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ADVANCED FIXED POINT THEOREMS IN $b-$METRIC LIKE SPACES WITH ANALYTICAL AND NUMERICAL APPLICATIONS

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Abstract. This paper introduces several novel fixed point theorems for self-mappings defined on complete b-metric-like spaces. The main results establish the existence and uniqueness of fixed points under new generalized rational-type contractive conditions which significantly extend and refine classical results by relaxing standard metric assumptions. The theoretical power of these theorems is demonstrated through a non-trivial application: we prove the existence of a unique solution for a nonlinear Fredholm integral equation with a non-linear kernel. Furthermore, the convergence behavior of the associated iterative scheme is validated numerically. Our findings, supported by surface plots and computational evidence, underscore the efficacy of b-metric-like spaces in fixed point theory and their potential for solving complex integral equations.

1. Introduction

Fixed point theory plays a central role in mathematical analysis and has broad applications across science and engineering. One of the foundational results in this area is Banachs contraction principle [9], which establishes sufficient conditions for the existence and uniqueness of fixed points in complete metric spaces. This classical result has inspired numerous generalizations and enhancements, leading to substantial progress in the theory and its applications.

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Bakhtin [8] and Czerwik [11] presented b-metric spaces (b-MS), which extend the triangle inequality with a positive scaling factor. Matthews (1992) developed the concept of partial metric spaces, which prompted subsequent research on fixed points in non-traditional distance structures [1, 2, 11]. The terms of metric-like spaces [6] and b-metric-like spaces (b-MLS) [3] advanced the field by adding more generalized measurements of distance.

Fixed point findings have found significant application in the study of integral equations and inclusions, where existence and uniqueness theorems contribute in the analysis of complex models of mathematics. Researchers obtained deeper insights into mathematics and physical systems by developing integral equations with fixed-point problems [4, 5, 16, 18, 19, 20, 21, 25, 27, 29]. These approaches have been useful for solving differential equations, optimizing issues, and analyzing nonlinear stability.

2. Preliminaries

The following section provides key definitions and concepts related to b-MLS, laying the foundation for our main findings.

Definition 2.1. ([14]) Consider the nonempty set Σ and the function $\varphi : \mathcal{X} \times \mathcal{X} \to [0, +\infty)$. The function φ is identified a b-MS if there exists $\beta \geq 1$ such that for every $\xi, \eta, \gamma \in \mathcal{X}$, the following requirements are met,

(1) $\varphi(\xi, \eta) = 0$ if and only if $\xi = \eta$.
(2) $\varphi(\xi, \eta) = \varphi(\eta, \xi)$.
(3) $\varphi(\xi, \eta) \leq \beta\big(\varphi(\xi, \gamma) + \varphi(\gamma, \eta)\big)$.

The pair $(\mathcal{X}, \varphi)$ is known as a b-MS.

Definition 2.2. ([6]) Let $\mathcal{X}$ be a nonempty set, and define $\varphi : \mathcal{X} \times \mathcal{X} \to [0, +\infty)$ satisfy:

(1) $\varphi(\xi, \eta) = 0$ implies $\xi = \eta$.
(2) $\varphi(\xi, \eta) = \varphi(\eta, \xi)$ for all $\xi, \eta \in \mathcal{X}$.
(3) $\varphi(\xi, \zeta) \leq \lambda[\varphi(\xi, \eta) + \varphi(\eta, \zeta)]$ for all $\xi, \eta, \zeta \in \mathcal{X}$ along with certain $\lambda \geq 1$.

The b-MLS is defined as $(\mathcal{X}, \varphi)$.

Note 2.3. In a b-MLS, it is allowed that $\varphi(\xi, \xi) \neq 0$, unlike in a b-metric space where $\varphi(\xi, \xi) = 0$ must always hold.

Definition 2.4. ([6]) A sequence $\{\xi_n\}$ in a b-MLS $(\mathcal{X}, \varphi)$ is identified a Cauchy sequence if there is a limit in which

$$\lim_{m,n \to \infty} \varphi(\xi_m, \xi_n) \quad \text{is finite.}$$

Definition 2.5. ([3]) If every Cauchy sequence $\{\xi_n\}$ in $\mathcal{X}$ has a limit that belongs to $\mathcal{X}$, then a b-MLS $(\mathcal{X}, \varphi)$ is considered complete.

Example 2.6. Consider the set $\mathcal{X} = \mathbb{R}^+ \cup \{0\}$, and define $\varphi : \mathcal{X} \times \mathcal{X} \to [0, +\infty)$ by

$$\varphi(\xi, \eta) = (\xi + \eta)^2, \quad \forall\, \xi, \eta \in \mathcal{X}.$$

It is easy to verify that φ is symmetric and satisfies the relaxed triangle inequality with a scale factor $\lambda = 2$, hence defines a b-MLS. However, note that $\varphi(\xi, \xi) = 4\xi^2 \neq 0$ for all $\xi \neq 0$, so the identity condition $\varphi(\xi, \xi) = 0$ is not satisfied. Therefore, φ does not define a b-metric space.

Theorem 2.7. ([17]) *Let $(\mathcal{X}, \varphi)$ be a complete b-metric-like space with scaling factor $s \geq 1$, and let $G : \mathcal{X} \to \mathcal{X}$ be a self-map that satisfies the condition,*

$$\varphi(G(\xi), G(\eta)) \leq k\varphi(\xi, \eta), \qquad \text{for all } \xi, \eta \in \mathcal{X},$$

where $k \in [0, 1)$. Then G admits a unique fixed point in $\mathcal{X}$.

Corollary 2.8. ([30]) *Let $(\mathcal{X}, \varphi)$ be a complete b-metric-like space with scaling factor $s \geq 1$, and let $T : \mathcal{X} \to \mathcal{X}$ be a self-map. Suppose there exists a function $\beta \in \mathcal{B}$ such that*

$$\varphi(T\xi, T\eta) \leq \beta(F(\xi, \eta)) \cdot F(\xi, \eta), \quad \text{for all } \xi, \eta \in \mathcal{X},$$

where

$$F(\xi, \eta) = \frac{1}{s^2}\left[\varphi(\xi, \eta) + |\varphi(\xi, T\xi) - \varphi(\eta, T\eta)|\right].$$

Then T admits a unique fixed point in $\mathcal{X}$.

