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NUMERICAL RADIUS INEQUALITIES FOR CERTAIN ACCRETIVE-DISSIPATIVE MATRICES

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Abstract. In this paper, we present interesting lower bounds inequalities for numerical radius involving accretive-dissipative matrices. We also present several useful results for two by two block matrices. An example to show sharpness of a proved theorem is also given.

1. Introduction

Let $\mathcal{M}_n(\mathbb{C})$ represents the algebra of size $n \times n$ matrices whose entries are in $\mathbb{C}$. A matrix $\mathcal{P} \in \mathcal{M}_n(\mathbb{C})$ is called positive semidefinite ($p.s.d.$) if $(\mathcal{P}x, x) \geq 0$ for all $x \in \mathbb{C}$. Moreover, a matrix $\mathcal{S}$ from this algebra is referred to as accretive-dissipative (acc-diss) if its Cartesian decomposition, given by $\mathcal{S} = \mathcal{S}_1 + i\mathcal{S}_2$, satisfies that both $\mathcal{S}_1$ and $\mathcal{S}_2$ are p.s.d. matrices. In this decomposition, the components are defined by $\mathcal{S}_1 = \mathrm{Re}(\mathcal{S}) = \frac{\mathcal{S}+\mathcal{S}^*}{2}$ and $\mathcal{S}_2 = \mathrm{Im}(\mathcal{S}) = \frac{\mathcal{S}-\mathcal{S}^*}{2i}$. For a matrix $\mathcal{S} \in \mathcal{M}_n(\mathbb{C})$, the numerical radius is given by

$$\omega(\mathcal{S}) = \max\left\{|(\mathcal{S}\xi, \xi)| : \xi \in \mathbb{C}^n, \|\xi\| = 1\right\}.$$

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The numerical radius is recognized as a norm on the space $\mathcal{M}_n(\mathbb{C})$. In fact, for every $\mathcal{S} \in \mathcal{M}_n(\mathbb{C})$, we have

$$\frac{1}{2}\|\mathcal{S}\| \leqslant \omega(\mathcal{S}) \leqslant \|\mathcal{S}\|. \tag{1.1}$$

The first inequality of (1.1) is sharp. It becomes an equality in the special case when $\mathcal{S}^2 = 0$, where $\|\mathcal{S}\|$ is the spectral norm of the matrix $\mathcal{S}$ which is defined as $\|\mathcal{S}\| = \max_{\|x\|=1} \|\mathcal{S}x\|$. The second inequality is also sharp and it becomes an equality when $\mathcal{S}$ is normal. Since every p.s.d is normal, for a p.s.d $\mathcal{S} \in \mathcal{M}_n(\mathbb{C})$, we must have

$$\omega(\mathcal{S}) = \|\mathcal{S}\|. \tag{1.2}$$

Also, we have

$$\|\mathcal{S}\| = \|\mathcal{S}^*\mathcal{S}\|^{1/2} = \|\mathcal{S}\mathcal{S}^*\|^{1/2}. \tag{1.3}$$

Two important inequalities are

$$\omega(\mathcal{S}) \geqslant \|\operatorname{Re}(\mathcal{S})\| \tag{1.4}$$

and

$$\omega(\mathcal{S}) \geqslant \|\operatorname{Im}(\mathcal{S})\|. \tag{1.5}$$

A notable characteristic of the numerical radius $\omega(\cdot)$ is its weak unitary invariance, which means that for a unitary matrix $\mathcal{U} \in \mathcal{M}_n(\mathbb{C})$ and any $\mathcal{S} \in \mathcal{M}_n(\mathbb{C})$, the equality

$$\omega(\mathcal{S}) = \omega\left(\mathcal{U}^*\mathcal{S}\mathcal{U}\right), \tag{1.6}$$

holds. Also,

$$\omega(\mathcal{S}) = \omega\left(\mathcal{S}^*\right). \tag{1.7}$$

The power inequality is also an important inequality. It states that

$$\omega^k(\mathcal{S}) \geqslant \omega\left(\mathcal{S}^k\right), \text{for every positive integer} \quad k. \tag{1.8}$$

Several inequalities involving numerical radius and matrix norm were investigated and studied in many books of inequalities such as Bhatia [2] and [3]. Moreover, several results concerning these concepts were established by El-Haddad and Kittaneh [5]. In [6] and [7], Hirzallah and others presented some inequalities for numerical radius of square block matrices of size two. Kittaneh also investigated many inequalities for numerical radius in [8]. Sakkijha, Al Dabbas and Yasin in [10] gave nice inequalities for numerical radius involving acc-diss matrices.

