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MULTI APPROXIMATION OF 3-MONOTONE BY CONVEX PIECEWISE POLYNOMIAL

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Abstract. We take into consideration piecewise polynomials with defined knots for the 3-monotone multi approximation. The aim of this paper is to demonstrate that any piecewise polynomial that is convex can be changed in a manner that makes it interpolate the function at the knots, the resulting difference between the two multi approximation errors is a constant factor that is dependent solely on the knots, in other words, we prove that every piecewise convex polynomial can be interpolatory while preserving its uniform multi approximation degree.

1. Introduction

Many researchers dealt with topic of convex and monotone approximations such as [1, 7, 9] where they used functions with one-variable on specific interval. In recent years there have been somewhat surprising results on the μ-monotone approximation, $\mu \geq 3$.

In [10] Shvedov proved these results when $\mu = 1, 2$, but in [5] these results have not been achieved when $\mu \geq 4$, see also [2, 3, 4, 6, 8]. Let's define the μ-th divided difference of a real function $\hbar$ at the points $\varkappa_0 =$

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$(\varkappa_{01},\ldots,\varkappa_{0\ell}),\ldots,\varkappa_{\mu}=(\varkappa_{\mu1},\ldots,\varkappa_{\mu\ell}),$

$$\hbar\left[(\varkappa_{01},\ldots,\varkappa_{0\ell}),\ldots,(\varkappa_{\mu1},\ldots,\varkappa_{\mu\ell})\right]=\sum_{r=0}^{\mu}\frac{\hbar(\varkappa_{r1},\ldots,\varkappa_{r\ell})}{\prod_{i=0,i\neq r}^{\mu}(\varkappa_{r1}-\varkappa_{i1})\ldots(\varkappa_{r\ell}-\varkappa_{i\ell})}.$$

On the interval $\Gamma=[E_1,F_1]\times\ldots\times[E_\ell,F_\ell]$, if

$$\hbar\left[(\varkappa_{01},\ldots,\varkappa_{0\ell}),\ldots,(\varkappa_{\mu1},\ldots,\varkappa_{\mu\ell})\right]\geq 0,$$

then $\hbar$ is said μ-monotone function. Also let's define Δ_{Γ}^{μ} the collection of all μ-monotone functions on Γ, specifically $\Delta_{\Gamma}^{1},\Delta_{\Gamma}^{2}$ are the collections of monotone and convex functions in Γ.

In this study, we establish that changing any convex piecewise polynomial with interpolation is possible with no loss in the degree of uniform multi approximation, this means we show that it is possible to change any convex piecewise polynomial so that it interpolates the function at the knots and the new approximation error is different from the old one by a constant factor that depends only on the knots, this is done by proving the following theorem:

Theorem 1.1. *Let* $\hbar\in\Delta_{\Gamma}^{2}, d\geq 2$ *and* $\varkappa_{-1\xi}=E_{\xi}=\varkappa_{\xi}<\varkappa_{1\xi}<\cdots<\varkappa_{m\xi}=F_{\xi}=\varkappa_{m\xi+1}$, $\xi=1,\ldots,\ell$. *Then for all piecewise polynomial* $\delta\in\Delta_{\Gamma}^{2}$ *of degree* $\leq d-1$ *with knots* $\varkappa_{r}=(\varkappa_{r1},\ldots,\varkappa_{r\ell})$, $r=1,\ldots,m-1$, *there exists a piecewise polynomial* $\delta_{*}\in\Delta_{\Gamma}^{2}$ *of the same degree and knots satisfying*

(a) $\hbar(\varkappa_{r1},\ldots,\varkappa_{r\ell})=\delta_{*}(\varkappa_{r1},\ldots,\varkappa_{r\ell}),\quad r=0,\ldots,m,$

(b) $\|\hbar-\delta_{*}\|_{\Gamma_1}\leq c(\psi)\|\hbar-\delta\|_{\Gamma_2},\quad r=1,\ldots,m$

such that

$$\Gamma_1=[\varkappa_{r1-1}-\varkappa_{r1}]\times\cdots\times[\varkappa_{r\ell-1}-\varkappa_{r\ell}],$$
$$\Gamma_2=[\varkappa_{r1-2}-\varkappa_{r1+1}]\times\cdots\times[\varkappa_{r\ell-2}-\varkappa_{r\ell+1}]$$

and a constant $c(\psi)$ *is depending on*

$$\psi=\max_{1\leq r\leq m-1}\left\{\frac{(\varkappa_{r1+1}-\varkappa_{r1})\ldots(\varkappa_{r\ell+1}-\varkappa_{r\ell})}{(\varkappa_{r1}-\varkappa_{r1-1})\ldots(\varkappa_{r\ell}-\varkappa_{r\ell-1})},\frac{(\varkappa_{r1}-\varkappa_{r1-1})\ldots(\varkappa_{r\ell}-\varkappa_{r\ell-1})}{(\varkappa_{r1+1}-\varkappa_{r1})\ldots(\varkappa_{r\ell+1}-\varkappa_{r\ell})}\right\}.$$

Proof. In order to prove Theorem 1.1, we ned the following lemmas in Section 2, and you can find the proof in Section 3. □

2. Basic lemmas

Let $\alpha(\varkappa_1,\ldots,\varkappa_\ell)=L((\varkappa_1,\ldots,\varkappa_\ell);\hbar;(E_1,\ldots,E_\ell),(F_1,\ldots,F_\ell))$ be the linear Lagrange interpolating polynomial of $\hbar$ at the points $E=(E_1,\ldots,E_\ell)$, $F=(F_1,\ldots,F_\ell)$ and $\alpha'(\varkappa_1,\ldots,\varkappa_\ell)=\hbar\left[(E_1,\ldots,E_\ell),(F_1,\ldots,F_\ell)\right]$, $(\varkappa_1,\ldots,\varkappa_\ell)\in\Gamma=[E_1,F_1]\times\ldots\times[E_\ell,F_\ell]$. To prove theorem 1.1, some of the following lemmas must be proven:

Lemma 2.1. *Let for* $\hbar \in \Delta^2_\Gamma$ *and* $\delta \in \Delta^2_\Gamma$, *there exist either*

$$\delta'\left(F_1-,\ldots,F_\ell-\right) \le \hbar\left[\left(E_1,\ldots,E_\ell\right),\ \left(F_1,\ldots,F_\ell\right)\right]$$

or

$$\delta'\left(E_1+,\ldots,E_\ell+\right) \ge \hbar\left[\left(E_1,\ldots,E_\ell\right),\left(F_1,\ldots,F_\ell\right)\right].$$

Then $\|\hbar-\alpha\| \le 2\|\hbar-\delta\|$.

Proof. Suppose that

$$\delta'\left(F_1-,\ldots,F_\ell-\right) \le \hbar\left[\left(E_1,\ldots,E_\ell\right),\left(F_1,\ldots,F_\ell\right)\right]$$

or

$$\delta'\left(E_1+,\ldots,E_\ell+\right) \ge \hbar\left[\left(E_1,\ldots,E_\ell\right),\left(F_1,\ldots,F_\ell\right)\right].$$

If

$$\varkappa_{0\xi} = \sup\left\{\varkappa_\xi \in \left(E_\xi,F_\xi\right) : \hbar'\left(\varkappa_\xi\right) \le \hbar\left[E_\xi,F_\xi\right], \xi = 1,\ldots,\ell\right\},$$

then

$$\begin{aligned}\delta'\left(\varkappa_1,\ldots,\varkappa_\ell\right) &\le \delta'\left(F_1-,\ldots,F_\ell-\right)\\ &\le \hbar\left[\left(E_1,\ldots,E_\ell\right),\left(F_1,\ldots,F_\ell\right)\right]\\ &\le \hbar'\left(\varkappa_1,\ldots,\varkappa_\ell\right), \quad \varkappa_{0\xi} \le \varkappa_\xi \le F_\xi.\end{aligned}$$

Thus,

$$\begin{aligned}\|\hbar-\alpha\| &= \alpha\left(\varkappa_{01},\ldots,\varkappa_{0\ell}\right) - \hbar\left(\varkappa_{01},\ldots,\varkappa_{0\ell}\right)\\ &= \int_{\varkappa_{01}}^{F_1}\cdots\int_{\varkappa_{0\ell}}^{F_\ell} \hbar'\left(\varkappa_1,\ldots,\varkappa_\ell\right) - \alpha'\left(\varkappa_1,\ldots,\varkappa_\ell\right) d\varkappa_1\ldots d\varkappa_\ell\\ &\le \int_{\varkappa_{01}}^{F_1}\cdots\int_{\varkappa_{0\ell}}^{F_\ell} \hbar'\left(\varkappa_1,\ldots,\varkappa_\ell\right) - \delta'\left(\varkappa_1,\ldots,\varkappa_\ell\right) d\varkappa_1\ldots d\varkappa_\ell\\ &\le \hbar\left(F_1,\ldots,F_\ell\right) - \delta\left(F_1,\ldots,F_\ell\right) - \left(\hbar\left(\varkappa_{01},\ldots,\varkappa_{0\ell}\right) - \delta\left(\varkappa_{01},\ldots,\varkappa_{0\ell}\right)\right)\\ &\le 2\|\hbar-\delta\|.\end{aligned}$$

□

Lemma 2.2. *Let* $\hbar$ *be a function defined on* $\Gamma_3 = [E_{11},F_{11}]\times\ldots\times[E_{1\ell},F_{1\ell}]$ *and* δ *be a piecewise polynomial of degree* $\le d-1$ *with knots* $E = (E_1,\ldots,E_\ell)$, $F = (F_1,\ldots,F_\ell)$ *and* $E = (E_1,\ldots,E_\ell)$ *and* $F = (F_1,\ldots,F_\ell), E_{1\xi} \le E_\xi < F_\xi \le F_{1\xi} \ni \delta'\left(E_1+,\ldots,E_\ell+\right) \le \hbar\left[\left(E_1,\ldots,E_\ell\right),\left(F_1,\ldots,F_\ell\right)\right] \le \delta'\left(F_1-,\ldots,F_\ell-\right)$. *If* $\hbar,\delta \in \Delta^2_{\Gamma_3}$ *then there exists a piecewise polynomial* $\delta_* \in \Delta^2_{\Gamma_3}$ *of the same degree and knots such that*

(a) $\delta'\left(E_1+,\ldots,E_\ell+\right) \le \delta'_*\left(E_1+,\ldots,E_\ell+\right)$,
$\delta'_*\left(F_1-,\ldots,F_\ell-\right) \le \delta'\left(F_1-,\ldots,F_\ell-\right)$,

(b) $\delta_*\left(E_1,\ldots,E_\ell\right) = \hbar\left(E_1,\ldots,E_\ell\right)$,
$\delta_*\left(F_1,\ldots,F_\ell\right) = \hbar\left(F_1,\ldots,F_\ell\right)$,

(c) $\|\hbar-\delta_*\|_{\Gamma} \leq 4\|\hbar-\delta\|_{\Gamma}$,
(d) $\|\hbar-\delta_*\|_{\Gamma_3} \leq 4\|\hbar-\delta\|_{\Gamma_3}$.

