Nonlinear Functional Analysis and Applications
Vol. 30, No. 4 (2025), pp. 1153-1170
ISSN: 1229-1595(print), 2466-0973(online)

https://doi.org/10.22771/nfaa.2025.30.04.11
http://nfaa.kyungnam.ac.kr/journal-nfaa

TIME-VARYING SCALED CONSENSUS: DYNAMIC MANIFOLD CONVERGENCE UNDER HYBRID SCALING PROFILES

Siriluk Donganont[1] and Mana Donganont[2]

[1]School of Science, University of Phayao, Phayao 56000, Thailand
e-mail: `siriluk.pa@up.ac.th`

[2]School of Science, University of Phayao, Phayao 56000, Thailand
e-mail: `mana.do@up.ac.th`

Abstract. This paper investigates the problem of time-varying scaled consensus in directed multi-agent systems, where agent states evolve under nonuniform and time-dependent influence. The core challenge lies in ensuring asymptotic synchronization of scaled states across a network with heterogeneous and dynamically evolving scaling functions. A distributed consensus protocol is proposed that incorporates agent-specific time-varying scaling profiles, subject to mild regularity and boundedness conditions. By leveraging Lyapunov-based analysis and a vector-valued extension of Barbalats Lemma, we establish convergence of the scaled states to a common dynamic manifold. The framework is further extended to accommodate global reference tracking on the evolving scaled-consensus manifold. Numerical simulations validate the theoretical results and demonstrate robustness under both static and time-varying scaling. The main innovation lies in unifying dynamic consensus and adaptive scaling within a tractable analytical framework, enabling coordination in networks with asymmetric and time-dependent agent roles.

1. Introduction

Consensus problems in multi-agent systems (MASs) have received considerable attention in recent years due to their theoretical significance and wide-ranging applications in cooperative control, distributed optimization, formation flying, and networked robotics. Within this broad framework, the notion

[0]Received June 3, 2025. Revised September 20, 2025. Accepted September 22, 2025.
[0]2020 Mathematics Subject Classification: 93C10, 93C28, 93C85, 93D40, 93D50.
[0]Keywords: Time-varying scaled consensus, multi-agent systems (MAS), directed graphs, Lyapunov sability, distributed tracking control.
[0]Corresponding authors: Mana Donganont(`mana.do@up.ac.th`).

of *scaled consensus* where agent states converge to a prescribed set of ratios rather than a common value has emerged as a powerful extension that accommodates heterogeneous roles and prioritized coordination. The seminal work of Roy [14] introduced the scaled consensus concept, providing linear protocols under which the state of each agent converges to a common affine manifold, weighted by agent-specific scaling coefficients. This framework was further extended by Meng and Jia [11], who studied scaled consensus over switching communication topologies and established convergence under dwell-time conditions and balance assumptions.

Subsequent studies have considered various complexities in the scaled consensus framework. For instance, Shang [15] addressed time-delayed interactions and introduced sufficient conditions for scaled consensus under uniform communication delay bounds. Zhang et al. [20] generalized these ideas to second-order nonlinear MASs with aperiodically intermittent control and time-varying delays. Additionally, Shang [16] analyzed switched MASs where the topology may vary among a finite set of directed graphs. The role of actuation constraints has also been considered; Wang [19] explored scaled consensus under output saturation and derived convergence results via projection methods.

A recent line of inquiry has focused on integrating stochastic or probabilistic models into scaled consensus. Phongchan et al. [13] proposed probabilistic protocols for MASs with dynamically evolving scaling profiles, thereby introducing new uncertainty dimensions into the coordination problem. Trinh et al. [18] proposed a matrix-scaled consensus protocol, highlighting the importance of anisotropic scaling structures in modeling state alignment within modular and hierarchical networks. More recently, Donganont [4] introduced a distributed complex consensus framework for hybrid networks, leveraging complex Laplacian operators to extend scaled coordination into complex-valued state spaces.

Beyond node-based consensus, a parallel development has taken place in the domain of edge-based and hybrid multi-agent systems (HMASs). Donganont and Liu [7] formulated impulsive protocols for scaled consensus in MASs and established sufficient conditions for finite-time convergence. Park et al. [12] proposed a hybrid edge consensus framework using scaled dynamics, offering a comprehensive treatment of both node and edge coordination. Donganont et al. [8] extended these ideas further by incorporating impulsive control into edge-dynamic systems. In parallel, pulse-modulated control strategies have been proposed for scaled edge consensus in hybrid MASs [6], offering a systematic mechanism to regulate edge dynamics with high-frequency modulation effects. Hybrid agent models comprising both continuous-time and discrete-time components were rigorously treated in [2], where convergence was demonstrated under periodic impulses. Additional studies by Donganont [3, 5] proposed finite-time scaled consensus protocols under leader-following structures and conjugate gradient methods.

The practical relevance of scaled consensus has also motivated studies involving reference tracking and output regulation. Zhang et al. [21] analyzed scaled tracking for high-order MASs with time delays and disturbances, while Shang [17] addressed constraint-driven tracking problems in networked settings. From a PDE perspective, Chen et al. [1] established scaled consensus over networks governed by wave equations. Furthermore, Li et al. [10] incorporated communication noise into the scaled consensus problem over switching topologies and proposed Lyapunov-based design criteria for stability in noisy environments.

Despite this rich body of work, several gaps remain. First, many existing studies assume static or piecewise-constant scaling functions, thereby limiting the flexibility of the consensus manifold. Second, the majority of convergence results are derived for autonomous systems and do not accommodate dynamic references or external inputs. Third, hybrid systems that combine continuous-time and discrete-time dynamics under time-varying scaling profiles are insufficiently treated in the literature, especially when synchronization must occur on a dynamically evolving manifold rather than a fixed linear subspace. Lastly, few studies rigorously characterize the asymptotic tracking behavior when global signals are injected into scaled coordination protocols.

