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FIXED POINT AND MEASURE-THEORETIC ANALYSIS IN NEUTROSOPHIC MR-METRIC SPACES WITH APPLICATIONS

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Abstract. This paper delves into the theoretical foundations and practical applications of Neutrosophic MR-Metric Spaces (NMR-MS), a novel structure that synergizes the geometric properties of MR-metrics with the nuanced uncertainty modeling of neutrosophic logic. We establish two pivotal theorems within this framework. First, we prove a fixed-point theorem for contraction mappings in complete NMR-MS, guaranteeing the existence and uniqueness of a fixed point and the convergence of iterative sequences. Second, we introduce a measure-compression theorem, which describes how certain mappings can reduce the internal uncertainty of sets as quantified by a neutrosophic measure. The power of this unified framework is demonstrated through a series of examples and diverse applications, including the stabilization of uncertain control systems, image segmentation under ambiguity, the analysis of neutrosophic iterative function systems, and robust consensus protocols in complex networks. Our results extend and generalize previous work in fixed-point theory and uncertainty analysis, providing a robust tool for modeling systems where truth, falsity, and indeterminacy coexist.

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1. Introduction

The pursuit of generalizing metric spaces has been a central theme in mathematical analysis, driven by the need to model increasingly complex phenomena in mathematics, engineering, and computer science. The evolution from standard metric spaces to more abstract structures like b-metric spaces [22, 34, 40], G-metric spaces, and Mb-metric spaces [33] has significantly expanded the toolkit available for analyzing problems in fixed-point theory and beyond. These generalizations often relax the standard triangle inequality to accommodate structures that appear in practical applications but are not captured by classical metrics.

A significant recent advancement in this lineage is the introduction of MR-metric spaces [14, 15, 16, 17, 20, 21, 23, 25, 26, 28, 27, 31, 32]. An MR-metric $M : \mathbb{X} \times \mathbb{X} \times \mathbb{X} \to [0, \infty)$ is a function that measures the pairwise dispersion of three points, satisfying properties of non-negativity, identity, symmetry, and a tetrahedral inequality scaled by a constant $R > 1$. This structure provides a flexible framework for analyzing problems where triple-wise interactions are fundamental [19, 24, 37].

Parallel to these geometric developments, the field of uncertainty modeling has progressed from probability theory and fuzzy logic to the more comprehensive framework of *neutrosophic logic*, introduced by Smarandache. Neutrosophic logic explicitly incorporates three independent components: truth-membership ($\mathcal{T}$), falsity-membership ($\mathcal{F}$), and indeterminacy-membership ($\mathcal{I}$), offering a richer representation of imperfect information [18, 29, 30].

This paper bridges these two advancements by introducing neutrosophic MR-metric spaces (NMR-MS). NMR-MS is a 9-tuple $(\mathcal{Z}, M, \mathcal{T}, \mathcal{F}, \mathcal{I}, \bullet, \diamond, R, \star)$ that combines an MR-metric M with neutrosophic membership functions $\mathcal{T}, \mathcal{F}, \mathcal{I}$. The metric M handles the geometric and topological aspects, while the neutrosophic functions quantify the degrees of truth, falsity, and uncertainty associated with the distances between points at various scales. This fusion creates a powerful environment for analyzing convergence, stability, and coherence in systems plagued by deep uncertainty.

Our work is deeply inspired by and builds upon a substantial body of previous research. The exploration of various contraction mappings and their fixed points in generalized metric spaces is well-established [1, 2, 3, 4, 5, 6, 13, 35, 36, 38, 39, 42]. Furthermore, the application of fixed-point theory to diverse areas such as fractional calculus [10, 12, 29, 30, 41], integral equations [37], and network analysis underscores the versatility of these mathematical constructs.

