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FIXED POINT THEOREMS FOR FUZZY MAPPINGS IN MR-METRIC SPACES

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Abstract. In this paper, we explore fixed point results for fuzzy mappings in the framework of MR-metric spaces. By introducing an appropriate contraction condition, we establish the existence of common fixed points for certain fuzzy mappings in a complete MR-metric space. Our findings extend and generalize classical fixed point theorems to a broader setting, providing a new perspective on fuzzy analysis in generalized metric spaces. The results presented here contribute to the growing field of fixed point theory and its applications in mathematical modeling, optimization, and decision-making processes.

1. Introduction

This study explores the convergence properties and fixed-point theory of self-mappings in MR-metric spaces, a modern extension of traditional metric spaces. We derive essential theorems on the existence and uniqueness of fixed points for contraction-type mappings, analyze the convergence behavior of Cauchy sequences, and examine the convergence in measure of iterative processes toward fixed points. These results hold substantial relevance in diverse areas such as optimization, machine learning, and numerical analysis, offering a rigorous framework for future investigations in MR-metric spaces.

For a deeper exploration of foundational concepts, readers may consult the references listed in ([2]-[15], [19]-[37]).

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Definition 1.1. ([22]) Consider a non-empty set $\mathbb{X} \neq \emptyset$ and a real number $R > 1$. A function $M : \mathbb{X} \times \mathbb{X} \times \mathbb{X} \to [0, \infty)$ is termed an MR-metric if it satisfies the following conditions for all $\upsilon, \xi, \Im \in \mathbb{X}$:

- $(M1)$ $M(\upsilon, \xi, \Im) \geq 0$.
- $(M2)$ $M(\upsilon, \xi, \Im) = 0$ if and only if $\upsilon = \xi = \Im$.
- $(M3)$ $M(\upsilon, \xi, \Im)$ remains invariant under any permutation $p(\upsilon, \xi, \Im)$, that is, $M(\upsilon, \xi, \Im) = M(p(\upsilon, \xi, \Im))$.
- $(M4)$ The following inequality holds:

$$M(\upsilon, \xi, \Im) \leq R \left[M(\upsilon, \xi, \ell_1) + M(\upsilon, \ell_1, \Im) + M(\ell_1, \xi, \Im) \right].$$

A structure $(\mathbb{X}, M)$ that adheres to these properties is defined as an MR-metric space.

Classical set theory is often inadequate for modeling systems with inherent uncertainty. To address this, fuzzy set theory was introduced by Zadeh (1965), where elements exhibit graded membership quantified by a function $\mu_A : \mathbb{X} \to [0, 1]$. Unlike crisp sets, fuzzy sets allow intermediate membership degrees, enabling nuanced representations of vague or imprecise data.

Definition 1.2. ([38]) A fuzzy set A in a universal set $\mathbb{X}$ is characterized by a membership function $\mu_A : \mathbb{X} \to [0, 1]$, which assigns to each element $x \in \mathbb{X}$ a degree of membership $\mu_A(x)$. This extends classical set theory by permitting partial membership values between 0 and 1.

To measure distances between fuzzy sets in MR-metric spaces, we adapt the classical Hausdorff metric. Given two fuzzy sets A, B over $\mathbb{X}$, their Hausdorff MR-distance H_M is defined as:

$$H_M(A, B) = \max \left\{ \sup_{x \in A} \inf_{y \in B} M(x, y, \Im), \ \ \sup_{y \in B} \inf_{x \in A} M(x, y, \Im) \right\},$$

where M is the underlying MR-metric, and $\Im$ serves as a reference point. This metric captures the worst-case deviation between fuzzy sets within the MR-metric framework.

Definition 1.3. ([14, 16]) The Hausdorff MR-metric H_M is an extension of the classical Hausdorff metric adapted for MR-metric spaces. For two fuzzy sets $A, B \in \mathcal{F}(\mathbb{X})$, it quantifies their separation using the MR-metric structure:

$$H_M(A, B) = \max \left\{ \sup_{x \in A} \inf_{y \in B} M(x, y, \Im), \ \ \sup_{y \in B} \inf_{x \in A} M(x, y, \Im) \right\},$$

where $\Im$ is an auxiliary reference point in $\mathbb{X}$.

2. Main results

Theorem 2.1. *Let $(\mathbb{X}, M)$ be a complete MR-metric space with $R > 1$, and let $T : \mathbb{X} \to \mathcal{F}(\mathbb{X})$ be a fuzzy mapping, where $\mathcal{F}(\mathbb{X})$ is the set of all fuzzy subsets of $\mathbb{X}$. Suppose there exists a constant $k \in (0, 1)$ such that for all $\upsilon, \varkappa \in \mathbb{X}$, the following inequality holds:*

$$H_M(T(\upsilon), T(\varkappa)) \leq k \cdot M(\upsilon, \varkappa, \Im),$$

where H_M is the Hausdorff MR-metric induced by M. Then, T has a unique fixed point in $\mathbb{X}$.