The theme of our work is to calculate numerical radius for power matrices and for two by two acc-diss matrices. Due to the difficulty in calculating it in numerous scientific applications, it is often enough to determine that the eigenvalues are confined to a specific region. So, we present new version for power inequality and we show its sharpness by a given example. Additionally, we present several useful results for two by two block matrices by examining the relationships between numerical radius and unitarily invariant norms.

2. Lemmas

To establish our targeted numerical radius inequalities, we need to revise the following lemmas.

Lemma 2.1. ([3]) *If $\mathcal{S}, \mathcal{T} \in \mathcal{M}_n(\mathbb{C})$ are p.s.d, then*

$$2\|\mathcal{S}\mathcal{T}\mathcal{S}\| \leqslant \left\|\mathcal{S}^2\mathcal{T} + \mathcal{T}\mathcal{S}^2\right\|.$$

Lemma 2.2. ([4]) *If $\mathcal{S}, \mathcal{T} \in \mathcal{M}_n(\mathbb{C})$ are p.s.d, then*

$$\|\mathcal{S} + \mathcal{T}\| \leq \sqrt{2}\|\mathcal{S} + i\mathcal{T}\|.$$

Lemma 2.3. ([9]) *If $\mathcal{S}, \mathcal{T} \in \mathcal{M}_n(\mathbb{C})$ are p.s.d, then*

$$\|\mathcal{S} - \mathcal{T}\| \geqslant \max(\|\mathcal{S}\|, \|\mathcal{T}\|) - \left\|\mathcal{S}^{1/2}\mathcal{T}^{1/2}\right\|.$$

Lemma 2.4. ([9]) *If $\mathcal{S}, \mathcal{T} \in \mathcal{M}_n(\mathbb{C})$ are p.s.d, then*

$$\|\mathcal{S} + \mathcal{T}\| \geqslant \max(\|\mathcal{S}\|, \|\mathcal{T}\|).$$

Lemma 2.5. ([1]) *If $\mathcal{S}, \mathcal{T}, \mathcal{V}, \mathcal{W} \in \mathcal{M}_n(\mathbb{C})$, then*

$$\left\|\begin{bmatrix} \mathcal{S} & \mathcal{V} \\ \mathcal{W} & \mathcal{T} \end{bmatrix}\right\| \geqslant \left\|\begin{bmatrix} \mathcal{S} & 0 \\ 0 & \mathcal{T} \end{bmatrix}\right\|$$

and

$$\left\|\begin{bmatrix} \mathcal{S} & \mathcal{V} \\ \mathcal{W} & \mathcal{T} \end{bmatrix}\right\| \geqslant \left\|\begin{bmatrix} 0 & \mathcal{V} \\ \mathcal{W} & 0 \end{bmatrix}\right\|.$$

3. Main results

This section is devoted to derive lower bounds for the numerical radius of accdiss matrices also for two by two operator matrix $\begin{bmatrix} \mathcal{S} & \mathcal{T} \\ \mathcal{T} & \mathcal{S} \end{bmatrix}$ involving accdiss matrices.

Theorem 3.1. *Let $\mathcal{S} \in \mathcal{M}_n(\mathbb{C})$ be acc-diss with Cartesian decomposition $\mathcal{S} = \mathcal{S}_1 + i\mathcal{S}_2$. Then*

$$\omega^2(S) \geq \max\left(2\omega\left(\mathcal{S}_1^{1/2}\mathcal{S}_2\mathcal{S}_1^{1/2}\right), \max\left(\omega^2(\mathcal{S}_1), \omega^2(\mathcal{S}_2)\right)\right).$$

Proof. By (1.8), (1.2) and Lemma 2.1,

$$\begin{aligned}
\omega^2(\mathcal{S}) &\geq \omega\left(\mathcal{S}^2\right) \\
&= \omega\left(\left(\mathcal{S}_1^2 - \mathcal{S}_2^2\right) + i(\mathcal{S}_1\mathcal{S}_2 + \mathcal{S}_2\mathcal{S}_1)\right) \\
&\geq \left\|Im\left(\mathcal{S}^2\right)\right\| \\
&= \|\mathcal{S}_1\mathcal{S}_2 + \mathcal{S}_2\mathcal{S}_1\| \\
&= \left\|\left(\mathcal{S}_1^{1/2}\right)\left(\mathcal{S}_1^{1/2}\right)\mathcal{S}_2 + \mathcal{S}_2\left(\mathcal{S}_1^{1/2}\right)\left(\mathcal{S}_1^{1/2}\right)\right\| \\
&= \left\|\left(\mathcal{S}_1^{1/2}\right)^2\mathcal{S}_2 + \mathcal{S}_2\left(\mathcal{S}_1^{1/2}\right)\right\| \\
&\geq 2\left\|\mathcal{S}_1^{1/2}\mathcal{S}_2\mathcal{S}_1^{1/2}\right\| \\
&= 2\omega\left(\mathcal{S}_1^{1/2}\mathcal{S}_2\mathcal{S}_1^{1/2}\right).
\end{aligned}$$