Proof. If $\hbar(F_1,\dots,F_\ell)-\hbar(E_1,\dots,E_\ell) = \delta(F_1,\dots,F_\ell)-\delta(E_1,\dots,E_\ell)$, we put $\delta_*(\varkappa_1,\dots,\varkappa_\ell) = \delta(\varkappa_1,\dots,\varkappa_\ell)+\hbar(E_1,\dots,E_\ell)-\delta(E_1,\dots,E_\ell), (\varkappa_1,\dots,\varkappa_\ell) \in \Gamma_3$, then (a) and (b) are verified and

$$\|\hbar-\delta_*\|_{\Gamma} \leq \|\hbar-\delta\|_{\Gamma} + |\hbar(E_1,\dots,E_\ell)-\delta(E_1,\dots,E_\ell)| \leq 2\|\hbar-\delta\|_{\Gamma}, \text{ and}$$
$$\|\hbar-\delta_*\|_{\Gamma_3} \leq \|\hbar-\delta\|_{\Gamma_3} + |\hbar(E_1,\dots,E_\ell)-\delta(E_1,\dots,E_\ell)| \leq 2\|\hbar-\delta\|_{\Gamma_3}.$$

Let $\hbar(F_1,\dots,F_\ell)-\hbar(E_1,\dots,E_\ell) < \delta(F_1,\dots,F_\ell)-\delta(E_1,\dots,E_\ell)$, the problem $\hbar(F_1,\dots,F_\ell)-\hbar(E_1,\dots,E_\ell) > \delta(F_1,\dots,F_\ell)-\delta(E_1,\dots,E_\ell)$ is similar. First we define δ_* on Γ and then generalize it to Γ_3. Suppose

$$\bar{\bar{\delta}}(\varkappa_1,\dots,\varkappa_\ell) = \delta(\varkappa_1,\dots,\varkappa_\ell) - \delta'(E_1+,\dots,E_\ell+)(\varkappa_1-E_1)\dots(\varkappa_\ell-E_\ell)$$

and

$$\begin{aligned}\bar{\bar{\hbar}}(\varkappa_1,\dots,\varkappa_\ell) =& \hbar(\varkappa_1,\dots,\varkappa_\ell) - \delta'(E_1+,\dots,E_\ell+)(\varkappa_1-E_1)\dots(\varkappa_\ell-E_\ell), \\ &(\varkappa_1,\dots,\varkappa_\ell) \in \Gamma.\end{aligned}$$

Then,

$$\begin{aligned}&\|\bar{\bar{\hbar}}-\bar{\bar{\delta}}\|_{\Gamma} = \|\hbar-\delta\|_{\Gamma}, \\ &\bar{\bar{\delta}}'(F_1-,\dots,F_\ell-) = \delta'(F_1-,\dots,F_\ell-) - \delta'(E_1+,\dots,E_\ell+), \\ &\bar{\bar{\delta}}'(E_1+,\dots,E_\ell+) = 0\end{aligned}$$

and

$$\begin{aligned}\bar{\bar{\hbar}}[(E_1,\dots,E_\ell),(F_1,\dots,F_\ell)] =& \hbar[(E_1,\dots,E_\ell),(F_1,\dots,F_\ell)] \\ &-\delta'(E_1+,\dots,E_\ell+) \geq 0.\end{aligned}$$

In reality $\bar{\bar{\hbar}}(F_1,\dots,F_\ell)-\bar{\bar{\hbar}}(E_1,\dots,E_\ell) \geq 0$ and because

$$\bar{\bar{\hbar}}(F_1,\dots,F_\ell)-\bar{\bar{\hbar}}(E_1,\dots,E_\ell) < \bar{\bar{\delta}}(F_1,\dots,F_\ell)-\bar{\bar{\delta}}(E_1,\dots,E_\ell)$$

this refers to

$$\bar{\bar{\delta}}(F_1,\dots,F_\ell)-\bar{\bar{\delta}}(E_1,\dots,E_\ell) > 0.$$

Now, we put

$$\bar{\bar{\delta}}_*(\varkappa_1,\dots,\varkappa_\ell) = \bar{\bar{\hbar}}(E_1,\dots,E_\ell) + \beta\left(\bar{\bar{\delta}}(\varkappa_1,\dots,\varkappa_\ell)-\bar{\bar{\delta}}(E_1,\dots,E_\ell)\right)$$

such that

$$\beta = \left(\bar{\bar{\hbar}}(F_1,\dots,F_\ell)-\bar{\bar{\hbar}}(E_1,\dots,E_\ell)\right)\left(\bar{\bar{\delta}}(F_1,\dots,F_\ell)-\bar{\bar{\delta}}(E_1,\dots,E_\ell)\right)^{-1},$$

then $0 \leq \beta < 1$ and $\bar{\bar{\delta}}_*$ satisfy convexity on Γ. Also,

$$\begin{aligned}
\bar{\bar{\delta}}_*\left(E_1,\ldots,E_\ell\right) &= \bar{\bar{\hbar}}\left(E_1,\ldots,E_\ell\right),\\
\bar{\bar{\delta}}_*\left(F_1,\ldots,F_\ell\right) &= \bar{\bar{\hbar}}\left(F_1,\ldots,F_\ell\right),\\
\bar{\bar{\delta}}'_*\left(F_1-,\ldots,F_\ell-\right) &= \beta\bar{\bar{\delta}}'_*\left(F_1-,\ldots,F_\ell-\right)\\
&< \delta'\left(F_1-,\ldots,F_\ell-\right) - \delta'\left(E_1+,\ldots,E_\ell+\right),\\
\bar{\bar{\delta}}'_*\left(E_1+,\ldots,E_\ell+\right) &= 0.
\end{aligned}$$

Since, $\bar{\bar{\delta}}' \geq 0$ on Γ so that $\bar{\bar{\delta}}$ is monotone and

$$\left\|\bar{\bar{\delta}}\left(\varkappa_1,\ldots,\varkappa_\ell\right) - \bar{\bar{\delta}}\left(E_1,\ldots,E_\ell\right)\right\|_\Gamma = \bar{\bar{\hbar}}\left(F_1,\ldots,F_\ell\right) - \bar{\bar{\delta}}\left(E_1,\ldots,E_\ell\right).$$

