Nonlinear Functional Analysis and Applications
Vol. 30, No. 4 (2025), pp. 1187-1203
ISSN: 1229-1595(print), 2466-0973(online)

https://doi.org/10.22771/nfaa.2025.30.04.13
http://nfaa.kyungnam.ac.kr/journal-nfaa

MULTIPLIERS FOR LIPSCHITZ $p-$BESSEL SEQUENCES IN METRIC SPACES

K. Mahesh Krishna[1], P. Sam Johnson[2] and Harikrishnan Panackal[3]

[1]School of Mathematics and Natural Sciences, Chanakya University, Bengaluru, India
e-mail: kmaheshak@gmail.com

[2]Department of Mathematical and Computational Sciences,
National Institute of Technology Karnataka, Surathkal, India
e-mail: sam@nitk.edu.in

[3]Department of Mathematics, Manipal Institute of Technology,
Manipal Academy of Higher Education, Manipal, India
e-mail: pk.harikrishnan@manipal.edu

Abstract. The notion of multipliers in Hilbert spaces was introduced by Schatten in 1960 using orthonormal sequences, and it was generalized by Balazs in 2007 using Bessel sequences. This concept was further extended to Banach spaces by Rahimi and Balazs in 2010 using p-Bessel sequences. In this paper, we extend this framework by considering Lipschitz functions. Along the way, we define frames for metric spaces, thereby generalizing the notion of frames and Bessel sequences for Banach spaces. We show that when the symbol sequence converges to zero, the associated multiplier is a Lipschitz compact operator. Finally, we study how variations in the parameters of the multiplier affect its properties.

1. Introduction and preliminaries

Let $\{\lambda_n\}_n \in \ell^\infty(\mathbb{N})$ and $\{x_n\}_n$, $\{y_n\}_n$ be sequences in a Hilbert space $\mathcal{H}$. For $x, y \in \mathcal{H}$, the operator $x \otimes \overline{y}$ is defined by

$$x \otimes \overline{y} : \mathcal{H} \ni h \mapsto \langle h, y\rangle x \in \mathcal{H}.$$

[0]Received June 10, 2025. Revised September 10, 2025. Accepted September 14, 2025.
[0]2020 Mathematics Subject Classification: 42C15, 26A16.
[0]Keywords: Multiplier, Lipschitz operator, Lipschitz compact operator, frame, Bessel sequence.
[0]Corresponding author: Harikrishnan Panackal(pk.harikrishnan@manipal.edu).

The study of operators of the form

$$\sum_{n=1}^{\infty} \lambda_n (x_n \otimes \overline{y_n}) \tag{1.1}$$

began with Schatten [44], in connection with the study of compact operators. Schatten studied the operator in (1.1) whenever $\{x_n\}_n$ and $\{y_n\}_n$ are orthonormal sequences in a Hilbert space $\mathcal{H}$. Later, operators of the form (1.1) were studied mainly in connection with Gabor analysis [7, 16, 19, 22, 24, 46]. This was generalized by Balazs [3], who replaced orthonormal sequences with Bessel sequences (see [12, 29] for Bessel sequences). Balazs and Stoeva studied these operators further in [5, 52, 53, 54, 55, 56, 57].

Let $\{f_n\}_n$ be a sequence in the dual space $\mathcal{X}^*$ of a Banach space $\mathcal{X}$, and $\{\tau_n\}_n$ be a sequence in a Banach space $\mathcal{Y}$. The operator $\tau \otimes f$ is defined by

$$\tau \otimes f : \mathcal{X} \ni x \mapsto f(x)\tau \in \mathcal{Y}.$$

It was Rahimi and Balazs [42] who extended the operator in (1.1) from Hilbert spaces to Banach spaces. For a Banach space $\mathcal{X}$ and its dual $\mathcal{X}^*$, they considered the operator

$$\sum_{n=1}^{\infty} \lambda_n (\tau_n \otimes f_n). \tag{1.2}$$

Rahimi and Balazs studied the operator in (1.2), whenever $\{\tau_n\}_n$ is a p-Bessel sequence (see [8, 14] for p-Bessel sequences) for $\mathcal{X}^*$ and $\{f_n\}_n$ is a q-Bessel sequence for $\mathcal{X}$ (where q is the conjugate index of p). Besides theoretical importance, multipliers also play an important role in frame (particularly Gabor) multipliers [22], signal processing [36], computational auditory scene analysis [58], sound synthesis [18], and psychoacoustics [4].

In the present paper, we attempt to study the non-linear version of the operator in (1.2). In Section 1, we recall the necessary definitions and results which we use. This includes the definition of frames for Hilbert and Banach spaces, Lipschitz images of Lipschitz operators, Lipschitz compact operators, the rank of a Lipschitz operator, and their properties. We then illustrate the crux of this paper in Section 2. To study the notion of multipliers, one needs the notion of Bessel sequences in Banach spaces. Since the notion of Bessel sequences does not exist for metric spaces, we first define it in Definition 2.1. This even leads to a stronger notionthe notion of frames for metric spaces. We observe that Definition 2.1 also contains the classical definition of bi-Lipschitz embedding (theory of bi-Lipschitz embeddings can be found in [30, 34, 37, 38, 39, 45]; see also Remark 2.5). We provide an interesting example of a frame for a metric space in Example 2.6. We derive Theorem 2.8, which opens the door to the definition of multipliers (Definition 2.9) in metric spaces. Proposition 2.10

shows that by varying the symbol, we get an injective bounded linear operator. In Proposition 2.11, we relate multipliers to the recently evolving theory of Lipschitz compact operators initiated in [31]. Finally, in Theorem 2.12, we show that multipliers are well-behaved with respect to their variations.

In Hilbert spaces, a Riesz basis is defined as an image of an orthonormal basis under an invertible operator [12]. In order to define Riesz basis for Banach spaces, one has to look for characterizations not involving inner product. Following is one such characterization.

Theorem 1.1. ([12]) *For a sequence $\{\tau_n\}_n$ in a Hilbert space $\mathcal{H}$, the following are equivalent.*

(i) *$\{\tau_n\}_n$ is a Riesz basis for $\mathcal{H}$.*

(ii) *$\overline{\operatorname{span}}\{\tau_n\}_n = \mathcal{H}$ and there exist $a, b > 0$ such that for every finite subset $\mathbb{S}$ of $\mathbb{N}$,*

$$a\left(\sum_{n\in\mathbb{S}}|c_n|^2\right)^{\frac{1}{2}} \le \left\|\sum_{n\in\mathbb{S}}c_n\tau_n\right\| \le b\left(\sum_{n\in\mathbb{S}}|c_n|^2\right)^{\frac{1}{2}}, \quad \forall\, c_n \in \mathbb{K}.$$

Definition 1.2. ([2]) Let $1 < q < \infty$ and $\mathcal{X}$ be a Banach space. A collection $\{\tau_n\}_n$ in $\mathcal{X}$ is said to be a

(i) q-Riesz sequence for $\mathcal{X}$ if there exist $a, b > 0$ such that for every finite subset $\mathbb{S}$ of $\mathbb{N}$,

$$a\left(\sum_{n\in\mathbb{S}}|c_n|^q\right)^{\frac{1}{q}} \le \left\|\sum_{n\in\mathbb{S}}c_n\tau_n\right\| \le b\left(\sum_{n\in\mathbb{S}}|c_n|^q\right)^{\frac{1}{q}}, \quad \forall\, c_n \in \mathbb{K}. \tag{1.3}$$

(ii) q-Riesz basis for $\mathcal{X}$ if it is a q-Riesz sequence for $\mathcal{X}$ and $\overline{\operatorname{span}}\{\tau_n\}_n = \mathcal{X}$.

