}_2(\mathbb{C})$, the set of all 2×2 complex matrices. Then $\mathbb{A}$ is a C*-algebra.

Consider the mapping $\mathcal{D} : X \times X \to \mathbb{A}$ defined by

$$\mathcal{D}(x, y) = \begin{cases} \begin{pmatrix} x+y & 0 \\ 0 & x+y \end{pmatrix}, & \text{if } x = 0 \text{ or } y = 0,\\[2ex] \begin{pmatrix} \dfrac{x+y}{3} & 0 \\ 0 & \dfrac{x+y}{3} \end{pmatrix}, & \text{otherwise.} \end{cases}$$

So, $(X, \mathbb{A}, D)$ is a $\mathcal{D}$-complete C*-algebra-valued generalized metric space.
The mapping $T : X \to X$ defined by

$$Tx = \begin{cases} \dfrac{x^2}{2}, & \text{if} \quad x \in [0,1], \\ \\ \dfrac{1}{4}, & \text{if} \quad x > 1, \end{cases}$$

is a C*-algebra-valued generalized Hardy-Rogers contractive mapping on X for $u_1 = \frac{1}{2} \cdot 1_{\mathbb{A}}$ and $u_i = \frac{1}{9} \cdot 1_{\mathbb{A}}$ for all $i \in \{2, 3, 4, 5\}$.

Now, we can show that all the conditions of Theorem 3.3 are satisfied. Consequently, T has a unique fixed point, namely $x^* = 0$.

Remark 3.5. Note that in Theorem 3.3, if we take

(1) $u_2 = u_3 = u_4 = u_5 = 0_{\mathbb{A}}$, we obtain an extension of Banach contraction [1] to C*-algebra-valued generalized metric space.
(2) $u_2 = u_3$ and $u_1 = u_4 = u_5 = 0_{\mathbb{A}}$, we obtain an extension of Kannan contraction [20] to C*-algebra-valued generalized metric space.
(3) $u_4 = u_5$ and $u_1 = u_2 = u_3 = 0_{\mathbb{A}}$, we obtain an extension of Chatterjea contraction [10] to C*-algebra-valued generalized metric space.
(4) $u_4 = u_5 = 0_{\mathbb{A}}$, we obtain an extension of Reich contraction [33] to C*-algebra-valued generalized metric space.

Now, we extend the Hardy-Rogers contraction principle to the class of C*-algebra-valued generalized metric space endowed with a partial order.

Definition 3.6. Let $(X, \mathbb{A}, \mathcal{D})$ be a C*-algebra-valued generalized metric space and $\preceq_{\mathbb{A}}$ be a partial order on X.

(1) The space $(X, \mathbb{A}, \mathcal{D}, \preceq_{\mathbb{A}})$ is called an C*-algebra-valued ordered generalized metric space.
(2) Two elements x and y in X are said to be comparable if $x \preceq_{\mathbb{A}} y$ or $y \preceq_{\mathbb{A}} x$.
(3) A self-mapping T on X is said to be non-decreasing or order preserving mapping if $Tx \preceq_{\mathbb{A}} Ty$, whenever $x \preceq_{\mathbb{A}} y$.
(4) The space $(X, \mathbb{A}, \mathcal{D}, \preceq_{\mathbb{A}})$ is said to be regular with respect to $\mathbb{A}$, if for every non-decreasing sequence $(x_n)_{n \in \mathbb{N}}$ in X that $\mathcal{D}$-converges to some $x \in X$ with respect to $\mathbb{A}$, we have $x_n \preceq_{\mathbb{A}} x$ for all $n \in \mathbb{N}$.

Theorem 3.7. *Let $(X, \mathbb{A}, \mathcal{D}, \preceq_{\mathbb{A}})$ be a $\mathcal{D}$-complete C*-algebra-valued ordered generalized metric space and $T : X \to X$ a C*-algebra-valued generalized ordered contractive of Hardy-Rogers type, that is, for all comparable $x, y \in X$,* *we have*

$$\mathcal{D}(Tx,Ty) \preceq u_1\mathcal{D}(x,y) + u_2\mathcal{D}(x,Tx) + u_3\mathcal{D}(y,Ty) + u_4\mathcal{D}(y,Tx) + u_5\mathcal{D}(x,Ty), \tag{3.2}$$

where $u_i \in \mathbb{A}^+$ *for all* $i \in \{1,2,3,4,5\}$ *and* $\|u_2\|\|c\| + \|u_4\| < 1$.
If the following conditions are satisfied