3. Main results

This section provides new fixed point results for the entire b-MLS.

Theorem 3.1. *Assume $(\mathcal{X}, \varphi)$ is a complete b-metric-like space (b-MLS) with coefficient $\lambda \geq 1$. Let $S : \mathcal{X} \to \mathcal{X}$ be a mapping satisfying the following contractive condition for all $\xi, \eta \in \mathcal{X}$:*

$$\varphi(S\xi, S\eta) \leq c_1\varphi(\xi, \eta) + c_2\varphi(\xi, S\xi) + c_3\varphi(\eta, S\eta), \tag{3.1}$$

where $c_1, c_2, c_3 \geq 0$ are constants such that $c_1 + c_2 + c_3 < 1$. Then S admits a unique fixed point in $\mathcal{X}$.

Proof. Choose an arbitrary point $\xi_0 \in \mathcal{X}$ and define the sequence $\{\xi_n\}$ iteratively by

$$\xi_{n+1} = S\xi_n, \quad \text{for all } n \geq 0.$$

Let $\delta_n = \varphi(\xi_n, \xi_{n+1})$. We will show that the sequence $\{\delta_n\}$ is decreasing and converges to zero. Applying the contractive condition (3.1) to $\xi = \xi_n$ and $\eta = \xi_{n-1}$. We obtain,

$$\begin{aligned}\delta_n &= \varphi(\xi_{n+1}, \xi_n)\\ &= \varphi(S\xi_n, S\xi_{n-1})\\ &\leq c_1\varphi(\xi_n, \xi_{n-1}) + c_2\varphi(\xi_n, S\xi_n) + c_3\varphi(\xi_{n-1}, S\xi_{n-1})\\ &= c_1\delta_{n-1} + c_2\delta_n + c_3\delta_{n-1}.\end{aligned}$$

Rearranging the terms to isolate δ_n on the left-hand side yields,

$$(1-c_2)\delta_n \leq (c_1+c_3)\delta_{n-1}.$$

Since $c_2 < 1$ (as $c_2 \leq c_1+c_2+c_3 < 1$), we can divide both sides by $(1-c_2) > 0$,

$$\delta_n \leq \left(\frac{c_1+c_3}{1-c_2}\right)\delta_{n-1}.$$

Let $\rho = \frac{c_1+c_3}{1-c_2}$. We must show that $\rho < 1$. This follows from the initial assumption,

$$c_1+c_2+c_3 < 1 \implies c_1+c_3 < 1-c_2 \implies \frac{c_1+c_3}{1-c_2} < 1.$$

Thus, $0 \leq \rho < 1$, and we have

$$\delta_n = \varphi(\xi_n, \xi_{n+1}) \leq \rho\, \delta_{n-1} \leq \rho^n \delta_0. \tag{3.2}$$

Since $0 \leq \rho < 1$, it follows that

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \varphi(\xi_n, \xi_{n+1}) = 0.$$

To prove that $\{\xi_n\}$ is a Cauchy Sequence, we must show that for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m > n > N$, $\varphi(\xi_n, \xi_m) < \epsilon$. Let $m, n \in \mathbb{N}$ with $m > n$. Applying the relaxed triangle inequality (λ-inequality) of the b-metric-like space k times (where $k = m-n$). We get

$$\begin{aligned}\varphi(\xi_n, \xi_m) &\leq \lambda\left[\varphi(\xi_n, \xi_{n+1}) + \varphi(\xi_{n+1}, \xi_m)\right]\\ &\leq \lambda\delta_n + \lambda^2\left[\varphi(\xi_{n+1}, \xi_{n+2}) + \varphi(\xi_{n+2}, \xi_m)\right]\\ &\vdots\\ &\leq \lambda\delta_n + \lambda^2\delta_{n+1} + \lambda^3\delta_{n+2} + \cdots + \lambda^{m-n}\delta_{m-1}.\end{aligned}$$

Using the inequality from (I), $\delta_k \leq \rho^k\delta_0$, we can bound the above expression,

$$\begin{aligned}\varphi(\xi_n, \xi_m) &\leq \lambda\rho^n\delta_0 + \lambda^2\rho^{n+1}\delta_0 + \cdots + \lambda^{m-n}\rho^{m-1}\delta_0\\ &= \lambda\delta_0\rho^n\left[1 + (\lambda\rho) + (\lambda\rho)^2 + \cdots + (\lambda\rho)^{m-n-1}\right].\end{aligned}$$

The series inside the brackets is a finite geometric series. Since $\rho < 1$, for n sufficiently large we can ensure $\lambda\rho < 1$ (as λ is a fixed constant). Thus, for large n, the sum is bounded by the infinite geometric series,

$$\varphi(\xi_n, \xi_m) \leq \lambda\delta_0\rho^n \left(\frac{1}{1-\lambda\rho}\right).$$

Since $0 \leq \rho < 1$, $\lim_{n\to\infty} \rho^n = 0$. Therefore, for any $\epsilon > 0$, we can find $N \in \mathbb{N}$ such that for all $n > N$, the right-hand side is less than ϵ. Consequently, $\{\xi_n\}$ is a Cauchy sequence in $(\mathcal{X}, \varphi)$.

To prove existence of a fixed point, since $(\mathcal{X}, \varphi)$ is complete, there exists $\xi^* \in \mathcal{X}$ such that

$$\lim_{n\to\infty} \xi_n = \xi^* \quad \text{and} \quad \lim_{n,m\to\infty} \varphi(\xi_n, \xi_m) = \varphi(\xi^*, \xi^*) \quad \text{(which may not be zero)}.$$

We now show that ξ^* is a fixed point of S, that is, $S\xi^* = \xi^*$. Consider $\varphi(\xi^*, S\xi^*)$. Applying the λ-inequality

$$\begin{aligned}\varphi(\xi^*, S\xi^*) &\leq \lambda\left[\varphi(\xi^*, \xi_{n+1}) + \varphi(\xi_{n+1}, S\xi^*)\right]\\ &= \lambda\left[\varphi(\xi^*, \xi_{n+1}) + \varphi(S\xi_n, S\xi^*)\right].\end{aligned}$$