Proof. **Step 1:** Construction of the Sequence. Let $\upsilon_0 \in \mathbb{X}$ be an arbitrary starting point. Since $T(\upsilon_0)$ is a fuzzy subset of $\mathbb{X}$, there exists an $\alpha_0 \in (0, 1]$ such that the α_0-cut $[T(\upsilon_0)]_{\alpha_0}$ is nonempty. Choose $\upsilon_1 \in [T(\upsilon_0)]_{\alpha_0}$.

Similarly, for υ_1, there exists an $\alpha_1 \in (0, 1]$ such that $[T(\upsilon_1)]_{\alpha_1}$ is nonempty. Choose $\upsilon_2 \in [T(\upsilon_1)]_{\alpha_1}$.

Continuing this process, we construct a sequence $\{\upsilon_n\}$ in $\mathbb{X}$ such that

$$\upsilon_{n+1} \in [T(\upsilon_n)]_{\alpha_n},$$

where $\alpha_n \in (0, 1]$ for all $n \in \mathbb{N}$.

Step 2: Establishing the Contraction Property. Using the given inequality for the Hausdorff MR-metric H_M, we have:

$$H_M(T(\upsilon_n), T(\upsilon_{n+1})) \leq k \cdot M(\upsilon_n, \upsilon_{n+1}, \Im).$$

Since $\upsilon_{n+1} \in [T(\upsilon_n)]_{\alpha_n}$ and $\upsilon_{n+2} \in [T(\upsilon_{n+1})]_{\alpha_{n+1}}$, it follows that:

$$M(\upsilon_{n+1}, \upsilon_{n+2}, \Im) \leq H_M(T(\upsilon_n), T(\upsilon_{n+1})) \leq k \cdot M(\upsilon_n, \upsilon_{n+1}, \Im).$$

This implies:

$$M(\upsilon_{n+1}, \upsilon_{n+2}, \Im) \leq k \cdot M(\upsilon_n, \upsilon_{n+1}, \Im).$$

Step 3: Inductive Argument. By induction, we can show that:

$$M(\upsilon_n, \upsilon_{n+1}, \Im) \leq k^n \cdot M(\upsilon_0, \upsilon_1, \Im).$$

For the base case ($n = 0$), the inequality holds trivially. Assume it holds for some $n \geq 0$. Then, for $n + 1$, we have

$$M(\upsilon_{n+1}, \upsilon_{n+2}, \Im) \leq k \cdot M(\upsilon_n, \upsilon_{n+1}, \Im) \leq k \cdot k^n \cdot M(\upsilon_0, \upsilon_1, \Im) = k^{n+1} \cdot M(\upsilon_0, \upsilon_1, \Im).$$

Thus, by induction, the inequality holds for all $n \in \mathbb{N}$.

Step 4: Proving the Sequence is Cauchy. To show that $\{\upsilon_n\}$ is a Cauchy sequence, consider $m, n \in \mathbb{N}$ with $m > n$. Using the MR-metric property (M4), we have

$$M(\upsilon_n, \upsilon_m, \Im) \leq R \left[M(\upsilon_n, \upsilon_{n+1}, \Im) + M(\upsilon_{n+1}, \upsilon_m, \Im) + M(\upsilon_m, \upsilon_n, \Im) \right].$$

By repeated application of the triangle inequality and the contraction property, we obtain:

$$M(\upsilon_n, \upsilon_m, \Im) \leq R\left[k^n + k^{n+1} + \cdots + k^{m-1}\right] \cdot M(\upsilon_0, \upsilon_1, \Im).$$

Since $k \in (0,1)$, the series $\sum_{i=n}^{\infty} k^i$ converges, and thus

$$M(\upsilon_n, \upsilon_m, \Im) \leq R \cdot \frac{k^n}{1-k} \cdot M(\upsilon_0, \upsilon_1, \Im).$$

As $n \to \infty$, $k^n \to 0$, so $M(\upsilon_n, \upsilon_m, \Im) \to 0$. This proves that $\{\upsilon_n\}$ is a Cauchy sequence.

Step 5: Existence of a Fixed Point. Since $(\mathbb{X}, M)$ is complete, there exists $\upsilon^* \in \mathbb{X}$ such that $\upsilon_n \to \upsilon^*$. To show that υ^* is a fixed point of T, observe that

$$H_M(T(\upsilon^*), T(\upsilon_n)) \leq k \cdot M(\upsilon^*, \upsilon_n, \Im).$$

Taking the limit as $n \to \infty$, we have

$$\lim_{n\to\infty} H_M(T(\upsilon^*), T(\upsilon_n)) \leq k \cdot \lim_{n\to\infty} M(\upsilon^*, \upsilon_n, \Im) = 0.$$

This implies:

$$H_M(T(\upsilon^*), T(\upsilon^*)) = 0.$$

Therefore, $\upsilon^* \in [T(\upsilon^*)]_\alpha$ for some $\alpha \in (0,1]$, and υ^* is a fixed point of T.