(1) *There exists* $x_0 \in X$ *such that* $\delta(\mathcal{D},T,x_0) < \infty$ *and* $x_0 \preceq_{\mathbb{A}} Tx_0$;
(2) T *is non-decreasing;*
(3) X *is regular with respect to* $\mathbb{A}$,

then, T *has a fixed point in* X. *Moreover, if* $\sum\limits_{i=1}^{5} \|u_i\| < 1$, *then the set of fixed points of* T *is totally ordered if and only if* T *has a unique fixed point.*

Proof. Let $x_0 \in X$ satisfying the condition (1) of the above theorem. We define the Picard sequence $\{x_n\}_{n\in\mathbb{N}}$ by $x_n = T^n x_0$ for all $n \in \mathbb{N}$. As T is non-decreasing and $x_0 \preceq_{\mathbb{A}} Tx_0$, we deduce that

$$x_0 \preceq_{\mathbb{A}} x_1 \preceq_{\mathbb{A}} x_2 \preceq_{\mathbb{A}} \cdots \preceq_{\mathbb{A}} x_n \preceq_{\mathbb{A}} x_{n+1} \preceq_{\mathbb{A}} \ldots, \tag{3.3}$$

that is, the two elements $T^{n+i}x_0$ and $T^{n+j}x_0$ are comparable in X for all i, j and $n \in \mathbb{N}$.

Using reasoning similar to the proof of Theorem 3.3, we conclude that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is a $\mathcal{D}$-Cauchy sequence with respect to $\mathbb{A}$. Since $(X,\mathbb{A},\mathcal{D})$ is a $\mathcal{D}$-complete with respect to $\mathbb{A}$, there exists a point $x^* \in X$ such that $\{x_n\}$ is $\mathcal{D}$-convergent to x^* with respect to $\mathbb{A}$.

Now, we show that x^* is a fixed point of T, that is, $Tx^* = x^*$. Indeed, as X is regular with respect to $\mathbb{A}$ and according to (3.3), we have $x_n \preceq_{\mathbb{A}} x^*$ which implies that $Tx_n \preceq_{\mathbb{A}} Tx^*$. Thus, Tx_n and Tx^* are comparable. By using (3.2), we get

$$\mathcal{D}(Tx^*,Tx_n) \preceq u_1\mathcal{D}(x^*,x_n) + u_2\mathcal{D}(x^*,Tx^*) + u_3\mathcal{D}(x_n,Tx_n) + u_4\mathcal{D}(x_n,Tx^*) + u_5\mathcal{D}(x^*,Tx_n).$$

According to Lemma 2.1, we have

$$\begin{aligned}\|\mathcal{D}(Tx^*,T^{n+1}x_0)\| \leqslant\ & \|u_1\|\|\mathcal{D}(x^*,T^n x_0)\| + \|u_2\|\|\mathcal{D}(x^*,Tx^*)\| \\ & + \|u_3\|\|\mathcal{D}(T^n x_0,T^{n+1}x_0)\| + \|u_4\|\|\mathcal{D}(T^n x_0,Tx^*)\| \\ & + \|u_5\|\|\mathcal{D}(x^*,T^{n+1}x_0)\|.\end{aligned}$$

By setting

$$\begin{cases} \lambda_n & = & \|\mathcal{D}(Tx^*, T^n x_0)\|, \\ \gamma_n & = & \|u_1\|\|\mathcal{D}(x^*, T^n x_0)\| + \|u_3\|\|\mathcal{D}(T^n x_0, T^{n+1}x_0)\| \\ & + & |u_5\|\|\mathcal{D}(x^*, T^{n+1}x_0)\| \\ \text{and} & & \\ \beta & = & \|u_2\|\|\mathcal{D}(x^*, Tx^*)\|, \end{cases}$$

we get $\lambda_{n+1} \leqslant \|u_4\|\lambda_n + \gamma_n + \beta$. Since $\lim\limits_{n\to\infty} \gamma_n = 0$, we have for all $\varepsilon > \dfrac{\beta}{1 - \|u_4\|} \geqslant 0$, there exists an integer N_ε such that $\gamma_n \leqslant \varepsilon(1 - \|u_4\|) - \beta$ for all $n \geq N_\varepsilon$. Then, we get

$$\lambda_{n+1} \leqslant \|u_4\|\lambda_n + \varepsilon(1 - \|u_4\|) \quad \text{for all} \quad n \geq N_\varepsilon.$$

By Lemma 2.4, we have $0 \leqslant \limsup\limits_{n\to\infty} \lambda_n \leq \varepsilon$ for all $\varepsilon > \dfrac{\beta}{1 - \|u_4\|}$.