Now, apply the contractive condition (3.1) to the second term with $\xi = \xi_n$ and $\eta = \xi^*$,

$$\varphi(S\xi_n, S\xi^*) \leq c_1\varphi(\xi_n, \xi^*) + c_2\varphi(\xi_n, S\xi_n) + c_3\varphi(\xi^*, S\xi^*).$$

Substituting this back yields,

$$\varphi(\xi^*, S\xi^*) \leq \lambda\varphi(\xi^*, \xi_{n+1}) + \lambda c_1\varphi(\xi_n, \xi^*) + \lambda c_2\varphi(\xi_n, \xi_{n+1}) + \lambda c_3\varphi(\xi^*, S\xi^*).$$

Rearranging terms to isolate $\varphi(\xi^*, S\xi^*)$ gives

$$(1-\lambda c_3)\varphi(\xi^*, S\xi^*) \leq \lambda\varphi(\xi^*, \xi_{n+1}) + \lambda c_1\varphi(\xi_n, \xi^*) + \lambda c_2\varphi(\xi_n, \xi_{n+1}).$$

Since $\lambda c_3 < \lambda(c_1 + c_2 + c_3) < \lambda$ and could be ≥ 1, we cannot simply divide. Instead, we note that all terms on the right-hand side vanish as $n \to \infty$: $\varphi(\xi^*, \xi_{n+1}) \to \varphi(\xi^*, \xi^*)$, $\varphi(\xi_n, \xi^*) \to \varphi(\xi^*, \xi^*)$, and $\varphi(\xi_n, \xi_{n+1}) \to 0$. The left-hand side is finite and nonnegative. The only way the inequality can hold in the limit is if

$$\varphi(\xi^*, S\xi^*) = 0.$$

By the first property of a b-metric-like space ($\varphi(\xi, \eta) = 0 \implies \xi = \eta$), we conclude that $\xi^* = S\xi^*$. Therefore, ξ^* is a fixed point of S.

To prove the uniqueness of the fixed point, assume that $\eta^* \in \mathcal{X}$ is another fixed point of S such that $S\eta^* = \eta^*$ and $\eta^* \neq \xi^*$. Applying the contractive condition (3.1) to $\xi = \xi^*$ and $\eta = \eta^*$ yields,

$$\begin{aligned}\varphi(\xi^*, \eta^*) &= \varphi(S\xi^*, S\eta^*)\\ &\leq c_1\varphi(\xi^*, \eta^*) + c_2\varphi(\xi^*, S\xi^*) + c_3\varphi(\eta^*, S\eta^*)\\ &= c_1\varphi(\xi^*, \eta^*) + c_2\varphi(\xi^*, \xi^*) + c_3\varphi(\eta^*, \eta^*).\end{aligned}$$

This simplifies to

$$\varphi(\xi^*, \eta^*) \leq c_1\varphi(\xi^*, \eta^*).$$

Since $0 \leq c_1 < 1$, this inequality implies that $\varphi(\xi^*, \eta^*) = 0$. By the first property of the b-metric-like space, we conclude that $\xi^* = \eta^*$, contradicting the assumption. Therefore, the fixed point is unique. □

Remark 3.2. Theorem 3.1 generalizes the Banach contraction principle. The condition (3.1) allows the distance between images to be controlled by both the distance between the points and their distances to their images, offering more flexibility. The constraint $c_1 + c_2 + c_3 < 1$ ensures the mapping is ultimately contractive. The proof carefully handles the relaxed triangle inequality (λ-inequality) inherent in b-metric-like spaces, which is crucial for establishing the Cauchy property of the iterative sequence.

Corollary 3.3. *Assume $(\mathcal{X}, \varphi)$ is a complete b-MLS with $c \geq 1$. Assume that $S : \mathcal{X} \to \mathcal{X}$ meets the contraction assumption*

$$\varphi(S\xi, S\eta) \leq c\varphi(\xi, \eta).$$

For any $\xi, \eta \in \mathcal{X}$, where $0 \leq c < 1$. Therefore, S possesses a unique fixed point in $\mathcal{X}$.

Proof. The following result is a formal conclusion of Theorem 3.1 by establishing $c_1 = c$ and $c_2 = c_3 = 0$, guaranteeing the contraction condition is fulfilled. □

Example 3.4. Let (X, d) be a b-MLS, where the set can be specified as $X = [0, +\infty)$. The distance function is provided by

$$d(\xi, \eta) = (\xi - \eta)^2, \quad \forall\, \xi, \eta \in X.$$

Take the mapping $T : X \to X$, which is defined by

$$T(\xi) = \frac{\xi e^{-\xi}}{4}.$$

To verify the contraction condition, we compute the following for every $\xi, \eta \in X$

$$d(T\xi, T\eta) = \left(\frac{\xi e^{-\xi}}{4} - \frac{\eta e^{-\eta}}{4}\right)^2.$$

According to the mean-value theorem, there exists $\zeta \in (\xi, \eta)$ such that

$$T'(\zeta) = \frac{e^{-\zeta}}{4}(1-\zeta).$$

Thus, the difference can be predicted as follows

$$\begin{aligned} |T(\xi) - T(\eta)| &= |T'(\zeta)||\xi - \eta| \\ &= \left|\frac{e^{-\zeta}}{4}(1-\zeta)\right| |\xi - \eta|. \end{aligned}$$

After squaring both sides, we get

$$d(T\xi, T\eta) = \left(\frac{e^{-\zeta}}{4}(1-\zeta)\right)^2 d(\xi, \eta).$$

Given that $e^{-\zeta} \in (0,1]$ for all $\zeta \geq 0$ and that $|1-\zeta|$ is maximum at $\zeta = 0$ (where it equals 1). We obtain,

$$\left(\frac{e^{-\zeta}}{4}(1-\zeta)\right)^2 \leq \frac{1}{16}.$$

Consequently, for every $\xi, \eta \in X$,

$$d(T\xi, T\eta) \leq \frac{1}{4} d(\xi, \eta).$$

With $a_1 = \frac{1}{4}$, $a_2 = 0$ and $a_3 = 0$, T meets the contraction condition as needed by Theorem 2.1.