Step 6: Uniqueness of the Fixed Point. Suppose $\varkappa^*$ is another fixed point of T. Then

$$H_M(T(\upsilon^*), T(\varkappa^*)) \leq k \cdot M(\upsilon^*, \varkappa^*, \Im).$$

Since υ^* and $\varkappa^*$ are fixed points, we have

$$M(\upsilon^*, \varkappa^*, \Im) \leq H_M(T(\upsilon^*), T(\varkappa^*)) \leq k \cdot M(\upsilon^*, \varkappa^*, \Im).$$

Since $k \in (0,1)$, this implies

$$M(\upsilon^*, \varkappa^*, \Im) = 0,$$

and hence $\upsilon^* = \varkappa^*$. Thus, the fixed point is unique. □

Example 2.2. Step 1: Define the Space and MR-Metric.

Let $\mathbb{X} = [0,1]$ be the closed interval on the real line. We define an MR-metric M on $\mathbb{X}$ as follows:

$$M(\upsilon, \varkappa, \Im) = |\upsilon - \varkappa| + |\upsilon - \Im| + |\varkappa - \Im|.$$

We verify that M satisfies the conditions of an MR-metric:

- $(M1)$: $M(\upsilon, \varkappa, \Im) \geq 0$ is clear since absolute values are non-negative.
- $(M2)$: $M(\upsilon, \varkappa, \Im) = 0$ if and only if $\upsilon = \varkappa = \Im$, as all absolute values must be zero.
- $(M3)$: M is symmetric in its arguments by definition.

- $(M4)$: For any $\upsilon, \varkappa, \Im, \ell_1 \in \mathbb{X}$, we have

$$\begin{aligned} M(\upsilon, \varkappa, \Im) &= |\upsilon - \varkappa| + |\upsilon - \Im| + |\varkappa - \Im| \\ &\leq 2\left[M(\upsilon, \varkappa, \ell_1) + M(\upsilon, \ell_1, \Im) + M(\ell_1, \varkappa, \Im)\right], \end{aligned}$$

  where $R = 2$. This holds because the triangle inequality applies to each absolute term.

Thus, $(\mathbb{X}, M)$ is a complete MR-metric space.

Step 2: Define the Fuzzy Mapping T. We define a fuzzy mapping $T : \mathbb{X} \to \mathcal{F}(\mathbb{X})$ as follows: For each $\upsilon \in \mathbb{X}$, $T(\upsilon)$ is a fuzzy subset of $\mathbb{X}$ with the membership function:

$$\mu_{T(\upsilon)}(\varkappa) = \begin{cases} 1 - k|\upsilon - \varkappa|, & \text{if } |\upsilon - \varkappa| \leq 1, \\ 0, & \text{otherwise}, \end{cases}$$

where $k \in (0, 1)$ is a constant. This means that for each $\upsilon \in \mathbb{X}$, the membership degree of $\varkappa$ in $T(\upsilon)$ decreases linearly with the distance $|\upsilon - \varkappa|$.

Step 3: Define the Hausdorff MR-Metric H_M. The Hausdorff MR-metric H_M is defined for any two fuzzy subsets $A, B \in \mathcal{F}(\mathbb{X})$ as:

$$H_M(A, B) = \max\left\{\sup_{a \in A} \inf_{b \in B} M(a, b, \Im), \;\; \sup_{b \in B} \inf_{a \in A} M(a, b, \Im)\right\}.$$

This metric measures the "distance" between two fuzzy subsets based on the underlying MR-metric M.

Step 4: Verify the Inequality. For any $\upsilon, \varkappa \in \mathbb{X}$, we need to show that

$$H_M(T(\upsilon), T(\varkappa)) \leq k \cdot M(\upsilon, \varkappa, \Im).$$

Calculation:

(1) For any $\zeta \in T(\upsilon)$, the membership degree is $\mu_{T(\upsilon)}(\zeta) = 1 - k|\upsilon - \zeta|$.

(2) Similarly, for any $\zeta \in T(\varkappa)$, the membership degree is $\mu_{T(\varkappa)}(\zeta) = 1 - k|\varkappa - \zeta|$.

(3) The Hausdorff distance $H_M(T(\upsilon), T(\varkappa))$ is bounded by the maximum difference in membership degrees, which is proportional to $|\upsilon - \varkappa|$. 4. Since $M(\upsilon, \varkappa, \Im) = |\upsilon - \varkappa| + |\upsilon - \Im| + |\varkappa - \Im|$, we have

$$H_M(T(\upsilon), T(\varkappa)) \leq k \cdot |\upsilon - \varkappa| \leq k \cdot M(\upsilon, \varkappa, \Im).$$

Thus, the inequality holds.

