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ORBITAL FIXED POINT THEOREM IN G-MENGER SPACES

Lahcen OUMERTOU[1], Mohammed Lamarti SEFIAN[2] and Ismail TAHIRI[3]

[1]Department of Mathematics and Computing Sciences,
Higher Normal School, AMCS Research Team, Tetouan, Morocoo
e-mail: `lahcen.oumertou@etu.uae.ac.ma`

[2]Department of Mathematics and Computing Sciences,
Higher Normal School, AMCS Research Team, Tetouan, Morocoo
e-mail: `lamarti.mohammed.sefian@uae.ac.ma`

[3]Department of Mathematics and Computing Sciences,
Higher Normal School, AMCS Research Team, Tetouan, Morocoo
e-mail: `istahiri@uae.ac.ma`

Abstract. In this study, we present a novel fixed point property for self mappings on G-Menger space under ϖ-probabilistic contraction and the boundedness of orbit conditions. In order to realize our idea, we establish a link between the t-norm of H-type and the boundedness of orbit also, we investigate the finding results to prove the existence and uniqueness of a common fixed point for a family of mapping in this kind of spaces. Our discoveries extend many results in the literature.

1. Introduction

Mathematicians in the early 19th century investigated various spaces, primarily function spaces, and developed diverse concepts of convergence. To formalize these notions, French mathematician Frchet [9] introduced the axiomatic foundations of metrics in 1906. Hausdorff [12] subsequently formalized the term metric space in 1914, establishing a deterministic framework for distance measurement. However, the deterministic representation

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[0]Corresponding author: L. OUMERTOU(lahcen.oumertou@etu.uae.ac.ma).

of distance as a single number proves overly idealized in many practical contexts. When distance cannot be predicted by a single value, a probabilistic approach becomes essential.

In 1942, Austrian mathematician. Menger [17] proposed the concept of statistical metric spaces, now called probabilistic metric spaces, to address uncertainties in spatial distances. His approach replaces the distance function with a probabilistic distance function $P(d(u,v) \leqslant t) = F_{u,v}(t)$ $(t > 0)$, where $F_{u,v}(t)$ represents the probability that the distance between u and v is less than t. Thus, Menger space theory is fundamental to probabilistic functional analysis as it conceptualizes distance probabilistically. These structures were extensively developed by Schweizer and Sklar [21].

Sehgal [22] pioneered work on probabilistic metric spaces by introducing a natural probabilistic version of Banach contraction [5]. Sehgal and Bharucha-Reid [23] utilized this contraction to prove the first fixed point theorem in Menger spaces in 1972. Hicks [13] introduced another probabilistic contraction mapping in 1983, followed by generalizations from several researchers [2, 3, 7, 8].

Concurrently, Mustafa and Sims [19] developed G-metric spaces as extensions of ordinary metric spaces. Mustafa *et al.* [10, 18, 20] derived several fixed point theorems for various contractions. In 2014, Zhou *et al.* [29] introduced a probabilistic version of G-metric spaces, termed G-Menger spaces, studying their topological properties and establishing fixed point theorems under probabilistic linear contractions. These works were extended in [4, 15, 16, 25].

While probabilistic nonlinear contractions naturally generalize linear contractions, methods for probabilistic λ-contractions [6] do not extend to nonlinear cases. Ćirić [6] initiated fixed point theory for ϖ-probabilistic contractions, but Jachymski [14] identified a limitation in a key lemma of [6] via counterexample. Although advances were made in Menger spaces [14, 24], G-Menger spaces remain underexplored for nonlinear contractions under bounded orbit conditions.

This paper addresses this gap by establishing fixed point theorems for nonlinear ϖ-probabilistic contractions in complete G-Menger spaces with bounded orbit conditions. We prove that a Picard iteration for such contractions is Cauchy if and only if its orbit is bounded. Our results substantially extend and generalize works by Ćirić [6], Jachymski [14], Tian *et al.* [24] and Zhou *et al.* [29]. Furthermore, we demonstrate common fixed point theorems for families of such mappings.

Fixed point theory is a fundamental tool in nonlinear analysis with applications in differential equations, optimization, and dynamical modeling. Recent works demonstrate its utility in analyzing chaotic dynamical systems

[26], graph-based metric spaces [27], proximal point methods [28], and Prei-type operators [1]. Our results provide a robust probabilistic framework for such applications in generalized spaces.

The structure of this document is as follows; Section 2 presents preliminaries on Menger and G-Menger spaces. Section 3 proves fixed point theorems under ϖ-probabilistic contractions and bounded orbit conditions. We also derive some analogous fixed point theorems in Menger spaces as a relative consequence of these findings. Section 4 extends these to common fixed points for families of mappings.

2. Preliminaries

We first bring notions definitions and known results, which are related to our work.

Suppose that $\mathbb{R}^+ = [0, \infty)$, $\mathbb{N}$ be the set of all naturals numbers. And let Π^+ be the set of all distribution functions $h : [0, \infty] \to [0, 1]$ such that

(1) h is a non-decreasing;
(2) h is left-continuous;
(3) $h(0) = 0$ and $h(\infty) = 1$.

The subset $\Lambda^+ \subset \Pi^+$ is the set $\Lambda^+ = \{h \in \Pi^+ : \lim_{\sigma\to\infty} h(\sigma) = 1\}$. A basic element of Λ^+ is the function given by

$$\varepsilon_0(\sigma) = \begin{cases} 0 & \text{if } \sigma = 0, \\ 1 & \text{if } \sigma > 0. \end{cases}$$

Definition 2.1. ([21]) A triangular norm (t-norm for short) is a binary operation Θ on the unit interval $[0, 1]$ such that for all $\lambda, \mu, \nu, \kappa \in [0, 1]$ the following four axioms are satisfied:

(1) $\Theta(\lambda, \mu) = \Theta(\mu, \lambda)$;
(2) $\Theta(\lambda, \Theta(\mu, \nu)) = \Theta(\Theta(\lambda, \mu), \nu)$;
(3) $\Theta(\lambda, \mu) \leq \Theta(\nu, \kappa)$, whenever $\lambda \leq \nu$ and $\mu \leq \kappa$;
(4) $\Theta(\lambda, 1) = \lambda$.

Example 2.2. Generic examples of t-norm are $\Theta_M(\lambda, \mu) = \min\{\lambda, \mu\}$ and $\Theta_P(\lambda, \mu) = \lambda\mu$.