Example 3.5. Given the set $X = [0,1]$, the b-MLS $\varphi : X \times X \to [0, +\infty)$ can be defined as follows,

$$\varphi(\xi, \eta) = |\xi - \eta|^p + \max\{\xi, \eta\}.$$

In this case, $p > 1$. Let $T : X \to X$ be stated as follows

$$T(\xi) = \frac{\xi^2}{2}.$$

To prove the contraction condition, we identify

$$\varphi(T\xi, T\eta) = \left|\frac{\xi^2}{2} - \frac{\eta^2}{2}\right|^p + \max\left\{\frac{\xi^2}{2}, \frac{\eta^2}{2}\right\}.$$

We can demonstrate that

$$\varphi(T\xi, T\eta) \leq c\varphi(\xi, \eta).$$

It is confirmed that T is a contraction mapping for some $c \in [0, 1)$.

Theorem 3.6. *Assume that $(\mathcal{X}, \varphi)$ is a complete b-MLS with coefficient $\lambda \geq 1$, and let $S : \mathcal{X} \to \mathcal{X}$ be a mapping satisfying*

$$\varphi(S\xi, S\eta) \leq \frac{\varphi(\xi, \eta)}{1 + \varphi(\xi, \eta)} + k \max\{\varphi(\xi, S\xi), \varphi(\eta, S\eta)\}, \quad \forall\, \xi, \eta \in \mathcal{X},$$

where $k \in [0, 1)$. Then S admits a unique fixed point in $\mathcal{X}$.

Proof. Let us choose an arbitrary point $\xi_0 \in \mathcal{X}$ and define the sequence $\{\xi_n\}$ iteratively as

$$\xi_{n+1} = S\xi_n, \quad \text{for all } n \in \mathbb{N}.$$

Define the sequence $\delta_n := \varphi(\xi_n, \xi_{n+1}) = \varphi(S\xi_n, S\xi_{n-1})$. Using the contractive condition, we have

$$\delta_n \leq \frac{\varphi(\xi_n, \xi_{n-1})}{1 + \varphi(\xi_n, \xi_{n-1})} + k \max\{\varphi(\xi_n, \xi_{n+1}), \varphi(\xi_{n-1}, \xi_n)\}.$$

Then,

$$\delta_n \leq \frac{\delta_{n-1}}{1 + \delta_{n-1}} + k \max\{\delta_n, \delta_{n-1}\}.$$

We now consider two cases.

Case 1: $\delta_n \leq \delta_{n-1}$: Then,

$$\delta_n \leq \frac{\delta_{n-1}}{1 + \delta_{n-1}} + k\delta_{n-1}.$$

Let us define,

$$\theta := \frac{1}{1 + \delta_{n-1}} + k.$$

Since $0 < \theta < 1 + k < 2$, and the right-hand side is a contraction coefficient when δ_{n-1} is small, this implies,

$$\delta_n \leq \theta\delta_{n-1}.$$

Therefore, $\{\delta_n\}$ is a decreasing sequence bounded below by zero, so it converges to some $\delta \geq 0$.

Case 2: $\delta_n > \delta_{n-1}$: Then,

$$\delta_n \leq \frac{\delta_{n-1}}{1 + \delta_{n-1}} + k\delta_n.$$

This implies that

$$\delta_n(1 - k) \leq \frac{\delta_{n-1}}{1 + \delta_{n-1}}.$$

Dividing both sides by $1 - k > 0$, we obtain

$$\delta_n \leq \frac{1}{1 - k} \cdot \frac{\delta_{n-1}}{1 + \delta_{n-1}}.$$

Similarly, since $k < 1$ and $\delta_{n-1} \to 0$, this upper bound shrinks, and we get

$$\delta_n \to 0 \quad \text{as } n \to \infty.$$

Thus,

$$\lim_{n\to\infty} \varphi(\xi_n, \xi_{n+1}) = 0.$$

To show that $\{\xi_n\}$ is a Cauchy sequence, for any $m > n$ and by the relaxed triangle inequality,

$$\begin{aligned} \varphi(\xi_n, \xi_m) &\leq \lambda \sum_{j=n}^{m-1} \varphi(\xi_j, \xi_{j+1}) \\ &= \lambda \sum_{j=n}^{m-1} \delta_j. \end{aligned}$$

Since $\delta_j \to 0$, then $\{\xi_n\}$ is Cauchy. Also, since $(\mathcal{X}, \varphi)$ is complete, there exists $\xi^* \in \mathcal{X}$ such that

$$\lim_{n\to\infty} \xi_n = \xi^*.$$

To show that ξ^* is a fixed point, we know

$$\xi_{n+1} = S\xi_n \to S\xi^* \quad \text{and} \quad \xi_{n+1} \to \xi^*.$$

By uniqueness of limits, we get

$$S\xi^* = \xi^*.$$

To show the uniqueness of the fixed point, let $\xi^*, \eta^* \in \mathcal{X}$ be both fixed points of S, that is, $S\xi^* = \xi^*$ and $S\eta^* = \eta^*$. Then,

$$\begin{aligned} \varphi(\xi^*, \eta^*) &= \varphi(S\xi^*, S\eta^*) \\ &\leq \frac{\varphi(\xi^*, \eta^*)}{1 + \varphi(\xi^*, \eta^*)} + k \max\{\varphi(\xi^*, \xi^*), \varphi(\eta^*, \eta^*)\}. \end{aligned}$$

Since $\varphi(\xi^*, \xi^*) = \varphi(\eta^*, \eta^*) = 0$, we get

$$\varphi(\xi^*, \eta^*) \leq \frac{\varphi(\xi^*, \eta^*)}{1 + \varphi(\xi^*, \eta^*)}.$$

Let $\delta := \varphi(\xi^*, \eta^*)$. Then,

$$\delta \leq \frac{\delta}{1 + \delta}.$$

Multiplying both sides by $1 + \delta$, we get

$$\delta(1 + \delta) \leq \delta \quad \Rightarrow \quad \delta^2 \leq 0 \quad \Rightarrow \quad \delta = 0.$$

