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LARGE MAXIMAL SECOND MODULES AND LARGE MAXIMAL SMALL SECOND MODULES

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Abstract. The notions of large maximal second modules and large maximal small second modules are submitted as generalizations of essential second modules and small second modules; respectively, where a T-module V is named a large maximal second module (simply LM second). If for a proper an ideal H of T, then either $VH = (0)$ or $VH <_{LM} V$ and a T-module V is named a large maximal small second module (simply LMS second).Further, if for a proper an ideal H of T, then either $VH = (0)$ or $VH <_{LMS} V$,with a number of characteristics and some fundamental features and theorems of these concepts are given. This study also shows the relationship between these ideas and other types of second modules.

1. Introduction

Every module in this article is a unitary right module, and every ring is associated to identity. r-semi simple modules were defined and examined by

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Agayev in [2], where a T-module V_T is named a r-semi simple if for any right ideal H of T, VH is a Direct summand of V(briefly $VH \leq^{\oplus} V$), the semi-simple module class is contained in the class of r-semi-simple modules.

In [19], Yassemi introduced the idea of the second module, where a T-module V is referred to as the second module if $V \neq 0$ and for each $t \in T$, either $Vt = (0)$ or $Vt = V$. Equivalently V is second module if for each ideal H of T, either $VH = (0)$ or $VH = V$ [19]. Kasch in [15] presented the concept of essential sub-module where a sub-module P of a T-module V is called essential (large) sub-module of V and denoted by $P \leq_e V$ if whenever $P \cap L = (0), L \leq V, then L = (0)$. Wisbauer in [18] presented the notion of small sub-module where a proper sub-module P of a T-module V, is called small sub-module of V and denoted by $P \ll V$ if $P + L \neq V$ for any proper submodule L of V [8].

In [12], Hadi et al. introduced the idea of the essential second module where a T-module V is an essentially second (simply ess second) if for each ideal H of T, either $VH = (0)$ or $VH \leq_e V$. In 2024 Hadi et al. presented the concept of small second sub-module where a submodule P of a T-module V is named small second sub-module if for each $t \in T$, either $Pt = P$ or $Pt \ll P$, [11].

As expansions of the concepts of essential second module and small second module, the purpose of this study is to introduce the ideas of large maximal second module and large maximal small second module as generalizations of previous concepts where the large maximal second module includes all the conditions of a large second submodule and maximality condition, also the concept of large maximal small second module must satisfy large maximality condition inside the class of all small second module. A first idea is a T-module V is named a large maximal second module (simply LM second) if for a proper an ideal H of T, then either $VH = (0)$ or $VH <_{LM} V$ where a proper sub-module P of V is a large maximal (simply LM) sub-module if there exists a sub-module L of V, $P < L \leq V$, then L is essential sub-module of V, [1].

The second idea A T-module V is named a large maximal small second module (simply LMS second) if for a proper an ideal H of T, then either $VH = (0)$ or $VH <_{LMS} V$ where a proper sub-module P of V is a large maximal small (simply LMS) sub-module if there exists a sub-module L of V $P + L = V$, then L is a large maximal sub-module of V [13].

This research is divided into two parts, in the first section the large maximal second modules and endo large maximal second modules are introduced along with their characteristics and relationships while in the second section, the large maximal small second modules and endo large maximal small second modules are introduced along with their relationships with other second modules.

2. Large maximal second modules

Along with the concept of large maximal second modules and endo large maximal second modules, additional definitions, and the necessary information to establish them, several propositions, relationships, remarks, examples, and claims are provided.

Definition 2.1. A proper sub-module P of a T-module V is named a large maximal second sub-module (simply LM second) if for a proper an ideal H of T, then either $PH = (0)$ or $PH <_{LM} P$. A T-module V is named LM second module if V is LM second sub-module of V. Furthermore, a ring T is named LM second ring if for a proper an ideal H of T, then either $TH = (0)$ or $TH <_{LM} T$.

Example 2.2. (1) Every second(visible) module is LM second module, but not the other way around, for instance: Take into consideration Z as Z-module is LM second module since for an ideal $H \neq (0), H = nZ, n \in Z_+$, then $Z(nZ) = nZ \leq_{LM} Z$, if $H = (0)$, then $Z(0) = (0)$, but Z as Z-module is not second module since $Z(nZ) \neq Z$ and $Z(nZ) \neq (0)$, and it is not visible module.