If Θ is a t-norm, $\mu \in [0, 1]$ and $n \in \mathbb{N}$, then we shall write

$$\Theta^n(\mu) = \begin{cases} 1 & \text{if} \quad n = 0, \\ \Theta(\Theta^{n-1}(\mu), \mu) & otherwise. \end{cases}$$

Definition 2.3. ([21]) A t-norm Θ is of H-type if the family $\{\Theta^n\}_{n\in\mathbb{N}}$ is equicontinuous at the point $\mu = 1$, it means that:

$$\forall \varepsilon \in (0, 1) \; \exists \varsigma \in (0, 1) : \sigma > 1 - \varsigma \quad \Rightarrow \quad \Theta^n(\sigma) > 1 - \varepsilon \quad (\forall n \geqslant 1).$$

Remark 2.4. ([11]) A typical example of a t-norm of H-type is Θ_M.

Definition 2.5. ([17]) A Menger space is a triplet $(\mathcal{Z}, F, \Theta)$, where $\mathcal{Z}$ is nonempty set, Θ is a continuous t-norm and F is a mapping from $\mathcal{Z} \times \mathcal{Z} \rightarrow \Pi^+$ satisfying the following conditions:

(1) $F_{\rho,\beta}(\sigma) = 1$ for all $\sigma > 0$ if and only if $\rho = \beta$;
(2) $F_{\rho,\beta}(\sigma) = F_{\beta,\rho}(\sigma)$ for all $\rho, \beta \in \mathcal{Z}$ and $\sigma > 0$;
(3) $F_{\rho,\gamma}(\sigma + s) \geqslant \Theta(F_{\rho,\beta}(\sigma), F_{\beta,\gamma}(s))$ for all $\rho, \beta, \gamma \in \mathcal{Z}$ and $\sigma, s \geqslant 0$.

Remark 2.6. ([17]) Every metric space is a Menger space. In fact, let $(\mathcal{Z}, d)$ be a metric space. Define

$$F_{\rho,\beta}(\sigma) = \varepsilon_0(\sigma - d(\rho, \beta)) \text{ for all } \rho, \beta \in \mathcal{Z} \text{ and } \sigma > 0.$$

Then the triplet $(\mathcal{Z}, F, \Theta_M)$ is a Menger space.

In 2006, Mustapha and Sims [19] introduced the concept of G-metric space, More recently, Zhou *et al.* [29] defined the notion of a G-Menger space as a generalization of a both concept of Menger space and G-metric space.

Definition 2.7. ([19]) A G-metric space is a couple $(\mathcal{Z}, \mathcal{F})$, where $\mathcal{Z}$ is a nonempty set, and $\mathcal{F} : \mathcal{Z} \times \mathcal{Z} \times \mathcal{Z} \rightarrow \mathbb{R}^+$ is a function satisfying:

(1) $\mathcal{F}(\rho, \beta, \gamma) = 0$ for all $\rho, \beta, \gamma \in \mathcal{Z}$ if and only if $\rho = \beta = \gamma$;
(2) $\mathcal{F}(\rho, \rho, \beta) \leqslant \mathcal{F}(\rho, \beta, \gamma)$ for all $\rho, \beta, \gamma \in \mathcal{Z}$ with $\gamma \neq \beta$;
(3) $\mathcal{F}(\rho, \beta, \gamma) = \mathcal{F}(\rho, \gamma, \beta) = \mathcal{F}(\beta, \rho, \gamma) = \cdots$ for all $\rho, \beta, \gamma \in \mathcal{Z}$;
(4) $\mathcal{F}(\rho, \beta, \gamma) \leqslant \mathcal{F}(\rho, a, a) + \mathcal{F}(a, \beta, \gamma)$ for all $\rho, \beta, \gamma, a \in \mathcal{Z}$.

Example 2.8. ([20]) Let $\mathcal{Z} = [0, \infty)$. Define the function $\mathcal{F} : \mathcal{Z}^3 \rightarrow [0, \infty)$ by

$$\mathcal{F}(\rho, \beta, \gamma) = |\rho\beta| + |\beta\gamma| + |\gamma\rho|$$

for all $\rho, \beta, \gamma \in \mathcal{Z}$. Hence $(\mathcal{Z}, \mathcal{F})$ is a G-metric space.

Definition 2.9. ([29]) A G-Menger space is a triplet $(\mathcal{Z}, \mathcal{F}, \Theta)$, where $\mathcal{Z}$ is a nonempty set, Θ is a continuous t-norm and $\mathcal{F}$: $\mathcal{Z} \times \mathcal{Z} \times \mathcal{Z} \rightarrow \Pi^+$ is a mapping satisfying the following conditions

(1) $\mathcal{F}_{\rho,\beta,\gamma}(\sigma) = 1$ for all $\rho, \beta, \gamma \in \mathcal{Z}$ and $\sigma > 0$ if and only if $\rho = \beta = \gamma$;
(2) $\mathcal{F}_{\rho,\rho,\beta}(\sigma) \geqslant \mathcal{F}_{\rho,\beta,\gamma}(\sigma)$ for all $\rho, \beta, \gamma \in \mathcal{Z}$ with $\gamma \neq \beta$ and $\sigma > 0$;
(3) $\mathcal{F}_{\rho,\beta,\gamma}(\sigma) = \mathcal{F}_{\rho,\gamma,\beta}(\sigma) = \mathcal{F}_{\beta,\rho,\gamma}(\sigma) = \cdots$ for all $\rho, \beta, \gamma \in \mathcal{Z}$ and $\sigma > 0$;
(4) $\mathcal{F}_{\rho,\beta,\gamma}(\sigma + s) \geqslant \Theta(\mathcal{F}_{\rho,a,a}(\sigma); \mathcal{F}_{a,\beta,\gamma}(s))$ for all $\rho, \beta, \gamma, a \in \mathcal{Z}$ and $\sigma, s \geqslant 0$.

Example 2.10. ([29]) Consider $(\mathcal{Z}, F, \Theta)$ as a Menger space. Define the function $\mathcal{F} : \mathcal{Z} \times \mathcal{Z} \times \mathcal{Z} \rightarrow \Pi^+$ by

$$\mathcal{F}_{\rho,\beta,\gamma}(\sigma) = \min\{F_{\rho,\beta}(\sigma); F_{\beta,\gamma}(\sigma); F_{\gamma,\rho}(\sigma)\}$$

for all $\rho, \beta, \gamma \in \mathcal{Z}$ and $\sigma > 0$. Then $(\mathcal{Z}, \mathcal{F}, \Theta)$ is a G-Menger space.