Now, from (1.4) and Lemma 2.3,

$$\begin{aligned}
\omega^2(\mathcal{S}) &\geq \omega\left(\mathcal{S}^2\right) \\
&\geqslant \left\|Re\left(\mathcal{S}^2\right)\right\| \\
&= \left\|\mathcal{S}_1^2 - \mathcal{S}_2^2\right\| \\
&\geqslant \max\left(\left\|\mathcal{S}_1^2\right\|, \left\|\mathcal{S}_2^2\right\|\right) - \|\mathcal{S}_1\mathcal{S}_2\| = \alpha.
\end{aligned}$$

Also, by (1.7), (1.8), (1.4) and Lemma 2.4,

$$\begin{aligned}
\omega^2(\mathcal{S}) &\geqslant \omega\left(\mathcal{S}^2\right) = \omega\left(\left(\mathcal{S}^2\right)^*\right) = \omega\left((\mathcal{S}^*)^2\right) \\
&= \omega\left(\left(\mathcal{S}_1^2 + \mathcal{S}_2^2\right) + i(-\mathcal{S}_1\mathcal{S}_2 - \mathcal{S}_2\mathcal{S}_1)\right) \\
&\geqslant \left\|\mathrm{Re}\left(\mathcal{S}^2\right)\right\| \\
&= \left\|\mathcal{S}_1^2 + \mathcal{S}_2^2\right\| \\
&\geqslant \max\left(\left\|\mathcal{S}_1^2\right\|, \left\|\mathcal{S}_2^2\right\|\right) = \beta.
\end{aligned}$$

Note that

$$\beta = \max\left(\left\|\mathcal{S}_1^2\right\|, \left\|\mathcal{S}_2^2\right\|\right) \geq \max\left(\left\|\mathcal{S}_1^2\right\|, \left\|\mathcal{S}_2^2\right\|\right) - \|\mathcal{S}_1\mathcal{S}_2\| = \alpha.$$

Thus

$$\begin{aligned}
\omega^2(\mathcal{S}) &\geqslant \max\left(\max\left(\left\|\mathcal{S}_1^2\right\|, \left\|\mathcal{S}_2^2\right\|\right), 2\|\mathcal{S}_1^{1/2}\mathcal{S}_2\mathcal{S}_1^{1/2}\|\right) \\
&= \max\left(\max\left(\omega^2(\mathcal{S}_1), \omega^2(\mathcal{S}_2)\right), 2\omega\left(\mathcal{S}_1^{1/2}\mathcal{S}_2\mathcal{S}_1^{1/2}\right)\right).
\end{aligned}$$

It can be seen that the inequality in Theorem 3.1 is sharp. This can be demonstrated by considering the acc-diss matrix

$$\mathcal{S} = \mathcal{S}_1 + i\mathcal{S}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + i \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1+i & 0 \\ 0 & 0 \end{bmatrix},$$

where $\mathcal{S}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\omega(\mathcal{S}_1) = \|\mathcal{S}_1\| = 1$, $\mathcal{S}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\omega(\mathcal{S}_2) = \|\mathcal{S}_2\| = 1$, $\mathcal{S}_1^{1/2}\mathcal{S}_2\mathcal{S}_1^{1/2} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $2\omega\left(\mathcal{S}_1^{1/2}\mathcal{S}_2\mathcal{S}_1^{1/2}\right) = 2$.