Therefore,

$$\begin{aligned}
\|\hbar - \delta_*\|_\Gamma &= \left\|\bar{\bar{\hbar}} - \bar{\bar{\delta}}_*\right\|_\Gamma\\
&= \Big\|\bar{\bar{\hbar}}(\cdot) - \bar{\bar{\delta}}(\cdot) + \bar{\bar{\delta}}\left(E_1,\ldots,E_\ell\right)\\
&\quad -\bar{\bar{\hbar}}\left(E_1,\ldots,E_\ell\right) + \bar{\bar{\delta}}(\cdot) - \bar{\bar{\delta}}\left(E_1,\ldots,E_\ell\right) + \bar{\bar{\hbar}}\left(E_1,\ldots,E_\ell\right) - \bar{\bar{\delta}}_*(\cdot)\Big\|_\Gamma\\
&\leq 2\|\bar{\bar{\hbar}} - \bar{\bar{\delta}}\|_\Gamma + \left\|\bar{\bar{\delta}}(\cdot) - \bar{\bar{\delta}}\left(E_1,\ldots,E_\ell\right) - \beta\left(\bar{\bar{\delta}}(\cdot) - \bar{\bar{\delta}}\left(E_1,\ldots,E_\ell\right)\right)\right\|_\Gamma\\
&\leq 2\|\bar{\bar{\hbar}} - \bar{\bar{\delta}}\|_\Gamma + (1-\beta)\left|\bar{\bar{\delta}}\left(F_1,\ldots,F_\ell\right) - \bar{\bar{\delta}}\left(E_1,\ldots,E_\ell\right)\right|\\
&= 2\|\bar{\bar{\hbar}} - \bar{\bar{\delta}}\|_\Gamma + \Big|\bar{\bar{\delta}}\left(F_1,\ldots,F_\ell\right) - \bar{\bar{\delta}}\left(E_1,\ldots,E_\ell\right)\\
&\quad - \left(\bar{\bar{\hbar}}\left(F_1,\ldots,F_\ell\right) - \bar{\bar{\hbar}}\left(E_1,\ldots,E_\ell\right)\right)\Big|\\
&\leq 4\|\bar{\bar{\hbar}} - \bar{\bar{\delta}}\|_\Gamma = 4\|\hbar - \delta\|_\Gamma.
\end{aligned}$$

Thus (c) is proved. Using setting

$$\delta_*\left(\varkappa_1,\ldots,\varkappa_\ell\right) = \begin{cases}
\delta\left(\varkappa_1,\ldots,\varkappa_\ell\right) + \hbar\left(E_1,\ldots,E_\ell\right) - \delta\left(E_1,\ldots,E_\ell\right),\\
\qquad \left(\varkappa_1,\ldots,\varkappa_\ell\right) \in [E_{11}, E_1) \times \cdots \times [E_{1\ell}, E_\ell),\\
\delta\left(\varkappa_1,\ldots,\varkappa_\ell\right) + \hbar\left(F_1,\ldots,F_\ell\right) - \delta\left(F_1,\ldots,F_\ell\right),\\
\qquad \left(\varkappa_1,\ldots,\varkappa_\ell\right) \in (E_1, E_{11}] \times \cdots \times (E_\ell, E_{1\ell}].
\end{cases}$$

We get δ_* is a convex piecewise polynomial of the same degree and knots on Γ_3, which achieves (a), (b) and (c). Since

$$\begin{aligned}
\|\hbar - \delta_*\|_{\Gamma_4} &\leq \|\hbar - \delta\|_{\Gamma_4} + \left|\hbar\left(F_1,\ldots,F_\ell\right) - \delta\left(F_1,\ldots,F_\ell\right)\right|\\
&\leq 2\|\hbar - \delta\|_{\Gamma_4}, \Gamma_4\\
&= [F_1, F_{11}] \times \cdots \times [F_\ell, F_{1\ell}]
\end{aligned}$$

and

$$\|\hbar - \delta_*\|_{\Gamma_5} \le 2\|\hbar - \delta\|_{\Gamma_5}, \quad \Gamma_5 = [E_{11}, E_1] \times \cdots \times [E_{1\ell}, E_\ell],$$

adding these with (c), we get (d). □

Lemma 2.3. *Let* $\hbar$ *and* $\delta \in \Delta^2_{\Gamma_6}$, $\Gamma_6 = [E_1, F_{11}] \times \cdots \times [E_\ell, F_{1\ell}]$, $E_\xi < F_\xi < F_{1\xi}$, *and* $\delta'(F_1-, \ldots, F_\ell-) - \hbar[(F_1, \ldots, F_\ell), (F_{11}, \ldots, F_{1\ell})] > 0$. *Then*

$$\begin{aligned}&\left(\delta'(F_1-, \ldots, F_\ell-) - \hbar[(F_1, \ldots, F_\ell), (F_{11}, \ldots, F_{1\ell})]\right)\\ &\qquad \times (F_{11} - F_1) \ldots (F_{1\ell} - F_\ell) \le 2\|\hbar - \delta\|_{\Gamma_4}.\end{aligned}$$