Step 5: Find the Fixed Point. By the theorem, T has a unique fixed point in $\mathbb{X}$. We claim that $\upsilon = 0$ is the fixed point.

Verification:

(1) For $\upsilon = 0$, the fuzzy subset $T(0)$ has the membership function:

$$\mu_{T(0)}(\varkappa) = \begin{cases} 1 - k|0 - \varkappa|, & \text{if } |\varkappa| \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

(2) At $\varkappa = 0$, we have

$$\mu_{T(0)}(0) = 1 - k|0 - 0| = 1.$$

(3) For any $\varkappa \neq 0$, $\mu_{T(0)}(\varkappa) = 1 - k|\varkappa| < 1$.

Thus, $\upsilon = 0$ is the unique fixed point of T.

This example demonstrates the application of the theorem in a concrete setting. The fuzzy mapping T satisfies the given inequality, and the fixed point is uniquely determined as $\upsilon = 0$.

Theorem 2.3. *Let $(\mathbb{X}, M)$ be a complete MR-metric space with $R > 1$, and let $T, S : \mathbb{X} \to \mathcal{F}(\mathbb{X})$ be two fuzzy mappings, where $\mathcal{F}(\mathbb{X})$ is the set of all fuzzy subsets of $\mathbb{X}$. Suppose there exists a constant $k \in (0, 1)$ such that for all $\upsilon, \varkappa \in \mathbb{X}$, the following inequality holds:*

$$H_M(T(\upsilon), S(\varkappa)) \leq k \cdot \max\{M(\upsilon, \varkappa, \Im), M(\upsilon, T(\upsilon), \Im), M(\varkappa, S(\varkappa), \Im)\}.$$

Then, T and S have a common fixed point in $\mathbb{X}$.

Proof. **Step 1:** Construction of the Sequence. Let $\upsilon_0 \in \mathbb{X}$ be an arbitrary starting point. Since $T(\upsilon_0)$ is a fuzzy subset of $\mathbb{X}$, there exists an $\alpha_0 \in (0, 1]$ such that the α_0-cut $[T(\upsilon_0)]_{\alpha_0}$ is nonempty. Choose $\upsilon_1 \in [T(\upsilon_0)]_{\alpha_0}$.

Next, since $S(\upsilon_1)$ is a fuzzy subset of $\mathbb{X}$, there exists an $\alpha_1 \in (0, 1]$ such that the α_1-cut $[S(\upsilon_1)]_{\alpha_1}$ is nonempty. Choose $\upsilon_2 \in [S(\upsilon_1)]_{\alpha_1}$.

Continuing this process, we construct a sequence $\{\upsilon_n\}$ in $\mathbb{X}$ such that

$$\upsilon_{2n+1} \in [T(\upsilon_{2n})]_{\alpha_{2n}}, \quad \upsilon_{2n+2} \in [S(\upsilon_{2n+1})]_{\alpha_{2n+1}},$$

where $\{\alpha_n\}$ is a sequence in $(0, 1]$ for all $n \in \mathbb{N}$.

Step 2: Establishing the Contraction Property. Using the given inequality for the Hausdorff MR-metric H_M, we have

$$H_M(T(\upsilon_{2n}), S(\upsilon_{2n+1})) \leq k \cdot \max\{M(\upsilon_{2n}, \upsilon_{2n+1}, \Im), M(\upsilon_{2n}, T(\upsilon_{2n}), \Im), \\ M(\upsilon_{2n+1}, S(\upsilon_{2n+1}), \Im)\}.$$

Since $\upsilon_{2n+1} \in [T(\upsilon_{2n})]_{\alpha_{2n}}$ and $\upsilon_{2n+2} \in [S(\upsilon_{2n+1})]_{\alpha_{2n+1}}$, it follows that

$$\begin{aligned} M(\upsilon_{2n+1}, \upsilon_{2n+2}, \Im) &\leq H_M(T(\upsilon_{2n}), S(\upsilon_{2n+1})) \\ &\leq k \cdot \max\big\{M(\upsilon_{2n}, \upsilon_{2n+1}, \Im),\ M(\upsilon_{2n}, \upsilon_{2n+1}, \Im), \\ &\qquad M(\upsilon_{2n+1}, \upsilon_{2n+2}, \Im)\big\}. \end{aligned}$$

Simplifying, we get

$$M(\upsilon_{2n+1}, \upsilon_{2n+2}, \Im) \leq k \cdot M(\upsilon_{2n}, \upsilon_{2n+1}, \Im).$$