Now, $\omega(\mathcal{S}) = \|\mathcal{S}\| = \|1+i\| = \sqrt{2}$, since $\mathcal{S}$ is normal matrix and so

$$\omega^2(S) = 2.$$

Hence,

$$2 = \omega^2(\mathcal{S}) = \max\left(\max\left(\omega^2(\mathcal{S}_1), \omega^2(\mathcal{S}_2)\right), 2\omega\left(\mathcal{S}_1^{\frac{1}{2}}\mathcal{S}_2\mathcal{S}_1^{\frac{1}{2}}\right)\right) = 2.$$

□

Theorem 3.2. *Let $\mathcal{S} \in \mathcal{M}_n(\mathbb{C})$ be acc-diss with Cartesian decomposition $\mathcal{S} = \mathcal{S}_1 + i\mathcal{S}_2$. Then*

$$\omega\left(\mathcal{S}^*\mathcal{S} + \mathcal{S}\mathcal{S}^*\right) \geqslant \max\left(\omega^2(\mathcal{S}_1), \omega^2(\mathcal{S}_2)\right).$$

Proof. From (1.1) and Lemma 2.1,

$$\begin{aligned}
\omega\left(\mathcal{S}^*\mathcal{S} + \mathcal{S}\mathcal{S}^*\right) &= \omega((\mathcal{S}_1 - i\mathcal{S}_2)(\mathcal{S}_1 + i\mathcal{S}_2) + (\mathcal{S}_1 + i\mathcal{S}_2)(\mathcal{S}_1 - i\mathcal{S}_2)) \\
&= \omega\left(2\left(\mathcal{S}_1^2 + \mathcal{S}_2^2\right)\right) \\
&= 2\omega\left(\mathcal{S}_1^2 + \mathcal{S}_2^2\right) \\
&\geqslant 2 \times \frac{1}{2}\left\|\mathcal{S}_1^2 + \mathcal{S}_2^2\right\| \\
&= \left\|\mathcal{S}_1^2 + \mathcal{S}_2^2\right\| \\
&\geqslant \max\left(\left\|\mathcal{S}_1^2\right\| \left\|\mathcal{S}_2^2\right\|\right) \\
&= \max\left(\|\mathcal{S}_1\|^2, \|\mathcal{S}_2\|^2\right) \\
&= \max\left(\omega^2(\mathcal{S}_1), \omega^2(\mathcal{S}_2)\right).
\end{aligned}$$

□

Theorem 3.3. *Let $\mathcal{S}, \mathcal{T} \in \mathcal{M}_n(\mathbb{C})$ be acc-diss with Cartesian decomposition $\mathcal{S} = \mathcal{S}_1 + i\mathcal{S}_2$ and $\mathcal{T} = \mathcal{T}_1 + i\mathcal{T}_2$. Then*

$$\omega\left(\begin{bmatrix} \mathcal{S} & 0 \\ \mathcal{T} & 0 \end{bmatrix}\right) \geqslant \max\left(\omega(\mathcal{S}_1), \omega(\mathcal{S}_2), \frac{1}{2\sqrt{2}}\omega(\mathcal{T}_1 + \mathcal{T}_2)\right).$$

Proof. By (1.4) and Lemma 2.5,

$$\begin{aligned}
\omega\left(\begin{bmatrix} \mathcal{S} & 0 \\ \mathcal{T} & 0 \end{bmatrix}\right) &\geqslant \left\|\operatorname{Re}\left(\begin{bmatrix} \mathcal{S} & 0 \\ \mathcal{T} & 0 \end{bmatrix}\right)\right\| \\
&= \frac{1}{2}\left\|\begin{bmatrix} \mathcal{S} & 0 \\ \mathcal{T} & 0 \end{bmatrix} + \begin{bmatrix} \mathcal{S}^* & \mathcal{T}^* \\ 0 & 0 \end{bmatrix}\right\| \\
&= \frac{1}{2}\left\|\begin{bmatrix} \mathcal{S}+\mathcal{S}^* & \mathcal{T}^* \\ \mathcal{T} & 0 \end{bmatrix}\right\| \\
&= \frac{1}{2}\left\|\begin{bmatrix} 2\mathcal{S}_1 & \mathcal{T}^* \\ \mathcal{T} & 0 \end{bmatrix}\right\| \\
&\geq \frac{1}{2}\left\|\begin{bmatrix} 2\mathcal{S}_1 & 0 \\ 0 & 0 \end{bmatrix}\right\| \\
&= \frac{1}{2}\|2\mathcal{S}_1\| \\
&= \|\mathcal{S}_1\| \\
&= \omega(\mathcal{S}_1).
\end{aligned}$$