(2) Each simple module is LM second module, but not the other way around, for instance: Take into consideration Z as Z-module is LM second module, but it is not simple module.

(3) Each essential second module is LM second module, but not the other way around. For instance: consider Z_6 as Z-module is LM second module since for $H = nZ$, n is even number, then $Z_6(nZ) =< \overline{2} ><_{LM} Z_6$ or $Z_6(nZ) =< \overline{0} ><_{LM} Z_6$ and if $H = nZ, n$ is odd number, then $Z_6(nZ) =< \overline{3} ><_{LM} Z_6$. Now, if $H = (0)$, then $Z_6(0) =< \overline{0} >$, but Z_6 is not essential second module since $< \overline{2} >\nleq_e Z_6$ and $< \overline{3} >\nleq_e Z_6$.

(4) Every small second module is LM second module, but not the other way around, for instance: Take into consideration Z_6 as Z- module is LM second module, but it is not small second module.

(5) Every essential sub-module is LM second sub-module but not the other way around, for instance: : Take into consideration Z_{12} as Z-module $P =< \overline{3} >$ is LM second sub-module since for $H = nZ$, n is even number, then $P(nZ) =< \overline{6} ><_{LM} P$ or $P(nZ) =< \overline{0} >$ and if $H = nZ$, n is odd number, then $P(nZ) = P <_{LM} P$, but it is not essential sub-module in Z_{12}.

(6) Every LM sub-module is LM second sub module, but not the other way around, for instance: Take into consideration Z_{12} as Z- module, $P =< \overline{6} >$ is LM second sub-module for $H = nZ$, n is even number, then $P(nZ) =< \overline{0} >$ and if $H = nZ$, n is odd number, then $P(nZ) = P$, but it is not LM sub-module since $P \subset < \overline{3} > \not\leq_e Z_{12}$.

(7) Every r-semi-simple module is LM second module, but not the other way around, for instance: Take into consideration Z_8 as Z-module is LM second module since for $H = nZ$, n is even number, then $Z_8(nZ) =< \overline{2} ><_{LM} Z_8$ or $Z_8(nZ) =< \overline{4} ><_{LM} Z_8$ or $Z_8(nZ) =< \overline{0} >$, and if $H = nZ$, n is odd number, then $Z_8(nZ) = Z_8$, but it is not r-semi-simple module.

(8) Every uniform module is LM second module, but not the other way around, for instance: Take into consideration Z_6 as Z-module is LM second module, but it is not uniform module.

(9) If T is LM second ring and H is any ideal of T, then H is LM second ideal, the proof is if we Let $H \neq 0$ be any ideal of T, $TH = (0)$ or $TH <_{LM} T$, but $TH = H$. Thus, $H <_{LM} T$.

(10) Let P, S be two sub-modules of T-module V such that $P \subset S$. If S is LM second sub-module, then P is LM second sub-module, but not conversely, for example: consider Z_{36} as Z-module and $< \overline{4} > \subset < \overline{2} >$ *where* $< \overline{4} >$ is LM second sub-module since for $H = nZ$, n is even number, then $< \overline{4} > (nZ) =< \overline{4} > <_{LM} < \overline{4} >$ or $< \overline{4} > (nZ) =< \overline{0} >$ and if $H = nZ$, n is odd number, then $< \overline{4} > (nZ) =< \overline{12} > <_{LM} < \overline{4} >$, but $< \overline{2} >$ is not LM second sub-module since for $H = 6Z$, then $< \overline{2} > H =< \overline{12} > \subset < \overline{4} > \not\leq_e < \overline{2} >$.

Proposition 2.3. *Let V be a T-module. In that case, the following claims are equivalent:*

(1) *V is LM second module.*
(2) *If $(0) \neq P < V, P = V(P :_T V)$, then $P <_{LM} V$.*
(3) *For all $t \in T$, either $Vt = (0)$ or $Vt <_{LM} V$.*

Proof. (2) $\Longrightarrow$ (1) Let H be an ideal of T and $VH \neq (0)$. Put $P = VH$, then $P = VH = V(VH :_T V)$, so that $P = V(P :_T V)$, hence by (2), we get $P = VH <_{LM} V$.

(1) $\Longrightarrow$ (3) is obvious.