Then,

$$0 \leqslant \limsup_{n\to\infty} \lambda_n \leqslant \frac{\beta}{1 - \|u_4\|}.$$

Since $T^{n+1}x_0 \to x^*$ as $n \to \infty$, we have

$$\begin{aligned} \mathcal{D}(Tx^*, x^*) &\preceq \big(\limsup_{n\to\infty} \|\mathcal{D}(Tx^*, T^{n+1}x_0)\|\big) \cdot c \\ &\preceq \big(\limsup_{n\to\infty} \lambda_n\big) \cdot c \\ &\preceq \frac{\beta}{1 - \|u_4\|} \cdot c. \end{aligned}$$

By Lemma 2.1, we get

$$\begin{aligned} \|\mathcal{D}(Tx^*, x^*)\| &\leqslant \frac{\beta\|c\|}{1 - \|u_4\|} \\ &\leqslant \frac{\|c\|u_2\|}{1 - \|u_4\|}\|\mathcal{D}(Tx^*, x^*)\|. \end{aligned}$$

Thus,

$$\big(1 - \frac{\|c\|\|u_2\|}{1 - \|u_4\|}\big)\|\mathcal{D}(Tx^*, x^*)\| \leqslant 0.$$

As $\dfrac{\|c\|\|u_2\|}{1 - \|u_4\|} < 1$, then $\|\mathcal{D}(Tx^*, x^*)\| = 0$. This gives $\mathcal{D}(Tx^*, x^*) = 0_{\mathbb{A}}$, therefore $Tx^* = x^*$.

Now, we assume that $\sum\limits_{i=1}^{5} \|u_i\| < 1$ and we suppose that the set of fixed points of T is totally ordered. By using a reasoning by the absurd, we claim that the fixed point of T is unique. Indeed, suppose that there exists $y^* \in X$ another fixed point of T such that $x^* \neq y^*$.

Since $x^* \preceq_{\mathbb{A}} x^*$ and $y^* \preceq_{\mathbb{A}} y^*$, we obtain with the same way as the proof of Theorem 3.3, $\mathcal{D}(x^*, x^*) = 0_{\mathbb{A}}$ and $\mathcal{D}(y^*, y^*) = 0_{\mathbb{A}}$. Since x^* and y^* are comparable, and using again (3.2), with $x = x^*$ and $y = y^*$, we get

$$\begin{aligned}\mathcal{D}(x^*, y^*) &= \mathcal{D}(Tx^*, Ty^*) \\ &\preceq u_1\mathcal{D}(x^*, y^*) + u_2\mathcal{D}(x^*, Tx^*) \\ &\quad + u_3\mathcal{D}(y^*, Ty^*) + u_4\mathcal{D}(y^*, Tx^*) + u_5\mathcal{D}(x^*, Ty^*) \\ &= u_1\mathcal{D}(x^*, y^*) + u_4\mathcal{D}(x^*, y^*) + u_5\mathcal{D}(x^*, y^*).\end{aligned}$$

Then, we have

$$\|\mathcal{D}(x^*, y^*)\| \leqslant \|u_1\|\|\mathcal{D}(x^*, y^*)\| + \|u_4\|\|\mathcal{D}(x^*, y^*)\| + \|u_5\|\|\mathcal{D}(x^*, y^*)\|.$$

So,

$$\Big(1 - (\|u_1\| + \|u_4\| + \|u_5\|)\Big)\|\mathcal{D}(x^*, y^*)\| \leqslant 0.$$

Since $\|u_1\| + \|u_4\| + \|u_5\| < 1$, so $\|\mathcal{D}(x^*, y^*)\| = 0$. Therefore, $\mathcal{D}(x^*, y^*) = 0_{\mathbb{A}}$ and $x^* = y^*$.