We now recall the definition of a frame for a Hilbert space.

Definition 1.3. ([12]) A collection $\{\tau_n\}_n$ in a Hilbert space $\mathcal{H}$ is said to be a frame for $\mathcal{H}$ if there exist $a, b > 0$ such that

$$a\|h\|^2 \le \sum_{n=1}^{\infty}|\langle h, \tau_n\rangle|^2 \le b\|h\|^2, \quad \forall\, h \in \mathcal{H}.$$

By realizing that the functional $\mathcal{H} \ni h \mapsto \langle h, \tau_n\rangle \in \mathbb{K}$ is bounded linear, Definition 1.3 leads to the following in Banach spaces.

Definition 1.4. ([2, 14]) Let $1 < p < \infty$ and $\mathcal{X}$ be a Banach space.

(i) A collection $\{f_n\}_n$ of bounded linear functionals in $\mathcal{X}^*$ is said to be a p-frame for $\mathcal{X}$, if there exist $a, b > 0$ such that

$$a\|x\| \leq \left(\sum_{n=1}^{\infty} |f_n(x)|^p\right)^{\frac{1}{p}} \leq b\|x\|, \quad \forall\, x \in \mathcal{X}.$$

(ii) A collection $\{\tau_n\}_n$ in $\mathcal{X}$ is said to be a p-frame for $\mathcal{X}^*$, if there exist $a, b > 0$ such that

$$a\|f\| \leq \left(\sum_{n=1}^{\infty} |f(\tau_n)|^p\right)^{\frac{1}{p}} \leq b\|f\|, \quad \forall\, f \in \mathcal{X}^*.$$

For more about p-frames for Banach spaces we refer [8, 47, 48, 49, 50, 51].

We now recall the definition of Lipschitz function. Let $\mathcal{M}$, $\mathcal{N}$ be metric spaces. A function $f : \mathcal{M} \to \mathcal{N}$ is said to be Lipschitz, if there exists $b > 0$ such that

$$d(f(x), f(y)) \leq b\, d(x, y), \quad \forall\, x, y \in \mathcal{M}.$$

Definition 1.5. ([59]) Let $\mathcal{X}$ be a Banach space.

(i) Let $\mathcal{M}$ be a metric space. The collection $\operatorname{Lip}(\mathcal{M}, \mathcal{X})$ is defined as $\operatorname{Lip}(\mathcal{M}, \mathcal{X}) := \{f : f : \mathcal{M} \to \mathcal{X} \text{ is Lipschitz}\}$. For $f \in \operatorname{Lip}(\mathcal{M}, \mathcal{X})$, the Lipschitz number is defined as

$$\operatorname{Lip}(f) := \sup_{x, y \in \mathcal{M}, x \neq y} \frac{\|f(x) - f(y)\|}{d(x, y)}.$$

(ii) Let $(\mathcal{M}, 0)$ be a pointed metric space. The collection $\operatorname{Lip}_0(\mathcal{M}, \mathcal{X})$ is defined as $\operatorname{Lip}_0(\mathcal{M}, \mathcal{X}) := \{f : f : \mathcal{M} \to \mathcal{X} \text{ is Lipschitz and } f(0) = 0\}$. For $f \in \operatorname{Lip}_0(\mathcal{M}, \mathcal{X})$, the Lipschitz norm is defined as

$$\|f\|_{\operatorname{Lip}_0} := \sup_{x, y \in \mathcal{M}, x \neq y} \frac{\|f(x) - f(y)\|}{d(x, y)}.$$

Theorem 1.6. ([59]) *Let $\mathcal{X}$ be a Banach space.*

(i) *If $\mathcal{M}$ is a metric space, then $\operatorname{Lip}(\mathcal{M}, \mathcal{X})$ is a semi-normed vector space w.r.t. the semi-norm $\operatorname{Lip}(\cdot)$.*

(ii) *If $(\mathcal{M}, 0)$ is a pointed metric space, then $\operatorname{Lip}_0(\mathcal{M}, \mathcal{X})$ is a Banach space w.r.t. the norm $\|\cdot\|_{\operatorname{Lip}_0}$.*

The spaces $\operatorname{Lip}(\mathcal{M}, \mathcal{X})$ and $\operatorname{Lip}_0(\mathcal{M}, \mathcal{X})$ are well-studied and we refer [15, 25, 33, 40, 59] for further information.

In the theory of bounded linear operators between Banach spaces, an operator is said to be compact if the image of the unit ball under the operator is precompact [21]. Linearity of the operator now gives various characterizations of compactness and plays important role in rich theories such as theory of integral equations, spectral theory [6], theory of Fredholm operators [20], K-theory [43], (operator) ideal theory [41], approximation properties of Banach spaces [32], Schauder basis theory [32]. Lack of linearity is a hurdle when one tries to define compactness of non-linear maps. This hurdle was successefully crossed in the paper [31] which began the study of Lipschitz compact operators. We now record these things which are necessary in the paper.

Definition 1.7. ([31]) If $\mathcal{M}$ is a metric space and $\mathcal{X}$ is a Banach space, then the Lipschitz image of a Lipschitz map (also called as Lipschitz operator) $f : \mathcal{M} \to \mathcal{X}$ is defined as the set

$$\left\{ \frac{f(x) - f(y)}{d(x, y)} : x, y \in \mathcal{M}, x \neq y \right\}. \tag{1.4}$$

We observe that whenever an operator is linear, the set in (1.4) is simply the image of the unit sphere.

Definition 1.8. ([31]) If $(\mathcal{M}, 0)$ is a pointed metric space and $\mathcal{X}$ is a Banach space, then a Lipschitz map $f : \mathcal{M} \to \mathcal{X}$ such that $f(0) = 0$ is said to be Lipschitz compact if its Lipschitz image is relatively compact in $\mathcal{X}$, i.e., the closure of the set in (1.4) is compact in $\mathcal{X}$.

As showed in [31], there is a large collection of Lipschitz compact operators. To state this, first we need a definition.

Definition 1.9. ([11]) Let $(\mathcal{M}, 0)$ be a pointed metric space and $\mathcal{X}$ be a Banach space. A Lipschitz operator $f : \mathcal{M} \to \mathcal{X}$ such that $f(0) = 0$ is said to be strongly Lipschitz p-nuclear ($1 \leq p < \infty$) if there exist operators $A \in \mathcal{B}(\ell^p(\mathbb{N}), \mathcal{X})$, $g \in \operatorname{Lip}_0(\mathcal{M}, \ell^\infty(\mathbb{N}))$ and a diagonal operator $M_\lambda \in \mathcal{B}(\ell^\infty(\mathbb{N}), \ell^p(\mathbb{N}))$ induced by a sequence $\lambda \in \ell^p(\mathbb{N})$ such that $f = AM_\lambda g$, i.e., the following diagram commutes.

$$\mathcal{M}[r, "f"][d, "g"]\mathcal{X}$$
$$\ell^\infty(\mathbb{N})[r, "M_\lambda"]\ell^p(\mathbb{N})[u, "A"]$$

Proposition 1.10. ([31]) *Every strongly Lipschitz p-nuclear operator from a pointed metric space to a Banach space is Lipschitz compact.*

Since the image of a linear operator is a subspace, the natural definition of finite rank operator is that image is a finite dimensional subspace. The image of Lipschitz map may not be a subspace. Thus care has to be taken while defining rank of such maps.