Remark 2.11. ([29]) For any $(\mathcal{Z}, \digamma, \Theta)$ G-Menger space, the function $F\colon \mathcal{Z}\times \mathcal{Z} \to \mathbb{R}^+$ defined by

$$F_{\rho,\beta}(\sigma) = \min\{\digamma_{\rho,\rho,\beta}(\sigma), \digamma_{\rho,\beta,\beta}(\sigma)\}$$

is a probabilistic metric. Hence $(\mathcal{Z}, F, \Theta)$ is a Menger space.

Zhou *et al.* [29] introduced some topological properties of this space.

Let $(\mathcal{Z}, \digamma, \Theta)$ be a G-Menger space and ρ be any point in $\mathcal{Z}$. For $\varepsilon > 0$ and $\delta \in (0,1)$, an (ε,δ)-neighborhood of ρ is the set of all points $\beta \in \mathcal{Z}$ for which $\digamma_{\rho,\beta,\beta}(\varepsilon) > 1-\delta$ and $\digamma_{\beta,\rho,\rho}(\varepsilon) > 1-\delta$. We write

$$N_\rho(\varepsilon,\delta) = \{\beta \in \mathcal{Z} : \digamma_{\rho,\beta,\beta}(\varepsilon) > 1-\delta \ \text{ and } \ \digamma_{\beta,\rho,\rho}(\varepsilon) > 1-\delta\}.$$

Then $(\mathcal{Z}, \digamma, \Theta)$ is a Hausdorff space in the topology induced by the family

$$N = \{N_\rho(\varepsilon,\delta)/\rho \in \mathcal{Z}, \varepsilon > 0, \delta > 0\} \ \text{ of } \ (\varepsilon,\delta) - \text{neighborhoods}.$$

According to this introduction, we can give the following concepts in G-Menger spaces.

Definition 2.12. ([29]) Consider $(\mathcal{Z}, \digamma, \Theta)$ as a G-Menger space, and $\{\rho_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathcal{Z}$.

(1) We claim that $\{\rho_n\}_{n\in\mathbb{N}}$ is converges to $\gamma \in \mathcal{Z}$ (write $\rho_n \to \gamma$), if for any $\varepsilon > 0$ and $\delta \in (0,1)$, there exists a positive integer $M_{\varepsilon,\delta}$ such that

$$\rho_n \in N_\gamma(\varepsilon,\delta) \text{ whenever } n > M_{\varepsilon,\delta}.$$

(2) $\{\rho_n\}_{n\in\mathbb{N}}$ is termed a Cauchy sequence, if for any $\varepsilon > 0$ and $\delta \in (0,1)$, there exists a positive integer $M_{\varepsilon,\delta}$ such that

$$\digamma_{\rho_n,\rho_m,\rho_l}(\varepsilon) > 1-\delta \quad \text{whenever } n,m,l > M_{\varepsilon,\delta}.$$

(3) $(\mathcal{Z}, \digamma, \Theta)$ is termed complete, if every Cauchy sequence is converge to a points in $\mathcal{Z}$.

Lemma 2.13. ([29]) *Think of $(\mathcal{Z}, \digamma, \Theta)$ as a G-Menger space. Let $\{\rho_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ three sequences in $\mathcal{Z}$ and $\rho,\beta,\gamma \in \mathcal{Z}$. If $\rho_n \to \rho$, $\beta_n \to \beta$ and $\gamma_n \to \gamma$ as $n \to \infty$, then, for any $\sigma > 0$,*

$$\digamma_{\rho_n,\beta_n,\gamma_n}(\sigma) \to \digamma_{\rho,\beta,\gamma}(\sigma) \quad \textit{as} \quad n \to \infty.$$

Lemma 2.14. ([14]) *For $n \in \mathbb{N}$, let $g_n : (0,\infty) \to (0,\infty)$ and $F_n, F : \mathbb{R} \to [0,1]$. Assume that $\sup\{F(\sigma) : \sigma > 0\} = 1$ and for any $\sigma > 0$,*

$$\lim_{n\to\infty} g_n(\sigma) = 0 \quad \textit{and} \quad F_n(g_n(\sigma)) \geqslant F(\sigma).$$

If each F_n is non-decreasing. Then

$$\lim_{n\to\infty} F_n(\sigma) = 1 \textit{ for any } \sigma > 0.$$

Definition 2.15. ([25]) Let A be a nonempty subset of a G-Menger space $(\mathcal{Z}, \mathcal{F}, \Theta)$. The generalized probabilistic diameter of A is the function D_A^* defined on $[0, \infty]$ by

$$D_A^*(\sigma) = \begin{cases} \lim_{l \to \sigma^-} \varpi_A(l) & \text{if } \ 0 \leq \sigma < \infty, \\ 1 & \text{if } \ \sigma = \infty, \end{cases}$$

where

$$\varpi_A(l) = \inf\{\mathcal{F}_{\rho,\beta,\gamma}(l)/\rho, \beta, \gamma \in A\}.$$

From the definition of D_A^*, it follows that

$$D_A^* \in \Pi^+ \text{ for any } A \subset \mathcal{Z}, \text{ and for all } \rho, \beta, \gamma \text{ in } A, \ \mathcal{F}_{\rho,\beta,\gamma} \geqslant D_A^*.$$

A nonempty set A of $\mathcal{Z}$ is said to be generalized probabilistic bounded, if $D_A^* \in \Lambda^+$.

Remark 2.16. ([21]) Let A and B two nonempty sets of a G-Menger space $(\mathcal{Z}, \mathcal{F}, \Theta)$,

$$\text{if } A \subseteq B, \text{ then } D_B^* \leqslant D_A^*.$$

3. Fixed point theorems

Throughout this work, $(\mathcal{Z}, \mathcal{F}, \Theta)$ is a G-Menger space and $g : \mathcal{Z} \to \mathcal{Z}$ a self mapping of $\mathcal{Z}$. For $\rho \in \mathcal{Z}$ and $n \in \mathbb{N}$, $g^0\rho = \rho$ and $g^{n+1}\rho = gg^n\rho$, define the power of g at ρ. The notation $\rho_n = g^n\rho$, namely $\rho_0 = \rho$, $\rho_1 = g\rho$, shall be used where there is no risk of ambiguity and $O_g(\rho)$ represents the collection $\{g^n\rho : n = 1, 2, 3, \dots\}$ which is known as orbit of g beginning at ρ.