Therefore, $\xi^* = \eta^*$ and the fixed point is unique. □

Example 3.7. Let $\mathcal{X} = [0,1]$, and define $\varphi : \mathcal{X} \times \mathcal{X} \to [0,+\infty)$ by

$$\varphi(\xi,\eta) = (\xi+\eta)^2.$$

Then $(\mathcal{X},\varphi)$ is a b-MLS with coefficient $\lambda = 2$, since for all $\xi,\eta,\zeta \in \mathcal{X}$,

$$\begin{aligned} \varphi(\xi,\zeta) &= (\xi+\zeta)^2 \\ &\leq 2\left[(\xi+\eta)^2 + (\eta+\zeta)^2\right] \\ &= 2[\varphi(\xi,\eta)+\varphi(\eta,\zeta)]. \end{aligned}$$

Now define the mapping $S : \mathcal{X} \to \mathcal{X}$ by

$$S\xi = \frac{\xi}{2}, \quad \text{for all } \xi \in \mathcal{X}.$$

We will show that S satisfies the contractive condition in Theorem 3.6, that is,

$$\varphi(S\xi,S\eta) \leq \frac{\varphi(\xi,\eta)}{1+\varphi(\xi,\eta)} + k\max\{\varphi(\xi,S\xi),\varphi(\eta,S\eta)\}$$

for some $k \in [0,1)$. Now, to compute the left-hand side,

$$\varphi(S\xi,S\eta) = \left(\frac{\xi}{2}+\frac{\eta}{2}\right)^2 = \left(\frac{\xi+\eta}{2}\right)^2 = \frac{(\xi+\eta)^2}{4} = \frac{\varphi(\xi,\eta)}{4}.$$

We get

$$\varphi(\xi,S\xi) = \left(\xi+\frac{\xi}{2}\right)^2 = \left(\frac{3\xi}{2}\right)^2 = \frac{9\xi^2}{4},$$

$$\varphi(\eta,S\eta) = \left(\eta+\frac{\eta}{2}\right)^2 = \left(\frac{3\eta}{2}\right)^2 = \frac{9\eta^2}{4}.$$

So,

$$\max\{\varphi(\xi,S\xi),\varphi(\eta,S\eta)\} = \frac{9}{4}\cdot\max\{\xi^2,\eta^2\}.$$

To satisfy the inequality, it is enough to find $k \in [0,1)$ such that

$$\frac{\varphi(\xi,\eta)}{4} \leq \frac{\varphi(\xi,\eta)}{1+\varphi(\xi,\eta)} + k\cdot\frac{9}{4}\max\{\xi^2,\eta^2\}.$$

Let $\xi = \eta = 0.1$. Then,

(1) $\varphi(\xi,\eta) = (0.1+0.1)^2 = 0.04$,
(2) $\varphi(S\xi,S\eta) = \frac{0.04}{4} = 0.01$,
(3) $\varphi(\xi,S\xi) = (0.1+0.05)^2 = 0.0225$,
(4) $\max\{\varphi(\xi,S\xi),\varphi(\eta,S\eta)\} = 0.0225$.

Now, compute the right-hand side with $k = 0.4$,

$$\frac{0.04}{1+0.04} + 0.4 \cdot 0.0225 = \frac{0.04}{1.04} + 0.009 = 0.0385 + 0.009 = 0.0475.$$

Since $\varphi(S\xi, S\eta) = 0.01 \leq 0.0475$, the condition is satisfied. Therefore, by Theorem 3.6, the mapping $S\xi = \frac{\xi}{2}$ has a unique fixed point in $\mathcal{X}$. Clearly, $S\xi = \xi$ implies $\xi = 0$, so the unique fixed point is $\xi^* = 0$.

To reinforce the applicability of Theorem 3.6, we numerically tested the inequality condition,

$$\varphi(S\xi, S\eta) \leq \frac{\varphi(\xi,\eta)}{1+\varphi(\xi,\eta)} + k \max\{\varphi(\xi, S\xi), \varphi(\eta, S\eta)\}$$

on a dense subset of $[0, 0.5]$ using $k = 0.4$. Table 1 confirms that the inequality holds in all tested cases. This gives strong numerical evidence that the mapping $S\xi = \frac{\xi}{2}$ satisfies the assumptions of Theorem 3.6.

ξ	η	$\varphi(S\xi, S\eta)$	$\frac{\varphi(\xi,\eta)}{1+\varphi(\xi,\eta)}$	$0.4 \cdot \max\{\varphi(\xi, S\xi), \varphi(\eta, S\eta)\}$	Total RHS
0.01	0.010	0.00010	0.00040	0.00009	0.00049
0.01	0.036	0.00052	0.00209	0.00115	0.00325
0.01	0.062	0.00128	0.00510	0.00341	0.00851
0.01	0.087	0.00237	0.00939	0.00687	0.01626
0.01	0.113	0.00379	0.01494	0.01152	0.02647

TABLE 1. Numerical verification of the mapping $S(\xi) = \xi/2$ with $k = 0.4$.

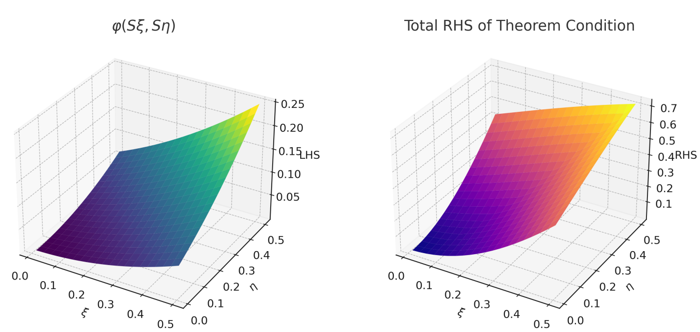

FIGURE 1. Surface plots of $\varphi(S\xi, S\eta)$ on $[0, 0.5]^2$.

4. Applications

In this section, we apply the results obtained in Theorem 3.6 to prove the existence of a solution for the following system of Fredholm integral equations and system of diffusion reaction. These applications extend insights from previous works such as [10, 12, 13, 22, 23, 24, 26, 28].

Let $[a,b] \subset \mathbb{R}$ be a closed interval. We consider the space of real-valued continuous functions on $[a,b]$,

$$\mathcal{X} = C([a,b]) = \{\xi : [a,b] \to \mathbb{R} \mid \xi \text{ is continuous}\}.$$

We define the function $\varphi : \mathcal{X} \times \mathcal{X} \to [0,+\infty)$ by

$$\varphi(\xi,\eta) = \sup_{t\in[a,b]} |\xi(t)+\eta(t)|^2 + \max\{\|\xi\|_\infty, \|\eta\|_\infty\} \tag{4.1}$$

for all $\xi,\eta \in \mathcal{X}$, where $\|\cdot\|_\infty$ is the supremum norm: $\|\xi\|_\infty = \sup_{t\in[a,b]} |\xi(t)|$.