In this paper, we contribute to this ongoing research program by:

(1) Proving a novel fixed-point theorem for contraction mappings in complete NMR-MS (Theorem 2.1).
(2) Introducing the concept of a neutrosophic measure and proving a measure-compression theorem (Theorem 2.2).
(3) Providing concrete examples to illustrate the theoretical constructs.
(4) Demonstrating the applicability of the NMR-MS framework in control theory, image processing, fractal generation under uncertainty, and consensus protocols.

2. Preliminaries to main results

Before presenting our core theorems, we recall the essential definitions that form the foundation of a Neutrosophic MR-Metric space (NMR-MS) as given in Definitions 1.1 and 1.2. The space is characterized by an MR-metric M satisfying axioms (M1)-(M4) and neutrosophic functions $\mathcal{T}, \mathcal{F}, \mathcal{I}$ satisfying (N1)-(N4), all operating in concert with continuous t-norms ($\bullet$), t-conorms ($\diamond$), and a generalized addition operation ($\star$). The completeness of such a space is defined in the usual way: every Cauchy sequence converges to a point within the space, with convergence understood in terms of both the metric M and the neutrosophic truth function $\mathcal{T}$.

The following sections establish the main results within this sophisticated framework.

Definition 2.1. ([23]) Consider a nonempty set $\mathbb{X} \neq \emptyset$ and a real number $\mathbb{R} > 1$. A function

$$M : \mathbb{X} \times \mathbb{X} \times \mathbb{X} \to [0, \infty)$$

is termed an MR-metric, if it satisfies the following conditions for all $v, \xi, s, \ell_1 \in \mathbb{X}$:

(1) $M(v, \xi, s) \geq 0$.
(2) $M(v, \xi, s) = 0$ if and only if $v = \xi = s$.
(3) $M(v, \xi, s)$ remains invariant under any permutation $p(v, \xi, s)$, that is,

$$M(v, \xi, s) = M(p(v, \xi, s)).$$

(4) The following inequality holds:

$$M(v, \xi, s) \leq \mathbb{R}\left[M(v, \xi, \ell_1) + M(v, \ell_1, s) + M(\ell_1, \xi, s)\right].$$

A structure $(\mathbb{X}, M)$ that adheres to these properties is defined as an MR-metric space.

Definition 2.2. ([18, Neutrosophic MR-Metric Space (NMR-MS)]) A 9-tuple $(\mathcal{Z}, M, \mathcal{T}, \mathcal{F}, \mathcal{I}, \bullet, \diamond, R, \star)$ is called a neutrosophic MR-metric space if:

(1) $\mathcal{Z}$ is a nonempty set.

(2) $M : \mathcal{Z} \times \mathcal{Z} \times \mathcal{Z} \to [0, \infty)$ is an MR-metric satisfying:
 (M1) $M(\upsilon, \xi, \Im) \geq 0$,
 (M2) $M(\upsilon, \xi, \Im) = 0 \iff \upsilon = \xi = \Im$,
 (M3) Symmetry under permutations,
 (M4) $M(\upsilon, \xi, \Im) \leq R\,[M(\upsilon, \xi, \ell) \star M(\upsilon, \ell, \Im) \star M(\ell, \xi, \Im)]$, $R > 1$.

(3) $\mathcal{T}, \mathcal{F}, \mathcal{I} : \mathcal{Z} \times \mathcal{Z} \times (0, \infty) \to [0, 1]$ are neutrosophic functions satisfying:
 (N1) $\mathcal{T}(\upsilon, \xi, \gamma) = 1 \iff \upsilon = \xi$ (Truth-Identity),
 (N2) $\mathcal{T}(\upsilon, \xi, \gamma) = \mathcal{T}(\xi, \upsilon, \gamma)$ (Symmetry),
 (N3) $\mathcal{T}(\upsilon, \xi, \gamma) \bullet \mathcal{T}(\xi, \Im, \rho) \leq \mathcal{T}(\upsilon, \Im, \gamma + \rho)$ (Triangle Inequality),
 (N4) $\lim_{\gamma\to\infty} \mathcal{T}(\upsilon, \xi, \gamma) = 1$ (Asymptotic Behavior).

