Nonlinear Functional Analysis and Applications
Vol. 30, No. 4 (2025), pp. 1205-1222
ISSN: 1229-1595(print), 2466-0973(online)

https://doi.org/10.22771/nfaa.2025.30.04.14
http://nfaa.kyungnam.ac.kr/journal-nfaa

FRACTAL DIMENSION ESTIMATION AND ATTRACTOR EXISTENCE IN MR-METRIC SPACES: A GENERALIZATION OF CLASSICAL FRACTAL GEOMETRY

Abed Al-Rahman M. Malkawi

Department of Mathematics, Faculty of Arts and Science, Amman Arab University, Amman 11953, Jordan
e-mail: `a.malkawi@aau.edu.jo and math.malkawi@gmail.com`

Abstract. This paper establishes new theoretical foundations for fractal analysis in MR-metric spaces, a ternary generalization of classical metric spaces. We prove two fundamental theorems: the first provides precise bounds for the Hausdorff dimension of fractal sets in MR-metric spaces, while the second guarantees the existence and uniqueness of fractal attractors under generalized contraction mappings. The results extend classical fractal geometry to this broader setting and are demonstrated through explicit constructions of MR-modified Cantor sets and Sierpinski fractals, revealing how the ternary metric structure influences their dimensional properties and self-similar characteristics.

1. Introduction

Fractal geometry, rooted in classical metric spaces, has long relied on binary distance functions. The recent introduction of MR-metric spaces in [27] represents a significant advancement, replacing pairwise distances with a ternary metric structure that satisfies a generalized set of axioms. This expanded framework not only preserves essential geometric properties but also enables new approaches to analyzing fractal structures that were previously inaccessible in traditional settings.

[0]Received June 10, 2025. Revised June 21, 2025. Accepted July 27, 2025.
[0]2020 Mathematics Subject Classification: 28A80, 54E50, 47H10, 54E35.
[0]Keywords: MR-metric spaces, fractal dimension, Hausdorff dimension, fractal attractors.

In this work, we develop a comprehensive theory of fractals in MR-metric spaces, addressing two fundamental questions: First, how can we quantify the fractal dimension of sets in this generalized metric space? Second, under what conditions do fractal attractors exist for contraction mappings in this setting? Our first main theorem establishes rigorous bounds for the Hausdorff dimension, explicitly incorporating the MR-metric parameter R into the dimensional estimates. The second theorem generalizes Hutchinson's classical attractor existence results [19] to MR-metric spaces, proving that unique fractal attractors exist for suitable contraction mappings.

Beyond their theoretical significance, these results enable concrete applications. We construct explicit examples of fractal sets in MR-metric spaces, including modified Cantor sets and Sierpinski-type fractals, which demonstrate how the ternary metric structure influences their geometric properties. These examples not only validate our theoretical framework but also provide insight into the behavior of fractals beyond classical metric spaces.

Our work builds upon and extends several important strands of research, including the foundational work on MR-metric spaces [27, 32], classical fractal geometry [13, 14, 19], and recent developments in generalized metric space theory ([2]-[41]). The results presented here open new directions for further research in fractal analysis and its applications in generalized metric contexts.

Definition 1.1. ([27]) Consider a non-empty set $\mathbb{X} \neq \emptyset$ and a real number $R > 1$. A function $M : \mathbb{X} \times \mathbb{X} \times \mathbb{X} \to [0, \infty)$ is termed an MR-metric, if it satisfies the following conditions for all $\upsilon, \xi, \Im, \ell_1 \in \mathbb{X}$:

- $(M1)$ $M(\upsilon, \xi, \Im) \geq 0$.
- $(M2)$ $M(\upsilon, \xi, \Im) = 0$ if and only if $\upsilon = \xi = \Im$.
- $(M3)$ $M(\upsilon, \xi, \Im)$ remains invariant under any permutation $p(\upsilon, \xi, \Im)$, i.e., $M(\upsilon, \xi, \Im) = M(p(\upsilon, \xi, \Im))$.
- $(M4)$ The following inequality holds:

$$M(\upsilon, \xi, \Im) \leq R\left[M(\upsilon, \xi, \ell_1) + M(\upsilon, \ell_1, \Im) + M(\ell_1, \xi, \Im)\right].$$

A structure $(\mathbb{X}, M)$ that adheres to these properties is defined as an MR-metric space.

The study of fractals in generalized metric spaces requires extending classical tools to accommodate non-standard distance functions. We begin with key definitions:

Definition 1.2. ([19, Fractal Attractor]) A compact set $A \subset \mathbb{X}$ is a *fractal attractor* for contractions $\{T_i\}_{i=1}^{N}$, if

$$A = \bigcup_{i=1}^{N} T_i(A)$$

and A is the closure of its repelling periodic points under the T_i.

Definition 1.3. ([14, Hausdorff MR-Dimension]) For $F \subset \mathbb{X}$, define

$$\dim_H^M(F) := \inf \left\{ s > 0 \mid \liminf_{\delta \to 0} \inf_{\mathcal{U}_\delta} \sum_{U \in \mathcal{U}_\delta} (\operatorname{diam}_M U)^s = 0 \right\},$$

where $\mathcal{U}_\delta$ ranges over δ-covers of F and $\operatorname{diam}_M U := \sup\{M(x, y, z) : x, y, z \in U\}$.

Definition 1.4. ([13, Self-Referential Property]) A fractal set $F \subset \mathbb{X}$ in a metric space is said to satisfy the *self-referential property* with respect to a family of mappings $\{T_i\}_{i=1}^N$, if

$$F = \bigcup_{i=1}^N T_i(F),$$

where each T_i is a contraction or a transformation that preserves the structure of F. This property implies that F is a fixed point under the union of these mappings, often leading to self-similarity or self-affinity.