Also if $\hbar$ *and* $\delta \in \Delta^2_{\widetilde{\Gamma}_6}, \widetilde{\Gamma}_6 = [E_{11}, F_1]\Big] \times \cdots \times [E_{1\ell}, F_\ell], E_{1\xi} < E_\xi < F_\xi$ *and* $\hbar[(E_{11}, \ldots, E_{1\ell}), (E_1, \ldots, E_\ell)] - \delta'(E_1+, \ldots, E_\ell+) > 0$, *then*

$$\begin{aligned}&\left(\hbar[(E_{11}, \ldots, E_{1\ell}), (E_1, \ldots, E_\ell)] - \delta'(E_1+, \ldots, E_\ell+)\right)\\ &\qquad (E_1 - E_{11}) \ldots (E_\ell - E_{1\ell}) \le 2\|\hbar - \delta\|_{\Gamma_5}.\end{aligned}$$

Proof. Let $\varkappa_{1\xi} = \sup\{\varkappa_\xi \in (F_\xi, F_{1\xi}) : \hbar'(\varkappa_\xi) \le \delta'(F_\xi-), \xi = 1, \ldots, \ell\}$. Then

$$\begin{aligned}&\left(\delta'(F_1-, \ldots, F_\ell-) - \hbar[(F_1, \ldots, F_\ell), (F_{11}, \ldots, F_{1\ell})]\right)(F_{11} - F_1) \ldots (F_{1\ell} - F_\ell)\\ &= \int_{F_1}^{F_{11}} \ldots \int_{F_\ell}^{F_{1\ell}} \left(\delta'(F_1-, \ldots, F_\ell-) - \hbar[(F_1, \ldots, F_\ell), (F_{11}, \ldots, F_{1\ell})]\right) d\varkappa_1 \ldots d\varkappa_\ell\\ &= \int_{F_1}^{F_{11}} \ldots \int_{F_\ell}^{F_{1\ell}} \left(\delta'(F_1-, \ldots, F_\ell-) - \hbar'(\varkappa_1, \ldots, \varkappa_\ell)\right) d\varkappa_1 \ldots d\varkappa_\ell\\ &\le \int_{F_1}^{\varkappa_{11}} \ldots \int_{F_\ell}^{\varkappa_{1\ell}} \left(\delta'(F_1-, \ldots, F_\ell-) - \hbar'(\varkappa_1, \ldots, \varkappa_\ell)\right) d\varkappa_1 \ldots d\varkappa_\ell.\end{aligned}$$

Since δ' is monotone $\hbar'(\varkappa_1, \ldots, \varkappa_\ell) \le \delta'(F_1-, \ldots, F_\ell-) \le \delta'(\varkappa_1, \ldots, \varkappa_\ell)$, $(\varkappa_1, \ldots, \varkappa_\ell) \in (F_1, \varkappa_{11}) \times \cdots \times (F_\ell, \varkappa_{1\ell})$, then

$$\begin{aligned}&\int_{F_1}^{\varkappa_{11}} \ldots \int_{F_\ell}^{\varkappa_{1\ell}} \left(\delta'(F_1-, \ldots, F_\ell-) - \hbar'(\varkappa_1, \ldots, \varkappa_\ell)\right) d\varkappa_1 \ldots d\varkappa_\ell\\ &\quad \le \int_{F_1}^{\varkappa_{11}} \ldots \int_{F_\ell}^{\varkappa_{1\ell}} \left(\delta'(\varkappa_1, \ldots, \varkappa_\ell) - \hbar'(\varkappa_1, \ldots, \varkappa_\ell)\right) d\varkappa_1 \ldots d\varkappa_\ell\\ &\quad = \delta(\varkappa_{11}, \ldots, \varkappa_{1\ell}) - \hbar(\varkappa_{11}, \ldots, \varkappa_{1\ell}) - (\delta(F_1, \ldots, F_\ell) - \hbar(F_1, \ldots, F_\ell))\\ &\quad \le 2\|\hbar - \delta\|_{\Gamma_4}.\end{aligned}$$

The proof for the second statement is similar. □

Lemma 2.4. *Suppose* $E_{1\xi} < E_\xi < F_\xi < F_{1\xi}$,

$$\psi = \max\left\{\frac{(F_1 - E_1) \ldots (F_\ell - E_\ell)}{(F_{11} - F_1) \ldots (F_{1\ell} - F_\ell)}; \frac{(F_1 - E_1) \ldots (F_\ell - E_\ell)}{(E_1 - E_{11}) \ldots (E_\ell - E_{1\ell})}\right\}$$

and $\hbar \in \Delta^2_{\Gamma_3}$*, and let* $\delta \in \Delta^2_{\Gamma_3}$ *be a piecewise polynomial of degree* $\leq d - 1$ *with knots* $E = (E_1, \ldots, E_\ell)$ *and* $F = (F_1, \ldots, F_\ell) \ni \hbar(E_1, \ldots, E_\ell) = \delta(E_1, \ldots, E_\ell), \hbar(F_1, \ldots, F_\ell) = \delta(F_1, \ldots, F_\ell)$*. Then there exists a piecewise polynomial* $\delta_* \in \Delta^2_\Gamma$ *of the same degree such that*

(a) $\delta'(E_1+, \ldots, E_\ell+) \leq \delta'_*(E_1+, \ldots, E_\ell+)$,
$\delta'_*(F_1-, \ldots, F_\ell-) \leq \delta'(F_1-, \ldots, F_\ell-)$,
(b) $\hbar[(E_1, \ldots, E_\ell), (E_{11}, \ldots, E_{1\ell})] = d_{E_1} \ldots d_{E_\ell} \leq \delta'_*(E_1+, \ldots, E_\ell+)$,
$\delta'_*(F_1-, \ldots, F_\ell-) \leq d_{F_1} \ldots d_{F_\ell} = \hbar[(F_1, \ldots, F_\ell), (F_{11}, \ldots, F_{1\ell})]$,
(c) $\delta_*(E_1, \ldots, E_\ell) = \hbar(E_1, \ldots, E_\ell)$,
$\delta_*(F_1, \ldots, F_\ell) = \hbar(F_1, \ldots, F_\ell)$,
(d) $\|\hbar - \delta_*\|_\Gamma \leq (2\psi + 1)\|\hbar - \delta\|_{\Gamma_3}$.