(3) $\Longrightarrow$ (2) Let $(0) \neq P = V(P :_T V)$. Hence there exists $t \in (P :_T V)$ such that $Vt \neq (0)$, so that by (3), we get $Vt <_{LM} V$, but $Vt \leq V(P :_T V) = P$. Thus, $P <_{LM} V$.

A sub-module P of a T-module V is pure if $VH \cap P = PH$ for each ideal H of T, [15]. □

Proposition 2.4. *Every pure sub-module of LM second module is LM second sub-module.*

Proof. Suppose that V is LM second T-module and P is pure sub-module in V. So that for any an ideal H of T, then $VH = (0)$ or $VH <_{LM} V$. If $VH = (0)$, then $PH = VH \cap P = (0)$. Now, if $VH <_{LM} V$, then there exists $C \leq_e V$ such that $VH < C \leq_e V$, hence $PH = VH \cap P < C \cap P \leq_e V \cap P = P$. Thus, P is LM second sub-module. □

Obviously, every direct summand submodule is a pure submodule so that by proposition 2.4, we get the following result.

Corollary 2.5. *Every direct summand of LM second module is LM second sub-module.*

Since every submodule of semisimple module is a direct summand, and by Corollary 2.5, that leads us to the following result.

Corollary 2.6. *Every sub-module of semi-simple LM second module is LM second sub-module.*

Remark 2.7. The direct sum of two LM second modules is not always LM second module, for example: Consider Z_6 as Z-module and Z_8 as Z-module are two LM second modules since for $H = nZ$, n is even number, then $Z_6(nZ) =< \overline{2} ><_{LM} Z_6$ or $Z_6(nZ) =< \overline{0} >$ and $Z_8(nZ) =< \overline{2} ><_{LM} Z_8$ or $Z_8(nZ) =< \overline{4} ><_{LM} Z_(8)$ or $Z_6(nZ) =< \overline{0} >$and if $H = nZ$, n is odd number, then $Z_6(nZ) =< \overline{3} ><_{LM} Z_6$ and $Z_8(nZ) = Z_8$, but $Z_6 \bigoplus Z_8 \cong Z_{48}$ is not LM second module since $Z_{48}(12Z) =< \overline{12} > \nleq_{LM} Z_{48}$ where $< \overline{12} > \subset < \overline{3} > \nleq_e Z_{48}$ and $Z_{48}(12Z) \neq < \overline{0} >$.

Proposition 2.8. ([1]) *Let P, S be proper sub-modules of a T-module V. If P and S are LM sub-modules of V, then $P + S$ is LM sub-module of V.*

If there is no proper essential extension for a sub-module P of a T-module V, then P is closed [10].

Proposition 2.9. *Let P be a proper sub-module of LM second T-module V such that P is closed and LM sub-module of V. Then $\frac{V}{P}$ is LM second module.*

Proof. Since V is LM second module then for any an ideal H, $VH = (0)$ or $VH <_{LM} V$. If $VH = (0)$, then $\frac{V}{P}H = \frac{VH+P}{P} = (0_{\frac{V}{P}})$. Now, if $VH <_{LM} V$, then $VH + P <_{LM} V$ by Proposition 2.8. Further, to prove $\frac{VH+P}{P} <_{LM} \frac{V}{P}$, let $\frac{VH+P}{P} < \frac{B}{P} < \frac{V}{P}$, we prove $\frac{B}{P} <_e \frac{V}{P}$, then $VH+P < B < V$, but $VH + P <_{LM} V$, so that $B <_e V$, also; since p is closed, hence $\frac{B}{P} <_e \frac{V}{P}$ by [9, Proposition 1.4]. Then $\frac{VH+P}{P} <_{LM} \frac{V}{P}$. Thus, $\frac{V}{P}$ is LM second module. □

Corollary 2.10. *If $\rho : V \to V'$ is an epimorphism such that $Ker(\rho)$ is closed and LM sub-module in LM second T-module V, then V' is LM second module.*

Proof. Since V is LM second and $Ker(\rho)$ is closed and LM sub-module in V, then by Proposition 2.9, we get $V/Ker(\rho)$ is LM second module, but by first fundamental isomorphism for module, we have $V/Ker(\rho) \cong V'$. So that V' is LM second module. □

Remark 2.11. The condition P is closed in V cannot be deleted from Proposition 2.9, for example: In Z as Z-module is LM second module, but $Z/24Z \cong Z_{24}$ is not LM second module and $24Z$ is not closed in Z.