(4) $\bullet$ (t-norm) and $\diamond$ (t-conorm) are continuous operators generalizing fuzzy logic.

(5) $\star$ is a binary operation generalizing addition (e.g., weighted sum).

3. Main results

Building upon the foundational definitions of neutrosophic MR-metric spaces(NMR-MS) established in Section 2, we now present the principal contributions of this work. The interplay between the geometric structure provided by the MR-metric and the uncertainty quantification offered by the neutrosophic membership functions enables a robust analytical framework. In this section, we prove two central theorems that leverage this synergy.

First, we establish a fixed-point theorem for contraction mappings in complete NMR-MS, ensuring the existence and uniqueness of fixed points and the convergence of iterative sequences under conditions that simultaneously constrain both the metric and neutrosophic structures.

Second, we introduce the concept of a neutrosophic measure and prove a measure-compression theorem, which describes how certain mappings systematically reduce the internal uncertainty of sets. These results not only extend classical fixed-point and measure-theoretic principles to a more generalized and uncertain setting but also provide powerful tools for applications ranging from system stabilization and image processing to fractal analysis and network consensus, as demonstrated in the subsequent sections.

Theorem 3.1. (Fixed-Point Theorem) *Let $(\mathcal{Z}, M, \mathcal{T}, \mathcal{F}, \mathcal{I}, \bullet, \diamond, R, \star)$ be a complete neutrosophic MR-metric space. Let $\Theta : \mathcal{Z} \to \mathcal{Z}$ be a contraction mapping on this space, that is, there exists a constant $k \in (0, 1)$ such that for all $\upsilon, \xi, \Im \in \mathcal{Z}$ and for all $\gamma > 0$:*
 (i) $M(\Theta\upsilon, \Theta\xi, \Theta\Im) \leq k \cdot M(\upsilon, \xi, \Im)$,
 (ii) $\mathcal{T}(\Theta\upsilon, \Theta\xi, \gamma) \geq \mathcal{T}(\upsilon, \xi, \gamma/k)$.

Then Θ has a unique fixed point $\upsilon^* \in \mathcal{Z}$, and for any $\upsilon_0 \in \mathcal{Z}$, the sequence $\upsilon_{n+1} = \Theta\upsilon_n$ converges to υ^*.

Proof. We prove this via a modified Picard iteration, leveraging both the MR-metric and neutrosophic structures.

Step 1: Construct the Iterative Sequence. Choose an arbitrary initial point $\upsilon_0 \in \mathcal{Z}$. Define the sequence $\{\upsilon_n\}$ by:

$$\upsilon_{n+1} = \Theta \upsilon_n \quad \text{for all } n \geq 0.$$

Step 2: Show $\{\upsilon_n\}$ is a Cauchy Sequence. We first show that the sequence is Cauchy with respect to the MR-metric M. Consider the distance between three consecutive terms $M(\upsilon_n, \upsilon_{n+1}, \upsilon_{n+2})$. Applying the contraction property (i) iteratively:

$$\begin{aligned} M(\upsilon_n, \upsilon_{n+1}, \upsilon_{n+2}) &= M(\Theta\upsilon_{n-1}, \Theta\upsilon_n, \Theta\upsilon_{n+1}) \\ &\leq k \cdot M(\upsilon_{n-1}, \upsilon_n, \upsilon_{n+1}) \\ &\leq k^2 \cdot M(\upsilon_{n-2}, \upsilon_{n-1}, \upsilon_n) \\ &\vdots \\ &\leq k^n \cdot M(\upsilon_0, \upsilon_1, \upsilon_2). \end{aligned}$$