Now, by Lemma 2.5, Lemma 2.2 and (1.2),

$$\begin{aligned}
\omega\left(\begin{bmatrix} \mathcal{S} & 0 \\ \mathcal{T} & 0 \end{bmatrix}\right) &\geqslant \left\|\operatorname{Re}\left(\begin{bmatrix} \mathcal{S} & 0 \\ \mathcal{T} & 0 \end{bmatrix}\right)\right\| \\
&= \frac{1}{2}\left\|\begin{bmatrix} 2\mathcal{S}_1 & \mathcal{T}^* \\ \mathcal{T} & 0 \end{bmatrix}\right\| \\
&\geqslant \frac{1}{2}\left\|\begin{bmatrix} 0 & \mathcal{T}^* \\ \mathcal{T} & 0 \end{bmatrix}\right\| \\
&= \frac{1}{2}\max\left(\|\mathcal{T}\|, \|\mathcal{T}^*\|\right) \\
&= \frac{1}{2}\|\mathcal{T}\| \\
&= \frac{1}{2}\|\mathcal{T}_1 + i\mathcal{T}_2\| \\
&\geqslant \frac{1}{2\sqrt{2}}\|\mathcal{T}_1 + \mathcal{T}_2\| \\
&= \frac{1}{2\sqrt{2}}\omega(\mathcal{T}_1 + \mathcal{T}_2).
\end{aligned}$$

Also, from (1.5),

$$\begin{aligned}\omega\left(\begin{bmatrix}\mathcal{S} & 0\\ \mathcal{T} & 0\end{bmatrix}\right) &\geqslant \left\|\operatorname{Im}\left(\begin{bmatrix}\mathcal{S} & 0\\ \mathcal{T} & 0\end{bmatrix}\right)\right\|\\ &= \frac{1}{2}\left\|\begin{bmatrix}\mathcal{S} & 0\\ \mathcal{T} & 0\end{bmatrix} - \begin{bmatrix}\mathcal{S}^* & \mathcal{T}^*\\ 0 & 0\end{bmatrix}\right\|\\ &= \frac{1}{2}\left\|\begin{bmatrix}2i\mathcal{S}_2 & \mathcal{T}^*\\ \mathcal{T} & 0\end{bmatrix}\right\|\\ &\geq \frac{1}{2}\left\|\begin{bmatrix}2i\mathcal{S}_2 & 0\\ 0 & 0\end{bmatrix}\right\|\\ &= \|\mathcal{S}_2\| = \omega(\mathcal{S}_2).\end{aligned}$$

Thus the result is obvious. □

Corollary 3.4. *Let $\mathcal{S},\mathcal{T} \in \mathcal{M}_n(\mathbb{C})$ be acc-diss with Cartesian decomposition $\mathcal{S} = \mathcal{S}_1 + i\mathcal{S}_2$ and $\mathcal{T} = \mathcal{T}_1 + i\mathcal{T}_2$. Then*

$$\omega\left(\begin{bmatrix}\mathcal{S} & 0\\ -\mathcal{T} & 0\end{bmatrix}\right) \geq \max\left(\omega(\mathcal{S}_1), \omega(\mathcal{S}_2), \frac{1}{2\sqrt{2}}\omega(\mathcal{T}_1 + \mathcal{T}_2)\right).$$

Proof. Let $\mathcal{U}$ be the unitary matrix $\begin{bmatrix}I & 0\\ 0 & -I\end{bmatrix}$. Then

$$\omega\left(\begin{bmatrix}\mathcal{S} & 0\\ -\mathcal{T} & 0\end{bmatrix}\right) = \omega\left(\mathcal{U}^*\begin{bmatrix}\mathcal{S} & 0\\ \mathcal{T} & 0\end{bmatrix}U\right) = \omega\left(\begin{bmatrix}\mathcal{S} & 0\\ \mathcal{T} & 0\end{bmatrix}\right).$$

The result is now obvious using (1.6) and Theorem 3.3. □

Corollary 3.5. *Let $\mathcal{S},\mathcal{T} \in \mathcal{M}_n(\mathbb{C})$ be acc-diss with Cartesian decomposition $\mathcal{S} = \mathcal{S}_1 + i\mathcal{S}_2$ and $\mathcal{T} = \mathcal{T}_1 + i\mathcal{T}_2$. Then*

$$\omega\left(\begin{bmatrix}0 & \mathcal{T}\\ 0 & \mathcal{S}\end{bmatrix}\right) \geqslant \max\left(\omega(\mathcal{S}_1), \omega(\mathcal{S}_2), \frac{1}{2\sqrt{2}}\omega(\mathcal{T}_1 + \mathcal{T}_2)\right).$$