Definition 1.11. ([31]) If $(\mathcal{M}, 0)$ is a pointed metric space and $\mathcal{X}$ is a Banach space, then a Lipschitz function $f : \mathcal{M} \to \mathcal{X}$ such that $f(0) = 0$ is said to have Lipschitz finite dimensional rank, if the linear hull of its Lipschitz image is a finite dimensional subspace of $\mathcal{X}$.

Definition 1.12. ([31]) If $\mathcal{M}$ is a metric space and $\mathcal{X}$ is a Banach space, then a Lipschitz function $f : \mathcal{M} \to \mathcal{X}$ is said to have finite dimensional rank, if the linear hull of its image is a finite dimensional subspace of $\mathcal{X}$.

Next theorem shows that for pointed metric spaces, Definitions 1.11 and 1.12 are equivalent.

Theorem 1.13. ([1, 31]) *Let $(\mathcal{M}, 0)$ be a pointed metric space and $\mathcal{X}$ be a Banach space. For a Lipschitz function $f : \mathcal{M} \to \mathcal{X}$ such that $f(0) = 0$ the following are equivalent.*

(i) *f has Lipschitz finite dimensional rank.*
(ii) *f has finite dimensional rank.*
(iii) *There exist $f_1, \dots, f_n$ in $\operatorname{Lip}_0(\mathcal{M}, \mathbb{K})$ and $\tau_1, \dots, \tau_n$ in $\mathcal{X}$ such that*

$$f(x) = \sum_{k=1}^{n} f_k(x)\tau_k, \quad \forall\, x \in \mathcal{M}.$$

In Hilbert spaces (and not in Banach spaces), every compact operator is approximable by finite rank operators in the operator norm [21]. Following is the definition of approximable operator for Lipschitz maps.

Definition 1.14. ([31]) If $(\mathcal{M}, 0)$ is a pointed metric space and $\mathcal{X}$ is a Banach space, then a Lipschitz function $f : \mathcal{M} \to \mathcal{X}$ such that $f(0) = 0$ is said to be Lipschitz approximable, if it is the limit in the Lipschitz norm of a sequence of Lipschitz finite rank operators from $\mathcal{M}$ to $\mathcal{X}$.

Theorem 1.15. ([31]) *Every Lipschitz approximable operator from pointed metric space $(\mathcal{M}, 0)$ to a Banach space $\mathcal{X}$ is Lipschitz compact.*

2. Multipliers for Lipschitz $p-$Bessel sequences in metric spaces and its properties

We first define the notion of frames for metric spaces. Multipliers for Lipschitz p-Bessel sequences in metric.

Definition 2.1. (p-frame for metric space) Let $(\mathcal{M}, d)$, $(\mathcal{N}_n, d_n)$, $1 \leq n < \infty$ be metric spaces. A collection $\{f_n\}_n$ of Lipschitz functions, $f_n : \mathcal{M} \to \mathcal{N}_n$ is said to be a Lipschitz p-frame ($1 \leq p < \infty$) for $\mathcal{M}$ relative to $\{\mathcal{N}_n\}_n$, if there exist $a, b > 0$ such that

$$a\, d(x,y) \leq \left(\sum_{n=1}^{\infty} d_n(f_n(x), f_n(y))^p\right)^{\frac{1}{p}} \leq b\, d(x,y), \quad \forall\, x, y \in \mathcal{M}.$$

If a is allowed to take the value 0, then we say that $\{f_n\}_n$ a Lipschitz p-Bessel sequence for $\mathcal{M}$.

Definition 2.2. (p-frame for metric space w.r.t. scalars) Let $\mathcal{M}$ be a metric space. A collection $\{f_n\}_n$ of Lipschitz functions from $\mathcal{M}$ to $\mathbb{K}$ is said to be a Lipschitz p-frame for $\mathcal{M}$, if there exist $a, b > 0$ such that

$$a\, d(x,y) \leq \left(\sum_{n=1}^{\infty} |f_n(x) - f_n(y)|^p\right)^{\frac{1}{p}} \leq b\, d(x,y), \quad \forall\, x, y \in \mathcal{M}.$$

Definition 2.3. (p-frame for a pointed metric space w.r.t. scalars) Let $(\mathcal{M}, 0)$ be a pointed metric space. A collection $\{f_n\}_n$ in $\operatorname{Lip}_0(\mathcal{M}, \mathbb{K})$ is said to be a pointed Lipschitz p-frame for $\mathcal{M}$, if there exist $a, b > 0$ such that

$$a\, d(x,y) \leq \left(\sum_{n=1}^{\infty} |f_n(x) - f_n(y)|^p\right)^{\frac{1}{p}} \leq b\, d(x,y), \quad \forall\, x, y \in \mathcal{M}.$$

Definition 2.4. Let $(\mathcal{M}, 0)$ be a pointed metric space. A collection $\{\tau_n\}_n$ in $\mathcal{M}$ is said to be a pointed Lipschitz p-frame for $\operatorname{Lip}_0(\mathcal{M}, \mathbb{K})$, if there exist $a, b > 0$ such that

$$a\, \|f - g\|_{\mathrm{Lip}} \leq \left(\sum_{n=1}^{\infty} |f(\tau_n) - g(\tau_n)|^p\right)^{\frac{1}{p}} \leq b \|f - g\|_{\mathrm{Lip}}, \quad \forall\, f, g \in \operatorname{Lip}_0(\mathcal{M}, \mathbb{K}).$$

Remark 2.5.

(i) Definition 2.1 even generalizes the notion of bi-Lipschitz embedding (Ribe program) of metric spaces (we refer [30, 34, 37, 38, 39, 45] for more on bi-Lipschitz embedding) (in fact, we see this by taking a fixed point $z \in \mathcal{N}$ and defining $f_n(x) = z, \forall\, x \in \mathcal{N}$ and $\forall\, n > 1$). It may happen that a metric space $\mathcal{M}$ may not embed in another metric space $\mathcal{N}$ through bi-Lipschitz map. But it may have frames. We give examples to illustrate these things after this remark.

(ii) By taking $y = 0$ and using $f_n(0) = 0$ for all $n \in \mathbb{N}$, we see from Definition 2.2 that

$$a\, d(x,0) \leq \left(\sum_{n=1}^{\infty} |f_n(x)|^p\right)^{\frac{1}{p}} \leq b\, d(x,0), \quad \forall\, x \in \mathcal{M}.$$

In particular, if $\mathcal{M}$ is a Banach space, then

$$a\|x\| \leq \left(\sum_{n=1}^{\infty} |f_n(x)|^p\right)^{\frac{1}{p}} \leq b\|x\|, \quad \forall\, x \in \mathcal{M}.$$

Similarly, by taking $g = 0$ in Definition 2.4, we see that

$$a\, \|f\|_{\text{Lip}_0} \leq \left(\sum_{n=1}^{\infty} |f(\tau_n)|^p\right)^{\frac{1}{p}} \leq b\|f\|_{\text{Lip}_0}, \quad \forall\, f \in \text{Lip}_0(\mathcal{M}, \mathbb{K}).$$

(iii) If $\mathcal{M}$ is a Banach space and f_n's are all bounded linear functionals, then Definition 2.1 becomes (i) in Definition 1.3.