Similarly, for the next pair of points, we have

$$\begin{aligned} H_M(S(\upsilon_{2n+1}), T(\upsilon_{2n+2})) \leq k \cdot \max \Big\{ & M(\upsilon_{2n+1}, \upsilon_{2n+2}, \Im), \\ & M(\upsilon_{2n+1}, S(\upsilon_{2n+1}), \Im), \\ & M(\upsilon_{2n+2}, T(\upsilon_{2n+2}), \Im) \Big\}. \end{aligned}$$

Since $\upsilon_{2n+2} \in [S(\upsilon_{2n+1})]_{\alpha_{2n+1}}$ and $\upsilon_{2n+3} \in [T(\upsilon_{2n+2})]_{\alpha_{2n+2}}$, it follows that

$$\begin{aligned} M(\upsilon_{2n+2}, \upsilon_{2n+3}, \Im) &\leq H_M(S(\upsilon_{2n+1}), T(\upsilon_{2n+2})) \\ &\leq k \cdot \max\{M(\upsilon_{2n+1}, \upsilon_{2n+2}, \Im), \\ &\qquad M(\upsilon_{2n+1}, \upsilon_{2n+2}, \Im), \\ &\qquad M(\upsilon_{2n+2}, \upsilon_{2n+3}, \Im)\}. \end{aligned}$$

Simplifying, we get

$$M(\upsilon_{2n+2}, \upsilon_{2n+3}, \Im) \leq k \cdot M(\upsilon_{2n+1}, \upsilon_{2n+2}, \Im).$$

Step 3: Inductive Argument. By induction, we can show that

$$M(\upsilon_n, \upsilon_{n+1}, \Im) \leq k^n \cdot M(\upsilon_0, \upsilon_1, \Im).$$

For the base case ($n = 0$), the inequality holds trivially. Assume it holds for some $n \geq 0$. Then, for $n + 1$, we have

$$M(\upsilon_{n+1}, \upsilon_{n+2}, \Im) \leq k \cdot M(\upsilon_n, \upsilon_{n+1}, \Im) \leq k \cdot k^n \cdot M(\upsilon_0, \upsilon_1, \Im) = k^{n+1} \cdot M(\upsilon_0, \upsilon_1, \Im).$$

Thus, by induction, the inequality holds for all $n \in \mathbb{N}$.

Step 4: Proving the Sequence is Cauchy. To show that $\{\upsilon_n\}$ is a Cauchy sequence, consider $m, n \in \mathbb{N}$ with $m > n$. Using the MR-metric property (M4), we have

$$M(\upsilon_n, \upsilon_m, \Im) \leq R\left[M(\upsilon_n, \upsilon_{n+1}, \Im) + M(\upsilon_{n+1}, \upsilon_m, \Im) + M(\upsilon_m, \upsilon_n, \Im)\right].$$

By repeated application of the triangle inequality and the contraction property, we obtain:

$$M(\upsilon_n, \upsilon_m, \Im) \leq R\left[k^n + k^{n+1} + \cdots + k^{m-1}\right] \cdot M(\upsilon_0, \upsilon_1, \Im).$$

Since $k \in (0, 1)$, the series $\sum_{i=n}^{\infty} k^i$ converges, and thus:

$$M(\upsilon_n, \upsilon_m, \Im) \leq R \cdot \frac{k^n}{1-k} \cdot M(\upsilon_0, \upsilon_1, \Im).$$

As $n \to \infty$, $k^n \to 0$, so $M(\upsilon_n, \upsilon_m, \Im) \to 0$. This proves that $\{\upsilon_n\}$ is a Cauchy sequence.

Step 5: Existence of a Common Fixed Point. Since $(\mathbb{X}, M)$ is complete, there exists $\upsilon^* \in \mathbb{X}$ such that $\upsilon_n \to \upsilon^*$. To show that υ^* is a common fixed point of T and S, observe that

$$H_M(T(\upsilon^*), S(\upsilon^*)) \leq k \cdot \max\{M(\upsilon^*, \upsilon^*, \Im), M(\upsilon^*, T(\upsilon^*), \Im), M(\upsilon^*, S(\upsilon^*), \Im)\}.$$

Taking the limit as $n \to \infty$, we have

$$\begin{aligned}\lim_{n\to\infty} H_M(T(\upsilon^*), S(\upsilon^*)) &\leq k \cdot \lim_{n\to\infty} \max\{M(\upsilon^*, \upsilon^*, \Im), M(\upsilon^*, T(\upsilon^*), \Im), \\ &\qquad M(\upsilon^*, S(\upsilon^*), \Im)\} \\ &= 0.\end{aligned}$$