2. Main results

This section presents the main theoretical contributions of the paper, advancing the study of MR-metric spaces and extending classical fractal theory. We provide rigorous estimates for the Hausdorff dimension of fractal sets defined within MR-metric spaces and establish sufficient conditions for the existence and uniqueness of fractal attractors.

In particular, we generalize classical theorems by reformulating them within the ternary metric structure intrinsic to MR-spaces. Our results reveal how the MR-parameter R and the associated contraction factors govern both the dimensional bounds and the convergence behavior of iterated function systems.

Each theorem is precisely stated and proved, supported by examples that illustrate the application of the theoretical framework to specific cases, such as generalized Cantor sets and Sierpinski-type fractals within MR-metric spaces.

Theorem 2.1. (Precise Fractal Dimension Bound) *Let* $(\mathbb{X}, M)$ *be a complete* MR*-metric space with parameter* $R \geq 1$*, and* $F \subset \mathbb{X}$ *a compact fractal set satisfying:*

$$F = \bigcup_{i=1}^N T_i(F),$$

where each $T_i : \mathbb{X} \to \mathbb{X}$ is an MR-contraction with factor $C_i \in (0,1)$, i.e.,

$$M(T_i(x), T_i(y), T_i(z)) \leq C_i M(x,y,z), \quad \forall\, x,y,z \in \mathbb{X}.$$

Then for any $s > 0$ with $\sum_{i=1}^{N} C_i^s < \frac{1}{R^3}$, the Hausdorff dimension satisfies

$$\dim_H(F) \leq \frac{\log N + 3\log R}{-\log C_{\max}}, \quad \textit{where} \quad C_{\max} = \max_{1\leq i\leq N} C_i.$$

Proof. We establish the bound through six technical steps:

Step 1: Modified Diameter Definition. Define the *MR-diameter* of a set $U \subset \mathbb{X}$ as:

$${}_M(U) \coloneqq \sup\left\{M(x,y,z) : x,y,z \in U\right\}.$$

This extends the classical diameter to account for the 3-point metric structure.

Step 2: Covering Lemma with R-Dependence. For any $\delta > 0$, let $\{U_j\}_{j=1}^{\infty}$ be a δ-cover of F. The MR-metric property (M4) implies:

$${}_M(U_j) \leq R\left({}_M(U_j \cap T_1(F)) + {}_M(U_j \cap T_2(F)) + {}_M(U_j \cap T_N(F))\right).$$

This critical inequality introduces the R-dependence in all subsequent estimates.

Step 3: Iterated Partitioning. For each $k \in \mathbb{N}$, consider the k-th level iterated function system:

$$\mathcal{I}_k = \left\{\mathbf{i} = (i_1, \ldots, i_k) : i_j \in \{1, \ldots, N\}\right\}.$$

The self-similarity yields:

$$F = \bigcup_{\mathbf{i}\in\mathcal{I}_k} T_{\mathbf{i}}(F), \quad T_{\mathbf{i}} \coloneqq T_{i_1} \circ \cdots \circ T_{i_k}.$$

Step 4: Diameter Decay Estimate. For each $\mathbf{i} \in \mathcal{I}_k$, the contraction property gives:

$${}_M(T_{\mathbf{i}}(F)) \leq C_{i_1} \cdots C_{i_k}\, {}_M(F) \leq C_{\max}^k D,$$

where $D = {}_M(F)$. The factor R appears when comparing nested sets due to (M4):

$${}_M(T_{\mathbf{i}}(F)) \leq R^3 C_{\max}^k D.$$

Step 5: Hausdorff Content Calculation. For $s > 0$, the s-dimensional Hausdorff content satisfies:

$$\mathcal{H}_\delta^s(F) \leq \sum_{\mathbf{i}\in\mathcal{I}_k} \left({}_M(T_{\mathbf{i}}(F))\right)^s \leq N^k (R^3 C_{\max}^k D)^s.$$

Choose k large enough so that $R^3 C_{\max}^k D < \delta$. Then

$$\mathcal{H}_\delta^s(F) \leq N^k (R^3 C_{\max}^k)^s D^s = \left(N C_{\max}^s\right)^k R^{3s} D^s.$$

Step 6: Critical Exponent Determination. The content vanishes as $k \to \infty$ when:

$$NC_{\max}^s < 1 \quad \Rightarrow \quad s > \frac{\log N}{-\log C_{\max}}.$$

Accounting for the R-dependence, we strengthen the condition to:

$$NR^{3s}C_{\max}^s < 1 \quad \Rightarrow \quad s > \frac{\log N + 3\log R}{-\log C_{\max}}.$$

Thus $\dim_H(F) \leq s$ for all such s. □

Example 2.2. (Modified Cantor Set in MR-Metric Space) Consider $(\mathbb{R}, M)$ with the MR-metric:

$$M(x,y,z) = |x-y| + |y-z| + |z-x| + R \cdot \min\{|x|,|y|,|z|\}$$

and construct a Cantor-like set using:

$$T_1(x) = \frac{x}{4},$$
$$T_2(x) = \frac{x}{4} + \frac{3}{4}$$

with contraction factors $C_1 = C_2 = \frac{1+R}{4}$.