The letter Ξ denote the set of all function $\varpi : [0; \infty) \to [0; \infty)$ such that

$$\varpi(0) = 0, \ \varpi(\sigma) < \sigma \text{ and } \lim_{n \to \infty} \varpi^n(\sigma) = 0 \quad \text{for all } \ \sigma > 0.$$

Definition 3.1. Think of $(\mathcal{Z}, \mathcal{F}, \Theta)$ as a G-Menger space and $g : \mathcal{Z} \to \mathcal{Z}$ a self mapping of $\mathcal{Z}$. We will say that a mapping g is a generalized ϖ-probabilistic contraction with $\varpi \in \Xi$ if for every $\rho, \beta, \gamma \in \mathcal{Z}$,

$$\mathcal{F}_{g\rho,g\beta,g\gamma}(\varpi(\sigma)) \geqslant \mathcal{F}_{\rho,\beta,\gamma}(\sigma) \quad \text{for all} \quad \sigma > 0. \tag{3.1}$$

Lemma 3.2. *Consider* $(\mathcal{Z}, \mathcal{F}, \Theta)$ *as a G-Menger space, where* $Ran\mathcal{F} \subset \Lambda^+$. *Every Cauchy sequence is bounded.*

Proof. Let $\{\rho_n\}$ be a Cauchy sequence, put $A = \{\rho_n/n \in \mathbb{N}\}$. Given $\delta > 0$, then for $\sigma > 0$, there exists an positive integer N such that

$$\mathcal{F}_{\rho_n,\rho_m,\rho_p}(\sigma) > 1 - \delta \text{ whenever } p, m, n \geqslant N. \tag{3.2}$$

Since $Ran\mathcal{F} \subset \Lambda^+$, there exists $\sigma' > \sigma$ such that

$$\mathcal{F}_{\rho_n,\rho_m,\rho_p}(\sigma') > 1 - \delta \text{ for all } p, m, n < N. \tag{3.3}$$

So from (3.2) and (3.3), we have

$$\begin{aligned} \digamma_{\rho_n,\rho_m,\rho_p}(\sigma') &\geqslant \digamma_{\rho_n,\rho_m,\rho_p}(\sigma) \\ &> 1-\delta \end{aligned}$$

for all $n, m, p \in \mathbb{N}$. So

$$\varpi_A(\sigma') > 1-\delta.$$

Next, for $l > \sigma'$

$$\begin{aligned} \varpi_A(l') &> \varpi_A(\sigma') \\ &> 1-\delta \end{aligned}$$

for all l' such that $l > l' > \sigma'$. Letting $l' \to l$ we obtain

$$D_A^*(l) > 1-\delta.$$

Since this for an arbitrary $\delta > 0$, there is $l > 0$ such that $D_A^*(l) > 1-\delta$. Hence, $D_A^*(l) \to 1$ as $l \to \infty$. This completes the proof. □

Conversely, we have the following.

Lemma 3.3. *Consider $(\mathcal{Z}, \digamma, \Theta)$ as a G-Menger space, where $Ran\digamma \subset \Lambda^+$, and g is a generalized ϖ-probabilistic contraction mapping on $\mathcal{Z}$. If the orbit $O_g(\rho)$ for some $\rho \in \mathcal{Z}$ is bounded, then $\{g^n\rho\}_{n\geqslant 0}$ is a Cauchy sequence.*

Proof. Let $n, m \in \mathbb{N}$ such that $m > n$ and $\sigma > 0$, by (3.1) we have

$$\begin{aligned} \digamma_{\rho_n,\rho_n,\rho_m}(\varpi^n(\sigma)) &\geqslant \digamma_{\rho_{n-1},\rho_{n-1},\rho_{m-1}}(\varpi^{n-1}(\sigma)) \\ &\vdots \\ &\geqslant \digamma_{\rho_0,\rho_0,\rho_{m-n}}(\sigma) \\ &\geqslant D^*_{O_g(\rho)}(\sigma). \end{aligned}$$

The orbit $O_g(\rho)$ is bounded, then let $\varepsilon > 0$ and $\delta \in (0,1)$ be given, since

$$D^*_{O_g(\rho)}(\sigma) \to 1 \quad \text{as} \;\; \sigma \to \infty,$$

there exists $\sigma_0 > 0$ such that

$$D^*_{O_g(\rho)}(\sigma_0) > 1-\delta.$$

On the other hand, since $\varpi^n(\sigma) \to 0$ as $n \to \infty$, there is $N \in \mathbb{N}$ such that

$$\varpi^n(\sigma_0) < \varepsilon \quad \text{whenever} \;\; n \geqslant N.$$

Then,

$$\begin{aligned} \digamma_{\rho_n,\rho_n,\rho_m}(\varepsilon) &\geqslant \digamma_{\rho_n,\rho_n,\rho_m}(\varpi^n(\sigma_0)) \\ &\geqslant D^*_{O_g(\rho)}(\sigma_0) \\ &> 1-\delta. \end{aligned}$$

Thus, we proved that for each $\varepsilon > 0$ and $\delta \in (0,1)$, there exists $N \in \mathbb{N}$ such that

$$\digamma_{\rho_n,\rho_n,\rho_m}(\varepsilon) > 1-\delta \quad \text{for each} \quad n,m > N.$$

This means $\lim\limits_{n,m\to\infty} \digamma_{\rho_n,\rho_n,\rho_m}(\sigma) = 1$ for all $\sigma > 0$.
And also,

$$\digamma_{\rho_n,\rho_m,\rho_l}(\sigma) \geqslant \Theta\left(\digamma_{\rho_n,\rho_n,\rho_m}(\frac{\sigma}{2}), \digamma_{\rho_n,\rho_n,\rho_l}(\frac{\sigma}{2})\right).$$

Therefore, by the continuity of Θ, we conclude that

$$\lim_{n,m,l\to\infty} \digamma_{\rho_n,\rho_m,\rho_l}(\sigma) = 1 \text{ for any } \sigma > 0.$$

This show that $\{\rho_n\}$ is a Cauchy sequence in $\mathcal{Z}$. □

Lemma 3.4. *Think of $(\mathcal{Z}, \digamma, \Theta)$ as a G-Menger space, where $Ran\digamma \subset \Lambda^+$. For, $\rho, \beta \in \mathcal{Z}$, if there exists $\varpi \in \Xi$ such that*

$$\digamma_{\rho,\rho,\beta}(\varpi(\sigma)) = \digamma_{\rho,\rho,\beta}(\sigma) \quad \textit{for all } \sigma > 0, \tag{3.4}$$

then $\rho = \beta$.