Proposition 2.12. *A T-module V is LM second module and simple module if and only if V is second module.*

Proof. For an ideal H of T. If $VH = (0)$, then the proof is completed. Now, if $VH \neq (0)$ and since V is LM second module, then $VH <_{LM} V$, but V is simple module, hence $VH = V$. Thus, V is second module. The converse part is obvious. □

Proposition 2.13. *Let P be essential in V, $ann(V) = ann(P)$. If P is LM second sub-module. Then V is LM second module.*

Proof. Since P is LM second sub-module, then there exists $t \in T$ such that $Pt = (0)$ or $Pt <_{LM} P$. If $Pt = (0)$, then $Vt = (0)$ since $ann(V) = ann(P)$. If $Pt <_{LM} P$, hence $Pt < B \leq_e P$, but $P \leq_e V$, so that $Pt < B \leq_e V$, then $Pt <_{LM} V$, but $Pt < Vt$, hence $Vt <_{LM} V$. Thus, V is LM second module. □

Recall that a non-zero T-module V is endo essential second module when every $\rho \in End(V)$, either $\rho(V) = (0)$ or $\rho(V) \leq_e V$, [3]. And a non-zero T-module V is endo small second module when every $\rho \in End(V)$, either $\rho(V) = V$ or $\rho(V) \ll V$, [11].

Definition 2.14. A non-zero T- module V is endo LM second module if for every $\rho \in End(V)$, either $\rho(V) = (0)$ or $\rho(V) <_{LM} V$.

Example 2.15. (1) Every endo LM second module is LM second module. In fact, suppose that V is endo LM second module over T. Then for every $\rho \in End(V)$. either $\rho(V) = (0)$ or $\rho(V) <_{LM} V$, hence for $t \in T$ and defined $\rho_t(V) = Vt$. It is clear ρ_t is well-defined and $\rho_t \in End(V)$. Hence $Vt = \rho_t(V) = Im\rho_t = (0)$ or $Vt = \rho_t(V) <_{LM} V$. So that V is LM second module.

(2) Every endo small second module is endo LM second module, but not the other way around, for instance: Take into consideration $Z_4 \oplus Z_2$ as Z-module and for $\rho \in End(Z_4 \oplus Z_2)$ by $\rho((u, v)) = (u, \overline{0})$ for $u \in Z_{(4)}, v \in Z_{(2)}$, where $\rho(Z_4 \oplus Z_2) = Z_4 \oplus < \overline{0} >$ is LM in $Z_4 \oplus Z_2$, hence $Z_4 \oplus Z_2$ is endo LM second module, but it is not endo small second module since $Z_4 \oplus < \overline{0} > \neq Z_4 \oplus Z_2$ and it is not small in $Z_4 \oplus Z_2$.

(3) Every endo essential second module is endo LM second module, but not the other way around, for instance: Take into consideration $Z_4 \oplus Z_4$ as Z-module is endo LM second module since for $\rho \in End(Z_4 \oplus Z_4)$ by $\rho((u, v)) = (u, 0)$ for $u, v \in Z_4$, implies that $\rho(Z_4 \oplus Z_4) = Z_4 \oplus < \overline{0} >$ is LM sub-module of $Z_4 \oplus Z_4$, but it is not essential in $Z_4 \oplus Z_4$, so that $Z_4 \oplus Z_4$ is not endo essential module.

Recall that a T-module V is multiplication if for each sub-module P of V, then $P = VH$ for H is an ideal of T, [6].

Proposition 2.16. *Every multiplication LM second module is endo LM second module.*

Proof. Let V be a multiplication LM second module over a ring T and $\rho \in End(V)$. Then $\rho(V) = VH$ for H is an ideal of T, but V is LM second module, hence $VH = (0)$ or $VH <_{LM} V$, so that $\rho(V) = 0$ or $\rho(V) <_{LM} V$. Thus, V is endo LM second module. □

Remember that a T-module V. For any $\rho \in End(V)$, $t \in T$ has $\rho(V) = Vt$ where V is a scalar module, [17].

Proposition 2.17. *Every scalar LM second module is endo LM second module.*

Proof. It is the same proof of Proposition 2.16. □

Keeping in mind that a T-module V is considered fully visible if every proper sub-module of it is visible, and that a proper sub-module P of a T-module V is said to be visible whenever P=HP for every nonzero ideal H of T, [12]. A T-module V is said to be fully 2-visible if any proper sub-module of it is 2-visible, and a proper sub-module P of a T-module V is said to be 2-visible whenever $P = H^2P$ for every nonzero ideal H of T, [13]. Now, an element t of a ring T is referred to as an idempotent if $t^2 = t$, [5].