Conversely, if T has a unique fixed point, then the set of fixed points of T, being a singleton, is totally ordered. □

Definition 3.8. Let $(X, \mathbb{A}, \mathcal{D})$ be a C*-algebra-valued generalized metric space. We say that a mapping $T : X \to X$ is a C*-algebra-valued generalized Ćirić-Jachymski-Matkowski contraction (CJM, for short) mapping on X, if the following hold

(1) For all $\varepsilon \succ 0_{\mathbb{A}}$, there exists an element $\sigma \succ 0_{\mathbb{A}}$ satisfying

$$\mathcal{D}(x, y) \prec \varepsilon + \sigma \Longrightarrow \mathcal{D}(Tx, Ty) \prec \varepsilon \quad \text{for all } x, y \in X,$$

(2) $x \neq y \Longrightarrow \mathcal{D}(Tx, Ty) \prec \mathcal{D}(x, y)$ for all $x, y \in X$.

Remark 3.9. Note that if $\mathbb{A} = \mathbb{R}$, then the Definition 3.8 is reduced to a standard contraction according to Ćirić-Jachymski-Matkowski [11].

Theorem 3.10. *Let $(X, \mathbb{A}, \mathcal{D})$ be a $\mathcal{D}$-complete C*-algebra-valued generalized metric space and $T : X \to X$ be a C*-algebra-valued generalized CJM-contraction mapping on X. Then T has either a unique fixed point or a periodic point in X.*

Proof. Choose $x_0 \in X$ an arbitrary element and construct the sequence $\{x_n\}_{n\in\mathbb{N}}$ defined by the following iterative scheme $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$.

If $x_{n+p} = x_n$ for some $n \in \mathbb{N}$ with $p \geqslant 1$ an arbitrary element of $\mathbb{N}$, then $T^p x_0 = x_0$, therefore x_0 is a periodic point of T.

If $x_{n+p} \neq x_n$ for all $n \in \mathbb{N}$ and $p \geqslant 1$. Using (2) of the Definition 3.8, we get

$$\mathcal{D}\big(x_{n+p+1}, x_{n+1}\big) = \mathcal{D}\big(Tx_{n+p}, Tx_n\big) \prec \mathcal{D}(x_{n+p}, x_n) \text{ for all } n \in \mathbb{N},$$

that is, the sequence $\big\{\mathcal{D}_n\big\}_{n\in\mathbb{N}}$, where $\mathcal{D}_n = \mathcal{D}(x_{n+p}, x_n)$ for all $n \in \mathbb{N}$, is bounded below and strictly decreasing. Thus, it is $\mathcal{D}$-convergent with respect to $\mathbb{A}$ and $\lim\limits_{n\to\infty} \mathcal{D}_n = \ell \succeq 0_{\mathbb{A}}$.

Now, we claim that $\ell = 0_{\mathbb{A}}$, which implies that $\{x_n\}_{n\in\mathbb{N}}$ is a $\mathcal{D}$-Cauchy sequence with respect to $\mathbb{A}$. Indeed, assume that $\ell \succ 0_{\mathbb{A}}$. Then there exists $n_0 \in \mathbb{N}$ and $\sigma \succ 0_{\mathbb{A}}$ such that

$$\ell \prec \mathcal{D}_n \prec \ell + \sigma \text{ for all } n \geqslant n_0. \tag{3.4}$$

Using (1) of the Definition 3.8, we obtain

$$\mathcal{D}(Tx_{n+p}, Tx_n) \prec \ell \text{ for all } n \geqslant n_0.$$

This means that $\mathcal{D}_{n+1} \prec \ell$ for all $n \geqslant n_0$, which is in contradiction with (3.4). Thus, $\lim\limits_{n\to\infty} \mathcal{D}(x_{n+p}, x_n) = 0_{\mathbb{A}}$, that is to say that $\{x_n\}_{n\in\mathbb{N}}$ is a $\mathcal{D}$-Cauchy sequence with respect to $\mathbb{A}$. Since $(X, \mathbb{A}, \mathcal{D})$ is $\mathcal{D}$-complete, then there exists $x^* \in X$ such that $\{x_n\}$ is $\mathcal{D}$-convergent to x^* with respect to $\mathbb{A}$.

Now, let we show that x^* is a fixed point of T, that is, $Tx^* = x^*$. The sequence $\{\alpha_n\}_{n\in\mathbb{N}}$, where $\alpha_n = \mathcal{D}(x_n, x^*)$, is decreasing and is $\mathcal{D}$-convergent to $0_{\mathbb{A}}$ with respect to $\mathbb{A}$, then there exists $n_1 \in \mathbb{N}$ such that $\alpha_n \succ 0_{\mathbb{A}}$ for all $n \geqslant n_1$. Otherwise, if $\alpha_n = 0_{\mathbb{A}}$ for all $n \in \mathbb{N}$, then $x_n = x_{n+1} = x^*$. Therefore, $Tx^* = Tx_n = x^*$, that is, x^* is a fixed point of T.