Proof. Suppose $\hbar(E_1, \ldots, E_\ell) = \hbar(F_1, \ldots, F_\ell)$ and if δ is a constant on Γ, put $\delta_*(\varkappa_1, \ldots, \varkappa_\ell) = \delta(\varkappa_1, \ldots, \varkappa_\ell), (\varkappa_1, \ldots, \varkappa_\ell) \in \Gamma$. Because convexity of δ and $\delta(F_1, \ldots, F_\ell) = \delta(E_1, \ldots, E_\ell)$, then $\delta'(F_1-, \ldots, F_\ell-) > 0 > \delta'(E_1+, \ldots, E_\ell+)$.

Let
$$\beta = \min\left\{\frac{d_{F_1} \ldots d_{F_\ell}}{\delta'(F_1-, \ldots, F_{\ell^-})}, \frac{d_{E_1} \ldots d_{E_\ell}}{\delta'(E_1+, \ldots, E_\ell+)}\right\} \geq 0.$$

If $\beta \geq 1$, then nothing needs to proven. If $\beta < 1$ and we can put $\beta = \frac{d_{F_1} \ldots d_{F_\ell}}{\delta'(F_1-, \ldots, F_{\ell}-)} < 1$. So let

$$\delta_*(\varkappa_1, \ldots, \varkappa_\ell) = \delta(E_1, \ldots, E_\ell) + \beta(\delta(\varkappa_1, \ldots, \varkappa_\ell) - \delta(E_1, \ldots, E_\ell))$$

such that the polynomial $\delta_* \in \Delta^2_\Gamma$ and $\delta_*(E_1, \ldots, E_\ell) = \hbar(E_1, \ldots, E_\ell) = \hbar(F_1, \ldots, F_\ell) = \delta_*(F_1, \ldots, F_\ell)$.

Because $\delta'(E_1+, \ldots, E_\ell+) < 0$ and $\frac{d_{E_1} \ldots d_{E_\ell}}{\delta'(E_1+, \ldots, E_\ell+)} \geq \frac{d_{F_1} \ldots d_{F_\ell}}{\delta'(F_1-, \ldots, F_\ell-)} > 0$, we get

$$\begin{aligned}
\delta'_*(E_1+, \ldots, E_\ell+) &= \beta\delta'(E_1+, \ldots, E_\ell+) \\
&= \frac{d_{F_1} \ldots d_{F_\ell}}{\delta'(F_1-, \ldots, F_\ell-)}\delta'(E_1+, \ldots, E_\ell+) \\
&\geq \frac{d_{E_1} \ldots d_{E_\ell}}{\delta'(E_1+, \ldots, E_\ell+)}\delta'(E_1+, \ldots, E_\ell+) \\
&= d_{E_1} \ldots d_{E_\ell}
\end{aligned}$$

and

$$\begin{aligned}
\delta'_*(F_1-, \ldots, F_\ell-) &= \beta\delta'(F_1-, \ldots, F_\ell-) \\
&= \frac{d_{F_1} \ldots d_{F_\ell}}{\delta'(F_1-, \ldots, F_{\ell^-})}\delta'(F_1-, \ldots, F_\ell-) \\
&= d_{F_1} \ldots d_{F_\ell}.
\end{aligned}$$

Suppose $\varkappa_{0\xi} = \sup\{\varkappa_\xi \in (E_\xi, F_\xi) : \delta'(\varkappa_\xi) \le 0, \xi = 1, \dots, \ell\}$. Because $0 = \delta(F_1, \dots, F_\ell) - \delta(E_1, \dots, E_\ell) = \int_{E_1}^{F_1} \dots \int_{E_\ell}^{F_\ell} \delta'(f_1, \dots, f_\ell)\, df_1 \dots df_\ell$, we get

$$\begin{aligned}\|\delta - \delta(E_1, \dots, E_\ell)\|_\Gamma &= \int_{\mathcal{H}_{01}}^{E_1} \dots \int_{\mathcal{H}_{0\ell}}^{E_\ell} \delta'(f_1, \dots, f_\ell)\, df_1 \dots df_\ell \\ &= \int_{\mathcal{H}_{01}}^{F_1} \dots \int_{\mathcal{H}_{0\ell}}^{F_\ell} \delta'(f_1, \dots, f_\ell)\, df_1 \dots df_\ell \\ &\le (F_1 - \varkappa_{01}) \dots (F_\ell - \varkappa_{0\ell})\, \delta'(F_1-, \dots, F_\ell-) \\ &\le (F_1 - E_1) \dots (F_\ell - E_\ell)\, \delta'(F_1-, \dots, F_\ell-).\end{aligned}$$

Using Lemma 2.3, we get

$$\begin{aligned}\|\delta - \delta_*\|_\Gamma &= \max|\delta(\varkappa_1, \dots, \varkappa_\ell) - \delta(E_1, \dots, E_\ell) - \beta(\delta(\varkappa_1, \dots, \varkappa_\ell) \\ &\qquad - \delta(E_1, \dots, E_\ell))| \\ &= (1 - \beta)\|\delta - \delta(E_1, \dots, E_\ell)\|_\Gamma \\ &\le (1 - \beta)(F_1 - E_1) \dots (F_\ell - E_\ell)\, \delta'(F_1-, \dots, F_\ell-) \\ &= \frac{\delta'(F_1-, \dots, F_\ell-) - d_{F_1} \dots d_{F_\ell}}{\delta'(F_1-, \dots, F_\ell-)}((F_1 - E_1) \dots (F_\ell - E_\ell)) \\ &\qquad \times \delta'(F_1-, \dots, F_\ell-) \\ &\le \left(\delta'(F_1-, \dots, F_\ell-) - d_{F_1} \dots d_{F_\ell}\right)((F_1 - E_1) \dots (F_\ell - E_\ell)) \\ &\le \psi\left(\delta'(F_1-, \dots, F_\ell-) - d_{F_1} \dots d_{F_\ell}\right)((F_{11} - F_1) \dots (F_{1\ell} - F_\ell)) \\ &\le 2\psi\|\hbar - \delta\|_{\Gamma_4}.\end{aligned}$$