In this paper, we address these shortcomings by developing a unified framework for *time-varying scaled consensus and tracking* over directed networks. Our setting allows each agent to possess a distinct, smooth, time-varying scaling function $\beta_i(t)$, subject to mild boundedness and regularity conditions. The resulting consensus manifold is thus dynamic and state-dependent, enabling a richer class of coordination behaviors. We rigorously prove asymptotic convergence of the scaled state trajectories using Lyapunov methods and a vector-valued extension of Barbalats Lemma [9]. Additionally, we design a modified protocol that allows all agents to track a global reference signal $f(t)$ while preserving their relative scaling structure. This dual functionalitydynamic scaling and global trackingmarks a significant advancement over static and autonomous consensus designs.

The main contributions of this work are threefold. First, we generalize the scaled consensus protocol of [14] to accommodate smoothly time-varying scaling functions over strongly connected directed graphs. Second, we extend the theoretical framework to include reference tracking on a time-varying manifold, offering a solution to problems previously considered only in limited forms [17, 21]. Third, we present numerical simulations, including comparisons with constant-scaling and unscaled cases, to validate the theoretical results and demonstrate the robustness of the proposed protocol under time-varying scaling and dynamic inputs. These results lay the groundwork for future studies on event-triggered, impulsive, and stochastic scaled consensus strategies in hybrid and uncertain environments.

The remainder of this paper is structured as follows. Section 2 introduces the mathematical preliminaries and formalizes the problem setting, including graph-theoretic constructs and the class of time-varying scaled consensus protocols. In Section 3, we rigorously analyze the asymptotic behavior of multi-agent systems under the proposed framework. This includes theoretical guarantees for both pure consensus and reference-tracking dynamics over strongly connected directed graphs, accompanied by sufficient conditions for convergence. Section 4 presents detailed numerical simulations that validate the theoretical results and compare the performance across time-varying, constant, and standard consensus strategies. Finally, Section 5 summarizes the main contributions and outlines potential directions for future research.

2. Preliminaries and problem formulations

2.1. **Notation and graph-theoretic preliminaries.** We consider a networked multi-agent system modeled by a directed graph (or digraph) denoted by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, 2, \dots, n\}$ represents the set of n agents (nodes), and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ denotes the set of directed communication links (edges) between agents. A directed edge $(j, i) \in \mathcal{E}$ indicates that agent i has access to the information state of agent j, that is, agent j can influence the update dynamics of agent i through a directed communication channel.

For each agent $i \in \mathcal{V}$, the set of its in-neighbors is defined as

$$\mathcal{N}_i := \{j \in \mathcal{V} \mid (j, i) \in \mathcal{E}\},$$

which consists of all agents from whom agent i can receive information. Throughout this work, we assume that the communication graph $\mathcal{G}$ is *strongly connected*, meaning that for any pair of distinct nodes $i, j \in \mathcal{V}$, there exists a finite sequence of directed edges forming a path from i to j. This structural property is crucial for ensuring network-wide information propagation and plays a central role in establishing consensus guarantees.

We adopt the following notational conventions: For any vector $x \in \mathbb{R}^n$, the i-th component is denoted by x_i, its transpose by $x^\top$, and its Euclidean norm by $\|x\| := \sqrt{x^\top x}$. The identity matrix of dimension n is denoted by

I_n, and the all-ones column vector of length n by $\mathbf{1}_n$. For any scalar-valued function $x_i(t)$, the time derivative is denoted $\dot{x}_i(t) = \frac{d}{dt}x_i(t)$ when it exists. The notation $\mathrm{diag}(x)$ represents the diagonal matrix whose diagonal entries are given by the components of the vector x. Finally, δ_{ij} denotes the Kronecker delta, equal to 1 if $i = j$ and 0 otherwise.

2.2. **Agent dynamics and scaled consensus objective.** We consider a class of continuous-time networked systems in which each agent $i \in \mathcal{V}$ maintains a scalar state $x_i(t) \in \mathbb{R}$ that evolves over time. The global network state is represented by the column vector $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^\top \in \mathbb{R}^n$. Unlike classical consensus problems where all agents aim to asymptotically agree on a common scalar value, this work focuses on a more general objective known as *scaled consensus*, in which agents converge to values in fixed or time-varying proportions.

To formally capture this behavior, we assign to each agent i a time-varying scaling function $\beta_i(t) : \mathbb{R}_{\geq 0} \to \mathbb{R} \setminus \{0\}$. These functions act as dynamic weights that shape the desired equilibrium ratios among the agents' states. The collection of all scaling functions is denoted $\beta(t) := [\beta_1(t), \beta_2(t), \ldots, \beta_n(t)]^\top$.

The central coordination goal is to design distributed interaction protocols under which the agents' state trajectories satisfy the asymptotic agreement condition

$$\lim_{t\to\infty} (\beta_i(t)x_i(t) - \beta_j(t)x_j(t)) = 0, \quad \forall\, i, j \in \mathcal{V}, \tag{2.1}$$

despite possible temporal variations in the scaling profiles $\beta_i(t)$. In other words, the scaled states $\beta_i(t)x_i(t)$ of all agents are required to synchronize in the long run, even if the agents themselves follow distinct trajectories or carry different weights over time.

This formulation generalizes traditional average consensus as a special case where $\beta_i(t) \equiv 1$ for all i, and extends naturally to scenarios with heterogeneous agent priorities, task allocations, or dynamic role hierarchies. Moreover, the time-varying nature of $\beta_i(t)$ allows modeling both exogenous signals (e.g., control inputs or mission profiles) and endogenous adaptations (e.g., time-triggered or event-driven adjustments in agent importance).

In the sections that follow, we will define a class of scalable and decentralized protocols that realize the objective (2.1) under appropriate assumptions on the communication graph structure and the regularity of the scaling functions $\beta_i(t)$.

2.3. **Time-varying scaled consensus protocol.** Building upon the linear scaled consensus framework introduced by Roy [14], we extend the formulation to accommodate time-dependent scaling profiles associated with each agent.