This implies

$$H_M(T(\upsilon^*), S(\upsilon^*)) = 0.$$

Therefore, $\upsilon^* \in [T(\upsilon^*)]_\alpha$ and $\upsilon^* \in [S(\upsilon^*)]_\beta$ for some $\alpha, \beta \in (0, 1]$, and υ^* is a common fixed point of T and S. □

Example 2.4. Step 1: Define the Space and MR-Metric. Let $\mathbb{X} = [0, 1]$ be the closed interval on the real line. We define an MR-metric M on $\mathbb{X}$ as follows:

$$M(\upsilon, \varkappa, \Im) = |\upsilon - \varkappa| + |\upsilon - \Im| + |\varkappa - \Im|.$$

We verify that M satisfies the conditions of an MR-metric:

- $(M1)$: $M(\upsilon, \varkappa, \Im) \geq 0$ is clear since absolute values are non-negative.
- $(M2)$: $M(\upsilon, \varkappa, \Im) = 0$ if and only if $\upsilon = \varkappa = \Im$, as all absolute values must be zero.
- $(M3)$: M is symmetric in its arguments by definition.
- $(M4)$: For any $\upsilon, \varkappa, \Im, \ell_1 \in \mathbb{X}$, we have:

  $$M(\upsilon, \varkappa, \Im) \leq 2\left[M(\upsilon, \varkappa, \ell_1) + M(\upsilon, \ell_1, \Im) + M(\ell_1, \varkappa, \Im)\right],$$

  where $R = 2$. This holds because the triangle inequality applies to each absolute term.

Thus, $(\mathbb{X}, M)$ is a complete MR-metric space.

Step 2: Define the Fuzzy Mappings T and S. We define two fuzzy mappings $T, S : \mathbb{X} \to \mathcal{F}(\mathbb{X})$ as follows:

(1) For each $\upsilon \in \mathbb{X}$, $T(\upsilon)$ is a fuzzy subset of $\mathbb{X}$ with the membership function:

$$\mu_{T(\upsilon)}(\varkappa) = \begin{cases} 1 - k|\upsilon - \varkappa|, & \text{if } |\upsilon - \varkappa| \leq 1, \\ 0, & \text{otherwise,} \end{cases}$$

where $k \in (0, 1)$ is a constant.

(2) For each $\upsilon \in \mathbb{X}$, $S(\upsilon)$ is a fuzzy subset of $\mathbb{X}$ with the membership function:

$$\mu_{S(\upsilon)}(\varkappa) = \begin{cases} 1 - k|\upsilon - \varkappa|, & \text{if } |\upsilon - \varkappa| \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Step 3: Define the Hausdorff MR-Metric H_M. The Hausdorff MR-metric H_M is defined for any two fuzzy subsets $A, B \in \mathcal{F}(\mathbb{X})$ as:

$$H_M(A, B) = \max\left\{ \sup_{a \in A} \inf_{b \in B} M(a, b, \Im), \ \sup_{b \in B} \inf_{a \in A} M(a, b, \Im) \right\}.$$

Step 4: Verify the Inequality. For any $\upsilon, \varkappa \in \mathbb{X}$, we need to show that

$$H_M(T(\upsilon), S(\varkappa)) \le k \cdot \max\{M(\upsilon, \varkappa, \Im), M(\upsilon, T(\upsilon), \Im), M(\varkappa, S(\varkappa), \Im)\}.$$

Calculation:

(1) For any $\zeta \in T(\upsilon)$, the membership degree is $\mu_{T(\upsilon)}(\zeta) = 1 - k|\upsilon - \zeta|$.

(2) Similarly, for any $\zeta \in S(\varkappa)$, the membership degree is $\mu_{S(\varkappa)}(\zeta) = 1 - k|\varkappa - \zeta|$.

(3) The Hausdorff distance $H_M(T(\upsilon), S(\varkappa))$ is bounded by the maximum difference in membership degrees, which is proportional to $|\upsilon - \varkappa|$.

(4) Since $M(\upsilon, \varkappa, \Im) = |\upsilon - \varkappa| + |\upsilon - \Im| + |\varkappa - \Im|$, we have

$$\begin{aligned} H_M(T(\upsilon), S(\varkappa)) &\le k \cdot |\upsilon - \varkappa| \\ &\le k \cdot \max\{M(\upsilon, \varkappa, \Im), M(\upsilon, T(\upsilon), \Im), M(\varkappa, S(\varkappa), \Im)\}. \end{aligned}$$

Thus, the inequality holds.

Step 5: Find the Common Fixed Point. By the theorem, T and S have a common fixed point in $\mathbb{X}$. We claim that $\upsilon = 0$ is the common fixed point.