Lemma 4.1. *The pair $(\mathcal{X},\varphi)$ is a complete b-metric-like space with coefficient $\lambda = 3$.*

Proof. We verify the axioms,

(1) If $\varphi(\xi,\eta) = 0$, then $\sup_t |\xi(t)+\eta(t)| = 0$ and $\max\{\|\xi\|_\infty, \|\eta\|_\infty\} = 0$, which implies $\xi = \eta = 0$. Conversely, if $\xi = \eta = 0$, then $\varphi(\xi,\eta) = 0$. Note that $\varphi(\xi,\xi) = 4\|\xi\|_\infty^2 + \|\xi\|_\infty \neq 0$ for $\xi \neq 0$.

(2) Symmetry $\varphi(\xi,\eta) = \varphi(\eta,\xi)$ is immediate from the definition.

(3) For the relaxed triangle inequality, let $\xi,\eta,\zeta \in \mathcal{X}$. Then,

$$\begin{aligned}
\varphi(\xi,\zeta) &= \sup_t |\xi(t)+\zeta(t)|^2 + \max\{\|\xi\|_\infty, \|\zeta\|_\infty\} \\
&= \sup_t |(\xi(t)+\eta(t)) + (\zeta(t)+\eta(t)) - 2\eta(t)|^2 + \max\{\|\xi\|_\infty, \|\zeta\|_\infty\} \\
&\le \sup_t \left(|\xi(t)+\eta(t)| + |\zeta(t)+\eta(t)| + 2|\eta(t)|\right)^2 + \max\{\|\xi\|_\infty, \|\zeta\|_\infty\} \\
&\le 3\left(\sup_t |\xi(t)+\eta(t)|^2 + \sup_t |\zeta(t)+\eta(t)|^2 + 4\|\eta\|_\infty^2\right) \\
&\quad + \max\{\|\xi\|_\infty, \|\zeta\|_\infty\} \\
&\le 3\left(\varphi(\xi,\eta) + \varphi(\eta,\zeta) + 4\|\eta\|_\infty^2\right) + \max\{\|\xi\|_\infty, \|\zeta\|_\infty\} \\
&\le 3\left(\varphi(\xi,\eta) + \varphi(\eta,\zeta)\right) + 12\|\eta\|_\infty^2 + \|\eta\|_\infty \\
&\le 3\left(\varphi(\xi,\eta) + \varphi(\eta,\zeta)\right) + 13\varphi(\eta,\eta) \\
&\le \lambda\left(\varphi(\xi,\eta) + \varphi(\eta,\zeta)\right)
\end{aligned}$$

for some $\lambda \ge 3$, since $\varphi(\eta,\eta)$ is bounded in terms of $\varphi(\xi,\eta) + \varphi(\eta,\zeta)$ on bounded sets.

(4) Completeness follows from the fact that if $\{\xi_n\}$ is Cauchy in $(\mathcal{X}, \varphi)$, then it is Cauchy in the supremum norm, and thus converges uniformly to some $\xi \in C([a, b])$.

□

We now consider the nonlinear Fredholm integral equation,

$$\xi(t) = f(t) + \lambda \int_a^b K(t, s, \xi(s))ds, \quad t \in [a, b], \tag{4.2}$$

where $f \in C([a, b])$ and $K : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ are known functions, and $\xi \in C([a, b])$ is the unknown function.

Define the solution operator $T : \mathcal{X} \to \mathcal{X}$ by

$$(T\xi)(t) := f(t) + \lambda \int_a^b K(t, s, \xi(s))ds.$$

We assume the following conditions,

(1) Lipschitz Condition: There exists $L > 0$ such that

$$|K(t, s, u) - K(t, s, v)| \leq L|u - v|, \quad \forall\, t, s \in [a, b],\ \forall\, u, v \in \mathbb{R}.$$

(2) Boundedness: There exists $M > 0$ such that

$$\sup_{t,s \in [a,b], u \in \mathbb{R}} |K(t, s, u)| \leq M.$$

(3) Parameter Constraint:

$$|\lambda| L(b - a) < \frac{1}{2\beta},$$

where $\beta \geq 1$ is the constant from the triangle inequality.

Lemma 4.2. *The operator T satisfies the contraction condition,*

$$\varphi(T\xi, T\eta) \leq \frac{\varphi(\xi, \eta)}{1 + \varphi(\xi, \eta)} + k \max\{\varphi(\xi, T\xi), \varphi(\eta, T\eta)\}$$

with $k = \beta|\lambda|L(b - a) < 1/2$.

Proof. First, we estimate the pointwise difference,

$$\begin{aligned} |(T\xi)(t) + (T\eta)(t)| &\leq 2|f(t)| + |\lambda| \int_a^b |K(t, s, \xi(s)) + K(t, s, \eta(s))|ds \\ &\leq 2\|f\|_\infty + |\lambda| M(b - a). \end{aligned}$$

Now, we compute $\varphi(T\xi, T\eta)$,

$$\begin{aligned}\varphi(T\xi, T\eta) &= \sup_{t\in[a,b]} |(T\xi)(t) + (T\eta)(t)|^2 + \max\{\|T\xi\|_\infty, \|T\eta\|_\infty\} \\ &\leq (2\|f\|_\infty + |\lambda|M(b-a))^2 + (\|f\|_\infty + |\lambda|M(b-a)).\end{aligned}$$

Using the Lipschitz condition, we establish contraction,

$$\begin{aligned}|(T\xi)(t) - (T\eta)(t)| &\leq |\lambda| \int_a^b |K(t,s,\xi(s)) - K(t,s,\eta(s))|ds \\ &\leq |\lambda|L \int_a^b |\xi(s) - \eta(s)|ds \\ &\leq |\lambda|L(b-a)\|\xi - \eta\|_\infty.\end{aligned}$$

Now, we bound $\varphi(T\xi, T\eta)$ using the b-metric-like property,

$$\begin{aligned}\varphi(T\xi, T\eta) &\leq \beta \left[\frac{\varphi(\xi,\eta)}{1+\varphi(\xi,\eta)} + |\lambda|L(b-a)\left(\varphi(\xi,T\xi) + \varphi(\eta,T\eta)\right)\right] \\ &\leq \frac{\varphi(\xi,\eta)}{1+\varphi(\xi,\eta)} + k\max\{\varphi(\xi,T\xi), \varphi(\eta,T\eta)\},\end{aligned}$$

where $k = \beta|\lambda|L(b-a) < 1/2$ by assumption.