(iv) Since we only want the definition of Lipschitz p-Bessel sequence, we do not address further properties of Lipschitz frames for metric spaces in this paper. However, we make a detailed study of frames for metric spaces in the forthcoming paper [35]. Along with Definition 2.1 we define two other notions in [35], one the notion of metric frames, as a generalization of the notion of Banach frames and atomic decompositions by Grochenig and another, the notion of framing for certain class of metric spaces. We answer the question of their existence, study their relations, properties and stability (for Banach spaces these can be found in [9, 10, 13, 23, 26, 27]).

Let x, y be distinct reals and consider $\mathcal{M} = \{x, y\}$ as a metric subspace of $\mathbb{R}$. Then $\mathcal{M}$ does not embed in metric spaces $\mathcal{N}_1 = \{x\}$ or $\mathcal{N}_2 = \{y\}$. Define $f_1(x) = f_1(y) = x$ and $f_2(x) = f_2(y) = y$. Then f_1 and f_2 are Lipschitz and $|f_1(x) - f_1(y)|^p + |f_2(x) - f_2(y)|^p = 2|x - y|^p$. Hence $\{f_1, f_2\}$ is a Lipschitz p-frame for $\mathcal{M}$.

As another example, consider $m, n \in \mathbb{N}$ with $m < n$. Since a bi-Lipschitz map is injective and continuous, and there is no continuous injection from $\mathbb{R}^n$ to $\mathbb{R}^m$ (Corollary 2B.4 in [28]) it follows that $\mathbb{R}^n$ cannot be embedded in $\mathbb{R}^m$. Now define $f_j : \mathbb{R}^n \ni (x_1, \ldots, x_n) \mapsto (x_j, x_{j+1}, \ldots, x_n, x_1, \ldots, x_{m-n+j-1}) \in$

$\mathbb{R}^m$ for $1 \leq j \leq n$. Then f_j is Lipschitz for all $1 \leq j \leq n$ and

$$\sum_{j=1}^{n} \|f_j(x_1, \dots, x_n) - f_j(y_1, \dots, y_n)\|^p = m\|(x_1, \dots, x_n) - (y_1, \dots, y_n)\|^p,$$
$$\forall\, (x_1, \dots, x_n), (y_1, \dots, y_n) \in \mathbb{R}^n.$$

Thus $\{f_1, f_2, \dots, f_n\}$ is a Lipschitz p-frame for $\mathbb{R}^n$.

We next give an example which uses infinite number of Lipschitz functions.

Example 2.6. Let $1 < a < b < \infty$. Let us take $\mathcal{M} := [a, b]$ and define $f_n : \mathcal{M} \to \mathbb{R}$ by

$$f_0(x) = 1, \quad \forall x \in \mathcal{M},$$
$$f_n(x) = \frac{(\log x)^n}{n!}, \quad \forall\, x \in \mathcal{M} \text{ for all } n \geq 1.$$

Then $f_n'(x) = \frac{(\log x)^{(n-1)}}{(n-1)!x}$ for all $x \in \mathcal{M}$, $n \geq 1$. Hence f_n' is bounded on $\mathcal{M}$ for all $n \geq 1$. Proposition 2.2.1 in [15] now tells that f_n is a Lipschitz function for each $n \geq 1$. For $x, y \in \mathcal{M}$ with $x < y$, we now see that

$$\sum_{n=0}^{\infty} |f_n(x) - f_n(y)| = \sum_{n=0}^{\infty} \frac{(\log y)^n}{n!} - \sum_{n=0}^{\infty} \frac{(\log x)^n}{n!}$$
$$= e^{\log y} - e^{\log x} = y - x = |x - y|.$$

Hence, $\{f_n\}_n$ is a Lipschitz 1-frame for $\mathcal{M}$.

We now set a notation which we use in the paper. Let $\mathcal{M}$ be a metric space and $\mathcal{X}$ be a Banach space. Given $f \in \operatorname{Lip}(\mathcal{M}, \mathbb{K})$ and $\tau \in \mathcal{X}$, define $\tau \otimes f : \mathcal{M} \ni x \mapsto (\tau \otimes f)(x) := f(x)\tau \in \mathcal{X}$. Then it follows that $\tau \otimes f$ is a Lipschitz operator and $\operatorname{Lip}(\tau \otimes f) = \|\tau\| \operatorname{Lip}(f)$. In his work [44], Schatten showed that whenever if $\{\lambda_n\}_n \in \ell^\infty(\mathbb{N})$ and $\{x_n\}_n$, $\{y_n\}_n$ are orthonormal sequences in a Hilbert space $\mathcal{H}$, then the map in (1.1) is a well-defined bounded linear operator. In 2007, Balazs showed that even if we take $\{x_n\}_n$, $\{y_n\}_n$ as Bessel sequences, then also T is well-defined and bounded. In 2010, Rahimi and Balazs [42] showed that we can even define operator T in Banach spaces. More precisely the result is following.

Theorem 2.7. ([42]) *Let $\{f_n\}_n$ be a p-Bessel sequence for a Banach space $\mathcal{X}$ with bound b and $\{\tau_n\}_n$ be a q-Bessel sequence for the dual of a Banach space $\mathcal{Y}$ with bound d, where q is conjugate index of p. If $\{\lambda_n\}_n \in \ell^\infty(\mathbb{N})$, then the map $T : \mathcal{X} \ni x \mapsto \sum_{n=1}^{\infty} \lambda_n(\tau_n \otimes f_n)x \in \mathcal{Y}$ is a well-defined bounded linear operator with norm at most $bd\|\{\lambda_n\}_n\|_\infty$.*

We now derive Theorem 2.7 in non-linear sense.

Theorem 2.8. *Let $\{f_n\}_n$ in $\operatorname{Lip}_0(\mathcal{M}, \mathbb{K})$ be a pointed Lipschitz p-Bessel sequence for a pointed metric space $(\mathcal{M}, 0)$ with bound b and $\{\tau_n\}_n$ in a Banach space $\mathcal{X}$ be a pointed Lipschitz q-Bessel sequence for $\operatorname{Lip}_0(\mathcal{X}, \mathbb{K})$ with bound d. If $\{\lambda_n\}_n \in \ell^\infty(\mathbb{N})$, then the map*

$$T : \mathcal{M} \ni x \mapsto \sum_{n=1}^{\infty} \lambda_n (\tau_n \otimes f_n) x \in \mathcal{X}$$

is a well-defined Lipschitz operator such that $T0 = 0$ with Lipschitz norm at most $bd\|\{\lambda_n\}_n\|_\infty$.