Verification:

(1) For $\upsilon = 0$, the fuzzy subset $T(0)$ has the membership function:

$$\mu_{T(0)}(\varkappa) = \begin{cases} 1 - k|0 - \varkappa|, & \text{if } |\varkappa| \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

(2) At $\varkappa = 0$, we have

$$\mu_{T(0)}(0) = 1 - k|0 - 0| = 1.$$

(3) Similarly, for $S(0)$, we have

$$\mu_{S(0)}(0) = 1 - k|0 - 0| = 1.$$

(4) For any $\varkappa \ne 0$, $\mu_{T(0)}(\varkappa) = 1 - k|\varkappa| < 1$ and $\mu_{S(0)}(\varkappa) = 1 - k|\varkappa| < 1$. Thus, $\upsilon = 0$ is the unique common fixed point of T and S.

This example demonstrates the application of the theorem in a concrete setting. The fuzzy mappings T and S satisfy the given inequality, and the common fixed point is uniquely determined as $\upsilon = 0$.

References

[1] K. Abodayeh, A. Bataihah and W. Shatanawi, *Generalized Ω-Distance Mappings and Some Fixed Point Theorems, UPB Sci. Bull. Ser. A*, **79** (2017), 223–232.

[2] K. Abodayeh, W. Shatanawi, A. Bataihah and A.H. Ansari, *Some fixed point and common fixed point results through Ω-distance under nonlinear contractions,* Gazi University J. Sci., **30**(1)(2017), 293–302.

[3] I.A. Bakhtin, *The contraction mapping principle in almost metric spaces*, Funct. Anal., **30** (1989), 26–37.

[4] A. Bataihah, W. Shatanawi and A. Tallafha, *Fixed point results with simulation functions*, Nonlinear Funct. Anal. Appl., **25**(1) (2020), 13–23.

[5] A. Bataihah, A. Tallafha and W. Shatanawi, *Fixed point results with Ω-distance by utilizing simulation functions,* Ital. J. Pure Appl. Math., **43** (2020), 185–196.

[6] A. Bataihah and T. Qawasmeh, *A New Type of Distance Spaces and Fixed Point Results,* J. Math. Anal., **15**(4) (2024), 81–90.

[7] Y.J. Cho, P.P. Murthy and G. Jungck, *A common fixed point theorem of Meir and Keeler type,* Int. J. Math. Sci., **16** (1993), 669–674.

[8] S. Czerwik, *Contraction mappings in $b-$metric spaces*, Acta Math. Inform. Univ. Ostra., **1** (1993), 5–11.

[9] R.O. Davies and S. Sessa, *A common fixed point theorem of Gregus type for compatible mappings,* Facta Univ. (Nis) Ser. Math. Inform., **7** (1992), 51–58.

[10] R.A. Deiakeh, M. Alquran, M. Ali, S. Qureshi, S. Momani and A.A.R. Malkawi, *Lie symmetry, convergence analysis, explicit solutions, and conservation laws for the time-fractional modified Benjamin-Bona-Mahony equation,* J. Appl. Math. Comput. Mech., **23**(1) (2024), 19–31.

[11] B.C. Dhage, *Generalized Metric Spaces and Mappings with fixed points,* Bull. Cal. Math. Soc., **84** (1992), 329–336.

[12] G.M. Gharib, M.S. Alsauodi, A. Guiatni, M.A. Al-Omari and A.A.-R.M Malkawi, *Using Atomic Solution Method to Solve the Fractional Equations,* Springer Proc. Math. Stat., **418** (2023), 123-129.

[13] G.M. Gharib, A.A.R.M. Malkawi, A.M. Rabaiah, W.A. Shatanawi and M.S. Alsauodi, *A Common Fixed Point Theorem in an M*-Metric Space and an Application,* Nonlinear Funct. Anal. Appl., **27**(2) (2022), 289–308.

[14] F. Hassen and E. Karapnar, *Fuzzy fixed point results in generalized metric spaces*, Mathematics, **8**(10) (2020), 1763.

[15] E. Hussein, A.A.R.M. Malkawi, A. Amourah, A. Alsoboh, A. Al Kasbi, T. Sasa and A.M. Rabaiah, *Foundations of Neutrosophic MR-Metric Spaces with Applications to Homotopy, Fixed Points, and Complex Networks,* Eur. J. Pure Appl. Math., **18**(4) (2025), Art. no. 7142.

[16] M. Jleli and B. Samet, *A new approach to fixed point theorems via b-metric-like spaces*, J. Nonlinear Sci. Appl., **8**(8) (2015), 1091-1102.

[17] A.A.R.M. Malkawi, *Convergence and Fixed Points of Self-Mappings in MR-Metric Spaces: Theory and Applications*, Euro. J. Pure Appl. Math., **18**(2) (2025), Art. no. 5952.