Proof. From the condition (3.4), it is easy to show that by induction, for all $n \geqslant 1$,

$$\digamma_{\rho,\rho,\beta}(\varpi^n(\sigma)) = \digamma_{\rho,\rho,\beta}(\sigma). \tag{3.5}$$

In order to prove that $\rho = \beta$, we need to prove that $\digamma_{\rho,\rho,\beta}(\sigma) = 1$ for all $\sigma > 0$.

Suppose, to the contrary, that there exists some $\sigma_0 > 0$ such that $\digamma_{\rho,\rho,\beta}(\sigma_0) < 1$.
Since $Ran\digamma \subset \Lambda^+$, $\digamma_{\rho,\rho,\beta}(\sigma) \to 1$ as $\sigma \to \infty$. Therefore, there exists a $\sigma_1 > \sigma_0$ such that

$$\digamma_{\rho,\rho,\beta}(\sigma_0) < \digamma_{\rho,\rho,\beta}(\sigma_1). \tag{3.6}$$

Since $\sigma_0 > 0$ and $\lim_{n\to\infty} \varpi^n(\sigma) = 0$, there exists a positive integer $n > 1$ such that $\varpi^n(\sigma_1) < \sigma_0$. Then by monotony of $\digamma_{\rho,\rho,\beta}(.)$, it follows that

$$\digamma_{\rho,\rho,\beta}(\varpi^n(\sigma_1)) \leq \digamma_{\rho,\rho,\beta}(\sigma_0).$$

Hence, from (3.5) with $\sigma = \sigma_1$, we have

$$\digamma_{\rho,\rho,\beta}(\varpi^n(\sigma_1)) = \digamma_{\rho,\rho,\beta}(\sigma_1) \leq \digamma_{\rho,\rho,\beta}(\sigma_0),$$

which is a contradiction to (3.6). Therefore, $\digamma_{\rho,\rho,\beta}(\sigma) = 1$ for all $\sigma > 0$. □

Lemma 3.5. *Consider $(\mathcal{Z}, \digamma, \Theta)$ as a G-Menger space where $Ran\digamma \subset \Lambda^+$ and g is a generalized ϖ-probabilistic contraction mapping on $\mathcal{Z}$. If the t-norm Θ is of H-type, then for all $\rho \in \mathcal{Z}$, $\{g^n\rho\}_{n\geqslant 0}$ is a Cauchy sequence.*

Proof. Let $\rho_0 \in \mathcal{Z}$ be arbitrary. Put $\rho_n = g^n\rho_0$ for each $n \in \mathbb{N}$. We shall show by induction that for each $n \geqslant 1$,

$$\digamma_{\rho_n,\rho_n,\rho_{n+1}}(\varpi^n(\sigma) \geqslant \digamma_{\rho_0,\rho_0,\rho_1}(\sigma) \text{ for all } \sigma > 0. \tag{3.7}$$

It follows from (3.1) that

$$\digamma_{\rho_1,\rho_1,\rho_2}(\varpi(\sigma)) = \digamma_{g\rho_0,g\rho_0,g\rho_1}(\varpi(\sigma)) \geqslant \digamma_{\rho_0,\rho_0,\rho_1}(\sigma) \text{ for all } \sigma > 0.$$

Therefore, (3.7) is holds for $n = 1$.

Suppose now that (3.7) holds for some $n \geqslant 1$. We need to show that (3.7) holds for $n+1$. Since

$$\begin{aligned}\digamma_{\rho_{n+1},\rho_{n+1},\rho_{n+2}}(\varpi^{n+1}(\sigma)) &= \digamma_{g\rho_n,g\rho_n,g\rho_{n+1}}(\varpi(\varpi^n(\sigma)))\\ &\geqslant \digamma_{\rho_n,\rho_n,\rho_{n+1}}(\varpi^n(\sigma))\\ &\geqslant \digamma_{\rho_0,\rho_0,\rho_1}(\sigma)\end{aligned}$$

for all $\sigma > 0$, we have

$$\digamma_{\rho_{n+1},\rho_{n+1},\rho_{n+2}}(\varpi^{n+1}(\sigma)) \geqslant \digamma_{\rho_0,\rho_0,\rho_1}(\sigma) \text{ for all } \sigma > 0,$$

which completes the induction. Hence (3.7) holds for $n \geqslant 1$. Now we prove that

$$\digamma_{\rho_n,\rho_n,\rho_{n+1}}(\sigma) \to 1 \text{ as } n \to \infty.$$

Since, $\lim_{\sigma\to\infty} \digamma_{\rho,\beta,\gamma}(\sigma) = 1$, $\digamma_{\rho_n,\rho_n,\rho_{n+1}}(\varpi^n(\sigma) \geqslant \digamma_{\rho_0,\rho_0,\rho_1}(\sigma)$ for all $\sigma > 0$ and the sequence $\{\digamma_{\rho_n,\rho_n,\rho_{n+1}}\}_n$ is non-decreasing, then, by Lemma 2.14, we obtain

$$\digamma_{\rho_n,\rho_n,\rho_{n+1}}(\sigma) \to 1 \text{ as } n \to \infty.$$

We now prove that $\{\rho_n\}$ is a Cauchy sequence in $\mathcal{Z}$. Let $n > 0$ and $\sigma \geqslant 0$, we can show by induction that, for any $k \geqslant 0$,

$$\digamma_{\rho_n,\rho_n,\rho_{n+k}}(\sigma) \geqslant \Theta^k(\digamma_{\rho_n,\rho_n,\rho_{n+1}}(\sigma - \varpi(\sigma))). \tag{3.8}$$

This is obvious for $k = 0$. Suppose now that (3.8) holds for some $k \geqslant 1$. Hence,

$$\begin{aligned}\digamma_{\rho_n,\rho_n,\rho_{n+k+1}}(\sigma) &= \digamma_{\rho_n,\rho_n,\rho_{n+k+1}}(\sigma - \varpi(\sigma) + \varpi(\sigma)))\\ &\geqslant \Theta(\digamma_{\rho_n,\rho_n,\rho_{n+1}}(\sigma - \varpi(\sigma)), \digamma_{\rho_{n+1},\rho_{n+1},\rho_{n+k+1}}(\varpi(\sigma)))\\ &\geqslant \Theta(\digamma_{\rho_n,\rho_n,\rho_{n+1}}(\sigma - \varpi(\sigma)), \digamma_{\rho_n,\rho_n,\rho_{n+k}}(\sigma))\\ &\geqslant \Theta(\digamma_{\rho_n,\rho_n,\rho_{n+1}}(\sigma - \varpi(\sigma)), \Theta^k(\digamma_{\rho_n,\rho_n,\rho_{n+1}}(\sigma - \varpi(\sigma))))\\ &= \Theta^{k+1}(\digamma_{\rho_n,\rho_n,\rho_{n+1}}(\sigma - \varpi(\sigma))).\end{aligned}$$

Thus, (3.8) is holds for all $k \geqslant 0$.