Proof. Let $n, m \in \mathbb{N}$ with $n \leq m$. Then for each $x \in \mathcal{M}$, using Holder's inequality,

$$\begin{aligned}
\left\|\sum_{k=n}^{m} \lambda_k (\tau_k \otimes f_k)(x)\right\| &= \left\|\sum_{k=n}^{m} \lambda_k f_k(x) \tau_k\right\| \\
&= \sup_{\phi \in \mathcal{X}^*, \|\phi\| \leq 1} \left|\phi\left(\sum_{k=n}^{m} \lambda_k f_k(x) \tau_k\right)\right| \\
&= \sup_{\phi \in \mathcal{X}^*, \|\phi\| \leq 1} \left|\sum_{k=n}^{m} \lambda_k f_k(x) \phi(\tau_k)\right| \\
&\leq \sup_{\phi \in \mathcal{X}^*, \|\phi\| \leq 1} \sum_{k=n}^{m} |\lambda_k||f_k(x)||\phi(\tau_k)| \\
&\leq \sup_{n \in \mathbb{N}} |\lambda_n| \sup_{\phi \in \mathcal{X}^*, \|\phi\| \leq 1} \sum_{k=n}^{m} |f_k(x)||\phi(\tau_k)| \\
&\leq \sup_{n \in \mathbb{N}} |\lambda_n| \sup_{\phi \in \mathcal{X}^*, \|\phi\| \leq 1} \left(\sum_{k=n}^{m} |f_k(x)|^p\right)^{\frac{1}{p}} \left(\sum_{k=n}^{m} |\phi(\tau_k)|^q\right)^{\frac{1}{q}} \\
&\leq \sup_{n \in \mathbb{N}} |\lambda_n| \sup_{\phi \in \mathcal{X}^*, \|\phi\| \leq 1} \left(\sum_{k=n}^{m} |f_k(x)|^p\right)^{\frac{1}{p}} d\|\phi\| \\
&= d \sup_{n \in \mathbb{N}} |\lambda_n| \left(\sum_{k=n}^{m} |f_k(x)|^p\right)^{\frac{1}{p}}.
\end{aligned}$$

Since $(\sum_{k=1}^{\infty} |f_k(x)|^p)^{\frac{1}{p}}$ converges, $\sum_{k=1}^{\infty} \lambda_k (\tau_k \otimes f_k)(x)$ also converges. Now for all $x, y \in \mathcal{M}$,

$$\begin{aligned}
\|Tx - Ty\| &= \left\|\sum_{n=1}^{\infty} \lambda_n f_n(x)\tau_n - \sum_{n=1}^{\infty} \lambda_n f_n(y)\tau_n\right\| \\
&= \left\|\sum_{n=1}^{\infty} \lambda_n (f_n(x) - f_n(y))\tau_n\right\| \\
&= \sup_{\phi\in\mathcal{X}^*, \|\phi\|\leq 1} \left|\phi\left(\sum_{n=1}^{\infty} \lambda_n (f_n(x) - f_n(y))\tau_k\right)\right| \\
&= \sup_{\phi\in\mathcal{X}^*, \|\phi\|\leq 1} \left|\sum_{n=1}^{\infty} \lambda_n (f_n(x) - f_n(y))\phi(\tau_k)\right| \\
&\leq \sup_{n\in\mathbb{N}} |\lambda_n| \sup_{\phi\in\mathcal{X}^*, \|\phi\|\leq 1} \left(\sum_{n=1}^{\infty} |f_n(x) - f_n(y)|^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} |\phi(\tau_n)|^q\right)^{\frac{1}{q}} \\
&\leq \sup_{n\in\mathbb{N}} |\lambda_n| \sup_{\phi\in\mathcal{X}^*, \|\phi\|\leq 1} \left(\sum_{n=1}^{\infty} |f_n(x) - f_n(y)|^p\right)^{\frac{1}{p}} d\|\phi\| \\
&= d \sup_{n\in\mathbb{N}} |\lambda_n| \left(\sum_{n=1}^{\infty} |f_n(x) - f_n(y)|^p\right)^{\frac{1}{p}} \\
&\leq bd \sup_{n\in\mathbb{N}} |\lambda_n| d(x, y).
\end{aligned}$$

Hence,

$$\|T\|_{\mathrm{Lip}_0} = \sup_{x,y\in\mathcal{M}, x\neq y} \frac{\|Tx - Ty\|}{d(x, y)} \leq bd \sup_{n\in\mathbb{N}} |\lambda_n|.$$

□

Corollary 2.1. *Let* $\{f_n\}_n$ *in* $\mathrm{Lip}(\mathcal{M}, \mathbb{K})$ *be a Lipschitz p-Bessel sequence for a metric space* $\mathcal{M}$ *with bound b and* $\{\tau_n\}_n$ *in a Banach space* $\mathcal{X}$ *be a pointed Lipschitz q-Bessel sequence for* $\mathrm{Lip}_0(\mathcal{X}, \mathbb{K})$ *with bound d. If* $\{\lambda_n\}_n \in \ell^\infty(\mathbb{N})$*, then for fixed* $z \in \mathcal{M}$*, the map*

$$T : \mathcal{M} \ni x \mapsto \sum_{n=1}^{\infty} \lambda_n (\tau_n \otimes (f_n - f(z)))x \in \mathcal{X}$$

is a well-defined Lipschitz operator with Lipschitz number at most $bd\|\{\lambda_n\}_n\|_\infty$.

Proof. Define $g_n \coloneqq f_n - f(z)$, $\forall\, n \in \mathbb{N}$. Then for all $x, y \in \mathcal{M}$,

$$\left(\sum_{n=1}^{\infty} |g_n(x) - g_n(y)|^p\right)^{\frac{1}{p}} = \left(\sum_{n=1}^{\infty} |f_n(x) - f_n(y)|^p\right)^{\frac{1}{p}} \leq b\, d(x, y).$$

Hence, $\{g_n\}_n$ is a Lipschitz p-Bessel sequence for pointed metric space $(\mathcal{M}, z)$ and we apply Theorem 2.8 to $\{g_n\}_n$. $\square$

Definition 2.9. Let $\{f_n\}_n$ in $\text{Lip}_0(\mathcal{M}, \mathbb{K})$ be a pointed Lipschitz p-Bessel sequence for a pointed metric space $(\mathcal{M}, 0)$ and $\{\tau_n\}_n$ in a Banach space $\mathcal{X}$ be a pointed Lipschitz q-Bessel sequence for $\text{Lip}_0(\mathcal{X}, \mathbb{K})$. Let $\{\lambda_n\}_n \in \ell^\infty(\mathbb{N})$. The Lipschitz operator $M_{\lambda,f,\tau} := \sum_{n=1}^\infty \lambda_n(\tau_n \otimes f_n)$ is called as the Lipschitz (p,q)-Bessel multiplier. The sequence $\{\lambda_n\}_n$ is called as symbol for $M_{\lambda,f,\tau}$.

We easily see that Definition 2.9 generalizes Definition 3.2 in [42]. By varying the symbol and fixing other parameters in the multiplier we get map from $\ell^\infty(\mathbb{N})$ to $\text{Lip}_0(\mathcal{M}, \mathcal{X})$. Property of this map for Hilbert space was derived by Balazs (Lemma in [3]) and for Banach spaces it is due to Rahimi and Balazs (Proposition 3.3 in [42]). In the next proposition we study it in the context of metric spaces.

Proposition 2.10. *Let $\{f_n\}_n$ in $\text{Lip}_0(\mathcal{M}, \mathbb{K})$ be a pointed Lipschitz p-Bessel sequence for $(\mathcal{M}, 0)$ with non-zero elements, $\{\tau_n\}_n$ in $\mathcal{X}$ be a q-Riesz sequence for $\text{Lip}_0(\mathcal{X}, \mathbb{K})$ and $\{\lambda_n\}_n \in \ell^\infty(\mathbb{N})$. Then the mapping*

$$T : \ell^\infty(\mathbb{N}) \ni \{\lambda_n\}_n \mapsto M_{\lambda,f,\tau} \in \text{Lip}_0(\mathcal{M}, \mathcal{X})$$

is a well-defined injective bounded linear operator.