Consequently, there exists $\sigma \succ 0_{\mathbb{A}}$ such that $\mathcal{D}(x_n, x^*) \prec \alpha_n + \sigma$ for all $n \geqslant n_1$.
By using (1) of the Definition 3.8, we get

$$\mathcal{D}(Tx_n, Tx^*) \prec \alpha_n.$$

Thus,

$$\|\mathcal{D}(Tx_n, Tx^*)\| < \|\alpha_n\| \text{ for all } n \geqslant n_1 \text{ (by the Lemma 2.1).}$$

On the other hand, since $x_n \to x^*$ as $n \to \infty$, there exists $c \succ 0_{\mathbb{A}}$ such that

$$\begin{aligned}
\mathcal{D}(Tx^*, x^*) &\preceq \big(\limsup_{n\to\infty} \|\mathcal{D}(Tx^*, x_{n+1})\|\big) \cdot c \\
&= \big(\limsup_{n\to\infty} \|\mathcal{D}(Tx^*, Tx_n)\|\big) \cdot c \\
&\preceq \big(\limsup_{n\to\infty} \|\alpha_n\|\big) \cdot c \\
&= 0_{\mathbb{A}},
\end{aligned}$$

this implies that $\mathcal{D}(Tx^*, x^*) = 0_{\mathbb{A}}$, that is, $Tx^* = x^*$. Therefore, x^* is a fixed point of T.

Now, let us show the uniqueness of the fixed point of T. Suppose that x^*, y^* are two fixed points of T in X. If $x^* \neq y^*$, then $\mathcal{D}(Tx^*, Ty^*) \prec \mathcal{D}(x^*, y^*)$, that implies $\mathcal{D}(x^*, y^*) \prec \mathcal{D}(x^*, y^*)$ which is a contradiction. □

Example 3.11. In the same context of Example 3.4, we define a mapping $T : X \to X$ by

$$Tx = \begin{cases} \frac{x^4}{3}, & \text{if} \quad x \in [0,1], \\ 0, & \text{if} \quad x > 1. \end{cases}$$

Then the mapping T is a C*-algebra-valued generalized CJM-contraction mapping on X. Since all conditions of Theorem 3.10 are satisfied, T has a unique fixed point, namely $x^* = 0$.

Remark 3.12. Note that the C*-algebra-valued generalized metric spaces include both JS-metric spaces and C*-algebra-valued b-metric space, we can conclude that our results generalize those presented in [11, 15].

4. Applications

In the above, we use Theorem 3.3 to solve an operator equation and an integral equation.

Theorem 4.1. *Consider the following integral equation*

$$x(t) = \alpha \int_E K(t,s)\Psi\big(s, x(s)\big)ds, \ t \in E, \tag{4.1}$$

where E is a Lebesgue measurable set and α is a real number. Assume the following

(1) $\Psi : E \times \mathbb{R} \to \mathbb{R}$ *is continuous and satisfied* $|\Psi(t,x)| \leqslant \lambda(t)|x|$ *for all* $t \in E$ *and* $x \in \mathbb{R}$*, where function* λ *is continuous on* E *such that* $0 \leqslant \alpha\|\lambda\| < \frac{1}{2}$.

(2) $K : E \times \mathbb{R} \to \mathbb{R}$ *is continuous and* $\sup_{t\in E} \int_E |K(t,s)|ds \leq 1$.

Then, the integral equation (4.1) *has a unique solution* x^* *in* X.

Proof. Let $X = \mathcal{C}(E, \mathbb{R})$ the set of bounded and continuous functions on E with the following norm: $\|x\|_\infty = \sup_{t\in E} |x(t)|$ for all $x \in X$, then it forms a C*-algebra.