Thus,

$$\|\hbar - \delta_*\|_\Gamma \le \|\hbar - \delta\|_\Gamma + \|\delta - \delta_*\|_\Gamma \le (2\psi + 1)\|\hbar - \delta\|_{\Gamma_3}.$$

□

Lemma 2.5. *Suppose that* $\hbar \in \Delta^2_{\Gamma_6}, E_\xi < F_\xi < F_{1\xi}$ *and* $\hat{\psi} = \frac{(F_1 - E_1) \dots (F_\ell - E_\ell)}{(F_{11} - F_1) \dots (F_{1\ell} - F_\ell)}$, *and let* $\delta \in \Delta^2_{\Gamma_6}$ *be a piecewise polynomial of degree* $\le d - 1$ *with knot* $F = (F_1, \dots, F_\ell) \ni \hbar(E_1, \dots, E_\ell) = \delta(E_1, \dots, E_\ell)$ *and* $\hbar(F_1, \dots, F_\ell) = \delta(F_1, \dots, F_\ell)$. *Then there exists a polynomial* $\delta_* \in \Delta^2_\Gamma$ *such that*

(a) $\delta'_*(F_1-, \dots, F_\ell-) \le \delta'(F_1-, \dots, F_\ell-)$,
(b) $\delta'_*(F_1-, \dots, F_\ell-) \le d_{F_1} \dots d_{F_\ell} = \hbar[(F_1, \dots, F_\ell), (F_{11}, \dots, F_{1\ell})]$,
(c) $\delta_*(E_1, \dots, E_\ell) = \hbar(E_1, \dots, E_\ell), \delta_*(F_1, \dots, F_\ell) = \hbar(F_1, \dots, F_\ell)$,
(d) $\|\hbar - \delta_*\|_\Gamma \le c(\widehat{\psi})\|\hbar - \delta\|_{\Gamma_6}, c(\widehat{\psi}) \le 2\widehat{\psi} + 1$.

Also, Suppose that $\hbar \in \Delta^2_{\Gamma_7}, \Gamma_7 = [E_{11}, F_1] \times \dots \times [E_{1\ell}, F_\ell], E_{1\xi} < E_\xi < F_\xi$ *and* $\hat{\psi} = \frac{(F_1 - E_1) \dots (F_\ell - E_\ell)}{(E_1 - E_{11}) \dots (E_\ell - E_{1\ell})}$, *and let* $\delta \in \Delta^2_{\Gamma_6}$ *be a piecewise polynomial of the same degree with knot* $E = (E_1, \dots, E_\ell)$ *such that* $\hbar(E_1, \dots, E_\ell) = \delta(E_1, \dots, E_\ell)$

and $\hbar(F_1,\ldots,F_\ell)=\delta(F_1,\ldots,F_\ell)$*. Then there exists a polynomial* $\delta_*\in\Delta^2_\Gamma$ *such that*

(a) $\delta'(E_1+,\ldots,E_\ell+)\le\delta'_*(E_1+,\ldots,E_\ell+)$,
(b) $\hbar[(E_1,\ldots,E_\ell),(E_{11},\ldots,E_{1\ell})]=d_{E_1}\ldots d_{E_\ell}\le\delta'_*(E_1+,\ldots,E_\ell+)$,
(c) $\delta_*(E_1,\ldots,E_\ell)=\hbar(E_1,\ldots,E_\ell),\quad\delta_*(F_1,\ldots,F_\ell)=\hbar(F_1,\ldots,F_\ell)$,
(d) $\|\hbar-\delta_*\|_\Gamma\le c(\widehat{\psi})\|\hbar-\delta\|_{\Gamma_7},\quad c(\widehat{\psi})\le 2\widehat{\psi}+1.$

Proof. It is enough to prove the first case since the second case is similar. The proof of this lemma is the same as the proof of Lemma 2.4, but here we take $\beta=\frac{d_{F_1}\ldots d_{F_\ell}}{\delta'(F_1-,\ldots,F_\ell-)}$. The same properties are at (c) and (d), but at (a) and (b) we only dealt with point $F=(F_1-,\ldots,F_\ell-)$. □

3. Proof of Theorem 1.1

Proof of Theorem 1.1: Let

$$\alpha_\tau(\varkappa_1,\ldots,\varkappa_\ell)=L\left((\varkappa_1,\ldots,\varkappa_\ell);\hbar;(\varkappa_{\tau1-1},\ldots,\varkappa_{\tau\ell-1}),(\varkappa_{\tau1},\ldots,\varkappa_{\tau\ell})\right),\quad \tau=0,\ldots,m+1.$$