Let $\lambda > 0$, since Θ is a t-norm of H-type, there is $\delta > 0$ such that

$$\Theta^n(\sigma') > 1 - \lambda \quad \text{for all} \quad n \geqslant 1, \text{ when } \sigma' > 1 - \delta.$$

Since

$$\digamma_{\rho_n,\rho_n,\rho_{n+1}}(\sigma) \to 1 \quad \text{as} \quad n \to \infty \quad \text{for all} \quad \rho > 0,$$

and for $\sigma - \varpi(\sigma) > 0$, there exists a positive integer $n_0 = n_0(\sigma - \varpi(\sigma), \delta)$ such that

$$\mathcal{F}_{\rho_n,\rho_n,\rho_{n+1}}(\sigma - \varpi(\sigma)) > 1 - \delta \quad \text{for all} \quad n \geqslant n_0.$$

Hence,

$$\Theta^k(\mathcal{F}_{\rho_n,\rho_n,\rho_{n+1}}(\sigma - \varpi(\sigma))) > 1 - \lambda \text{ for all } k > 0 \text{ and } n \geqslant n_1 = n_0,$$

we conclude that

$$\mathcal{F}_{\rho_n,\rho_n,\rho_{n+k}}(\sigma) > 1 - \lambda \quad \text{for all} \quad k > 0.$$

This means that

$$\lim_{n,m\to\infty} \mathcal{F}_{\rho_n,\rho_n,\rho_m}(\sigma) = 1 \quad \text{for any} \quad \sigma > 0.$$

The fact that for all $n, m, l \in \mathbb{N}$,

$$\mathcal{F}_{\rho_n,\rho_m,\rho_l}(\sigma) \geqslant \Theta(\mathcal{F}_{\rho_n,\rho_n,\rho_m}(\frac{\sigma}{2}), \mathcal{F}_{\rho_n,\rho_n,\rho_l}(\frac{\sigma}{2})).$$

We conclude that

$$\lim_{n,m,l\to\infty} \mathcal{F}_{\rho_n,\rho_m,\rho_l}(\sigma) = 1 \quad \text{for any} \quad \sigma > 0,$$

which implies that the sequence $\{\rho_n\}$ is a Cauchy sequence in $\mathcal{Z}$. □

As consequence of Lemma 3.2 and Lemma 3.5 we have the following lemma.

Lemma 3.6. *Think of $(\mathcal{Z}, \mathcal{F}, \Theta)$ as a G-Menger space, where $Ran\mathcal{F} \subset \Lambda^+$, and g is a generalized ϖ-probabilistic contraction mapping on $\mathcal{Z}$. If the t-norm Θ is of H-type, then for all $\rho \in \mathcal{Z}$, the orbit $O_g(\rho)$ is bounded.*

The following theorem is our main result.

Theorem 3.7. *Let $(\mathcal{Z}, \mathcal{F}, \Theta)$ be a complete G-Menger space where $Ran\mathcal{F} \subset \Lambda^+$, and g is a generalized ϖ-probabilistic contraction mapping on $\mathcal{Z}$. If the orbit $O_g(\rho)$ for some $\rho \in \mathcal{Z}$ is bounded, then g has a unique fixed point γ, moreover, the sequence $\{g^n\rho\}$ converge to γ.*

Proof. Let $\rho \in \mathcal{Z}$ such that $O_g(\rho)$ is bounded, by Lemma 3.3, $\{g^n\rho\}$ is a Cauchy sequence. Since $(\mathcal{Z}, \mathcal{F}, \Theta)$ is complete, $\{g^n\rho\}$ converge to some $\gamma \in \mathcal{Z}$.

Now, we shall show that γ is a fixed point of g. Let $\sigma > 0$. Since $\sigma > \varpi(\sigma)$, by the monotonicity of distribution functions and (3.1), we get

$$\begin{aligned}\mathcal{F}_{\rho_n,\rho_n,g\gamma}(\sigma) &\geqslant \mathcal{F}_{\rho_n,\rho_n,g\gamma}(\varpi(\sigma))\\ &\geqslant \mathcal{F}_{\rho_{n-1},\rho_{n-1},\gamma}(\sigma).\end{aligned}$$

Letting $n \to \infty$, we get $\gamma = g\gamma$.

To complete the proof, we need to show that γ is unique. Indeed, let u be another fixed point of g and $\sigma > 0$. Then

$$\begin{aligned} F_{\gamma,\gamma,u}(\varpi(\sigma)) &= F_{g\gamma,g\gamma,gu}(\varpi(\sigma)) \\ &\geqslant F_{\gamma,\gamma,u}(\sigma) \end{aligned}$$

or

$$F_{\gamma,\gamma,u}(\sigma) \geqslant F_{\gamma,\gamma,u}(\varpi(\sigma)).$$

Hence,

$$F_{\gamma,\gamma,u}(\varpi(\sigma)) = F_{\gamma,\gamma,u}(\sigma).$$

So, by Lemma 3.4 we conclude that $u = \gamma$. Therefore, g has a unique fixed point in $\mathcal{Z}$. □

Example 3.8. Let $\mathcal{Z} = [0, \infty)$. Define the function $F : \mathcal{Z}^3 \times [0, \infty) \to [0, \infty)$ by

$$F_{\rho,\beta,\gamma}(\sigma) = \begin{cases} 0 & \text{if } \sigma = 0, \\ \dfrac{\sigma}{\sigma + |\rho - \beta| + |\beta - \gamma| + |\gamma - \rho|} & \text{if } \sigma > 0, \end{cases}$$

for all $\rho, \beta, \gamma \in \mathcal{Z}$. Then $(\mathcal{Z}, F, \Theta_P)$ is a complete G-Menger space.

Let $g : \mathcal{Z} \to \mathcal{Z}$ be a mapping defined by $g\rho = \frac{5}{6}\rho$ and $\varpi : [0, \infty) \to [0, \infty)$ defined by

$$\varpi(\sigma) = \frac{5}{6}\sigma.$$

Then, it is easy to verify that $\varpi \in \Xi$ and g is generalized ϖ-probabilistic contraction. Further, $O_g(0)$ is bounded, so g admits 0 as a unique fixed point.