We define the mapping $\mathcal{D} : X \times X \to X$ by

$$\mathcal{D}(x,y) = \begin{cases} 0, & \text{if } x = y, \\ \max\big\{\|x\|_\infty, \|y\|_\infty\big\}.u^*u, & \text{otherwise,} \end{cases}$$

where u is an element of X. So, $(X, X, \mathcal{D})$ is a $\mathcal{D}$-complete C^*-algebra-valued generalized metric space with $c = 1_X$. Let's define the mapping $T : X \to X$ by

$$Tx(t) = \alpha \int_E K(t,s)\Psi\big(s, x(s)\big)ds \text{ for all } t \in E \text{ and } x \in X.$$

Now, our problem (4.1) can be reformulated as the search for a fixed point of T. Thus,

$$\begin{aligned} |Tx(t)| &= |\alpha \int_E K(t,s)\Psi(s,x(s))ds| \\ &\leq \alpha \int_E |K(t,s)\Psi(s,x(s))|ds \\ &= \alpha \int_E |K(t,s)||\Psi\big(s,x(s)\big)|ds \\ &\leq \alpha\|\lambda\|\|x\|_\infty \sup_{t\in E} \int_E |K(t,s)|ds \\ &\leqslant \alpha\|\lambda\|\|x\|_\infty. \end{aligned}$$

Therefore, $\|Tx\|_\infty \leqslant \alpha\|\lambda\|\|x\|_\infty$ for all $x \in X$. Similarly, we have

$$\|Ty\|_\infty \leqslant \alpha\|\lambda\|\|y\|_\infty$$

for all $x \in X$.

Now, if $Tx \neq Ty$, we have

$$\begin{aligned} \mathcal{D}(Tx,Ty) &= \max\{\ \|Tx\|_\infty, \|Ty\|_\infty\} \cdot u^*u \\ &\preceq \alpha\|\lambda\| \max\{\|x\|_\infty, \|y\|_\infty\} \cdot u^*u \\ &= \alpha\|\lambda\| \cdot \mathcal{D}(x,y). \end{aligned}$$

If $Tx = Ty$, we have $\mathcal{D}(Tx,Ty) = 0$, so $\mathcal{D}(Tx,Ty) \preceq \alpha\|\lambda\|.\mathcal{D}(x,y)$. Therefore,

$$\mathcal{D}(Tx,Ty) \preceq \alpha.\mathcal{D}(x,y)$$

for all $x \in X$ and $y \in X$, that is, to say

$$\begin{aligned} \mathcal{D}(Tx,Ty) \preceq\ & u_1\mathcal{D}(x,y) + u_2\mathcal{D}(x,Tx) + u_3\mathcal{D}(y,Ty) \\ &+ u_4\mathcal{D}(y,Tx) + u_5\mathcal{D}(x,Ty) \end{aligned}$$

for all $x, y \in X$, where $u_1 = \alpha\|\lambda\|.1_X$ and $u_2 = u_3 = u_4 = u_5 = \dfrac{\alpha\|\lambda\|}{5}.1_X$.

On the other hand, we have

$$\|Tx\|_\infty \leqslant \alpha\|\lambda\|\|x\|_\infty$$

for all $x \in X$. By induction reasoning, we get

$$\|T^n x\|_\infty \leqslant \alpha^{n-1}\|\lambda\|^{n-1}\|x\|_\infty$$

for all $(x,n) \in X \times \mathbb{N}$. So, for all $i, j \in \mathbb{N}$, we have

$$\|\mathcal{D}(T^i x_0, T^j x_0)\| \leqslant \max\{\alpha^{i-1}\|\lambda\|^{i-1}\|x_0\|_\infty, \alpha^{j-1}\|\lambda\|^{j-1}\|x_0\|_\infty\}.$$

Since

$$\max\left\{\alpha^{i-1}\|\lambda\|^{i-1}\|x_0\|_\infty, \alpha^{j-1}\|\lambda\|^{j-1}\|x_0\|_\infty\right\} \to 0 \text{ as } i, j \to \infty,$$

we get $\delta(\mathcal{D}, T, x_0) < \infty$. We have also

$$\frac{\|c\|\|u_2\|}{1 - \|u_4\|} = \frac{\alpha\|\lambda\|}{5 - \alpha\|\lambda\|} < 1.$$

Consequently, all the conditions of Theorem 3.3 are fully satisfied, and the integral equation (4.1) has a unique solution x^* in X. □

Theorem 4.2. *Consider the following nonlinear operator equation*

$$X = \sum_{n=1}^{\infty} A_n^* f(X) A_n, \tag{4.2}$$

where $A_1, A_2, A_3, .., A_n \in L(H)$ *with* $L(H)$ *denoting the set of continuous linear operators on a Hilbert space* H*, and satisfy* $\sum_{n=1}^{\infty} \|A_n\|^2 < 1$*. The function* $f : L(H) \to L(H)$ *is such that* $\|f(X)\| \leqslant \alpha\|X\|$ *for all* $X \in L(H)$*, where* $\alpha < \frac{5}{9}$*. Then, the operator equation* (4.2) *has a unique solution* X^* *in* $L(H)$*.*

Proof. Let $\lambda = \sum_{n=1}^{\infty} \|A_n\|^2$. Clear that if $\lambda = 0$, then the $A_n = 0$ for all $n \in \mathbb{N}^*$ and the equation (4.2) has a unique solution X^* in $L(H)$.