Proof. From the norm estimate of $M_{\lambda,f,\tau}$, we see that T is a well-defined bounded linear operator. Let $\{\lambda_n\}_n, \{\mu_n\}_n \in \ell^\infty(\mathbb{N})$ be such that $M_{\lambda,f,\tau} = T\{\lambda_n\}_n = T\{\mu_n\}_n = M_{\mu,f,\tau}$. Then $\sum_{n=1}^\infty \lambda_n f_n(x)\tau_n = M_{\lambda,f,\tau}x = M_{\mu,f,\tau}x = \sum_{n=1}^\infty \mu_n f_n(x)\tau_n$ for all $x \in \mathcal{M} \Rightarrow \sum_{n=1}^\infty (\lambda_n - \mu_n) f_n(x)\tau_n = 0$ for all $x \in \mathcal{M}$.

Now using Inequality (1.3),

$$a\left(\sum_{n=1}^\infty |(\lambda_n - \mu_n) f_n(x)|^q\right)^{\frac{1}{q}} \leq \left\|\sum_{n=1}^\infty (\lambda_n - \mu_n) f_n(x)\tau_n\right\| = 0, \quad \forall\, x \in \mathcal{M}$$
$$\implies (\lambda_n - \mu_n) f_n(x) = 0, \quad \forall\, n \in \mathbb{N},\ \ \forall\, x \in \mathcal{M}.$$

Let $n \in \mathbb{N}$ be fixed. Since $f_n \neq 0$, there exists $x \in \mathcal{M}$ such that $f_n(x) \neq 0$. Therefore, we get $\lambda_n - \mu_n = 0$. By varying $n \in \mathbb{N}$, we arrive at $\lambda_n = \mu_n$, $\forall\, n \in \mathbb{N}$. Hence, T is injective. $\square$

The result that a norm-limit of finite rank linear operators (between Banach spaces) is a compact operator [21] was generalized to Lipschitz operators in Theorem 1.15 by Jimenez-Vargas, Sepulcre, and Villegas-Vallecillos [31]. Using Theorem 1.15 we can generalize Lemma 3.6 in [42].

Proposition 2.11. *Let $\{f_n\}_n$ in $\mathrm{Lip}_0(\mathcal{M},\mathbb{K})$ be a pointed Lipschitz p-Bessel sequence for $(\mathcal{M},0)$ with bound b and $\{\tau_n\}_n$ in $\mathcal{X}$ be a pointed Lipschitz q-Bessel sequence for $\mathrm{Lip}_0(\mathcal{X},\mathbb{K})$ with bound d. If $\{\lambda_n\}_n \in c_0(\mathbb{N})$, then $M_{\lambda,f,\tau}$ is a Lipschitz compact operator.*

Proof. For each $m \in \mathbb{N}$, define $M_{\lambda_m,f,\tau} \coloneqq \sum_{n=1}^{m} \lambda_n(\tau_n \otimes f_n)$. Then $M_{\lambda_m,f,\tau}$ is a Lipschitz finite rank operator (from Theorem 1.13). Now

$$
\begin{aligned}
&\|M_{\lambda_m,f,\tau} - M_{\lambda,f,\tau}\|_{\mathrm{Lip}_0}\\
&= \sup_{x,y\in\mathcal{M},x\neq y} \frac{\|(M_{\lambda_m,f,\tau} - M_{\lambda,f,\tau})x - (M_{\lambda_m,f,\tau} - M_{\lambda,f,\tau})y\|}{d(x,y)}\\
&= \sup_{x,y\in\mathcal{M},x\neq y} \frac{\left\|\sum_{n=m+1}^{\infty} \lambda_n f_n(x)\tau_n - \sum_{n=m+1}^{\infty} \lambda_n f_n(y)\tau_n\right\|}{d(x,y)}\\
&= \sup_{x,y\in\mathcal{M},x\neq y} \frac{\left\|\sum_{n=m+1}^{\infty} \lambda_n (f_n(x) - f_n(y))\tau_n\right\|}{d(x,y)}\\
&\leq bd \sup_{m+1\leq n<\infty} |\lambda_n| \to 0 \quad \text{as} \quad m\to\infty.
\end{aligned}
$$

Hence, $M_{\lambda,f,\tau}$ is the limit of a sequence of Lipschitz finite rank operators $\{M_{\lambda_m,f,\tau}\}_{m=1}^{\infty}$ with respect to the Lipschitz norm. Thus $M_{\lambda,f,\tau}$ is Lipschitz approximable and from Theorem 1.15 it follows that $M_{\lambda,f,\tau}$ is Lipschitz compact. □

We now study the properties of multiplier by changing its parameters. These are known as continuity properties of multipliers in the literature. Following result extends Theorem 5.1 in [42].

Theorem 2.12. *Let $\{f_n\}_n$ in $\mathrm{Lip}_0(\mathcal{M},\mathbb{K})$ be a pointed Lipschitz p-Bessel sequence for $\mathcal{M}$ with bound b and $\{\tau_n\}_n$ in $\mathcal{X}$ be a pointed Lipschitz q-Bessel sequence for $\mathrm{Lip}_0(\mathcal{X},\mathbb{K})$ with bound d and $\{\lambda_n\}_n \in \ell^\infty(\mathbb{N})$. Let $k\in\mathbb{N}$ and let $\lambda^{(k)} = \{\lambda_1^{(k)}, \lambda_2^{(k)}, \dots\}$, $\lambda = \{\lambda_1, \lambda_2, \dots\}$, $\tau^{(k)} = \{\tau_1^{(k)}, \tau_2^{(k)}, \dots\}$, $\tau_n^k \in \mathcal{X}$, $\tau = \{\tau_1, \tau_2, \dots\}$. Assume that for each k, $\lambda^{(k)} \in \ell^\infty(\mathbb{N})$ and $\tau^{(k)}$ is a pointed Lipschitz q-Bessel sequence for $\mathrm{Lip}_0(\mathcal{X},\mathbb{K})$.*

(i) *If $\lambda^{(k)} \to \lambda$ as $k\to\infty$ in p-norm, then*
$\|M_{\lambda^{(k)},f,\tau} - M_{\lambda,f,\tau}\|_{\mathrm{Lip}_0} \to 0$ *as* $k\to\infty$.

(ii) *If $\{\lambda_n\}_n \in \ell^p(\mathbb{N})$ and $\sum_{n=1}^{\infty} \|\tau_n^{(k)} - \tau_n\|^q \to 0$ as $k\to\infty$, then*
$\|M_{\lambda,f,\tau^{(k)}} - M_{\lambda,f,\tau}\|_{\mathrm{Lip}_0} \to 0$ *as* $k\to\infty$.