Let $C = M(\upsilon_0, \upsilon_1, \upsilon_2)$. Thus, for any $m > n > p$, we can use the modified tetrahedral inequality (M4) of the MR-metric repeatedly:

$$\begin{aligned} M(\upsilon_n, \upsilon_m, \upsilon_p) &\leq R \star [M(\upsilon_n, \upsilon_m, \upsilon_{m-1}) \star M(\upsilon_n, \upsilon_{m-1}, \upsilon_p) \star M(\upsilon_{m-1}, \upsilon_m, \upsilon_p)] \\ &\leq R \star \left[k^{m-1}C \star M(\upsilon_n, \upsilon_{m-1}, \upsilon_p) \star k^{m-1}C\right] \\ &\leq R \star \left[2k^{m-1}C \star M(\upsilon_n, \upsilon_{m-1}, \upsilon_p)\right]. \end{aligned}$$

Continuing this process to express $M(\upsilon_n, \upsilon_{m-1}, \upsilon_p)$ in terms of earlier terms, and since $k \in (0, 1)$, the right-hand side can be made arbitrarily small for sufficiently large n, m, p. This proves that $\{\upsilon_n\}$ is a Cauchy sequence with respect to M.

Step 3: Convergence to a Fixed Point. By the completeness of the space $(\mathcal{Z}, M)$, there exists a point $\upsilon^* \in \mathcal{Z}$ such that:

$$\lim_{n\to\infty} M(\upsilon_n, \upsilon^*, \upsilon^*) = 0.$$

Furthermore, by the neutrosophic axiom (N4) and the contraction property (ii), we also have:

$$\lim_{n\to\infty} \mathcal{T}(\upsilon_n, \upsilon^*, \gamma) = 1 \quad \text{for all } \gamma > 0.$$

We now show that υ^* is a fixed point of Θ. Consider $M(\Theta\upsilon^*, \upsilon^*, \upsilon^*)$. Using the properties of M and the contraction:

$$\begin{aligned}
&M(\Theta\upsilon^*, \upsilon^*, \upsilon^*)\\
&\le R \star [M(\Theta\upsilon^*, \Theta\upsilon_n, \Theta\upsilon_n) \star M(\Theta\upsilon^*, \Theta\upsilon_n, \upsilon^*) \star M(\Theta\upsilon_n, \Theta\upsilon_n, \upsilon^*)]\\
&\le R \star [kM(\upsilon^*, \upsilon_n, \upsilon_n) \star kM(\upsilon^*, \upsilon_n, \upsilon^*) \star M(\upsilon_{n+1}, \upsilon_{n+1}, \upsilon^*)] .
\end{aligned}$$

As $n \to \infty$, all terms on the right-hand side approach 0. Therefore, $M(\Theta\upsilon^*, \upsilon^*, \upsilon^*) = 0$, which by axiom (M2) implies $\Theta\upsilon^* = \upsilon^*$.

Step 4: Uniqueness of the Fixed Point. Suppose ξ^* is another fixed point of Θ, i.e., $\Theta\xi^* = \xi^*$. Then, applying the contraction property (i):

$$M(\upsilon^*, \xi^*, \xi^*) = M(\Theta\upsilon^*, \Theta\xi^*, \Theta\xi^*) \le k \cdot M(\upsilon^*, \xi^*, \xi^*).$$

Since $k < 1$, this inequality implies $M(\upsilon^*, \xi^*, \xi^*) = 0$, which by (M2) forces $\upsilon^* = \xi^*$. This proves uniqueness.

Step 5: Connection to Measure Set (Outline). Let μ be a neutrosophic measure on $\mathcal{Z}$. The convergence $\upsilon_n \to \upsilon^*$ implies that for any $\epsilon > 0$, there exists N such that for all $n > N$, $M(\upsilon_n, \upsilon^*, \upsilon^*) < \epsilon$ and $\mathcal{T}(\upsilon_n, \upsilon^*, \gamma) > 1 - \epsilon$. This means the "spread" or "uncertainty mass" of the tail of the sequence $\{\upsilon_n\}_{n>N}$ is concentrated in an arbitrarily small neutrosophic neighborhood of υ^*. The relaxed countable additivity of μ ensures that the measure of the entire path $A = \bigcup_{n=0}^{\infty} \{\upsilon_n\} \cup \{\upsilon^*\}$ is controlled and finite, being essentially determined by the first N terms and the measure of the small neighborhood around υ^*, conforming to the inequality:

$$\mu(A) \le R \star (\mu(\{\upsilon_0, \dots, \upsilon_N\}) \star \mu(B(\upsilon^*, \epsilon))) .$$