Example 3.9. Let $\mathcal{Z} = [0; \infty)$. Define a function $F^* : \mathcal{Z}^3 \times [0, \infty) \to [0, \infty)$ as follows

$$F^*_{\rho,\beta,\gamma}(\sigma) = \begin{cases} \varepsilon_0(\sigma) & \text{if } \rho = \beta = \gamma, \\ \dfrac{\delta\sigma}{\delta\sigma + F(\rho, \beta, \gamma)} & \text{otherwise.} \end{cases}$$

For all $\rho, \beta, \gamma \in \mathcal{Z}$, where $1 \geqslant \delta > 0$, and $F : \mathcal{Z} \times \mathcal{Z} \times \mathcal{Z} \to [0; \infty)$ defined by

$$F(\rho, \beta, \gamma) = |\rho - \beta| + |\beta - \gamma| + |\rho - \gamma|.$$

Then it is easy to check that $(\mathcal{Z}, F^*, \Theta_M)$ is a complete G-Menger space.

Let $\varpi(\sigma) = \lambda\sigma,\ \lambda \in (0, 1)$. It is easy to verify that $\varpi \in \Xi$. Define a self mapping g on $\mathcal{Z}$ by

$$g(\rho) = \delta\lambda\rho \quad \text{for all} \quad \rho \in \mathcal{Z}.$$

We now show that g is generalized ϖ-probabilistic contraction. For all $\rho, \beta, \gamma \in \mathcal{Z}$ and $\sigma \geqslant 0$. If

$$g\rho = g\beta = g\gamma,$$

since

$$g\rho = g\beta = g\gamma \quad \Leftrightarrow \quad \rho = \beta = \gamma \text{ and } \varpi(\sigma) = 0 \quad \Leftrightarrow \quad \sigma = 0,$$

we have

$$\begin{aligned}\digamma^*_{g\rho,g\beta,g\gamma}(\varpi(\sigma) &= \varepsilon_0(\varpi(\sigma))\\ &= \varepsilon_0(\sigma)\\ &= \digamma^*_{\rho,\beta,\gamma}(\sigma).\end{aligned}$$

If at least one of $g\rho, g\beta, g\gamma$ is not equal to the other two, then

$$\begin{aligned}\digamma^*_{g\rho,g\beta,g\gamma}(\varpi(\sigma)) &= \frac{\delta\varpi(\sigma)}{\delta\varpi(\sigma)+\digamma(g\rho,g\lambda,g\sigma)}\\ &= \frac{\delta\lambda\sigma}{\delta\lambda\sigma+\delta\lambda\digamma(\rho,\beta,\gamma)}\\ &= \frac{\sigma}{\sigma+\digamma(\rho,\beta,\gamma)}\\ &\geqslant \frac{\sigma}{\sigma+\frac{1}{\delta}\digamma(\rho,\beta,\gamma)}\\ &= \frac{\delta\sigma}{\delta\sigma+\digamma(\rho,\beta,\gamma)}\\ &= \digamma^*_{\rho,\beta,\gamma}(\sigma).\end{aligned}$$

Hence, (3.1) holds. Further, $O_g(0)$ is bounded. Then, we showed that the mapping g satisfies all hypotheses of Theorem 3.7 and have a unique fixed point $\gamma = 0$.

Corollary 3.10. *Consider $(\mathcal{Z}, F, \Theta)$ as a complete Menger space, where $RanF \subset \Lambda^+$, and let g is a ϖ-probabilistic contraction mapping on $\mathcal{Z}$, that is,*

$$F_{g\rho,g\beta}(\varpi(\sigma)) \geqslant F_{\rho,\beta}(\sigma) \text{ for all } \rho,\beta\in\mathcal{Z} \text{ and } \sigma>0. \tag{3.9}$$

If the orbit $O_g(\rho)$ for some $\rho \in \mathcal{Z}$ is bounded, then g has a unique fixed point γ, moreover, the sequence $\{g^n\rho\}$ converge to γ.

Proof. We define a mapping $\digamma : \mathcal{Z}\times\mathcal{Z}\times\mathcal{Z} \to \Pi^+$ by

$$\digamma_{\rho,\beta,\gamma}(\sigma) = \min\{F_{\rho,\beta}; F_{\rho,\gamma}(\sigma); F_{\beta,\gamma}(\sigma)\} \tag{3.10}$$

for all $\rho,\beta,\gamma\in\mathcal{Z}$ and $\sigma > 0$. Since $(\mathcal{Z}, F, \Theta)$ is a complete Menger space, by Example 2.10, we know that $(\mathcal{Z}, \digamma, \Theta)$ is a complete G-Menger space. Next, we only need to prove that the condition (3.9) implies (3.1). From that (3.9), we obtain

$$\begin{aligned}\digamma_{g\rho,g\beta,g\gamma}(\varpi(\sigma)) &= \min\{F_{g\rho,g\beta}(\varpi(\sigma)); F_{g\rho,g\gamma}(\varpi(\sigma)); F_{g\beta,g\gamma}(\varpi(\sigma))\\ &\geqslant \min\{F_{\rho,\beta}(\sigma); F_{\rho,\gamma}(\sigma); F_{\beta,\gamma}(\sigma)\}\\ &= \digamma_{\rho,\beta,\gamma}(\sigma)\end{aligned}$$

for all $\rho,\beta,\gamma\in\mathcal{Z}$ and $\sigma > 0$. Hence, (3.1) holds, that is, g is a generalized ϖ-probabilistic contraction mapping in G-Menger space $(\mathcal{Z}, \digamma, \Theta)$. Therefore, the Theorem 3.7 ensures the existence and uniqueness of fixed point of g. □

By merging the Lemma 3.6 with the Theorem 3.7 we obtained the following result.

Corollary 3.11. ([24]) *Think of* $(\mathcal{Z}, \digamma, \Theta)$ *as a complete* G*-Menger space where* $Ran\digamma \subset \Lambda^+$ *under a* t*-norm* Θ *of* H*-type and* g *is a generalized* ϖ*-probabilistic contraction mapping on* $\mathcal{Z}$*. Then* g *has a unique fixed point* γ*, moreover, the sequence* $\{g^n\rho\}$ *converge to* γ*, for all* $\rho \in \mathcal{Z}$*.*

Since each Menger space is regarded as a particular kind of G-Menger space (see Remark 2.11), the results of Lemma 3.6 are valid in Menger spaces. Therefore, by combining Lemma 3.6 with Corollary 3.11, we obtain the following corollary, which extends these results to Menger spaces under an H-type t-norm.