Specifically, we consider a class of non-autonomous distributed protocols in which each agent $i \in \mathcal{V}$ updates its scalar state $x_i(t) \in \mathbb{R}$ according to a time-varying weighted combination of relative differences with its neighbors, scaled by agent-specific functions $\beta_i(t)$.

The proposed time-varying scaled consensus protocol is defined by the differential equation

$$\dot{x}_i(t) = \mathrm{sgn}(\beta_i(t)) \sum_{j \in \mathcal{N}_i} k_{ij} \left(\beta_j(t) x_j(t) - \beta_i(t) x_i(t) \right), \quad i \in \mathcal{V}, \tag{2.2}$$

where $k_{ij} > 0$ represents the interaction weight associated with the directed edge $(j, i) \in \mathcal{E}$, and $\mathrm{sgn}(\cdot)$ denotes the standard sign function, ensuring correct directional influence even when $\beta_i(t)$ takes negative values.

This protocol can be interpreted as a diffusion-type process wherein each agent dynamically seeks to minimize discrepancies in scaled state values with respect to its neighbors. The direction and intensity of the interaction are governed by the local time-varying scales $\beta_i(t)$ and the static network structure encoded by the weights k_{ij}.

For analytical convenience and to facilitate compact representation, we define a generalized time-varying interaction matrix $K(t) \in \mathbb{R}^{n \times n}$ as follows:

$$[K(t)]_{ij} = \begin{cases} k_{ij}, & \text{if } j \in \mathcal{N}_i, \\ -\sum_{j \in \mathcal{N}_i} k_{ij}, & \text{if } i = j, \\ 0, & \text{otherwise,} \end{cases}$$

which satisfies the zero row-sum property commonly associated with Laplacian-type matrices.

We then write the vector form of the agent dynamics as

$$\dot{x}(t) = A(t)x(t), \quad A(t) := \mathrm{diag}\left(\frac{1}{\beta_i(t)} \right) \cdot \mathrm{diag}\left(\mathrm{sgn}(\beta_i(t)) \right) \cdot K(t) \cdot \mathrm{diag}(\beta_i(t)), \tag{2.3}$$

where $x(t) \in \mathbb{R}^n$ is the global state vector, and the time-varying matrix $A(t)$ governs the non-autonomous dynamics induced by the evolving scale functions.

The primary analytical objective is to determine conditions under which the dynamic system (2.3) achieves scaled consensus in the sense of Equation (2.1), despite potential non-stationarity introduced by time-varying and heterogeneous scaling profiles $\beta_i(t)$. This requires a careful examination of the spectral and structural properties of $A(t)$, which are inherently shaped by both the underlying communication topology and the temporal behavior of the scaling functions.

To ensure that the system (2.3) is well-posed, robust to perturbations, and analytically tractable, we introduce a set of regularity conditions on the agent-specific scaling functions $\beta_i(t)$. These conditions enable the use of Lyapunov-based techniques, perturbation analysis, and hybrid system theory for both continuous-time and discrete-time components.

Assumption 2.1. Each agent $i \in \mathcal{V}$ is assigned a time-varying scaling function $\beta_i(t) : \mathbb{R}_{\geq 0} \to \mathbb{R} \setminus \{0\}$ satisfying

(i) There exist constants $0 < \beta_{\min} \leq \beta_{\max} < \infty$ such that

$$\beta_{\min} \leq |\beta_i(t)| \leq \beta_{\max}, \quad \forall\, t \geq 0.$$

(ii) For agents $i \in \mathcal{I}_C \subseteq \mathcal{V}$, the function $\beta_i(t)$ is continuously differentiable with uniformly bounded derivative

$$|\dot{\beta}_i(t)| \leq \gamma \quad \text{for some } \gamma > 0.$$

(iii) For $i \in \mathcal{I}_D := \mathcal{V} \setminus \mathcal{I}_C$, $\beta_i(t)$ is piecewise constant with sampling period $h > 0$:

$$\beta_i(t) = \beta_i(t_k), \quad \forall\, t \in [t_k, t_{k+1}), \quad t_k = kh.$$

(iv) There exists $C_\beta > 0$ such that for all $i, j \in \mathcal{V}$ and $t \in [t_k, t_{k+1})$,

$$\left| \frac{\beta_i(t)}{\beta_j(t)} - \frac{\beta_i(t_k)}{\beta_j(t_k)} \right| \leq C_\beta h.$$

Remark 2.2. Assumption 2.1 is foundational to the stability and convergence analysis of the proposed time-varying scaled consensus framework. The boundedness condition (i) ensures that the transformed system remains well-conditioned over time. Smoothness condition (ii) is critical for ensuring the validity of Lyapunov and input-to-state stability (ISS) arguments in continuous-time settings. Piecewise constancy (iii) naturally models digital agents in networked systems, facilitating hybrid analytical techniques. Lastly, the inter-agent ratio constraint (iv) provides a uniform bound on the rate of change of relative scales, thereby preventing topological and functional discontinuities that could lead to consensus failure or oscillatory behavior. Collectively, these conditions enable a unified and robust treatment of consensus dynamics under heterogeneous, time-dependent, and hybrid scaling environments.

Barbalat's lemma and its vector-valued extension. Barbalats Lemma is a classical tool in the stability analysis of nonlinear and time-varying systems. It provides a foundational criterion for establishing asymptotic convergence when direct Lyapunov methods yield integrable but non-strictly decreasing derivatives. In the context of multi-agent coordination, it is particularly useful when analyzing perturbed or time-varying consensus dynamics. We recall both the scalar and vector-valued forms below, following the formulation in [9].

Lemma 2.3. ([9, Barbalats Lemma]) *Let $f : [0, \infty) \to \mathbb{R}$ be uniformly continuous. If*

$$\int_0^\infty f(t)\, dt < \infty,$$

then $\lim_{t\to\infty} f(t) = 0$.