The collection of all integer i is $D\subset\{1,\ldots,m\}$ such that

$$\delta'(\varkappa_{i1-1}+,\ldots,\varkappa_{i\ell-1}+)\le\alpha'_i\le\delta'(\varkappa_{i1}-,\ldots,\varkappa_{i\ell}-),\quad\forall i\notin D,$$

then,

$$\delta_*(\varkappa_1,\ldots,\varkappa_\ell)=\alpha_i(\varkappa_1,\ldots,\varkappa_\ell),$$
$$(\varkappa_1,\ldots,\varkappa_\ell)\in\Gamma_8=[\varkappa_{i1-1},\varkappa_{i1}]\times\ldots\times[\varkappa_{i\ell-1},\varkappa_{i\ell}],$$

using Lemma 2.1

$$\|\hbar-\delta_*\|_{\Gamma_8}\le 2\|\hbar-\delta\|_{\Gamma_8}.$$

To define δ_* on $\Gamma_8, i\in D$, we suppose $1<i<m$ and use $\Gamma_9=[\varkappa_{i1-2},\varkappa_{i1+1}]\times\ldots\times[\ \varkappa_{i\ell-2},\varkappa_{i\ell+1}]$, by Lemma 2.2 and Lemma 2.4 with $E_\xi=\varkappa_{i\xi-1}$ and $F_\xi=\varkappa_{i\xi}$, we get $\delta_*\in\Delta^2_{\Gamma_8}$ satisfies

$$\|\hbar-\delta_*\|_{\Gamma_8}\le 4(2\psi+1)\|\hbar-\delta\|_{\Gamma_9}\tag{3.1}$$

and

$$\hbar(\varkappa_{i1-1},\ldots,\varkappa_{i\ell-1})=\delta_*(\varkappa_{i1-1},\ldots,\varkappa_{i\ell-1}),\hbar(\varkappa_{i1},\ldots,\varkappa_{i\ell})=\delta_*(\varkappa_{i1},\ldots,\varkappa_{i\ell}).$$

Let $i=1$ or $i=m$ such that $i\in D$. To prove it, assume $1\in D$ and $m\in D$ is similar, such that $\delta'(E_1+,\ldots,E_\ell+)\le\hbar[(E_1,\ldots,E_\ell),(\varkappa_{11},\ldots,\varkappa_{1\ell})]\le\delta'(\varkappa_{11}-,\ldots,\varkappa_{1\ell}-)$. Using Lemma 2.2 we get a convex piecewise polynomial $\overline{\overline{\delta}}_*\in\Gamma_{10}=[E_1,\varkappa_{21}]\times\ldots\times[E_\ell,\varkappa_{2\ell}]$ that interpolating $\hbar$ at $E=(E_1,\ldots,E_\ell)$ and $\varkappa_1=(\varkappa_{11},\ldots,\varkappa_{1\ell})$ such that $\overline{\overline{\delta}}'_*(\varkappa_{11}-,\ldots,\varkappa_{1\ell}-)\le\delta'(\varkappa_{11}-,\ldots,\varkappa_{1\ell}-)$ and

$$\left\|\hbar-\overline{\overline{\delta}}_*\right\|_{\Gamma_{10}}\le 4\|\hbar-\delta\|_{\Gamma_{10}}.$$

Using Lemma 2.5 and we have $\delta_* \in \Gamma_{11} = [E_1, \varkappa_{11}] \times \ldots \times [E_\ell, \varkappa_{1\ell}]$ that interpolating $\hbar$ at $E = (E_1, \ldots, E_\ell)$ and $\varkappa_1 = (\varkappa_{11}, \ldots, \varkappa_{1\ell})$ such that

$$\delta_*'(\varkappa_{11}-, \ldots, \varkappa_{1\ell}-) \leq \overline{\overline{\delta}}_*'(\varkappa_{11}-, \ldots, \varkappa_{1\ell}-)$$

and

$$\|\hbar - \delta_*\|_{\Gamma_{11}} \leq 4(2\psi + 1)\|\hbar - \delta\|_{\Gamma_{10}}. \tag{3.2}$$

Now, to prove

$$\delta_*'(\varkappa_{i1}-, \ldots, \varkappa_{i\ell}-) \leq \delta_*'(\varkappa_{i1}+, \ldots, \varkappa_{i\ell}+), \quad i = 1, \ldots, m-1. \tag{3.3}$$

If $i, i+1 \notin D$, then $\delta_*'(\varkappa_{i1}-, \ldots, \varkappa_{i\ell}-) = \alpha_i'$ and $\delta_*'(\varkappa_{i1}+, \ldots, \varkappa_{i\ell}+) = \alpha_{i+1}'$ and $\alpha_i' \leq \alpha_{i+1}'$ satisfies convexity with $\hbar$.

If $i, i+1 \in D$, then using Lemma 2.2 and Lemma 2.4, we get

$$\begin{aligned}\delta_*'(\varkappa_{i1}-, \ldots, \varkappa_{i\ell}-) &\leq \delta'(\varkappa_{i1}-, \ldots, \varkappa_{i\ell}-)\\ &\leq \delta'(\varkappa_{i1}+, \ldots, \varkappa_{i\ell}+)\\ &\leq \delta_*'(\varkappa_{i1}+, \ldots, \varkappa_{i\ell}+).\end{aligned}$$

If $i \in D, i+1 \notin D$, then using Lemma 2.4, we get

$$\delta_*'(\varkappa_{i1}-, \ldots, \varkappa_{i\ell}-) \leq \alpha_i' = \delta_*'(\varkappa_{i1}+, \ldots, \varkappa_{i\ell}+)$$

and for$i \notin D, i+1 \in D$, it is similar.

Finally, δ_* is a convex piecewise polynomial such that $\delta_*(\varkappa_{i1}, \ldots, \varkappa_{i\ell}) = \hbar(\varkappa_{i1}, \ldots, \varkappa_{i\ell}), i = 0, \ldots, m$ and using Eqs. (3.1) and (3.2), we get

$$\|\hbar - \delta_*\|_{\Gamma_1} \leq 4(2\psi + 1)\|\hbar - \delta\|_{\Gamma_2}.$$

This is the proof of Theorem 1.1.

4. Conclusion

This paper includes many basic concepts and facts in the 3-monotone multi approximation by piecewise polynomials with prescribed knots. We show that any piecewise convex polynomial can be changed to be interpolatory as well, while keeping the degree of the uniform multi approximation for multivariate the same.

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