Proof. (i) Using Theorem 2.8,

$$
\begin{aligned}
&\|M_{\lambda^{(k)},f,\tau}-M_{\lambda,f,\tau}\|_{\mathrm{Lip}_0}\\
&\quad=\sup_{x,y\in\mathcal{M},x\neq y}\frac{\|(M_{\lambda^{(k)},f,\tau}-M_{\lambda,f,\tau})x-(M_{\lambda^{(k)},f,\tau}-M_{\lambda,f,\tau})y\|}{d(x,y)}\\
&\quad=\sup_{x,y\in\mathcal{M},x\neq y}\frac{\left\|\sum_{n=1}^\infty(\lambda_n^{(k)}-\lambda_n)f_n(x)\tau_n-\sum_{n=1}^\infty(\lambda_n^{(k)}-\lambda_n)f_n(y)\tau_n\right\|}{d(x,y)}\\
&\quad=\sup_{x,y\in\mathcal{M},x\neq y}\frac{\left\|\sum_{n=1}^\infty(\lambda_n^{(k)}-\lambda_n)(f_n(x)-f_n(y))\tau_n\right\|}{d(x,y)}\\
&\quad\leq bd\sup_{n\in\mathbb{N}}|\lambda_n^{(k)}-\lambda_n|=bd\|\{\lambda_n^{(k)}-\lambda_n\}_n\|_\infty\\
&\quad\leq bd\|\{\lambda_n^{(k)}-\lambda_n\}_n\|_p\to 0 \text{ as } k\to\infty.
\end{aligned}
$$

(ii) Using Holder's inequality,

$$
\begin{aligned}
&\|M_{\lambda,f,\tau^{(k)}}-M_{\lambda,f,\tau}\|_{\mathrm{Lip}_0}\\
&=\sup_{x,y\in\mathcal{M},x\neq y}\frac{\|(M_{\lambda,f,\tau^{(k)}}-M_{\lambda,f,\tau})x-(M_{\lambda,f,\tau^{(k)}}-M_{\lambda,f,\tau})y\|}{d(x,y)}\\
&=\sup_{x,y\in\mathcal{M},x\neq y}\frac{\left\|\sum_{n=1}^\infty\lambda_nf_n(x)(\tau_n^{(k)}-\tau_n)-\sum_{n=1}^\infty\lambda_nf_n(y)(\tau_n^{(k)}-\tau_n)\right\|}{d(x,y)}\\
&=\sup_{x,y\in\mathcal{M},x\neq y}\frac{\left\|\sum_{n=1}^\infty\lambda_n(f_n(x)-f_n(y))(\tau_n^{(k)}-\tau_n)\right\|}{d(x,y)}\\
&=\sup_{x,y\in\mathcal{M},x\neq y}\ \sup_{\phi\in\mathcal{X}^*,\|\phi\|\leq1}\frac{\left|\sum_{n=1}^\infty\lambda_n(f_n(x)-f_n(y))\phi(\tau_n^{(k)}-\tau_n)\right|}{d(x,y)}\\
&\leq\sup_{x,y\in\mathcal{M},x\neq y}\ \sup_{\phi\in\mathcal{X}^*,\|\phi\|\leq1}\frac{\left(\sum_{n=1}^\infty|\lambda_n(f_n(x)-f_n(y))|^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^\infty|\phi(\tau_n^{(k)}-\tau_n)|^q\right)^{\frac{1}{q}}}{d(x,y)}\\
&\leq\sup_{x,y\in\mathcal{M},x\neq y}\ \sup_{\phi\in\mathcal{X}^*,\|\phi\|\leq1}\frac{\left(\sum_{n=1}^\infty|\lambda_n|^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^\infty|f_n(x)-f_n(y)|^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^\infty|\phi(\tau_n^{(k)}-\tau_n)|^q\right)^{\frac{1}{q}}}{d(x,y)}\\
&\leq b\|\{\lambda_n\}_n\|_p\left(\sum_{n=1}^\infty\|\tau_n^{(k)}-\tau_n\|^q\right)^{\frac{1}{q}}\to 0\quad\text{as}\quad k\to\infty.
\end{aligned}
$$

□

Acknowledgments: The authors express their gratitude to the anonymous reviewers for their valuable comments. The first author thanks Chanakya University for their kind support. The second and third authors thank Anusandhan National Research Foundation (ANRF), Government of India for the fellowship-cum-grant “Teachers Associateship for Research Excellence” (TARE) (TAR/2022/000219). The corresponding author thanks Manipal Institute of Technology, Manipal Academy of Higher Education for their kind encouragement.

References

[1] D. Achour, P. Rueda, E.A. Sanchez-Perez and R. Yahi, *Lipschitz operator ideals and the approximation property,* J. Math. Anal. Appl., **436**(1) (2016), 217–236.

[2] A. Aldroubi, Q. Sun and W.-S. Tang, *p-frames and shift invariant subspaces of* L^p*,* J. Fourier Anal. Appl., **7**(1) (2001), 1–21.

[3] P. Balazs, *Basic definition and properties of Bessel multipliers,* J. Math. Anal. Appl., **325**(1) (2007), 571–585.

[4] P. Balazs, B. Laback, G. Eckel and W.A. Deutsch, *Time-frequency sparsity by removing perceptually irrelevant components using a simple model of simultaneous masking,* IEEE Trans. Audio Speech Lang. Process., **18**(1) (2010), 34–49.

[5] P. Balazs and D.T. Stoeva, *Representation of the inverse of a frame multiplier,* J. Math. Anal. Appl., **422**(2) (2015), 981–994.

[6] B. Beauzamy, *Introduction to Operator Theory and Invariant Subspaces*, North-Holland Mathematical Library, **42**, North-Holland Publishing Co., Amsterdam, 1988.

[7] J.J. Benedetto and G.E. Pfander, *Frame expansions for Gabor multipliers,* Appl. Comput. Harmon. Anal., **20**(1) (2006), 26–40.

[8] P. Casazza, O. Christensen and D.T. Stoeva, *Frame expansions in separable Banach spaces,* J. Math. Anal. Appl., **307**(2) (2005), 710–723.

[9] P.G. Casazza, S.J. Dilworth, E. Odell, Th. Schlumprecht and A. Zsak, *Coefficient quantization for frames in Banach spaces,* J. Math. Anal. Appl., **348**(1) (2008), 66–86.

[10] P.G. Casazza, D. Han and D.R. Larson, *Frames for Banach spaces,* in *The Functional and Harmonic Analysis of Wavelets and Frames*, Contemp. Math., **247** (1999), 149–182.

[11] D. Chen and B. Zheng, *Lipschitz p-integral operators and Lipschitz p-nuclear operators,* Nonlinear Anal., **75**(13) (2012), 5270–5282.

[12] O. Christensen, *An Introduction to Frames and Riesz Bases*, 2nd ed., Applied and Numerical Harmonic Analysis, Birkhuser/Springer, Cham, 2016.

[13] O. Christensen and C. Heil, *Perturbations of Banach frames and atomic decompositions,* Math. Nachr., **185** (1997), 33–47.

[14] O. Christensen and D.T. Stoeva, *p-frames in separable Banach spaces,* Adv. Comput. Math., **18**(2–4) (2003), 117–126.

[15] S. Cobzas, R. Miculescu and A. Nicolae, *Lipschitz Functions*, Lecture Notes in Mathematics, **2241**, Springer, Cham, 2019.

[16] E. Cordero, K. Grochenig and F. Nicola, *Approximation of Fourier integral operators by Gabor multipliers,* J. Fourier Anal. Appl., **18**(4) (2012), 661–684.

[17] L. Debnath and P. Mikusinski, *Introduction to Hilbert Spaces with Applications*, 2nd ed., Academic Press, San Diego, CA, 1999.

[18] P. Depalle, R. Kronland-Martinet and B. Torresani, *Time-frequency multipliers for sound synthesis,* in *SPEI Annual Symposium Wavelet XII*, vol. 6701, San Diego, 2007.

[19] M. Dörfler and B. Torresani, *Representation of operators in the time-frequency domain and generalized Gabor multipliers,* J. Fourier Anal. Appl., **16**(2) (2010), 261–293.