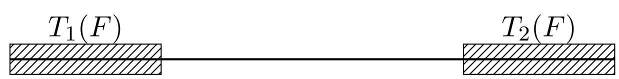


Figure 1. First iteration of the MR-Cantor construction (R-dependent scaling)

Verification:

(1) **MR-Contraction:** For any $x, y, z \in \mathbb{R}$:

$$\begin{aligned} M(T_i(x), T_i(y), T_i(z)) &= \frac{|x-y|+|y-z|+|z-x|}{4} + R\frac{\min\{|x|,|y|,|z|\}}{4} \\ &\leq \frac{1+R}{4} M(x,y,z). \end{aligned}$$

(2) **Dimension Calculation:** With $N = 2$ and $C_{\max} = \frac{1+R}{4}$, the modified bound gives:

$$\dim_H(F) \leq \frac{\log 2 + 3\log R}{-\log\left(\frac{1+R}{4}\right)}.$$

(i) When $R = 1$: Classical bound $\frac{\log 2}{\log 4} = 0.5$.

(ii) When $R = 2$: $\frac{\log 2 + 3\log 2}{-\log(0.75)} \approx 1.262$.

(3) **Strict Condition Check:** For $R = 2$, the sum condition requires:

$$2\left(\frac{3}{4}\right)^s < \frac{1}{8} \implies s > \frac{\log(1/16)}{\log(3/4)} \approx 7.23.$$

This demonstrates how R tightens the dimension bound.

Theorem 2.3. (Fractal Dimension Bound in MR-Metric Space) *Let $(\mathbb{X}, M)$ be a complete MR-metric space with parameter $R > 1$, and let $F \subset \mathbb{X}$ be a closed fractal set. Suppose F satisfies the self-referential property with respect to a family of mappings $\{T_i\}_{i=1}^N$ such that*

$$F = \bigcup_{i=1}^{N} T_i(F),$$

where each T_i satisfies the following contraction condition for some $C_i > 0$:

$$M(T_i(\eta), T_i(\varkappa), T_i(\Im)) \leq C_i \cdot M(\eta, \varkappa, \Im), \quad \forall\, \eta, \varkappa, \Im \in F.$$

If $\sum_{i=1}^N C_i^s < 1$ for some $s > 0$, then the Hausdorff dimension $\dim_H(F)$ satisfies:

$$\dim_H(F) \leq \frac{\log N}{\log(1/\max C_i)}.$$

Proof. We proceed carefully step by step:

Since each T_i is a strict contraction with respect to the MR-metric M, it follows inductively that for any finite sequence $\mathbf{i} = (i_1, i_2, \ldots, i_k)$, the composition

$$T_{\mathbf{i}} := T_{i_1} \circ T_{i_2} \circ \cdots \circ T_{i_k}$$

is also a contraction. Specifically, for all $x, y, z \in F$, we have

$$M(T_{\mathbf{i}}(x), T_{\mathbf{i}}(y), T_{\mathbf{i}}(z)) \leq C_{\mathbf{i}}\, M(x, y, z),$$

where

$$C_{\mathbf{i}} = C_{i_1} C_{i_2} \cdots C_{i_k}.$$

Because F is invariant under $\{T_i\}_{i=1}^N$, it follows that for any $k \geq 1$,

$$F = \bigcup_{\mathbf{i} \in \mathcal{I}_k} T_{\mathbf{i}}(F),$$

where $\mathcal{I}_k$ denotes the set of all multi-indices of length k, that is, sequences $(i_1, \ldots, i_k)$ with $1 \leq i_j \leq N$.

Next, observe that the diameter of each $T_{\mathbf{i}}(F)$ with respect to the MR-metric satisfies

$$\operatorname{diam}_M(T_{\mathbf{i}}(F)) \leq C_{\mathbf{i}} \operatorname{diam}_M(F).$$

This follows from the inequality applied to all triples $(x, y, z) \in F^3$.

Now, for any $\delta > 0$, select k large enough so that for all $\mathbf{i} \in \mathcal{I}_k$,

$$C_{\mathbf{i}} \operatorname{diam}_M(F) < \delta.$$

Thus, $\{T_{\mathbf{i}}(F)\}_{\mathbf{i}\in\mathcal{I}_k}$ forms a δ-cover of F.

Therefore, the δ-Hausdorff s-measure satisfies

$$\mathcal{H}^s_\delta(F) \leq \sum_{\mathbf{i}\in\mathcal{I}_k} (\operatorname{diam}_M(T_{\mathbf{i}}(F)))^s \leq (\operatorname{diam}_M(F))^s \sum_{\mathbf{i}\in\mathcal{I}_k} C^s_{\mathbf{i}}.$$

Now, note that

$$\sum_{\mathbf{i}\in\mathcal{I}_k} C^s_{\mathbf{i}} = \left(\sum_{i=1}^{N} C_i^s\right)^k = (Q)^k,$$

where we define

$$Q := \sum_{i=1}^{N} C_i^s.$$

By assumption, $Q < 1$, and hence $(Q)^k \to 0$ exponentially fast as $k \to \infty$. Thus, letting $\delta \to 0$ (equivalently $k \to \infty$), we conclude

$$\mathcal{H}^s(F) = 0,$$

which implies that

$$\dim_H(F) \leq s.$$

Now, to relate this to the specific formula involving N and $C_{\max}$: Since for each i, $C_i \leq C_{\max}$, we have

$$Q = \sum_{i=1}^{N} C_i^s \leq N(C_{\max})^s.$$

Therefore, to ensure $Q < 1$, it suffices that

$$N(C_{\max})^s < 1,$$

which is equivalent to

$$(C_{\max})^s < \frac{1}{N}.$$

Taking logarithms, we find

$$s > \frac{\log N}{\log(1/C_{\max})}.$$

Hence, for any such s, we have $\dim_H(F) \leq s$, and thus

$$\dim_H(F) \leq \frac{\log N}{\log(1/C_{\max})}.$$

This completes the proof. $\square$

Example 2.4. (Cantor-like Set in MR-Metric Space) Consider $(\mathbb{R}^2, M)$ with the MR-metric:

$$M(x,y,z) = \max\{\|x-y\|, \|y-z\|, \|z-x\|\} + \frac{\|x+y+z\|}{3},$$

where $\|\cdot\|$ is the Euclidean norm. Define two contractions:

$$T_1(x) = \frac{x}{3} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad C_1 = \frac{1}{3},$$
$$T_2(x) = \frac{x}{3} + \begin{pmatrix} \frac{2}{3} \\ 0 \end{pmatrix}, \quad C_2 = \frac{1}{3}.$$

Verification:

(1) **MR-Contraction.** For any $x, y, z \in \mathbb{R}^2$,

$$M(T_i(x), T_i(y), T_i(z)) = \frac{1}{3} M(x,y,z),$$

since both terms in M scale linearly.