Lemma 2.4. ([9, Vector-Valued Barbalats Lemma]) *Let $x : [0, \infty) \to \mathbb{R}^n$ be continuously differentiable. Suppose*

(1) *$x(t)$ is bounded on $[0, \infty)$;*
(2) *$\dot{x}(t)$ is uniformly continuous;*
(3) *$\dot{x}(t) \in L^1([0, \infty))$.*

Then $\lim_{t\to\infty} \dot{x}(t) = 0$.

Remark 2.5. Lemma 2.4 plays a critical role in consensus stability proofs, ensuring that velocity-like terms $\dot{x}(t)$ vanish asymptotically. This facilitates the derivation of convergence results on time-varying consensus manifolds, particularly in systems subject to smooth perturbations or nonautonomous coupling.

3. Main results

This section presents the theoretical foundations of time-varying scaled consensus and tracking protocols for directed multi-agent systems. By incorporating agent-specific time-varying scaling functions, we generalize the classical consensus framework to allow dynamic heterogeneity in agent influence. The proposed protocols preserve convergence properties while enabling each agent to modulate its role in the collective dynamics. We begin by establishing asymptotic synchronization results for the time-varying scaled consensus protocol in the absence of exogenous input, and subsequently extend the analysis to include tracking of a global reference signal along a time-dependent consensus manifold. All results are derived under graph-theoretic and regularity assumptions that ensure well-posedness of the system and enable Lyapunov-based stability analysis.

3.1. **Time-varying scaled consensus.**

We begin by analyzing the asymptotic behavior of a multi-agent system governed by a time-varying scaled consensus protocol. In this framework, each agent adjusts its influence via a smooth, bounded scaling function $\beta_i(t)$, allowing dynamic heterogeneity in the collective evolution. The following theorem establishes that, under appropriate graph-theoretic and regularity conditions, the scaled agent states $\beta_i(t)x_i(t)$ achieve asymptotic agreement.

Theorem 3.1. *Consider the time-varying scaled consensus protocol* (2.2) *over a strongly connected directed graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ *with positive edge weights* $k_{ij} > 0$ *for all* $(j, i) \in \mathcal{E}$. *Assume that the agent-specific scaling functions* $\beta_i(t)$ *satisfy all conditions in Assumption* 2.1. *Then, for any initial condition* $x(0) \in \mathbb{R}^n$, *the corresponding solution* $x(t)$ *satisfies*

$$\lim_{t\to\infty} (\beta_i(t)x_i(t) - \beta_j(t)x_j(t)) = 0, \quad \forall\, i, j \in \mathcal{V},$$

that is, the scaled states $\beta_i(t)x_i(t)$ *asymptotically synchronize across the network.*

Proof. Define the scaled state variable $z_i(t) := \beta_i(t)x_i(t)$ for each $i \in \mathcal{V}$ and collect these into the vector $z(t) = [z_1(t), \dots, z_n(t)]^\top \in \mathbb{R}^n$. Differentiating $z_i(t)$ gives

$$\dot{z}_i(t) = \dot{\beta}_i(t)x_i(t) + \beta_i(t)\dot{x}_i(t).$$

Substituting the dynamics from (2.2) and using the identity $\beta_j(t)x_j(t) = z_j(t)$, we obtain

$$\dot{z}_i(t) = \frac{\dot{\beta}_i(t)}{\beta_i(t)} z_i(t) + \operatorname{sgn}(\beta_i(t)) \sum_{j\in\mathcal{N}_i} k_{ij}(z_j(t) - z_i(t)).$$

Letting $\eta_i(t) := \frac{\dot{\beta}_i(t)}{\beta_i(t)} z_i(t)$, the dynamics become

$$\dot{z}_i(t) = \operatorname{sgn}(\beta_i(t)) \sum_{j\in\mathcal{N}_i} k_{ij}(z_j(t) - z_i(t)) + \eta_i(t),$$

and in vector form,

$$\dot{z}(t) = -\mathcal{L}(t)z(t) + \eta(t),$$

where $\mathcal{L}(t)$ is a time-varying Laplacian-like matrix and $\eta(t) = [\eta_1(t), \dots, \eta_n(t)]^\top$ is the perturbation term. Consider the disagreement function

$$V(t) := \frac{1}{2} \sum_{i=1}^{n} \sum_{j\in\mathcal{N}_i} k_{ij}(z_i(t) - z_j(t))^2.$$

This function is positive definite with respect to the disagreement space $\{z \in \mathbb{R}^n \mid z_i = z_j, \forall i, j\}$ and satisfies

$$\dot{V}(t) = \sum_{i,j} k_{ij}(z_i - z_j)(\dot{z}_i - \dot{z}_j) = -\sum_{i,j} k_{ij}(z_i - z_j)^2 + \sum_{i,j} k_{ij}(z_i - z_j)(\eta_i - \eta_j).$$

Due to Assumption 2.1(ii) and (i), each $\eta_i(t)$ is bounded as

$$|\eta_i(t)| \le \frac{\gamma}{\beta_{\min}} |z_i(t)| \quad \Rightarrow \quad \|\eta(t)\| \le \frac{\gamma}{\beta_{\min}} \|z(t)\|.$$

Since $\mathcal{L}(t)$ is associated with a strongly connected graph and has zero row-sum structure, it follows that the term $-\sum (z_i - z_j)^2$ ensures $V(t)$ is strictly decreasing up to perturbation. Because $\dot{V}(t)$ is bounded and $V(t)$ is non-increasing and lower bounded, it is uniformly continuous. Moreover, from the inequality

$$\dot{V}(t) \leq -\lambda_2 \|z(t) - \bar{z}(t)\mathbf{1}_n\|^2 + C\|z(t)\|^2,$$

and boundedness of $z(t)$ (due to bounded $\beta_i(t)$ and $x(t)$), we conclude that $\dot{V}(t) \in L^1$ and uniformly continuous. Thus, by applying the vector-valued version of Barbalats Lemma (Lemma 2.4) to $\dot{z}(t)$, it follows that $\lim_{t\to\infty} \dot{z}(t) = 0$, and hence $\dot{V}(t) \to 0$. Therefore, $z_i(t) - z_j(t) \to 0$ for all $i, j \in \mathcal{V}$, implying $\beta_i(t)x_i(t) - \beta_j(t)x_j(t) \to 0$. This completes the proof. □