Proof. Let $\mathcal{U}$ be the unitary matrix $\begin{bmatrix}0 & I\\ I & 0\end{bmatrix}$. Then

$$\omega\left(\begin{bmatrix}0 & \mathcal{T}\\ 0 & \mathcal{S}\end{bmatrix}\right) = \omega\left(\mathcal{U}^*\begin{bmatrix}\mathcal{S} & 0\\ \mathcal{T} & 0\end{bmatrix}\mathcal{U}\right) = \omega\left(\begin{bmatrix}\mathcal{S} & 0\\ \mathcal{T} & 0\end{bmatrix}\right).$$

The result is then obvious using Theorem 3.3. □

Theorem 3.6. *Let $\mathcal{S} \in \mathcal{M}_n(\mathbb{C})$ be acc-diss with Cartesian decomposition $\mathcal{S} = \mathcal{S}_1 + i\mathcal{S}_2$. Then*

$$\omega\left(\begin{bmatrix}\mathcal{S} & 0\\ \mathcal{S}^* & 0\end{bmatrix}\right) \geq \frac{1}{\sqrt{2}}\sqrt{\max\left(\omega^2(\mathcal{S}_1), \omega^2(\mathcal{S}_2)\right)} = \alpha.$$

Proof. By (1.1), (1.3) and Lemma 2.4, we have

$$\begin{aligned}
\omega\left(\begin{bmatrix} \mathcal{S} & 0 \\ \mathcal{S}^* & 0 \end{bmatrix}\right) &\geq \frac{1}{2}\left\|\begin{bmatrix} \mathcal{S} & 0 \\ \mathcal{S}^* & 0 \end{bmatrix}\right\| \\
&= \frac{1}{2}\left\|\begin{bmatrix} \mathcal{S}^* & \mathcal{S} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \mathcal{S} & 0 \\ \mathcal{S}^* & 0 \end{bmatrix}\right\|^{1/2} \\
&= \frac{1}{2}\left\|\mathcal{S}^*\mathcal{S} + \mathcal{S}\mathcal{S}^*\right\|^{1/2} \\
&= \frac{1}{2}\left\|2\left(\mathcal{S}_1^2 + \mathcal{S}_2^2\right)\right\|^{1/2} \\
&= \frac{1}{\sqrt{2}}\left\|\mathcal{S}_1^2 + \mathcal{S}_2^2\right\|^{1/2} \\
&= \frac{1}{\sqrt{2}}\sqrt{\max\left(\left\|\mathcal{S}_1^2\right\|, \left\|\mathcal{S}_2^2\right\|\right)} \\
&= \frac{1}{\sqrt{2}}\sqrt{\max\left(\omega^2(\mathcal{S}_1), \omega^2(\mathcal{S}_2)\right)}.
\end{aligned}$$

□

Theorem 3.7. *Let $\mathcal{S} \in \mathcal{M}_n(\mathbb{C})$ be acc-diss with Cartesian decomposition $\mathcal{S} = \mathcal{S}_1 + i\mathcal{S}_2$. Then*

$$\omega\left(\begin{bmatrix} \mathcal{S} & 0 \\ \mathcal{S}^* & 0 \end{bmatrix}\right) \geqslant \max\left(\omega(\mathcal{S}_1), \omega(\mathcal{S}_2), \frac{1}{2\sqrt{2}}\omega(\mathcal{S}_1 + \mathcal{S}_2)\right) = \beta.$$

Proof. From (1.4),

$$\begin{aligned}
\omega\left(\begin{bmatrix} \mathcal{S} & 0 \\ \mathcal{S}^* & 0 \end{bmatrix}\right) &\geqslant \left\|Re\left(\begin{bmatrix} \mathcal{S} & 0 \\ \mathcal{S}^* & 0 \end{bmatrix}\right)\right\| \\
&= \frac{1}{2}\left\|\begin{bmatrix} \mathcal{S} + \mathcal{S}^* & \mathcal{S} \\ \mathcal{S}^* & 0 \end{bmatrix}\right\| \\
&= \frac{1}{2}\left\|\begin{bmatrix} 2\mathcal{S}_1 & \mathcal{S} \\ \mathcal{S}^* & 0 \end{bmatrix}\right\| \\
&\geqslant \frac{1}{2}\left\|\begin{bmatrix} 2\mathcal{S}_1 & 0 \\ 0 & 0 \end{bmatrix}\right\| \\
&= \|\mathcal{S}_1\| \\
&= \omega(\mathcal{S}_1).
\end{aligned}$$