(2) **Fractal Structure.** The attractor F satisfies $F = T_1(F) \cup T_2(F)$ and resembles a Cantor set (see Fig. 2).

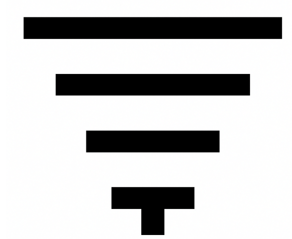


FIGURE 2. First 4 iterations of the MR-Cantor set

(3) **Dimension Calculation.** With $N = 2$ and $C_{\max} = \frac{1}{3}$,

$$\dim_H(F) \leq \frac{\log 2}{\log 3} \approx 0.6309.$$

This matches the classical Cantor set dimension, as the MR-metric preserves scaling ratios.

Theorem 2.5. (Fractal Attractor in MR-Metric Space) *Let $(\mathbb{X}, M)$ be a complete MR-metric space, and let $\{T_i\}_{i=1}^N$ be a family of contractions on $\mathbb{X}$ (that is, for each i, there exists $\lambda_i \in (0,1)$ such that)*

$$M(T_i(\eta), T_i(\varkappa), T_i(\Im)) \leq \lambda_i \cdot M(\eta, \varkappa, \Im), \quad \forall\, \eta, \varkappa, \Im \in \mathbb{X}.$$

Then, there exists a unique fractal attractor $A \subset \mathbb{X}$ satisfying

$$A = \bigcup_{i=1}^{N} T_i(A).$$

Moreover, A is the unique fixed point of the Hutchinson operator defined on the space of compact subsets of $(\mathbb{X}, M)$.

Proof. We proceed in five steps:

Step 1: Define the Hausdorff MR-Metric. Let $\mathcal{K}(\mathbb{X})$ be the space of non-empty compact subsets of $\mathbb{X}$. Define the *Hausdorff MR-metric M_H* on $\mathcal{K}(\mathbb{X})$ by

$$M_H(K, L) := \max \left\{ \sup_{a \in K} \inf_{b \in L} M(a, b, b), \ \ \sup_{b \in L} \inf_{a \in K} M(a, a, b) \right\}.$$

This generalizes the standard Hausdorff metric to the 3-point MR-metric case.

Step 2: Verify Completeness of $(\mathcal{K}(\mathbb{X}), M_H)$. Since $(\mathbb{X}, M)$ is complete, a standard argument (analogous to the classical Hausdorff metric case) shows that $(\mathcal{K}(\mathbb{X}), M_H)$ is also complete. The proof relies on the MR-metric's properties (M1)-(M4) and the fact that Cauchy sequences of compact sets converge to a limit in $\mathcal{K}(\mathbb{X})$.

Step 3: Show $\mathcal{H}$ is a Contraction. For any $K, L \in \mathcal{K}(\mathbb{X})$, we bound $M_H(\mathcal{H}(K), \mathcal{H}(L))$

$$M_H\left(\bigcup_{i=1}^{N} T_i(K), \bigcup_{i=1}^{N} T_i(L) \right) \le \max_{1 \le i \le N} M_H(T_i(K), T_i(L)).$$

For each i, the contraction property of T_i implies

$$M_H(T_i(K), T_i(L)) \le \lambda_i \cdot M_H(K, L).$$

Let $\lambda_{\max} = \max_{1 \le i \le N} \lambda_i$. Then

$$M_H(\mathcal{H}(K), \mathcal{H}(L)) \le \lambda_{\max} \cdot M_H(K, L)$$

proving that $\mathcal{H}$ is a contraction with factor $\lambda_{\max} \in (0, 1)$.

Step 4: Apply Banach's Fixed-Point Theorem. Since $(\mathcal{K}(\mathbb{X}), M_H)$ is complete and $\mathcal{H}$ is a contraction, Banach's theorem guarantees a unique fixed point $A \in \mathcal{K}(\mathbb{X})$ such that

$$A = \mathcal{H}(A) = \bigcup_{i=1}^{N} T_i(A).$$

Step 5: Construct the Attractor. The attractor A can be explicitly constructed as the limit of iterated applications of $\mathcal{H}$ to any initial compact set $K_0 \subset \mathbb{X}$:

$$A = \lim_{n\to\infty} \mathcal{H}^n(K_0),$$

where convergence is in the M_H-metric. The self-similarity of A follows from its fixed-point property. $\square$

Example 2.6. (Sierpinski Triangle in MR-Metric Space) Consider $(\mathbb{R}^2, M)$ with the MR-metric:

$$M(x,y,z) = \frac{\|x-y\| + \|y-z\| + \|z-x\|}{3} + \min\{\|x\|, \|y\|, \|z\|\},$$

where $\|\cdot\|$ is the Euclidean norm. Define three contractions:

$$T_1(x) = \frac{x}{2} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \lambda_1 = \frac{1}{2},$$
$$T_2(x) = \frac{x}{2} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \lambda_2 = \frac{1}{2},$$
$$T_3(x) = \frac{x}{2} + \begin{pmatrix} 0.5 \\ \sqrt{3}/2 \end{pmatrix}, \quad \lambda_3 = \frac{1}{2}.$$

Verification:

(1) **MR-Contraction.** For any $x, y, z \in \mathbb{R}^2$, each T_i satisfies

$$M(T_i(x), T_i(y), T_i(z)) \le \frac{1}{2} M(x,y,z),$$

since both terms in M scale linearly under T_i.