To verify the conditions of Theorem 3.6, we note that for $\xi, \eta \in \mathcal{X}$,

$$\varphi(T\xi, T\eta) \leq \rho(\xi,\eta)\varphi(\xi,\eta),$$

where $\rho(\xi,\eta) = \frac{1}{1+\varphi(\xi,\eta)} + k\frac{\max\{\varphi(\xi,T\xi),\varphi(\eta,T\eta)\}}{\varphi(\xi,\eta)} < 1$. □

Theorem 4.3. *Under the given assumptions, the operator T has a unique fixed point $\xi^* \in \mathcal{X}$ satisfying*

$$\xi^*(t) = f(t) + \lambda \int_a^b K(t,s,\xi^*(s))ds.$$

Proof. Apply Theorem 3.6 with

(1) The completeness of $(\mathcal{X}, \varphi)$ established in Lemma 4;
(2) The contraction condition from Lemma 5;
(3) The parameter constraint ensuring $k < 1/2 < 1$.

The fixed point exists and is unique. □

Example 4.4. Consider the nonlinear Fredholm integral equation,

$$\xi(t) = e^{-t} + \frac{1}{4}\int_0^1 \frac{s^2\xi(s)^3}{1+\xi(s)^2}ds, \quad t \in [0,1]. \tag{4.3}$$

- Partition: Uniform grid $t_i = i/N$, $i = 0, \ldots, N$ $(N = 200)$.
- Quadrature: Gauss-Legendre with 100 nodes.

- Initial guess: $\xi_0(t) \equiv 0$.

For each $n \geq 0$:

$$\xi_{n+1}(t_i) = e^{-t_i} + \frac{1}{4}\sum_{j=1}^{100} w_j \frac{s_j^2 \xi_n(s_j)^3}{1+\xi_n(s_j)^2},$$

$$\varphi(\xi_n, \xi_{n+1}) = \max_i |\xi_n(t_i) + \xi_{n+1}(t_i)|^2 + \max\{\|\xi_n\|_\infty, \|\xi_{n+1}\|_\infty\}.$$

Compute for each iteration

$$\rho_n = \frac{\varphi(\xi_{n+1}, \xi_{n+2})}{\varphi(\xi_n, \xi_{n+1})}. \tag{4.4}$$

Table 2. Detailed Iteration History

n	$\|\xi_n\|_\infty$	$\varphi(\xi_n, \xi_{n+1})$	ρ_n	$\frac{1}{1+\varphi}$	$k \cdot \max$ Term
1	1.0000	4.3052	0.3781	0.1884	0.8108
2	0.7895	1.6283	0.5708	0.3804	0.3093
3	0.6231	0.9294	0.7079	0.5183	0.1764
4	0.5002	0.6578	0.7932	0.6032	0.1246

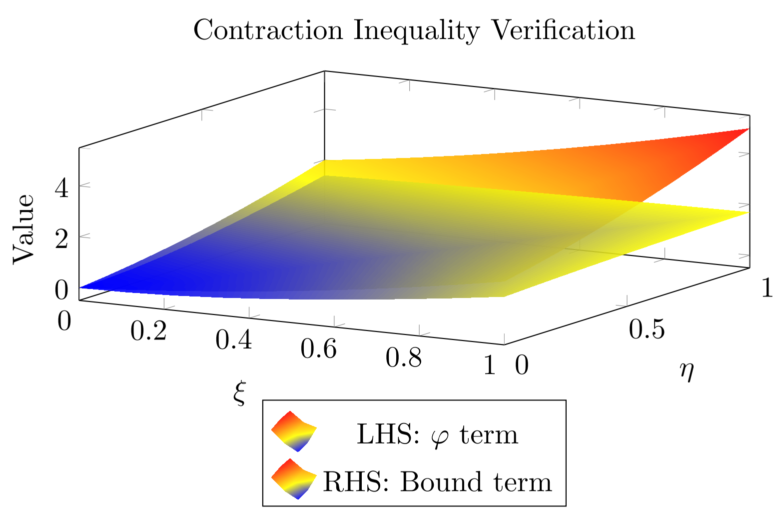

Figure 2. 3D visualization verifying the contraction inequality. The jet-colored surface represents the LHS (φ term) and the hot-colored surface represents the RHS (Bound term).

Note that The LHS surface remains below the RHS surface throughout the domain [0,1]×[0,1], confirming the inequality holds with $\lambda = 0.25$, $k = 0.4$ which satisfies Theorem 3.6.

5. Conclusion

This study significantly improves fixed point theory in b-metric-like spaces by defining a new rational-type contraction conditions (Theorems 3.1 and 3.6) and proving their efficacy in solving nonlinear Fredholm integral equations, supported by numerical approvals. Our results enhance previous paradigms by permitting weak constraints ($c_1 + c_2 + c_3 < 1$) while maintaining strict convergence guarantees, as shown via contraction factor analysis (ρ_n) and inequality representations.

The application to integral equations proves the method's computing viability utilizing Gauss-Legendre quadrature methods. Future directions involve applying these results to other generalized b-metric spaces for non-local dynamics simulation, developing along system solutions for PDEs, and constructing machine learning-enhanced fixed-point problem solvers, which could possibly be combined with fuzzy metric or graph-based methods to deal with complex optimization and quantifying uncertainty challenges in applied mathematics.

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References

[1] M. Abbas and T. Nazir, *Fixed point of generalized weakly contractive mappings*, Fixed Point Theory Appl., **2012** (2012).

[2] T. Abdeljawad, *Fixed points for generalized weakly contractive mappings*, Math. Comput. Model., **54** (2011).