Remark 3.2. Theorem 3.1 establishes that the time-varying scaled consensus protocol (2.2) asymptotically enforces alignment of agent states on a dynamic manifold defined by

$$\mathcal{M}(t) := \{x \in \mathbb{R}^n \mid \beta_i(t)x_i = \beta_j(t)x_j,\ \forall\, i, j \in \mathcal{V}\}.$$

This manifold generalizes the classical consensus subspace to a time-evolving structure governed by agent-specific scaling functions. Rather than converging to a common scalar value, the agents synchronize to a time-dependent configuration characterized by dynamically weighted ratios. This flexibility accommodates heterogeneous roles, adaptive priorities, and functional asymmetries within the network, and extends the applicability of consensus strategies to a broader class of coordination tasks.

Corollary 3.3. *Suppose the scaling functions are constant in time, i.e.,* $\beta_i(t) \equiv \alpha_i \in \mathbb{R} \setminus \{0\}$ *for all* $i \in \mathcal{V}$*. Then the time-varying scaled consensus protocol* (2.2) *reduces to the static scaled consensus protocol introduced in* [14]*, and the solution* $x(t)$ *satisfies*

$$\lim_{t\to\infty} x(t) = \left[\frac{1}{\alpha_1}, \dots, \frac{1}{\alpha_n}\right]^\top c, \quad \textit{for some constant } c \in \mathbb{R}.$$

Proof. When $\beta_i(t) \equiv \alpha_i$ for all i, it follows that $\dot{\beta}_i(t) = 0$ identically. In this case, Assumption 2.1 is trivially satisfied, and the scaled state $z_i(t) := \alpha_i x_i(t)$ evolves according to the linear autonomous system

$$\dot{z}(t) = -\mathcal{L}z(t),$$

where $\mathcal{L}$ is the constant Laplacian-like matrix induced by the communication topology and the static weights. Under strong connectivity, $\mathcal{L}$ has a simple zero eigenvalue and the remaining spectrum lies in the open right half-plane. It follows from standard consensus theory that $z_i(t) \to c$ for some $c \in \mathbb{R}$, and hence $x_i(t) \to \frac{c}{\alpha_i}$ as $t \to \infty$. The result follows. □

3.2. **Tracking Consensus on a time-varying scaled manifold.** In many coordination and control tasks such as formation control, distributed sensing, and resource allocation agents must not only reach consensus but also track a time-varying reference while maintaining prescribed inter-agent relationships. Extending the concept of scaled consensus, where state convergence respects static ratios, we introduce a framework that integrates time-varying scaling profiles with dynamic reference tracking. The core idea is to augment the scaled consensus protocol with a global input term, ensuring that trajectories evolve on a time-dependent consensus manifold defined by heterogeneous, smoothly varying scaling functions. This approach enables all agents to synchronize their scaled states to a common signal while preserving individualized roles.

The following theorem demonstrates that, under suitable regularity and connectivity conditions, the proposed protocol ensures both asymptotic synchronization and accurate tracking of the reference trajectory.

Theorem 3.4. *Consider a multi-agent system governed by the tracking-modified scaled consensus dynamics*

$$\dot{x}_i(t) = \operatorname{sgn}(\beta_i(t)) \sum_{j \in \mathcal{N}_i} k_{ij} \left(\beta_j(t)x_j(t) - \beta_i(t)x_i(t)\right) + \frac{1}{\beta_i(t)} \dot{f}(t), \quad i \in \mathcal{V}, \tag{3.1}$$

where $f(t)$ is a continuously differentiable global reference signal and $\beta_i(t)$ are agent-specific scaling functions satisfying Assumption 2.1*. If the communication graph $\mathcal{G}$ is strongly connected, then the solution $x(t)$ to* (3.1) *satisfies*

$$\lim_{t\to\infty} \left(\beta_i(t)x_i(t) - \beta_j(t)x_j(t)\right) = 0, \quad \forall\, i, j \in \mathcal{V},$$

and the scaled state $\beta_i(t)x_i(t)$ asymptotically tracks the signal $w^\top x(0) + f(t)$, where $w^\top$ is the left eigenvector of the limiting state matrix associated with eigenvalue zero.

Proof. Define the deviation variable $z_i(t) := \beta_i(t)x_i(t) - f(t)$ and let $z(t) = [z_1(t), \ldots, z_n(t)]^\top$. Differentiating yields

$$\dot{z}_i(t) = \dot{\beta}_i(t)x_i(t) + \beta_i(t)\dot{x}_i(t) - \dot{f}(t).$$

Substituting the tracking dynamics (3.1) and using $\beta_j(t)x_j(t) = z_j(t) + f(t)$, we obtain

$$\dot{z}_i(t) = \frac{\dot{\beta}_i(t)}{\beta_i(t)}(z_i(t) + f(t)) + \operatorname{sgn}(\beta_i(t)) \sum_{j \in \mathcal{N}_i} k_{ij}(z_j(t) - z_i(t)).$$

Letting $\eta_i(t) := \frac{\dot{\beta}_i(t)}{\beta_i(t)}(z_i(t) + f(t))$, the dynamics reduce to

$$\dot{z}_i(t) = \operatorname{sgn}(\beta_i(t)) \sum_{j \in \mathcal{N}_i} k_{ij}(z_j(t) - z_i(t)) + \eta_i(t)$$

or compactly,

$$\dot{z}(t) = -\mathcal{L}(t)z(t) + \eta(t)$$

with $\eta(t) = [\eta_1(t), \ldots, \eta_n(t)]^\top$ and $\mathcal{L}(t)$ defined as before. Since $\dot{\beta}_i(t)$ is bounded and $\beta_i(t)$ is bounded away from zero (Assumption 2.1(i)-(ii)), and $f(t)$ is smooth, we have

$$\|\eta(t)\| \leq \frac{\gamma}{\beta_{\min}}(\|z(t)\| + |f(t)|),$$

showing that $\eta(t)$ grows at most linearly in $\|z(t)\|$ and remains bounded. Using the same Lyapunov function

$$V(t) = \frac{1}{2}\sum_{i,j} k_{ij}(z_i - z_j)^2,$$

its derivative satisfies

$$\dot{V}(t) \leq -\lambda_2\|z(t) - \bar{z}(t)\mathbf{1}_n\|^2 + C\|\eta(t)\|\|z(t)\|.$$