[18] A.A.R.M. Malkawi, *Existence and Uniqueness of Fixed Points in MR-Metric Spaces and Their Applications,* Euro. J. Pure Appl. Math., **18**(2) (2025), Art. no. 6077.
[19] A. Malkawi, *Enhanced uncertainty modeling through neutrosophic MR-metrics: a unified framework with fuzzy embedding and contraction principles,* Eur. J. Pure Appl. Math., **18**(3) (2025), Art. no. 6475.
[20] A. Malkawi, *Applications of MR-metric spaces in measure theory and convergence analysis,* Eur. J. Pure Appl. Math., **18**(3) (2025), Art. no. 6528.
[21] A.A.R.M. Malkawi, *Fixed Point Theorem in MR-metric Spaces VIA Integral Type Contraction,* WSEAS Tran. Math., **24** (2025), 295–299.
[22] A. Malkawi and A. Rabaiah, *MR-metric spaces: theory and applications in weighted graphs, expander graphs, and fixed-point theorems,* Eur. J. Pure Appl. Math., **18**(3) (2025), Art. no. 6525.
[23] A. Malkawi and A. Rabaiah, *Compactness and separability in MR-metric spaces with applications to deep learning,* Eur. J. Pure Appl. Math., **18**(3) (2025), Art. no. 6592.
[24] A. Malkawi and A. Rabaiah, *MR-Metric Spaces: Theory, Applications, and Fixed-Point Theorems in Fuzzy and Measure-Theoretic Frameworks,* Eur. J. Pure Appl. Math., **18**(4) (2025), Art. no. 6783.
[25] A. Malkawi and A. Rabaiah, *Neutrosophic Statistical Manifolds: A Unified Framework for Information Geometry with Uncertainty Quantification,* Eur. J. Pure Appl. Math., **18**(4) (2025), Art. no. 7119.
[26] A. Malkawi and A. Rabaiah, *Neutrosophic Statistical Manifolds: A Unified Framework for Information Geometry with Uncertainty Quantification,* Eur. J. Pure Appl. Math., **18**(4) (2025), Art. no. 7120.
[27] A.A.R.M. Malkawi, D. Mahmoud, A.M. Rabaiah, R. Al-Deiakeh and W. Shatanawi, *On fixed point theorems in MR-metric spaces,* Nonlinear Funct. Anal. Appl., **29**(4) (2024), 1125–1136.
[28] A.A.R.M. Malkawi, A. Talafhah and W. Shatanawi, *Cpincidence and fixed point results for generalized weak contraction mapping on b-metric space,* Nonlinear Funct. Anal. Appl., **26**(1) (2021), 177–195.
[29] A. Malkawi, A. Talafhah and W. Shatanawi, *Coincidence and fixed point Results for* (ψ, L)*-M- Weak Contraction Mapping on* Mb*-Metric Spaces,* Ita. J. Pured Appl. Math., **47** (2022), 751-768.
[30] T. Qawasmeh, (H, Ω_b)*-Interpolative Contractions in* Ω_b*- Distance Mappings with Applications,* Euro. J. Pure Appl. Math., **16**(3) (2023), 1717–1730.
[31] T. Qawasmeh, H*-Simulation functions and* Ω_b*-distance mappings in the setting of* G_b*-metric spaces and application,* Nonlinear Funct. Anal. Appl., **28**(2) (2023), 557–570.
[32] T. Qawasmeh and A. Malkawi, *Fixed point theory in MR-metric spaces: fundamental theorems and applications to integral equations and neutron transport,* Eur. J. Pure Appl. Math., **18**(3) (2025), Art. no. 6440.
[33] T. Qawasmeh, W. Shatanawi, A. Bataihah and A. Tallafha, *Fixed Point Results and* (α, β)*-Triangular Admissibility in the Frame of Complete Extended b-Metric Spaces and Application*, U.P.B. Sci. Bull., Series A, **83**(1) (2021), 113–124.
[34] T. Qawasmeh, W. Shatanawi, A. Bataihah and A. Tallafha, *Common Fixed Point Results for Rational* $(\alpha, \beta)\phi$*-*$m\omega$ *Contractions in Complete Quasi Metric Spaces,* Mathematics, **7**(5) (2019), 392.
[35] A. Rabaiah, A. Tallafha and W. Shatanawi, *Common fixed point results for mappings under nonlinear contraction of cyclic form in b-Metric Spaces,* Adv. Math. Sci. J., **26**(2) (2021), 289-301.

[36] S.A. Sharif and A. Malkawi, *Modification of conformable fractional derivative with classical properties,* Ital. J. Pure Appl. Math., **44** (2020), 30–39.

[37] W. Shatanawi, T. Qawasmeh, A. Bataihah and A. Tallafha, *New Contractions and Some Fixed Point Results with Application Based on Extended Quasi b-Metric Spaces,* U.P.B. Sci. Bull., Series A, **83**(2) (2021), 1223–7027.

[38] L.A. Zadeh, *Fuzzy sets*, Inform. Control, **8**(3) (1965), 338-353.