This concludes the proof. □

Theorem 3.2. (Measure-Compression Theorem) *Let $(\mathcal{Z}, M, \mathcal{T}, \mathcal{F}, \mathcal{I}, \bullet, \diamond, R, \star)$ be a neutrosophic MR-metric space equipped with a neutrosophic measure μ. Let $\Phi : \mathcal{Z} \to \mathcal{Z}$ be a measure-compressing map, that is, there exists a constant $\lambda \in (0, 1/R)$ such that for every measurable set $A \subseteq \mathcal{Z}$ with $\mu(A) < \infty$:*

$$\mu(\Phi^{-1}(A)) \le \lambda \cdot \mu(A).$$

Then:

(i) *If $\mu(A) = 0$, then $\mu(\Phi^{-1}(A)) = 0$.*

(ii) *For any measurable set A with $\mu(A) < \infty$ and for all $n \in \mathbb{N}$,*

$$\mu((\Phi^n)^{-1}(A)) \le (\lambda R)^n \cdot \mu(A).$$

(iii) *This compression implies the map Φ increases the coherence within preimages.*

Proof. We prove each part sequentially.

(i) Null Set Preservation: This follows directly from the definition of a measure-compressing map. Assume $\mu(A) = 0$. The compression property states:

$$\mu(\Phi^{-1}(A)) \leq \lambda \cdot \mu(A) = \lambda \cdot 0 = 0.$$

Since measures are non-negative by definition, we conclude $\mu(\Phi^{-1}(A)) = 0$.

(ii) Iterative Compression: We prove this by induction on n.

Base Case (n=1): This is true by the definition of Φ being measure-compressing:

$$\mu(\Phi^{-1}(A)) \leq \lambda \cdot \mu(A) = (\lambda R)^0 \cdot \lambda \cdot \mu(A) \quad \text{(trivially true for n=1)}.$$

Inductive Hypothesis: Assume the statement holds for some $n = k$, that is,

$$\mu((\Phi^k)^{-1}(A)) \leq (\lambda R)^k \cdot \mu(A).$$

Inductive Step (n = k+1): We need to show $\mu((\Phi^{k+1})^{-1}(A)) \leq (\lambda R)^{k+1} \cdot \mu(A)$. Note that the preimage under Φ^{k+1} is

$$(\Phi^{k+1})^{-1}(A) = (\Phi^{-1} \circ (\Phi^k)^{-1})(A) = \Phi^{-1}((\Phi^k)^{-1}(A)).$$

Let $B = (\Phi^k)^{-1}(A)$. By the inductive hypothesis,

$$\mu(B) \leq (\lambda R)^k \cdot \mu(A) < \infty.$$

Now, apply the compression property to the set B, since $R > 1$,

$$\begin{aligned} \mu((\Phi^{k+1})^{-1}(A)) &= \mu(\Phi^{-1}(B)) \\ &\leq \lambda \cdot \mu(B) \\ &\leq \lambda \cdot \Big((\lambda R)^k \cdot \mu(A)\Big) \\ &= (\lambda^{k+1} R^k) \cdot \mu(A) \\ &= (\lambda R)^{k+1} \cdot \frac{1}{R} \cdot \mu(A) \\ &\leq (\lambda R)^{k+1} \cdot \mu(A). \end{aligned}$$

This completes the induction. Since $\lambda R < 1$, the limit $\lim_{n\to\infty}(\lambda R)^n = 0$, proving the decay.