Since $z(t)$ and $f(t)$ are bounded, $\dot{V}(t)$ is uniformly continuous and integrable. Applying Lemma 2.4, we conclude that $\lim_{t\to\infty} \dot{z}(t) = 0$ and hence $\lim_{t\to\infty}$ $(z_i(t) - z_j(t)) = 0$. This implies that all $\beta_i(t)x_i(t) - f(t)$ converge to a common limit trajectory, establishing synchronized tracking of the reference $f(t)$ on the time-varying scaled manifold. □

Remark 3.5. Theorem 3.4 demonstrates that the proposed tracking protocol not only enforces convergence on a time-varying scaled-consensus manifold, but also enables all agents to asymptotically follow a common reference trajectory. This result unifies dynamic consensus and global tracking within a flexible framework, where each agent maintains heterogeneous, time-varying influence via its scaling function $\beta_i(t)$. The agents' states need not converge to the same value, but rather evolve in synchrony with a common signal modulated by local scale profiles.

Corollary 3.6. *Suppose the scaling functions are constant in time, i.e., $\beta_i(t) \equiv \alpha_i \in \mathbb{R} \setminus \{0\}$ for all $i \in \mathcal{V}$. Then the tracking dynamics* (3.1) *reduce to*

$$\dot{x}_i(t) = \operatorname{sgn}(\alpha_i) \sum_{j \in \mathcal{N}_i} k_{ij}\left(\alpha_j x_j(t) - \alpha_i x_i(t)\right) + \frac{1}{\alpha_i}\dot{f}(t),$$

and the scaled state $\alpha_i x_i(t)$ satisfies

$$\lim_{t\to\infty}(\alpha_i x_i(t) - \alpha_j x_j(t)) = 0 \quad \textit{and} \quad \alpha_i x_i(t) \to w^\top x(0) + f(t)$$

for all $i, j \in \mathcal{V}$, where $w^\top$ is the left eigenvector of the static consensus matrix associated with eigenvalue zero.

Proof. Under constant scaling $\beta_i(t) \equiv \alpha_i$, the system becomes a static scaled consensus process with an additive global forcing term. The transformation $z_i(t) := \alpha_i x_i(t) - f(t)$ leads to dynamics of the form $\dot{z}(t) = -\mathcal{L}z(t)$, where $\mathcal{L}$ is a fixed Laplacian-like matrix. Standard consensus theory ensures $z_i(t) \to c$ for some $c \in \mathbb{R}$, implying that $\alpha_i x_i(t) \to c + f(t)$, and hence tracking is achieved. The conclusion follows. □

4. Simulations and discussion

To illustrate the theoretical results established in Theorem 3.4 and Corollary 3.6, we provide a comparative simulation study involving multiple consensus protocols with and without time-varying scaling and reference tracking. The objective is to evaluate the dynamic behavior of agents under different scaling strategies, and to verify convergence of both raw and scaled states to the desired consensus or tracking manifold. All scaling functions and network conditions are chosen to satisfy the structural assumptions required for well-posedness and convergence, including strong connectivity of the underlying graph and boundedness of scaling functions.

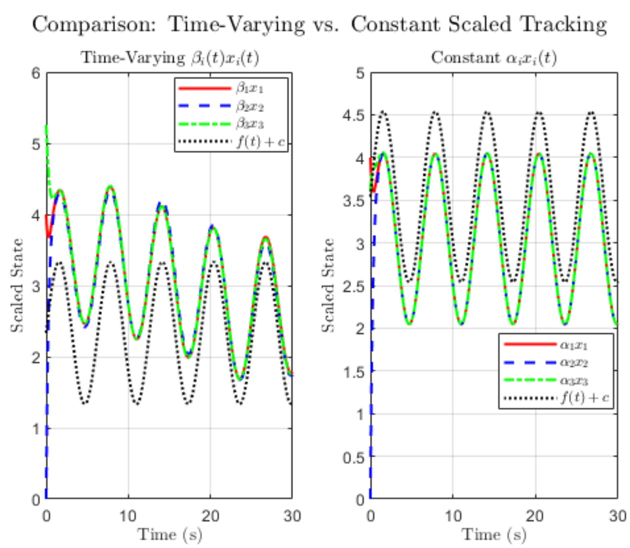

FIGURE 1. Comparison of time-varying scaled tracking (left) and constant scaled tracking (right). The scaled states $\beta_i(t)x_i(t)$ and $\alpha_i x_i(t)$ synchronize and track the global signal $f(t) + c$.

Example 4.1. Consider a multi-agent system composed of $n = 3$ agents interacting over a directed and strongly connected communication topology defined by the Laplacian matrix

$$K = \begin{bmatrix} -1 & 1 & 0 \\ 2 & -3 & 1 \\ 3 & 0 & -3 \end{bmatrix}.$$

The initial state vector is given by $x(0) = [4, 0, 3]^\top$. We investigate consensus behavior under three distinct distributed interaction protocols:

(i) time-varying scaled consensus tracking,

(ii) constant scaled consensus tracking, and

(iii) standard consensus with uniform influence weights.

All simulations satisfy the structural assumptions required for well-posedness and convergence, including strong connectivity of the graph and positive scaling bounds.