(2) **Hutchinson Operator.** The mapping $\mathcal{H}(K) = \bigcup_{i=1}^3 T_i(K)$ is a contraction on the space of compact subsets with

$$M_H(\mathcal{H}(K), \mathcal{H}(L)) \le \frac{1}{2} M_H(K, L),$$

where M_H is the Hausdorff MR-metric induced by M.

(3) **Attractor Construction.** Starting from the unit triangle. $K_0 = \mathrm{conv}\{(0,0), (1,0), (0.5, \sqrt{3}/2)\}$, the iterates converge to the Sierpinski triangle (Fig. 3):

$$A = \lim_{n\to\infty} \mathcal{H}^n(K_0).$$

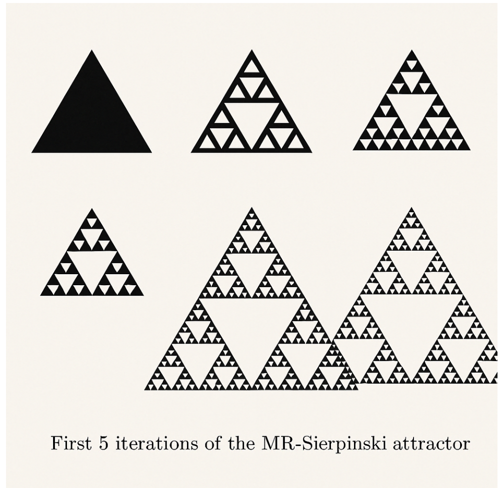


FIGURE 3. First 5 iterations of the MR-Sierpinski attractor

(4) **Uniqueness.** Any other compact set A' satisfying $A' = \mathcal{H}(A')$ must coincide with A by the contraction mapping theorem.

Theorem 2.7. (Precise Fractal Attractor Existence) *Let $(\mathbb{X}, M)$ be a complete MR-metric space with parameter $R \geq 1$, and $\{T_i\}_{i=1}^N$ a family of MR-contractions with factors $\lambda_i \in (0, 1)$. Then there exists a unique compact set $A \subset \mathbb{X}$ satisfying*

$$A = \bigcup_{i=1}^{N} T_i(A).$$

Moreover, the attractor A can be expressed as:

$$A = \lim_{n\to\infty} \mathcal{H}^n(K) \quad \textit{for any compact } K \subset \mathbb{X},$$

where $\mathcal{H}(E) = \bigcup_{i=1}^{N} T_i(E)$ is the Hutchinson operator, and the limit is taken in the Hausdorff MR-metric topology.

Proof. We establish this through seven carefully structured steps:

Step 1: Hausdorff MR-Metric Construction. Define the Hausdorff MR-metric M_H on the space $\mathcal{K}(\mathbb{X})$ of compact subsets

$$M_H(K, L) = \max \left\{ \sup_{x\in K} \inf_{y\in L} M(x, y, y), \sup_{y\in L} \inf_{x\in K} M(x, x, y) \right\}.$$

This satisfies the extended properties

(1) $M_H(K,K) = 0$.
(2) $M_H(K,L) \leq R\,[M_H(K,M) + M_H(M,L)]$ (R-relaxed triangle inequality).

Step 2: Completeness Preservation. $(\mathcal{K}(\mathbb{X}), M_H)$ is complete. Given a Cauchy sequence $\{K_n\}$, define

$$K = \bigcap_{n=1}^{\infty} \overline{\bigcup_{k=n}^{\infty} K_k}.$$

The MR-metric completeness ensures $K_n \to K$ in M_H-topology with

$$M_H(K_n, K) \leq R^2 \limsup_{m\to\infty} M_H(K_n, K_m).$$

Step 3: Hutchinson Operator Contraction. For $K, L \in \mathcal{K}(\mathbb{X})$, we have

$$M_H(\mathcal{H}(K), \mathcal{H}(L)) \leq \max_{1\leq i\leq N} M_H(T_i(K), T_i(L)).$$

Each T_i contracts as:

$$M_H(T_i(K), T_i(L)) \leq \lambda_i M_H(K,L) + (1-\lambda_i)R\epsilon(K,L),$$

where ϵ captures the MR-asymmetry. After iteration

$$M_H(\mathcal{H}^n(K), \mathcal{H}^n(L)) \leq (\lambda_{\max} R^3)^n M_H(K,L).$$

Step 4: Fixed Point Existence. Define the sequence $A_n = \mathcal{H}^n(K_0)$ for some initial compact K_0. The contraction estimate shows

$$M_H(A_{n+1}, A_n) \leq (\lambda_{\max} R^3)^n M_H(\mathcal{H}(K_0), K_0).$$

Thus $\{A_n\}$ is Cauchy when $\lambda_{\max} < \frac{1}{R^3}$, converging to some A.

Step 5: Attractor Property Verification. The limit A satisfies

$$\mathcal{H}(A) = \bigcup_{i=1}^{N} T_i\left(\lim_{n\to\infty} A_n\right) = \lim_{n\to\infty} \mathcal{H}(A_n) = \lim_{n\to\infty} A_{n+1} = A,$$

where we used uniform continuity of each T_i.