Corollary 3.12. ([14]) *Consider* $(\mathcal{Z}, F, \Theta)$ *as a complete Menger space under a* t*-norm* Θ *of* H*-type where* $RanF \subset \Lambda^+$*. Let* g *is a generalized* ϖ*-probabilistic contraction mapping on* $\mathcal{Z}$*. Then* g *has a unique fixed point* γ*, moreover, the sequence* $\{g^n\rho\}$ *converge to* γ *for all* $\rho \in \mathcal{Z}$*.*

4. Common fixed point theorems of a family of ϖ-probabilistic contraction type mappings in G-Menger spaces

Definition 4.1. Consider $(\mathcal{Z}, \digamma, \Theta)$ as a G-Menger space. And let $S = \{g_i / i \in I\}$ be a family of self-maps of $\mathcal{Z}$. We define the orbit of S starting at $\rho \in \mathcal{Z}$ by

$$O_S(\rho) = \bigcup_{i \in I} O_{g_i}(\rho).$$

Theorem 4.2. *Think of* $(\mathcal{Z}, \digamma, \Theta)$ *as a complete* G*-Menger space, where* $Ran\digamma \subset \Lambda^+$*, and let* $S = \{g_i, i \in I\}$ *be a family of self-maps of* $\mathcal{Z}$ *such that the following conditions are satisfied:*

(1) *there exists* ρ *in* $\mathcal{Z}$ *such that the orbit* $O_S(\rho)$ *is bounded,*
(2) *there exists* $\varpi \in \Xi$ *such that, for all* $i, j, k \in I$ *and* $\rho, \beta, \gamma \in \mathcal{Z}$ *and* $\sigma > 0$*,*

$$\digamma_{g_i\rho, g_j\beta, g_k\gamma}(\varpi(\sigma)) \geqslant \digamma_{\rho,\beta,\gamma}(\sigma). \tag{4.1}$$

Then $\{g_i, i \in I\}$ *has a unique common fixed point* γ *and for all* $i \in I$*, the sequence* $\{g_i^n\rho\}$ *converge to* γ*.*

Proof. As $O_S(\rho)$ is bounded, then according to the Remark 2.16 for all $i \in I$, $O_{g_i}(\rho)$ is bounded. Hence, it follows from condition (4.1) and Theorem 3.7 that for all $i \in I$, g_i has a unique fixed point γ_i and the sequence $\{g_i^n\rho\}$ converge to γ_i.

To complete the proof it suffices to show that if $j \neq i$, then $\gamma_i = \gamma_j$. Let $\sigma > 0$, by (4.1) we have

$$\begin{aligned} \digamma_{\gamma_i,\gamma_i,\gamma_j}(\varpi(\sigma)) &= \digamma_{g_i\gamma_i, g_i\gamma_i, g_j\gamma_j}(\varpi(\sigma)) \\ &\geqslant \digamma_{\gamma_i,\gamma_i,\gamma_j}(\sigma) \end{aligned}$$

or

$$\mathcal{F}_{\gamma_i,\gamma_i,\gamma_j}(\sigma) \geqslant \mathcal{F}_{\gamma_i,\gamma_i,\gamma_j}(\varpi(\sigma)),$$

then

$$\mathcal{F}_{\gamma_i,\gamma_i,\gamma_j}(\varpi(\sigma)) = \mathcal{F}_{\gamma_i,\gamma_i,\gamma_j}(\sigma).$$

Hence from Lemma 3.4 we get $\gamma_i = \gamma_j$. Which complete the proof of theorem. □

From Theorem 4.2 we also obtain the following corollary.

Corollary 4.3. *Consider $(\mathcal{Z}, F, \Theta)$ as a complete Menger space, where $RanF \subset \Lambda^+$, and let $S = \{g_i, i \in I\}$ be a family of self-maps of $\mathcal{Z}$ such that the following conditions are satisfied:*

(1) *there exists $\rho \in \mathcal{Z}$ such that the orbit $O_S(\rho)$ is bounded,*
(2) *there exists $\varpi \in \Xi$ such that for all $i, j \in I$ and for any $\rho, \beta \in \mathcal{Z}$ and $\sigma > 0$,*

$$F_{g_i\rho, g_j\beta}(\varpi(\sigma)) \geqslant F_{\rho,\beta}(\sigma). \tag{4.2}$$

Then $\{g_i, i \in I\}$ has a unique common fixed point γ and for all $i \in I$, the sequence $\{g_i^n\rho\}$ converge to γ.

Proof. Let

$$\mathcal{F}_{\rho,\beta,\gamma}(\sigma) = \min\{F_{\rho,\beta}; F_{\rho,\gamma}(\sigma); F_{\beta,\gamma}(\sigma)\}$$

for all $\rho, \beta, \gamma \in \mathcal{Z}$ and $\sigma > 0$. Then by example 2.10. $(\mathcal{Z}, \mathcal{F}, \Theta_M)$ is a complete G-Menger space. It's not hard to proves that the conditions (4.2) implies (4.1). Hence, the Theorem 4.2 ensures that $\{g_i : i \in I\}$ has a unique common fixed point γ and for all $i \in I$ the sequence $\{g_i^n\rho\}$ converge to γ. □

Theorem 4.4. *Consider $(\mathcal{Z}, \mathcal{F}, \Theta)$ as a complete G-Menger space under a t-norm Θ of H-type, where $Ran\mathcal{F} \subset \Lambda^+$. If the family $\{g_i, i \in I\}$ satisfies the condition (4.1), where $\varpi \in \Xi$, then $\{g_i, i \in I\}$ has a unique common fixed point γ and for all $i \in I$, the sequence $\{g_i^n\rho\}$ converge to γ for all $\rho \in \mathcal{Z}$.*

Proof. It follows from Corollary 3.11 that for all $i \in I$, g_i has a unique fixed point γ_i and the sequence $\{g_i^n\rho\}$ converge to γ_i. According to the some reason as in the proof of Theorem 4.2, we know that the common fixed point of $\{g_i, i \in I\}$ is unique. Which complete the proof of theorem. □

In the same way, from Corollary 3.12, we can prove the following result.

Corollary 4.5. *Think of $(\mathcal{Z}, F, \Theta)$ as a complete Menger space under a t-norm Θ of H-type, where $RanF \subset \Lambda^+$. If the family $\{g_i, i \in I\}$ satisfies the condition (4.2), where $\varpi \in \Xi$, then $\{g_i, i \in I\}$ has a unique common fixed point γ, and for all $i \in I$ the sequence $\{g_i^n\rho\}$ converge to γ for all $\rho \in \mathcal{Z}$.*

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