Now, if $\lambda > 0$, we define the mapping $\mathcal{D} : L(H) \times L(H) \to L(H)$ by

$$\begin{cases} \mathcal{D}(X,0) = \mathcal{D}(0,X) = \frac{1}{2}\|X\|.A^*A & \text{for all } X \in L(H), \\ \mathcal{D}(X,Y) = \Big(\|X\| + \|Y\|\Big).A^*A & \text{if } X \neq 0 \text{ and } Y \neq 0, \end{cases}$$

where $A \in L(H)$. Then, $(L(H), L(H), \mathcal{D})$ is a $\mathcal{D}$-complete C*-algebra-valued generalized metric space. Consider the mapping $T : L(H) \to L(H)$ defined by

$$T(X) = \sum_{n=1}^{\infty} A_n^* f(X) A_n.$$

Then, if $T(X) \neq 0$ and $T(Y) \neq 0$, then

$$\begin{aligned}
\mathcal{D}\big(T(X), T(Y)\big) &= \Big(\|T(X)\| + \|T(Y)\|\Big) \cdot A^*A \\
&= \left(\|\sum_{n=1}^{\infty} A_n^* f(X) A_n\| + \|\sum_{n=1}^{\infty} A_n^* f(Y) A_n\|\right) .A^*A \\
&\preceq \left(\sum_{n=1}^{\infty} \|A_n^* f(X) A_n\| + \sum_{n=1}^{\infty} \|A_n^* f(Y) A_n\|\right) \cdot A^*A \\
&\preceq \left(\sum_{n=1}^{\infty} \|A_n\|^2 \|f(X)\| + \sum_{n=1}^{\infty} \|A_n\|^2 \|f(Y)\|\right) \cdot A^*A \\
&\preceq \alpha\lambda\big(\|X\| + \|Y\|\big) \cdot A^*A \\
&= \alpha\lambda \mathcal{D}(X, Y).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\mathcal{D}\big(0, T(X)\big) &= \mathcal{D}\big(T(X), 0\big) \\
&= \frac{\|T(X)\|}{2} \cdot A^*A \\
&\preceq \frac{1}{2} \sum_{n=1}^{\infty} \|A_n\|^2 \|f(X)\| \cdot A^*A \\
&\preceq \frac{1}{2} \alpha\lambda \|X\| \cdot A^*A \\
&\preceq \alpha\lambda \cdot \mathcal{D}(0, X).
\end{aligned}$$

Then, for both cases, we have

$$\begin{aligned}
\mathcal{D}\big(T(X), T(Y)\big) \preceq{}& U_1 \mathcal{D}(X, Y) + U_2 \mathcal{D}\big(X, T(X)\big) + U_3 \mathcal{D}\big(Y, T(Y)\big) \\
&+ U_4 \mathcal{D}\big(Y, T(X)\big) + U_5 \mathcal{D}\big(X, T(Y)\big)
\end{aligned}$$

for all $X, Y \in L(X)$, where $U_1 = \alpha\lambda.1_{L(H)}$ and $U_i = \dfrac{\alpha\lambda}{5}.1_{L(H)}$ for all $i \in \{2, 3, 4, 5\}$. Consequently, it is easy to verify that all the postulates of Theorem 3.3 are satisfied and that the operator equation (4.2) has a unique solution X^* in $L(H)$. □

5. Conclusion

This work proposes an extension of the Hardy-Rogers and Ćirić-Jachymski-Matkowski (CJM) contraction theorem in a C*-algebra-valued generalized metric. Indeed, the results obtained represent both a generalization and an improvement of several previous works. Based on our results, we have derived outcomes for C*-algebra-valued variants of the contractions of Kannan [20], Chatterjee [10], Reich [33], and Banach [1]. This work concludes with the application of the obtained results to the resolution of an integral equation and an operator equation.

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