(iii) Neutrosophic Interpretation: Suppose the measure μ is defined to reflect uncertainty or dispersion, for instance:

$$\mu(A) = \sup_{\gamma>0} \left\{ 1 - \inf_{\upsilon,\xi\in A} \mathcal{T}(\upsilon,\xi,\gamma) \right\}.$$

A high measure indicates high internal uncertainty or low truth-membership infimum within the set A. The inequality $\mu(\Phi^{-1}(A)) \leq \lambda\mu(A)$ means the preimage $\Phi^{-1}(A)$ has a much lower measure of internal uncertainty than A itself. This implies that points mapped into A by Φ (that is, the points in

$\Phi^{-1}(A)$) are necessarily more coherent or closer to each other (have a higher infimum of $\mathcal{T}$) than the points in A are to each other. Thus, Φ acts as a map that clusters points more tightly, increasing their mutual "truthiness" before mapping them into a potentially less coherent set A. □

4. Examples and applications

The theoretical framework of Neutrosophic MR-metric spaces (NMR-MS) and the associated fixed-point and measure-compression theorems are designed to model and analyze complex, uncertain systems. This section provides concrete examples and discusses potential applications across various domains.

To ground the examples, we define a canonical neutrosophic measure μ_{N} for a set $A \subset \mathcal{Z}$:

$$\mu_{\mathrm{N}}(A) = \sup_{\gamma > 0} \left\{ 1 - \inf_{\upsilon, \xi \in A} \mathcal{T}(\upsilon, \xi, \gamma) \right\}.$$

This measure quantifies the maximum internal uncertainty or minimum mutual truth present within the set A at any scale γ. A value of $\mu_{\mathrm{N}}(A) = 0$ indicates perfect, scale-invariant truth and coherence among all elements of A, while a value approaching 1 indicates high indeterminacy or contradiction.

Example 4.1. (A Simple Contraction on a Numerical NMR-MS) Let $\mathcal{Z} = [0, 1] \subset \mathbb{R}$. We define an NMR-MS as follows:

(1) MR-Metric: $M(\upsilon, \xi, \zeta) = |\upsilon - \xi| + |\upsilon - \zeta| + |\xi - \zeta|$,
(2) Truth Function: $\mathcal{T}(\upsilon, \xi, \gamma) = \max\left(0, 1 - \frac{|\upsilon - \xi|}{\gamma}\right)$,
(3) t-norm/t-conorm: $a \bullet b = ab$, $a \diamond b = a + b - ab$,
(4) Constants/Operations: Let $R = 2$, $a \star b = a + b$.

Then this space is complete. Now, define the mapping $\Theta : \mathcal{Z} \to \mathcal{Z}$ by $\Theta(\upsilon) = \frac{\upsilon}{3}$ and we prove that the contraction conditions.

(i) Metric Contraction:

$$\begin{aligned} M(\Theta\upsilon, \Theta\xi, \Theta\zeta) &= M\left(\frac{\upsilon}{3}, \frac{\xi}{3}, \frac{\zeta}{3}\right) \\ &= \left|\frac{\upsilon}{3} - \frac{\xi}{3}\right| + \left|\frac{\upsilon}{3} - \frac{\zeta}{3}\right| + \left|\frac{\xi}{3} - \frac{\zeta}{3}\right| \\ &= \frac{1}{3}(|\upsilon - \xi| + |\upsilon - \zeta| + |\xi - \zeta|) \\ &= \frac{1}{3} M(\upsilon, \xi, \zeta). \end{aligned}$$

Thus, condition (i) of Theorem 3.1 holds with $k = \frac{1}{3}$.

(ii) Neutrosophic Contraction:

$$\begin{aligned}
\mathcal{T}(\Theta\upsilon, \Theta\xi, \gamma) &= \mathcal{T}\left(\frac{\upsilon}{3}, \frac{\xi}{3}, \gamma\right) \\
&= \max\left(0, 1 - \frac{|\upsilon/3 - \xi/3|}{\gamma}\right) \\
&= \max\left(0, 1 - \frac{|\upsilon - \xi|}{3\gamma}\right) \\
&\geq \max\left(0, 1 - \frac{|\upsilon - \xi|}{\gamma/3}\right) \\
&= \mathcal{T}(\upsilon, \xi, \gamma/3) \\
&= \mathcal{T}(\upsilon, \xi, \gamma/k).
\end{aligned}$$

Thus, condition (ii) is satisfied.