For the tracking protocol, we assume a global signal $f(t) = \sin(t)$ with derivative $\dot{f}(t) = \cos(t)$ and study the scaled consensus dynamics of the form

$$\dot{x}_i(t) = \operatorname{sgn}(\beta_i(t)) \sum_{j \in \mathcal{N}_i} k_{ij} \big(\beta_j(t) x_j(t) - \beta_i(t) x_i(t)\big) + \frac{1}{\beta_i(t)} \dot{f}(t),$$

where the time-varying scaling functions $\beta_i(t)$ are smooth and satisfy

$$\begin{aligned} \beta_1(t) &= 1.0 + 0.2 \sin(0.2t), \\ \beta_2(t) &= 0.5 + 0.1 \cos(0.3t), \\ \beta_3(t) &= 1.5 + 0.3 \sin(0.1t + 1). \end{aligned}$$

Each $\beta_i(t)$ is continuously differentiable, bounded above and below by positive constants, and hence satisfies Assumption 2.1. The agents' scaled states $\beta_i(t)x_i(t)$ converge asymptotically to $f(t) + c$, where $c \approx w^\top x(0)$ is the offset defined by the left eigenvector $w^\top$ of the limiting system matrix associated with the zero eigenvalue.

Figure 1 shows the comparison between time-varying scaled consensus and constant scaling $\alpha = [1.0, 0.5, 1.2]^\top$. The left panel illustrates that despite heterogeneity in scaling profiles, the scaled states synchronize and track $f(t)+c$ over time. The right panel demonstrates the constant scaled protocol leads to similar synchronization behavior, confirming Corollary 3.6.

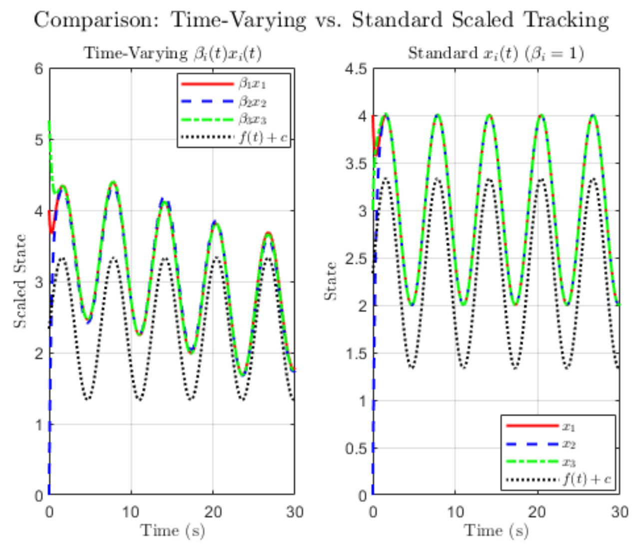

FIGURE 2. Comparison of time-varying scaled tracking (left) versus standard consensus with uniform scaling $\beta_i = 1$ (right). The scaled trajectories synchronize and track $f(t) + c$ in both cases, but exhibit different transient behavior.

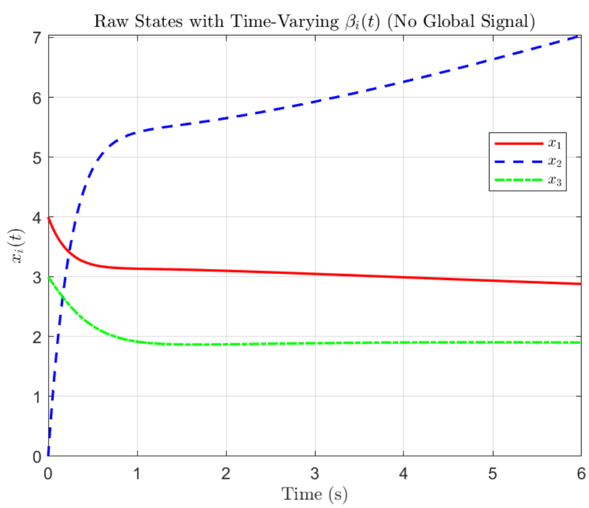

FIGURE 3. Raw agent states under time-varying scaled consensus protocol without external signal $f(t)$. The system exhibits coordinated evolution toward a scaled consensus configuration.

In Figure 2, we compare time-varying scaled tracking with the standard unscaled consensus protocol, where $\beta_i \equiv 1$ and the system evolves under

$$\dot{x}_i(t) = \sum_{j \in \mathcal{N}_i} k_{ij}(x_j(t) - x_i(t)) + \dot{f}(t).$$

Although all protocols track the same signal, the time-varying scaling allows for local heterogeneity in influence, consistent with adaptive or weighted coordination mechanisms in practice. To demonstrate the effect of removing the global reference signal, we consider the same network dynamics but set $f(t) \equiv 0$. Under this condition, the system reduces to a scaled consensus problem with pure coupling. Figures 3 and 4 show the raw states $x_i(t)$ for time-varying and standard scaling protocols respectively. In the standard case, consensus is achieved at the average of the initial conditions, whereas in the time-varying case, the system still reaches an agreement in the scaled domain, but the raw state trajectories evolve differently due to the scaling asymmetry.

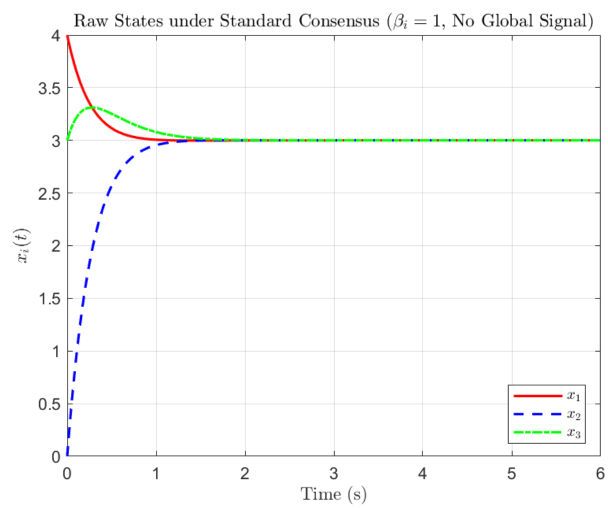

FIGURE 4. Raw agent states under standard consensus protocol with $\beta_i = 1$ and no external signal. Consensus is achieved at the average of the initial conditions.