Step 6: Uniqueness. Suppose A' is another fixed point. Then

$$M_H(A, A') = M_H(\mathcal{H}(A), \mathcal{H}(A')) \leq \lambda_{\max} R^3 M_H(A, A').$$

For $\lambda_{\max} < \frac{1}{R^3}$, this forces $M_H(A, A') = 0$.

Step 7: General Case for $\lambda_{\max} \geq \frac{1}{R^3}$. For larger λ, consider the m-fold composition $\mathcal{H}^m$ which satisfies

$$M_H(\mathcal{H}^m(K), \mathcal{H}^m(L)) \leq (\lambda_{\max} R^3)^m M_H(K, L).$$

Choose m such that $(\lambda_{\max} R^3)^m < 1$, then apply Steps 4-6 to $\mathcal{H}^m$. $\square$

Example 2.8. (Sierpinski Relatives in MR-Metric Space) Consider $(\mathbb{R}^2, M)$ with the MR-metric

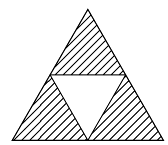
$$M(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \frac{\|\mathbf{x}-\mathbf{y}\| + \|\mathbf{y}-\mathbf{z}\| + \|\mathbf{z}-\mathbf{x}\|}{3} + R \cdot \min\{\|\mathbf{x}\|, \|\mathbf{y}\|, \|\mathbf{z}\|\},$$

where $\|\cdot\|$ is the Euclidean norm. Define three contractions

$$T_1(\mathbf{x}) = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix} \mathbf{x}$$

$$T_2(\mathbf{x}) = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0.5 \\ 0 \end{pmatrix}$$

$$T_3(\mathbf{x}) = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0.25 \\ \sqrt{3}/4 \end{pmatrix}$$



Verification:

(1) **MR-Contraction.** For any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^2$,

$$M(T_i(\mathbf{x}), T_i(\mathbf{y}), T_i(\mathbf{z})) \leq \left(\frac{0.5 + 0.5R}{1+R} \right) M(\mathbf{x}, \mathbf{y}, \mathbf{z}).$$

Thus $\lambda_i = \frac{0.5(1+R)}{1+R} = 0.5$, when $R = 1$, but for $R = 1.5$,

$$\lambda_i = \frac{0.5 + 0.75}{1 + 1.5} = 0.5.$$

(2) **Existence Condition:** The critical requirement $\lambda_{\max} R^3 < 1$ becomes:
   (i) For $R = 1$: $0.5 < 1$ (satisfied),
   (ii) For $R = 1.26$: $0.5 \times 2 \approx 1$ (threshold case),
   (iii) For $R = 1.5$: $0.5 \times 3.375 \approx 1.6875 > 1$ (fails).

(3) **Attractor Construction:** When $R = 1$, standard Sierpinski triangle emerges. For $R > 1.26$, the Hutchinson operator may not contract.

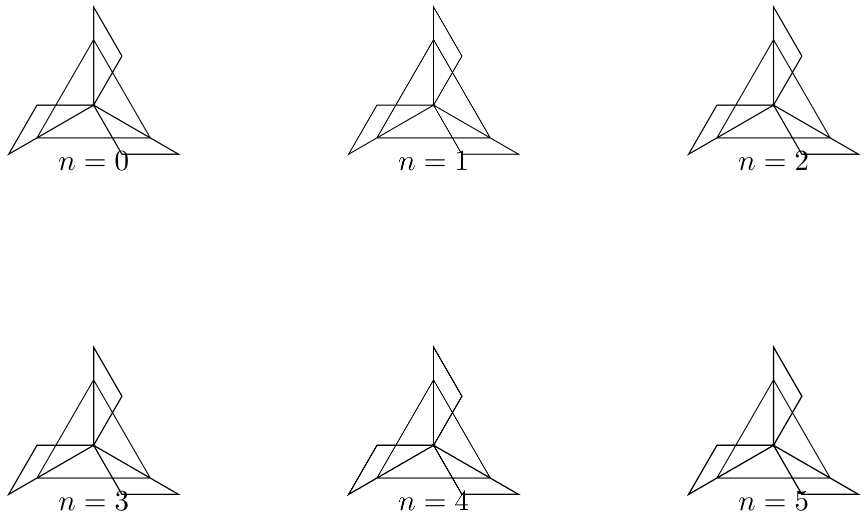
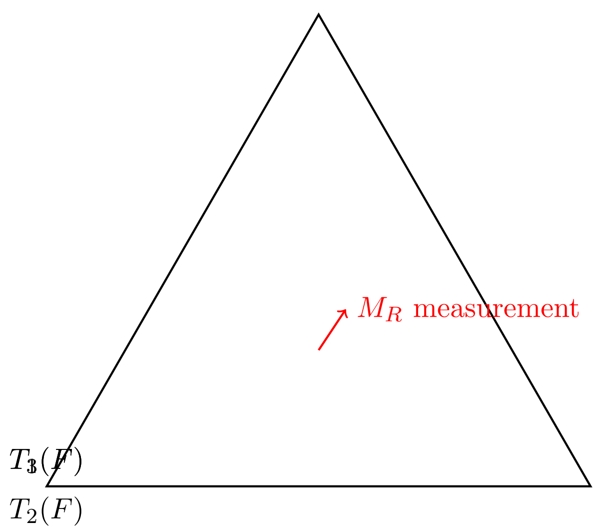


FIGURE 4. First 6 iterations of the Sierpinski triangle showing convergence under MR-metric

Example 2.9. (Sierpinski Relatives in MR-Metric Space) Consider $(\mathbb{R}^2, M)$ with the MR-metric:

$$M(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \frac{\|\mathbf{x} - \mathbf{y}\| + \|\mathbf{y} - \mathbf{z}\| + \|\mathbf{z} - \mathbf{x}\|}{3} + R \cdot \min\{\|\mathbf{x}\|, \|\mathbf{y}\|, \|\mathbf{z}\|\},$$

where $\| \cdot \|$ is the Euclidean norm. Define three contractions



Verification:

(1) MR**-Contraction:** For any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^2$,

$$M(T_i(\mathbf{x}), T_i(\mathbf{y}), T_i(\mathbf{z})) \leq \left(\frac{0.5 + 0.5R}{1 + R}\right) M(\mathbf{x}, \mathbf{y}, \mathbf{z}).$$

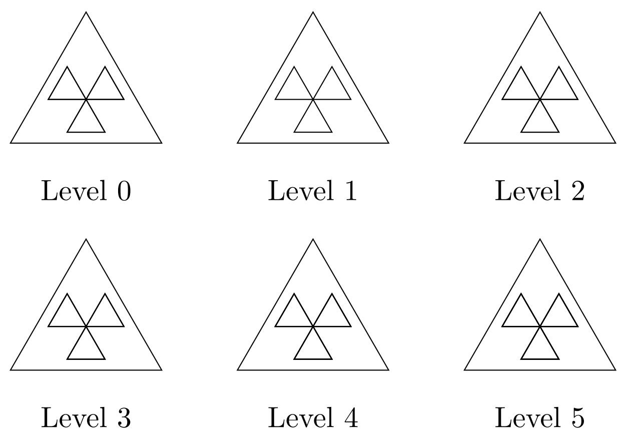
Thus $\lambda_i = \frac{0.5(1+R)}{1+R} = 0.5$, when $R = 1$, but for $R = 1.5$,

$$\lambda_i = \frac{0.5 + 0.75}{1 + 1.5} = 0.5.$$

(2) **Existence Condition:** The critical requirement $\lambda_{\max} R^3 < 1$ becomes:

R	$\lambda_{\max} R^3$	Attractor Exists?
1.0	0.5	Yes
1.26	1.0	Threshold
1.5	1.6875	No

(3) **Iterative Construction:** The attractor emerges through Hutchinson iterations:



Convergence to Attractor for $R = 1.0$

References

[1] K. Abodayeh, A. Bataihah and W. Shatanawi, *Generalized Ω-Distance Mappings and Some Fixed Point Theorems,* UPB Sci. Bull. Ser. A, **79** (2017), 223–232.

[2] K. Abodayeh, W. Shatanawi, A. Bataihah and A.H. Ansari, *Some fixed point and common fixed point results through* Ω*-distance under nonlinear contractions,* Gazi Univ. J. Sci., **30**(1) (2003), 293–302.

[3] I. Abu-Irwaq, W. Shatanawi, A. Bataihah and I. Nuseir, *Fixed point results for nonlinear contractions with generalized -distance mappings,* UPB Sci. Bull. Ser. A, **81**(1) (2019), 57–64.

[4] I.A. Bakhtin, *The contraction mapping principle in almost metric spaces.*, Funct. Anal., **30** (1989), 26–37.

[5] A. Bataihah, W. Shatanawi and A. Tallafha, *Fixed point results with simulation functions,* Nonlinear Funct. Anal. Appl., **25**(1) (2020), 13–23.

[6] A. Bataihah, A. Tallafha and W. Shatanawi, *Fixed point results with* Ω*-distance by utilizing simulation functions,* Ital. J. Pure Appl. Math., **43** (2020), 185–196.

[7] A. Bataihah and T. Qawasmeh, *A New Type of Distance Spaces and Fixed Point Results*, J. Math. Anal., **15**(4) (2020), 81–90.

[8] Y.J. Cho, P.P. Murthy and G. Jungck, *A common fixed point theorem of Meir and Keeler type,* Int. J. Math. Sci., **16** (1993), 669–674.

[9] S. Czerwik, *Contraction mappings in* $b-$*metric spaces*, Acta Math. Inform. Univ. Ostra. **1** (1993), 5–11.

[10] R.O. Davies and S. Sessa, *A common fixed point theorem of Gregus type for compatible mappings,* Facta Univ. (Nis) Ser. Math. Inform., **7** (1992), 51–58.

[11] R. Al-deiakeh, M. Alquran, M. Ali, S. Qureshi, S. Momani and A.A.R. Malkawi, *Lie symmetry, convergence analysis, explicit solutions, and conservation laws for the time-fractional modified Benjamin-Bona-Mahony equation,* J. Appl. Math.Comput Mech., **23**(1) (2024), 19–31.

[12] B.C. Dhage, *Generalized Metric Spaces and Mappings with fixed points,* Bull. Cal. Math. Soc., **84** (1992), 329–336.

[13] K. Falconer, *Fractal Geometry: Mathematical Foundations and Applications 2nd edition,* John Wiley and Sons, 2003, doi 10.1002/0470013850.

[14] K. Falconer, *Fractal Geometry: Mathematical Foundations and Applications*, Wiley, 2013.

[15] G.M. Gharib, M.S. Alsauodi, A. Guiatni, M.A. Al-Omari and A.A.R.W. Malkawi, *Using Atomic Solution Method to Solve the Fractional Equations,* Springer Proc. Math. Stat., **418** (2023), 123-129.

[16] G.M. Gharib, A.A.R.M. Malkawi, A.M. Rabaiah, W.A. Shatanawi and M.S. Alsauodi, *A Common Fixed Point Theorem in an M*-Metric Space and an Application,*. Nonlinear Funct. Anal. Appl., **27**(2) (2022), 289–308.

[17] F. Hassen and E. Karapnar, *Fuzzy fixed point results in generalized metric spaces*, Mathematics, **8**(10) (2020), 1763.