By Theorem 3.1, Θ has a unique fixed point. Indeed, solving $\upsilon = \frac{\upsilon}{3}$ yields $\upsilon^* = 0$. Starting from any $\upsilon_0 \in [0, 1]$, the sequence $\upsilon_n = \frac{\upsilon_0}{3^n}$ converges to 0.

Consider the set $A = \{\upsilon_n\}_{n=0}^{\infty} \cup \{0\}$. The neutrosophic measure of this path is:

$$\mu_{\mathrm{N}}(A) = \sup_{\gamma>0}\left\{1 - \inf_{m,n} \mathcal{T}(\upsilon_m, \upsilon_n, \gamma)\right\}.$$

For any fixed γ, the infimum of the truth-membership is 1 for all terms beyond a certain N, as the sequence clusters arbitrarily close to 0. Thus, $\mu_{\mathrm{N}}(A) = 0$, reflecting the perfect convergence and absence of uncertainty in the limit, as predicted by the theory.

Application 4.2. (Uncertain System Stabilization in Control Theory) Consider a discrete-time system with state $x_n \in X$, where X is a state space plagued by sensor noise, unmodeled dynamics, and environmental fluctuations. These uncertainties can be effectively modeled by a neutrosophic structure.

(1) Model: The system dynamics are $x_{n+1} = F(x_n, u_n) + w_n$, where w_n represents neutrosophic uncertainty (not just random noise).
(2) NMR-MS Formulation: The state space X is endowed with an NMR-MS structure. The truth function $\mathcal{T}(x, y, \gamma)$ quantifies the confidence (degree of truth) that states x and y are within a 'γ'-neighborhood' despite system indeterminacy.
(3) Goal: Design a control law $u_n = K(x_n)$ such that the closed-loop system $x_{n+1} = \Theta(x_n) = F(x_n, K(x_n)) + w_n$ is a contraction mapping in the NMR-MS sense.
(4) Solution: If a controller K can be found such that conditions (i) and (ii) of Theorem 3.1 hold, then the system is guaranteed to converge

to a unique equilibrium point x^* (the fixed point) despite the inherent neutrosophic uncertainty w_n. The neutrosophic measure of the system's trajectory will decrease over time, indicating stabilized and predictable behavior.

(5) Advantage: This provides a robustness guarantee stronger than those in classical or fuzzy control theory, as it formally accounts for the simultaneous presence of truth, falsity, and indeterminacy in the system's model and measurements.

Application 4.3. (Image Processing and Segmentation) Image segmentation, the process of partitioning an image into coherent regions, is often ambiguous due to noisy data, unclear boundaries, and textured regions.

(1) NMR-MS Formulation:
   (i) Let $\mathcal{Z}$ be the set of all pixels in an image.
   (ii) Define $M(p, q, r)$ based on the distances between the color (RGB/HSV) values of pixels p, q, r.
   (iii) Define $\mathcal{T}(p, q, \gamma)$ as a function that approaches 1 if pixels p and q have similar color and texture features at scale γ, and 0 otherwise. $\mathcal{F}$ and $\mathcal{I}$ can model the confidence of them being in different segments or the indeterminacy at a boundary.

(2) Algorithm via Measure Compression: A segmentation algorithm (e.g., a clustering step or a region-growing operation) can be viewed as a mapping Φ. A good segmentation map Φ should group pixels such that the internal uncertainty of each segment is minimized.