Also, by Lemma 2.5 and Lemma 2.2, we have

$$\begin{aligned}
\omega\left(\begin{bmatrix} \mathcal{S} & 0 \\ \mathcal{S}^* & 0 \end{bmatrix}\right) &\geq \frac{1}{2}\left\|\begin{bmatrix} 2\mathcal{S}_1 & \mathcal{S} \\ \mathcal{S}^* & 0 \end{bmatrix}\right\| \\
&\geq \frac{1}{2}\left\|\begin{bmatrix} 0 & \mathcal{S} \\ \mathcal{S}^* & 0 \end{bmatrix}\right\| \\
&= \frac{1}{2}\max\left(\|\mathcal{S}^*\|, \|\mathcal{S}\|\right) \\
&= \frac{1}{2}\|\mathcal{S}\| = \frac{1}{2}\|\mathcal{S}_1 + i\mathcal{S}_2\| \\
&\geq \frac{1}{2\sqrt{2}}\|\mathcal{S}_1 + \mathcal{S}_2\|.
\end{aligned}$$

Next, by (1.5) and Lemma 2.5, we have

$$\begin{aligned}
\omega\left(\begin{bmatrix} \mathcal{S} & 0 \\ \mathcal{S}^* & 0 \end{bmatrix}\right) &\geq \left\|\operatorname{Im}\left(\begin{bmatrix} \mathcal{S} & 0 \\ \mathcal{S}^* & 0 \end{bmatrix}\right)\right\| \\
&= \frac{1}{2}\left\|\begin{bmatrix} 2i\mathcal{S}_2 & \mathcal{S} \\ \mathcal{S}^* & 0 \end{bmatrix}\right\| \\
&\geqslant \frac{1}{2}\left\|\begin{bmatrix} 2i\mathcal{S}_2 & 0 \\ 0 & 0 \end{bmatrix}\right\| \\
&= \frac{1}{2}\|2i\mathcal{S}_2\| \\
&= \|\mathcal{S}_2\| \\
&= \omega(\mathcal{S}_2).
\end{aligned}$$

Thus the result is obvious. By Using Theorem 3.6 and Theorem 3.7, we get that

$$\omega\left(\begin{bmatrix} \mathcal{S} & 0 \\ \mathcal{S} & 0 \end{bmatrix}\right) \geqslant \max(\alpha, \beta).$$

□

Theorem 3.8. *Let $\mathcal{S}, \mathcal{T} \in \mathcal{M}_n(\mathbb{C})$ be acc-diss with Cartesian decomposition $\mathcal{S} = \mathcal{S}_1 + i\mathcal{S}_2$ and $\mathcal{T} = \mathcal{T}_1 + i\mathcal{T}_2$. Then*

$$\omega\left(\begin{bmatrix} \mathcal{T} & \mathcal{S} \\ \mathcal{S} & \mathcal{T} \end{bmatrix}\right) \geqslant \max(\omega(\mathcal{S}_1), \omega(\mathcal{S}_2), \omega(\mathcal{T}_1), \omega(\mathcal{T}_2)).$$

Proof. From (1.4) and Lemma 2.5, we have

$$\begin{aligned}
\omega\left(\begin{bmatrix} \mathcal{T} & S \\ S & \mathcal{T}\mathcal{T} \end{bmatrix}\right) &\geq \left\|\mathrm{Re}\left(\begin{bmatrix} \mathcal{T} & S \\ S & T \end{bmatrix}\right)\right\| \\
&= \frac{1}{2}\left\|\begin{bmatrix} \mathcal{T} & S \\ S & \mathcal{T} \end{bmatrix} + \begin{bmatrix} \mathcal{T}^* & S^* \\ S^* & \mathcal{T}^* \end{bmatrix}\right\| \\
&= \frac{1}{2}\left\|\begin{bmatrix} \mathcal{T}+\mathcal{T}^* & S+S^* \\ S+S^* & \mathcal{T}+\mathcal{T}^* \end{bmatrix}\right\| \\
&= \frac{1}{2}\left\|\begin{bmatrix} 2\mathcal{S}_1 & 2\mathcal{T}_1 \\ 2\mathcal{T}_1 & 2\mathcal{S}_1 \end{bmatrix}\right\| \\
&= \left\|\begin{bmatrix} \mathcal{S}_1 & \mathcal{T}_1 \\ \mathcal{T}_1 & \mathcal{S}_1 \end{bmatrix}\right\| \\
&\geqslant \left\|\begin{bmatrix} \mathcal{S}_1 & 0 \\ 0 & \mathcal{S}_1 \end{bmatrix}\right\| \\
&= \|\mathcal{S}_1\| = \omega(\mathcal{S}_1).
\end{aligned}$$