Lemma 2.18. ([14]) *T-module V is a fully visible module if it is a fully 2-visible module and all its elements are idempotent.*

Remember that if a T-module V has a finite generating set, let's say X, then its name is finitely generated, that is, $V = \langle X \rangle$ [15].

Lemma 2.19. ([17]) *Every finitely generated multiplication T-module V is scalar module.*

Keep in mind that if every ideal of an integral domain $(T, +, \cdot)$ is a product of prime ideals, then the domain is Dedekind, [16].

Proposition 2.20. *Let V be finitely generated fully 2-visible LM second module over a Dedekind domain T where all elements of T are idempotent. Then V is endo LM second module.*

Proof. Via Lemma 2.18, we get T-module V is a fully visible module, then by [7, corollary 2.6.26] we have T- module V is multiplication, hence by Lemma 2.19 and Proposition 2.17, we get result. □

3. Large maximal small second modules

Several statements, relationships, notes, instances, and claims are presented, along with the notions of large maximal small second modules and endo large maximal small second modules, further definitions, and the data required to verify them.

Definition 3.1. Large maximal small second is the name of a proper sub-module P of T-module V(briefly LMS second) if for any an ideal H of T, then either $PH = (0)$ or $PH <_{LMS} P$ If V is the LMS second submodule of V, then a T-module V is called the LMS second module. Additionally, if $TH = (0)$ or $TH <_{LMS} T$ for a proper an ideal H of T, then a ring T is called the LMS second ring.

Example 3.2. (1) The second (simple) module and LMS second module are independent, for instance: Z_6 as Z-module is LMS second module since $Z_6\ (nZ) =< \overline{2} > or < \overline{0} >$, n is even and $< \overline{2} ><_{LMS} Z_6$ and $Z_6(nZ) =< \overline{3} >$,$n$ is odd and $< \overline{3} ><_{LMS} Z_6$ but, it is not second (simple) module. Also Z_5 as Z-module is second (simple module), but it is not LMS second module.

(2) The essential second module and LMS second module are independent, for instance: Z_6 as Z-module is LMS second, but it is not essential second and Z_4 as Z-module is essential second, but it is not LMS second module.

(3) Every LMS second module is LM second module, but not in reverse, for instance: Take Z_8 as Z-module is LM second module, but it is not LMS second module since $Z_8(nz) =< \overline{2} > or < \overline{4} >$ or $< \overline{0} >$ if n is even where $< \overline{2} >, < \overline{4} >$ is not LMS submodule in Z_8.

(4) Every r-semi simple module is LMS second module, but not in reverse, for instance: Take Z as Z-module is LMS second module since $Z(nZ) = nZ$, $n \in Z_+, nZ <_{LMS} Z$ since if $n = 2, 2Z + 3Z = Z$ and $3Z <_{LM} Z$ since $3Z \leq_e Z$ also if $n = 5$, then $5Z + 4Z = Z, 4Z <_{LM} Z$ since $4Z < 2Z \leq_e Z$, but it is not r-semi simple module.

(5) The small second module and LM second module are independent, for instance: Z_8 as Z-module is small second module, but it is not LM second module. Although, Z_6 as Z- module is LM second module; it is not small second module.

Proposition 3.3. *If P, S are two sub-modules of T-module V such that $S < P < V$ and $P <^{\oplus} V$. If S is LMS second sub-module in V, then S is LMS second sub-module in P.*

Proof. Since $P <^{\oplus} V$, then there exists a sub-module U of V such that $V = P \oplus U$, to prove S is LMS second sub-module in P, but S is LMS second in V, hence $SH = (0)$ or $SH <_{LMS} V$. Now to prove $SH <_{LMS} P$, so assume that $SH + C = P$ for $C < P$, then $(SH + C) + U = V$, so $SH + (C + U) = V$, but $SH <_{LMS} V$, hence $C + V <_{LM} V$, that is $C + U < N \leq_e V$, for $N \leq V$, then $(C + U) \cap P < N \cap P \leq_e V \cap P = P$. So that S is LMS second sub-module in P. □

A T-module V is called faithful if $ann(V) = (0)$, where $ann(V) = \{t \in T : Vt = (0)\}$, so $((0) :_T V) = ann(V)$, [15].