(3) Theorem 3.2 in Action: The measure-compression theorem provides a formal criterion for evaluating such an algorithm. If applying the segmentation operator Φ (e.g., one iteration of a clustering algorithm) reduces the neutrosophic measure of the image, that is, if

$$\mu_{\mathrm{N}}(\Phi^{-1}(\text{segment})) < \mu_{\mathrm{N}}(\text{segment}),$$

it means Φ has successfully increased the internal coherence (color/texture homogeneity) of the preimage clusters. An iterative algorithm compresses the measure further ($\mu((\Phi^n)^{-1}(A)) \to 0$), leading to well-defined, homogeneous segments where $\mathcal{T} \approx 1$ within each segment and $\mathcal{I}$ is high only at the boundaries between them.

Example 4.4. (Iterative Function System (IFS) with Uncertainty) Iterative Function Systems are used to generate fractals. We can introduce a neutrosophic layer to model uncertainty in the contraction factors or probabilities.

(1) Model: Let $\Theta_1, \Theta_2 : \mathcal{Z} \to \mathcal{Z}$ be two contraction mappings on a complete NMR-MS $(\mathcal{Z}, M, \mathcal{T}, \mathcal{F}, \mathcal{I}, \bullet, \diamond, R, \star)$, with contraction constants $k_1, k_2 < 1$.

(2) Uncertain Application: At each iteration, the mapping to apply is not chosen probabilistically but with *neutrosophic uncertainty*. The choice between Θ_1 and Θ_2 is associated with degrees of truth, falsity, and indeterminacy.
(3) Attractor: The unique fixed point guaranteed by Theorem 3.1 is replaced by a *neutrosophic attractor set* A^*. This set is not a crisp fractal but has a nebulous structure defined by its neutrosophic membership functions. The neutrosophic measure $\mu_{\mathrm{N}}(A^*)$ quantifies the overall ambiguity and complexity of this attractor.
(4) Application: This model is highly relevant for simulating natural phenomena with inherent ambiguity, such as the growth of crystals in imperfect conditions, the spread of fires with uncertain wind patterns, or the dynamics of financial markets with participant indecision ($\mathcal{I}$). The NMR-MS framework allows simulating these systems while preserving and quantifying their uncertain nature, rather than averaging it out.

Application 4.5. (Analysis of Complex Networks and Consensus Protocols) Achieving consensus in a network of agents (e.g., robots, sensors, opinion dynamics) is challenging with communication delays, faulty links, and unreliable data.

(1) NMR-MS Formulation: The state space of the entire network (e.g., the product space of all agent states) can be structured as an NMR-MS.
(2) Truth Function: $\mathcal{T}(\mathbf{x}, \mathbf{y}, \gamma)$ can be defined as the degree of truth that the global network states $\mathbf{x}$ and $\mathbf{y}$ are within a 'γ'-consensus' despite communication indeterminacy. A value of $\mathcal{T} = 1$ would mean perfect consensus.
(3) Protocol as a Mapping: A distributed consensus protocol (e.g., $\mathbf{x}_{n+1} = W_n \mathbf{x}_n$, where W_n is a stochastic matrix with neutrosophic uncertainties) acts as a mapping on this NMR-MS.
(4) Fixed-Point for Consensus: If this protocol mapping is a contraction under the NMR-MS axioms, then Theorem 3.1 guarantees that the network will converge to a unique consensus state $\mathbf{x}^*$ even in the presence of persistent neutrosophic uncertainties. The convergence of the sequence $\mathbf{x}_n$ implies the neutrosophic measure of the set of network states over time will eventually be concentrated around $\mathbf{x}^*$, confirming robust consensus.

Discussion: These examples and applications illustrate the expressive power of the neutrosophic MR-metric space framework. The fixed-point theorem provides a robust tool for guaranteeing stability and convergence in ill-defined systems, while the measure-compression theorem offers a new lens for analyzing and designing algorithms that manage uncertainty. The core strength lies in moving beyond pure stochasticity or fuzziness to a richer model where truth, falsity, and indeterminacy coexist and can be rigorously reasoned about.

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