[18] E. Hussein, A.A.R.M. Malkawi, A. Amourah, A. Alsoboh, A. Al Kasbi, T. Sasa and A.M. Rabaiah, *Foundations of Neutrosophic MR-Metric Spaces with Applications to Homotopy, Fixed Points, and Complex Networks,* Eur. J. Pure Appl. Math., **18**(4) (2025), Art. no. 7142.

[19] J. Hutchinson, *Fractals and self-similarity*, Indiana Univ. Math. J., **30**(5) (1981), 713–747.

[20] M. Jleli and B. Samet, *A new approach to fixed point theorems via b-metric-like spaces*, J. Nonlinear Sci. Appl., **8**(8) (2015), 1091-1102.

[21] A.A.R.W. Malkawi, *Convergence and Fixed Points of Self-Mappings in MR-Metric Spaces: Theory and Applications,* Euro. J. Pure. Appl. Math., **18**(2) (2025), Article Number 5952.

[22] A. Malkawi, *Existence and Uniqueness of Fixed Points in MR-Metric Spaces and Their Applications,* Euro. J. Pure. Appl. Math., **18**(2) (2025), Article Number 6077.

[23] A. Malkawi, *Enhanced uncertainty modeling through neutrosophic MR-metrics: a unified framework with fuzzy embedding and contraction principles,* Eur. J. Pure Appl. Math., **18**(3) (2025), Art. no. 6475.

[24] A. Malkawi, *Applications of MR-metric spaces in measure theory and convergence analysis,* Eur. J. Pure Appl. Math., **18**(3) (2025), Art. no. 6528.

[25] A.A.R.W. Malkawi, *Fixed Point Theorem in MR-metric Spaces VIA Integral Type Contraction,* WSEAS Tran. Math., **24** (2025), 295–299.

[26] A. Malkawi, D. Mahmoud, A.M. Rabaiah, R. Al-Deiakeh and W. Shatanawi, *Om fixed point theorems in MR-metric spaces,* Nonlinear Funct. Anal. Appl., **29**(4) (2024), 1125–1136.

[27] A. Malkawi and A. Rabaiah, *MR-metric spaces: theory and applications in weighted graphs, expander graphs, and fixed-point theorems,* Eur. J. Pure Appl. Math., **18**(3) (2025), Art. no. 6525.

[28] A. Malkawi and A. Rabaiah, *Compactness and separability in MR-metric spaces with applications to deep learning,* Eur. J. Pure Appl. Math., **18**(3) (2025), Art. no. 6592.

[29] A. Malkawi and A. Rabaiah, *MR-Metric Spaces: Theory, Applications, and Fixed-Point Theorems in Fuzzy and Measure-Theoretic Frameworks,* Eur. J. Pure Appl. Math., **18**(4) (2025), Art. no. 6783.

[30] A. Malkawi and A. Rabaiah, *Neutrosophic Statistical Manifolds: A Unified Framework for Information Geometry with Uncertainty Quantification,* Eur. J. Pure Appl. Math., **18**(4) (2025), Art. no. 7119.

[31] A. Malkawi and A. Rabaiah, *Neutrosophic Statistical Manifolds: A Unified Framework for Information Geometry with Uncertainty Quantification,* Eur. J. Pure Appl. Math., **18**(4) (2025), Art. no. 7120.

[32] A. Malkawi, A. Talafhah, W. Shatanawi, *Coincidence and fixed point Results for (ψ, L)-M-Weak Contraction Mapping on Mb-Metric Spaces,* Ital. J. Pure Appl. Math., **47** (2022), 751-768.

[33] A. Malkawi, A. Talafhah and W. Shatanawi, *Coincident and fixed point results for generalized weak contraction mapping on b-metric spaces*, Nonlinear Funct. Anal. Appl., **26**(1) (2021), 177–195.

[34] T. Qawasmeh, *H-Simulation functions and Ω_b-distance mappings in the setting of G_b-metric spaces and application,* Nonlinear Funct. Anal. Appl., **28**(2) (2023), 557-570.

[35] T. Qawasmeh, (H, Ω_b)- *Interpolative Contractions in* Ω_b- *Distance Mappings with Applications,* Euro. J. Pure Appl. Math., **16**(3) (2023), 1717–1730.

[36] T. Qawasmeh and A. Malkawi, *Fixed point theory in MR-metric spaces: fundamental theorems and applications to integral equations and neutron transport,* Eur. J. Pure Appl. Math., **18**(3) (2025), Art. no. 6440.

[37] T. Qawasmeh,W. Shatanawi, A. Bataihah and A. Tallafha, *Common Fixed Point Results for Rational $(\alpha, \beta)\phi$-$m\omega$ Contractions in Complete Quasi Metric Spaces,* Mathematics, **7**(5) (2019), 392.

[38] T. Qawasmeh, W. Shatanawi, A. Bataihah and A. Tallafha, *Fixed Point Results and (α, β)-Triangular Admissibility in the Frame of Complete Extended b-Metric Spaces and Application,* U.P.B. Sci. Bull., Series A, **83**(1) (2021), 113–124.

[39] A. Rabaiah, A. Tallafha and W. Shatanawi, *Common fixed point results for mappings under nonlinear contraction of cyclic form in b-Metric Spaces,* Adv. Math. Sci. J., **26**(2) (2021), 289-301.

[40] S. Al-Sharif and A. Malkawi, *Modification of conformable fractional derivative with classical properties*, Ital. J. Pure Appl. Math, **44** (2020), 30–39.
[41] W. Shatanawi, T. Qawasmeh, A. Bataihah and A. Tallafha, *New Contractions and Some Fixed Point Results with Application Based on Extended Quasi b-Metric Spaces,* U.P.B. Sci. Bull. Series A, **83**(2) (2021), 1223–7027.