Proposition 3.4. *A T-module V is a faithful finitely generated multiplication and H is an ideal of T. Then T is LMS second ring if and only if V LMS second module.*

Proof. ($\Rightarrow$) Since T is LMS second ring, then $TH = (0)$ or $TH <_{LMS} T$, hence $VTH = (0)$ or $VTH <_{LMS} V$, then V is LMS second module.

($\Leftarrow$) To prove T is LMS second ring, that is to prove $TH = (0)$ or $TK <_{LMS} T$ for any ideal H in T. Suppose that $TH + TK = T$ for any ideal K of T, then $V(TH) = (0)$ or $V(TH) + V(TK) = VT$, but V is LMS second module, then $V(TH) = (0)$ or $V(TK) <_{LM} VT$. Since V is faithful finitely generated multiplication module, then $TH = (0)$ or $TK <_{LM} T$ by [15], so that TH is LMS ideal in T. Thus, T is LMS second ring. □

Proposition 3.5. *Let P be sub-module of T-module V and V/P is LMS second module. Then V is LMS second module.*

Proof. since V/P is LMS second module for any an ideal H of T, $\frac{V}{P}H = (0_{\frac{V}{P}})$ or $\frac{V}{P}H <_{LMS} \frac{V}{P}$, so that $VH = (0)$. Now, to prove $VH <_{LMS} V$, let $VH + S = V$ for $S < V$, then $\frac{VH}{P} + \frac{S+P}{P} = \frac{V}{P}$, but $\frac{V}{P}$ is LMS second module, so that $\frac{S+P}{P} <_{LM} \frac{V}{P}$ that is $\frac{S+P}{P} < \frac{U}{P} \leq_e \frac{V}{P}$, so we have $S < U \leq_e V$ hence $S <_{LM} V$, then we get $VH <_{LMS} V$. Thus, V is LMS second module. □

Definition 3.6. A non-zero T-module V is endo LMS second module if for every $\rho \in End(V)$ either $\rho(v) = (0)$ or $\rho(V) <_{LMS} V$.

Example 3.7. (1) Every endo LMS second module is endo LM second module, but not in reverse, for instance: Take Z_8 as Z-module and for $\rho \in End(Z_8), \rho(u) = 4u$ where $\rho(Z_8) = 4Z_8 =< \overline{4} >$ is LM sub-module in Z_8 and $\rho(< \overline{2} >) =< \overline{0} >$ $\rho(< \overline{4} >) =< \overline{0} >, \rho(< \overline{0} >) =< \overline{0} >$, so that Z_8 as Z-module is endo LM second module, but it is not LMS second since $\rho(Z_8) = 4Z_8$ is not LMS sub-module in Z_8.

(2) The endo small second module and endo LMS second module are independent, for instance: Z_8 as Z- module and for $\rho \in End(Z_8)$ by $\rho(u) = 2u$ where $\rho(Z_8) =< \overline{2} >$ is small sub- module in Z_8 and $\rho(< \overline{2} >) =< \overline{4} >$ is small sub-module in Z_8 also $\rho(< \overline{4} >) =< \overline{0} >$ $, \rho(< \overline{0} >) =< \overline{0} >$, so that Z_8 as Z-module is end small second module, but it is not end LMS second module. Now, Z_6 as Z- module by $\rho(u) = 2u$ where $\rho(Z_6) < \overline{2} >$ is LMS sub-module in Z_6 and $\rho(< \overline{2} >) =< \overline{2} >, \rho(< \overline{3} >) =< \overline{0} >, \rho(< \overline{0} >) =< \overline{0} >$, so that Z_6 as Z-module endo LMS second module, but it is not endo small second module since $\rho(Z_6) =< \overline{2} >$ is not small in Z_6.

(3) The end essential second module and end LMS second module are independent see example in part (2) where Z_8 as Z-module in endo

essential module, but it is not endo LMS second module, also Z_6 as Z- module is endo LMS second module, but it is not end essential second module.

According to the following proposition, the LMS second module is endo LMS second module when the multiplication module is used.

Proposition 3.8. *Every multiplication LMS second module is endo LMS second module.*

Proof. By similarity of Proposition 2.16, we get results. □

Corollary 3.9. *Every scalar LMS second module is endo LMS second module.*

Proof. It is same proof of Proposition 2.16. □

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