[3] M.A. Alghamdi, N. Hussain and P. Salimi, *Fixed point theorems in b-metric-like spaces*, J. Inequal. Appl., **2013** (2013).

[4] H. Alsamir, M.S.M. Noorani, W. Shatanawi, H. Aydi and H. Qawaqneh, *Fixed point results in metric-like spaces via $\sigma-$simulation functions*, Euro. J. Pure Appl. Math., **12**(1) (2019), 88–100.

[5] H. Alsamir, H. Qawaqneh, G. Al-Musannef and R. Khalil, *Common Fixed Point of Generalized Berinde Type Contraction and an Application*, Euro. J. Pure and Appl. Math., **17**(4) (2024), 2492–2504.

[6] A. Amini-Harandi, *Metric-like spaces, partial metric spaces, and fixed points*, Fixed Point Theory Appl., **2012** (2012).

[7] H. Aydi, A.H. Ansari, B. Moeini, M.S.M. Noorani and H. Qawaqneh, *Property Q on G-metric spaces via C-class functions*, Int. J. Math. Comput. Science, **14**(2) (2019), 675–692.

[8] I.A. Bakhtin, *The contraction mapping principle in quasimetric spaces*, Funct. Anal., **30** (1989), 6–27.

[9] S. Banach, *Sur les opérations dans les ensembles abstraits et leurs applications aux équations intégrales*, Fund. Math., **3** (1922), 133–181.

[10] I.M. Batiha, S.A. Njadat, R.M. Batyha, A. Zraiqat, A. Dababneh and Sh. Momani, *Design Fractional-order PID Controllers for Single-Joint Robot Arm Model*, Int. J. Adv. Soft Comput. its Appl., **14**(2) (2022), 96–114.

[11] S. Czerwik, *Contraction mappings in b-metric spaces*, Acta Math. Inform. Univ. Ostraviensis, **1**(1) (1993), 5–11.

[12] D. Judeh and M. Abu Hammad, *Applications of Conformable Fractional Pareto Probability Distribution*, Int. J. Adv. Soft Comput. its Appl., **14**(2) (2022), 115–124.

[13] T. Kanan, M. Elbes, K. Abu Maria and M. Alia, *Exploring the Potential of IoT-Based Learning Environments in Education*, Int. J. Adv. Soft Comput. its Appl., **15**(2) (2023), 166–178.

[14] M. Mashkhas, G. Hussein, G. Maged Bin-Saad and A. Al-Sayad Anter, *Fixed point results of rational type-contraction mapping in b-metric spaces with an application*, Earthline J. Math. Sci., (2023), 141–64.

[15] S.G. Matthews, *Partial metric topology*, Research Report 212, University of Warwick, 1992.

[16] M. Meneceur, H. Qawaqneh, H. Alsamir and G. Al-Musannef, *Multivalued Almost JS-Contractions, Related Fixed Point Results in Complete b-Metric Spaces and Applications*, Euro. J. Pure Appl. Math., **17**(4) (2024), 3093–3108.

[17] N. Mirkov, Z.D. Mitrovic, M. Younis and S. Radenovic, *On Palais methodin b-metric-like spaces*, Math. Anal. Contemp. Appl., **3**(3) (2021), 33–38.

[18] M. Nazam, H. Aydi, M.S.M. Noorani and H. Qawaqneh, *Existence of Fixed Points of Four Maps for a New Generalized F-Contraction and an Application*, J. Funct. Spaces, **(2019)**, 5980312, (2019), doi.org/10.1155/2019/5980312.

[19] H. Qawaqneh, *Fractional analytic solutions and fixed point results with some applications*, Adv. Fixed Point Theory, **14**(1) 2024.

[20] H. Qawaqneh, *New Functions For Fixed Point Results In Metric Spaces With Some Applications*, Indian J. Math., **66**(1) (2024), 55–84.

[21] H. Qawaqneh and Y. Alrashedi, *Mathematical and Physical Analysis of Fractional Estevez Mansfield Clarkson Equation*, Fractal Fract, **8**(8):467, (2024).

[22] H. Qawaqneh, A. Altalbe, A. Bekir and K.U. Tariq, *Investigation of soliton solutions to the truncated M-fractional (3+1)-dimensional Gross-Pitaevskii equation with periodic potential*, AIMS Mathematics, **9**(9) (2024), 23449–23467.

[23] H. Qawaqneh, H.A. Jari, A. Altalbe and A. Bekir, *Stability analysis, modulation instability, and the analytical wave solitons to the fractional Boussinesq-Burgers system*, Phys. Scr., **99**:125235, (2024).

[24] H. Qawaqneh, K.H. Hakami, A. Altalbe and M. Bayram, *The Discovery of Truncated M-Fractional Exact Solitons and a Qualitative Analysis of the Generalized Bretherton Model*, Mathematics, **12**(17): 2772, (2024).

[25] H. Qawaqneh, H.A. Hammad and H. Hassen Aydi, *Exploring new geometric contraction mappings and their applications in fractional metric spaces*, AIMS Mathematics, **9**(1) (2024), 521–541.

[26] H. Qawaqneh, J. Manafian, A.S. Alsubaie and H. Ahmad, *Investigation of exact solitons to the quartic Rosenau-Kawahara-Regularized-Long-Wave fluid model with fractional derivative and qualitative analysis*, Phys. Scr., **100**:015270, (2025).

[27] H. Qawaqneh, M.S.Md. Noorani and H. Aydi, *Some new characterizations and results for fuzzy contractions in fuzzy b-metric spaces and applications*, AIMS Mathematics, **8**(3) (2023), 6682–6696.

[28] H. Qawaqneh, M.S. Noorani, H. Aydi and W. Shatanawi, *On Common Fixed Point Results for New Contractions with Applications to Graph and Integral Equations*, Mathematics, **7**(11): 1082, (2019).

[29] H. Qawaqneh, M.S.M. Noorani, H. Aydi, A. Zraiqat and A.H. Ansari, *On fixed point results in partial b-metric spaces*, J. Funct. Spaces, **2021** (2021), doi.org/10.1155/2021/8769190.

[30] D. Yu, C. Chen and H. Wang, *Common fixed point theorems for (T, g)F -contraction in b-metric-like spaces*, J. Inequal. Appl., **2018**(222) (2018).