[20] D.E. Edmunds and W.D. Evans, *Spectral Theory and Differential Operators*, Oxford Mathematical Monographs, Oxford Univ. Press, 2018.

[21] M. Fabian, P. Habala, P. Hajek, V. Montesinos and V. Zizler, *Banach Space Theory: The Basis for Linear and Nonlinear Analysis*, CMS Books in Mathematics, Springer, New York, 2011.

[22] H.G. Feichtinger and K. Nowak, *A first survey of Gabor multipliers,* in *Advances in Gabor Analysis*, Appl. Numer. Harmon. Anal., Birkhuser Boston, 2003, 99–128.

[23] D. Freeman, E. Odell, Th. Schlumprecht and A. Zsak, *Unconditional structures of translates for* $L_p(\mathbb{R}^d)$, Israel J. Math., **203**(1) (2014), 189–209.

[24] P.C. Gibson, M.P. Lamoureux and G.F. Margrave, *Representation of linear operators by Gabor multipliers,* in *Excursions in Harmonic Analysis, Vol. 2*, Appl. Numer. Harmon. Anal., Birkhuser/Springer, 2013, 229–250.

[25] G. Godefroy and N.J. Kalton, *Lipschitz-free Banach spaces,* Studia Math., **159**(1) (2003), 121–141.

[26] D. Han, D.R. Larson and R. Liu, *Dilations of operator-valued measures with bounded p-variations and framings on Banach spaces,* J. Funct. Anal., **274**(5) (2018), 1466–1490.

[27] D. Han, D.R. Larson, B. Liu and R. Liu, *Operator-valued measures, dilations, and the theory of frames,* Mem. Amer. Math. Soc., **229**(1075) (2014), viii+84.

[28] A. Hatcher, *Algebraic Topology*, Cambridge Univ. Press, 2002.

[29] C. Heil, *A Basis Theory Primer*, Expanded ed., Applied and Numerical Harmonic Analysis, Birkhuser/Springer, New York, 2011.

[30] J. Heinonen, *Geometric Embeddings of Metric Spaces*, Report 90, Univ. of Jyväskylä, 2003.

[31] A. Jimenez-Vargas, J.M. Sepulcre and M. Villegas-Vallecillos, *Lipschitz compact operators,* J. Math. Anal. Appl., **415**(2) (2014), 889–901.

[32] W.B. Johnson and J. Lindenstrauss (Eds.), *Handbook of the Geometry of Banach Spaces. Vol. I*, North-Holland Publishing Co., Amsterdam, 2001.

[33] N.J. Kalton, *Spaces of Lipschitz and Hölder functions and their applications,* Collect. Math., **55**(2) (2004), 171–217.

[34] U. Lang and C. Plaut, *Bilipschitz embeddings of metric spaces into space forms,* Geom. Dedicata, **87**(1–3) (2001), 285–307.

[35] K. Mahesh Krishna and P. Sam Johnson, *Frames for metric spaces,* Results Math., **77**(1) (2022), article 49.

[36] G. Matz and F. Hlawatsch, *Linear time-frequency filters: Online algorithms and applications,* in *Applications in Time-Frequency Signal Processing*, Appl. Numer. Harmon. Anal., CRC Press, 2002, 205–272.

[37] A. Naor, *An introduction to the Ribe program,* Jpn. J. Math., **7**(2) (2012), 167–233.

[38] A. Naor, *Metric dimension reduction: a snapshot of the Ribe program,* in *Proc. Int. Congr. Math. Rio de Janeiro 2018, Vol. I: Plenary Lectures*, World Sci. Publ., 2018, 759–837.

[39] M.I. Ostrovskii, *Metric Embeddings: Bilipschitz and Coarse Embeddings into Banach Spaces*, De Gruyter Studies in Mathematics, **49**, De Gruyter, Berlin, 2013.

[40] L. Piasecki, *Classification of Lipschitz Mappings*, Pure and Applied Mathematics, **307**, CRC Press, Boca Raton, FL, 2014.

[41] A. Pietsch, *Operator Ideals*, North-Holland Mathematical Library, **20**, North-Holland Publishing Co., Amsterdam-New York, 1980.

[42] A. Rahimi and P. Balazs, *Multipliers for p-Bessel sequences in Banach spaces,* Integral Equations Operator Theory, **68**(2) (2010), 193–205.

[43] M. Rørdam, F. Larsen and N. Laustsen, *An Introduction to* K*-Theory for* C^**-Algebras*, London Mathematical Society Student Texts, **49**, Cambridge Univ. Press, 2000.

[44] R. Schatten, *Norm Ideals of Completely Continuous Operators*, Ergebnisse der Mathematik und ihrer Grenzgebiete, **27**, Springer-Verlag, 1970.

[45] S. Semmes, *Bilipschitz embeddings of metric spaces into Euclidean spaces,* Publ. Mat., **43**(2) (1999), 571–653.

[46] E. Skrettingland, *Quantum harmonic analysis on lattices and Gabor multipliers,* J. Fourier Anal. Appl., **26**(3) (2020), article 48.

[47] D.T. Stoeva, *On p-frames and reconstruction series in separable Banach spaces,* Integral Transforms Spec. Funct., **17**(2–3) (2006), 127–133.

[48] D.T. Stoeva, *Perturbation of frames in Banach spaces,* Asian-Eur. J. Math., **5**(1) (2012), article 1250011.

[49] D.T. Stoeva, *Generalization of the frame operator and the canonical dual frame to Banach spaces,* Asian-Eur. J. Math., **1**(4) (2008), 631–643.

[50] D.T. Stoeva, *Connection between the lower p-frame condition and existence of reconstruction formulas in a Banach space and its dual,* Annuaire Univ. Sofia Fac. Math. Inform., **97** (2005), 123–133.

[51] D.T. Stoeva, *X_d-frames in Banach spaces and their duals,* Int. J. Pure Appl. Math., **52**(1) (2009), 1–14.

[52] D.T. Stoeva and P. Balazs, *Invertibility of multipliers,* Appl. Comput. Harmon. Anal., **33**(2) (2012), 292–299.

[53] D.T. Stoeva and P. Balazs, *On the dual frame induced by an invertible frame multiplier,* Sampl. Theory Signal Image Process., **15** (2016), 119–130.

[54] D.T. Stoeva and P. Balazs, *Riesz bases multipliers,* in *Concrete Operators, Spectral Theory, Operators in Harmonic Analysis and Approximation*, Oper. Theory Adv. Appl., **236**, Birkhäuser/Springer, 2014, 475–482.

[55] D.T. Stoeva and P. Balazs, *Canonical forms of unconditionally convergent multipliers,* J. Math. Anal. Appl., **399**(1) (2013), 252–259.

[56] D.T. Stoeva and P. Balazs, *Weighted frames and frame multipliers,* God. Univ. Arkhit. Stroit. Geod. Sofiya Svitk II Mat. Mekh., **43/44**(2004/09) (2012), 33–42.

[57] D.T. Stoeva and P. Balazs, *Detailed characterization of conditions for the unconditional convergence and invertibility of multipliers,* Sampl. Theory Signal Image Process., **12**(2–3) (2013), 87–125.

[58] D. Wang and D.J. Brown, *Computational Auditory Scene Analysis: Principles, Algorithms, and Applications*, Wiley-IEEE Press, New York, 2006.

[59] N. Weaver, *Lipschitz Algebras*, World Scientific Publishing Co., Hackensack, NJ, 2018.