Also, from Lemma 2.5, we obtain

$$\begin{aligned}
\omega\left(\begin{bmatrix} \mathcal{T} & S \\ S & \mathcal{T} \end{bmatrix}\right) &\geq \left\|\begin{bmatrix} \mathcal{S}_1 & \mathcal{T}_1 \\ \mathcal{T}_1 & \mathcal{S}_1 \end{bmatrix}\right\| \\
&\geq \left\|\begin{bmatrix} 0 & \mathcal{T}_1 \\ \mathcal{T}_1 & 0 \end{bmatrix}\right\| \\
&= \|\mathcal{T}_1\| = \omega(\mathcal{T}_1).
\end{aligned}$$

Next,

$$\begin{aligned}
\omega\left(\begin{bmatrix} \mathcal{T} & S \\ S & \mathcal{T} \end{bmatrix}\right) &\geqslant \left\|\mathrm{Im}\left(\begin{bmatrix} \mathcal{T} & S \\ S & \mathcal{T} \end{bmatrix}\right)\right\| \\
&= \frac{1}{2}\left\|\begin{bmatrix} \mathcal{T} & S \\ S & \mathcal{T} \end{bmatrix} - \begin{bmatrix} \mathcal{T}^* & S^* \\ S^* & \mathcal{T}^* \end{bmatrix}\right\| \\
&= \frac{1}{2}\left\|\begin{bmatrix} 2i\mathcal{S}_2 & 2i\mathcal{T}_2 \\ 2i\mathcal{T}_2 & 2i\mathcal{S}_2 \end{bmatrix}\right\| \\
&= \left\|\begin{bmatrix} \mathcal{S}_2 & \mathcal{T}_2 \\ \mathcal{T}_2 & \mathcal{S}_2 \end{bmatrix}\right\|.
\end{aligned}$$

Now, using the same argument, we conclude that

$$\omega\left(\begin{bmatrix} \mathcal{T} & \mathcal{S} \\ \mathcal{S} & \mathcal{T} \end{bmatrix}\right) \geqslant \omega(\mathcal{S}_2)$$

and

$$\omega\left(\begin{bmatrix} \mathcal{T} & \mathcal{S} \\ \mathcal{S} & \mathcal{T} \end{bmatrix}\right) \geqslant \omega(\mathcal{T}_2).$$

Thus, our result is obvious. $\square$

Corollary 3.9. *Let $\mathcal{S} \in \mathcal{M}_n(\mathbb{C})$ be acc-diss with Cartesian decomposition $\mathcal{S} = \mathcal{S}_1 + i\mathcal{S}_2$. Then*

$$\omega\left(\begin{bmatrix} \mathcal{S} & I \\ I & \mathcal{S} \end{bmatrix}\right) \geq \max(\omega(\mathcal{S}_1), \omega(\mathcal{S}_2), 1),$$

where I is the identity matrix.

Proof. Using the same procedure in the previous theorem,

$$\omega\left(\begin{bmatrix} \mathcal{S} & I \\ I & \mathcal{S} \end{bmatrix}\right) \geq \left\| Re\left(\begin{bmatrix} \mathcal{S} & I \\ I & \mathcal{S} \end{bmatrix}\right)\right\| \geq \omega(\mathcal{S}_1).$$

Also,

$$\omega\left(\begin{bmatrix} \mathcal{S} & I \\ I & \mathcal{S} \end{bmatrix}\right) \geq \left\| Re\left(\begin{bmatrix} \mathcal{S} & I \\ I & \mathcal{S} \end{bmatrix}\right)\right\| = \|I\| = 1.$$

Now,

$$\omega\left(\begin{bmatrix} \mathcal{S} & I \\ I & \mathcal{S} \end{bmatrix}\right) \geq \left\| Im\left(\begin{bmatrix} \mathcal{S} & I \\ I & \mathcal{S} \end{bmatrix}\right)\right\| \geq \omega(\mathcal{S}_2),$$

which completes the proof. □

4. Conclusion

In this work, we introduced inequalities giving lower bounds for numerical radius of two by two block matrices and for power of numerical radius using acc-diss matrices. Of course we benefited from the properties of numerical radius and from inequalities of p.s.d. matrices.

The plan in the future is to examine our inequalities for accretive matrices and normal matrices.

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