The simulation results confirm that both the time-varying and constant scaled consensus tracking protocols lead to synchronization of scaled agent states onto the desired manifold $\mathcal{M}(t)$, with convergence to $f(t) + c$ as predicted. The time-varying scaling enables heterogeneous and adaptive weighting, while preserving convergence guarantees. In the absence of a global signal, the scaled consensus protocol still guides the agents toward a coordinated state, though the raw trajectories differ due to scaling asymmetry. This demonstrates the effectiveness and flexibility of the proposed protocols in shaping multi-agent dynamics under both tracking and pure consensus scenarios.

5. Conclusion

This work has developed a rigorous framework for time-varying scaled consensus in directed multi-agent systems, extending classical consensus to a dynamic manifold defined by agent-specific scaling functions. By analyzing both consensus and tracking protocols, we demonstrated that agents synchronize their scaled states asymptotically under mild regularity conditions and strong connectivity. Numerical simulations confirmed the theoretical predictions and revealed how time-varying scaling introduces adaptive heterogeneity while preserving convergence. These results provide new insight into coordination over dynamic networks with heterogeneous influence profiles. Future research may extend the proposed methodology to impulsive, sampled-data, or event-triggered settings, and explore robustness under communication delays, switching topologies, or stochastic perturbations, broadening the applicability of scaled consensus in real-world networked systems.

Acknowledgments: This work was supported by the University of Phayao and the Thailand Science Research and Innovation Fund (Fundamental Fund 2026, Grant No. 2268/2568).

References

[1] Y. Chen, Z. Zuo and Y. Wang, *Scaled consensus over a network of wave equations*, IEEE Tran. Control of Network Syst., **9**(3) (2022), 1385–1396.

[2] M. Donganont, *Scaled consensus of hybrid multi-agent systems via impulsive protocols*, J. Math. Comput. Sci., **36**(3) (2025), 275–289.

[3] M. Donganont, *Leader-following finite-time scaled consensus problems in multi-agent systems*, J. Math. Comput. Sci., **38**(4) (2025), 464–478.

[4] M. Donganont, *Distributed complex consensus in hybrid networks: A complex Laplacian-based approach*, Asian J. Control, (2025), 1–12, `https://doi.org/10.1002/asjc.3808`.

[5] M. Donganont, S. Intawichai and S. Phongchan, *Finite-Time Scaled Consensus in Hybrid Multi-Agent Systems via Conjugate Gradient Methods*, Euro. J. Pure Appl. Math., **18**(2) (2025), 6163–6163.

[6] M. Donganont, S. Donganont and H. Zhang, *Pulse-modulated Control for Scaled Consensus of Edge Dynamics in Hybrid Multi-agent Systems*, Int. J. Control, Auto. Syst. **23** (2025), 2503–2513, `https://doi.org/10.1007/s12555-025-0181-8`.

[7] M. Donganont and X. Liu, *Scaled consensus problems of multi-agent systems via impulsive protocols*, Appl. Math. Model., **116** (2023), 532–546.

[8] S. Donganont, U. Witthayarat and M. Donganont, *Impulsive Protocols for Scaled Consensus in Edge-Dynamic Multi-Agent Systems*, Euro. J. Pure Appl. Math., **18**(1) (2025), 5755–5755.

[9] H.K. Khalil, *Nonlinear Systems*, 3rd ed., Prentice Hall, Upper Saddle River, NJ, 2002.

[10] M. Li, F. Deng and H. Ren, *Scaled consensus of multi-agent systems with switching topologies and communication noises*, Nonlinear Anal. Hybrid Syst., **36** (2020), 100839.

[11] D. Meng and Y. Jia, *Scaled consensus problems on switching networks*, IEEE Trans. Autom. Control, **61**(6) (2016), 1664–1669.

[12] C. Park, S. Donganont and M. Donganont, *Achieving Edge Consensus in Hybrid Multi-Agent Systems: Scaled Dynamics and Protocol Design*, Euro. J. Pure Appl. Math., **18**(1) (2025), 5549–5549.

[13] S. Phongchan, S. Donganont and M. Donganont, *Probabilistic Approaches to Scaled Consensus in Dynamic Multi-Agent Networks*, J. Appl. Probab. Stat., **20**(1) (2025), 71–89.

[14] S. Roy, *Scaled consensus*, Automatica, **51** (2015), 259–262.

[15] Y. Shang, *On the delayed scaled consensus problems*, Appl. Sci., **7**(7) (2017), 713.

[16] Y. Shang, *Scaled consensus of switched multi-agent systems*, IMA J. Math. Control Informa., **36**(2) (2019), 639–657.

[17] Y. Shang, *Scaled consensus and reference tracking in multiagent networks with constraints*, IEEE Tran. Network Sci. Eng., **9**(3) (2022), 1620–1629.

[18] M.H. Trinh, D. Van Vu, Q. Van Tran, H.-S. Ahn, *Matrix-scaled consensus*, In: *2022 IEEE 61st Conference on Decision and Control (CDC)*, IEEE (2022), 346–351.

[19] Q. Wang, *Scaled consensus of multi-agent systems with output saturation*, J. Franklin Insti. **354**(14) (2017), 6190–6199.

[20] Z. Zhang, S. Chen and H. Su, *Scaled consensus of second-order nonlinear multiagent systems with time-varying delays via aperiodically intermittent control*, IEEE Trans. Cybernetics, **50**(8) (2020), 3503–3516.

[21] Z. Zhang, S. Chen and Y. Zheng, *Fully distributed scaled consensus tracking of high-order multiagent systems with time delays and disturbances*, IEEE Trans. Indus. Inform., **18**(1) (2021), 305–314.