Nonlinear Functional Analysis and Applications
Vol. 31, No. 1 (2026), pp. 285-301
ISSN: 1229-1595(print), 2466-0973(online)
https://doi.org/10.22771/nfaa.2026.31.01.18
http://nfaa.kyungnam.ac.kr/journal-nfaa

FUZZY STABILITY OF THE M-DIMENSIONAL ADDITIVE FUNCTIONAL EQUATION

Ismaail Essalih[1] and Nordine Bounader[2]

[1]Department of Mathematics, Faculty of Sciences, Ibn Tofail University, Kénitra, Morocco
e-mail: ismaail.essalih@uit.ac.ma

[2]Department of Mathematics, Faculty of Sciences, Ibn Tofail University, Kénitra, Morocco
e-mail: n.bounader@live.fr

Abstract. In this article, we investigate the stability and hyperstability of the following additive functional equation:

$$f\left(\sum_{i=1}^{m} \xi_i\right) = \sum_{i=1}^{m} f(\xi_i)$$

within the framework of fuzzy normed vector spaces. Motivated by the concept of Hyers-Ulam stability and its generalizations, we adopt a fixed point alternative method to analyze the behavior of functions that approximately satisfy this equation. Using appropriate control functions, we derive sufficient conditions ensuring the existence and uniqueness of additive mappings that closely approximate the given function in a fuzzy sense. We also establish hyperstability results under natural asymptotic assumptions on the control functions. The results presented here extend and refine earlier stability studies of additive functional equations, by embedding them in the context of fuzzy analysis and non-classical norm structures. Several corollaries are provided to demonstrate the applicability of our main theorems.

1. Introduction

A fundamental question in the theory of functional equations, posed by Ulam in 1940, concerns the stability of group homomorphisms: whether a

[0]Received November 7, 2025. Revised December 9, 2025. Accepted December 13, 2025.
[0]2020 Mathematics Subject Classification: 39B22, 39B52, 39B82, 46S10, 47S10, 46S40.
[0]Keywords: Hyers-Ulam stability, fuzzy normed space, additive functional equation, fixed point theorem, mathematical operators, fuzzy control, hyperstability.
[0]Corresponding author: Ismaail Essalih(ismaail.essalih@uit.ac.ma).

function that approximately satisfies a functional equation must be close to an exact solution [26]. The first affirmative answer to this question was given by Hyers in 1941 [15], who proved the stability of the Cauchy functional equation in the context of Banach spaces. Subsequent generalizations were introduced by Aoki [3], Bourgin [7], and Rassias [25], leading to what is now referred to as Hyers–Ulam–Rassias stability.

Among the most studied equations in this context is the additive functional equation, defined as

$$f\left(\sum_{i=1}^{m} \xi_i\right) = \sum_{i=1}^{m} f(\xi_i)$$

for an integer $m \geq 2$. This identity generalizes the classical Cauchy equation and plays a central role in the study of additive and linear mappings. Its stability has been investigated in various frameworks, including normed vector spaces and topological groups [12, 14]. Recently, Bodaghi introduced the notion of d-dimensional multi-additive mappings in neutrosophic normed spaces, providing a genuine multi-variable generalization of the additive functional equation considered here [6]. From this viewpoint, our m-dimensional additive equation can be regarded as a basic building block for such multi-additive structures, but in the setting of fuzzy normed spaces rather than neutrosophic ones.

In recent decades, the theory of fuzzy normed spaces has provided new tools for analyzing approximate functional behavior under uncertainty. These spaces, introduced by Katsaras [17] and further developed by Bag and Samanta [4], generalize classical normed spaces by incorporating fuzziness into the definition of norm and convergence.

In the fuzzy setting, Mirmostafaee and Moslehian [21] established fuzzy analogues of the Hyers–Ulam–Rassias theorem for the Cauchy equation, providing several stability results for approximately additive mappings in fuzzy normed spaces. In a related direction, Mirmostafaee, Mirzavaziri, and Moslehian [20] proved the fuzzy stability of the Jensen functional equation, again within fuzzy normed spaces. Park [23] later investigated the fuzzy stability of the Cauchy–Jensen functional equation in fuzzy Banach algebras by means of fixed point methods. More recently, Park, Bodaghi, and Donganont [24] obtained fuzzy stability results for multi-additive mappings in fuzzy normed spaces, combining both the direct Hyers method and the fixed point alternative approach. Alsahli and Bodaghi [2] studied generalized quadratic functional equations and derived fuzzy difference results, which emphasize the role of control functions in various stability phenomena.

In this direction, an important reference is the paper by Alanazi et al., who introduced a finite-variable additive functional equation and established fuzzy HyersUlam and HyersUlamRassias type stability results in fuzzy normed spaces by combining direct estimates with a fixed point method.[1] In their setting, the perturbations are controlled through several specific families of inequalities involving scalar parameters and no hyperstability phenomena are considered. The present paper complements and extends these results in two main directions. First, we work with the same m-variable additive equation in the more general framework of fuzzy Banach spaces and allow abstract control functions $\varphi : \chi^m \to Z$ that may depend simultaneously on all variables and are treated in a unified way, thereby covering and generalizing the special cases studied in [1]. Second, we go beyond stability and derive hyperstability results under suitable asymptotic assumptions on φ, showing that every approximate solution must in fact be exactly additive in the fuzzy sense, a phenomenon which, to the best of our knowledge, has not been previously investigated for this multi-variable additive equation in the fuzzy normed/Banach setting.

In parallel, fixed point theory, especially the fixed point alternative introduced by Diaz and Margolis [13], has emerged as a powerful technique to investigate the stability of functional equations. Brzdęk and Ciepliński [8] applied this idea in non-Archimedean metric spaces to obtain stability results for a broad class of functional equations by means of a fixed point argument, while Cădariu and Radu [9] used a related fixed point approach to study the Jensen functional equation and other classical equations in Banach spaces. These works illustrate the effectiveness of fixed point methods in deriving Hyers–Ulam type stability results.

In this paper, we apply the fixed point alternative method to study the stability and hyperstability of an m-dimensional additive functional equation in fuzzy Banach spaces. We construct a generalized metric space of functions and establish sufficient conditions under which a function approximately satisfying the equation admits a unique additive mapping close to it. We also demonstrate that, under certain asymptotic conditions on the control function, the approximate solutions must coincide with the exact ones, thus proving hyperstability.

The main advantages of the present study can be summarized as follows. We treat an m-dimensional additive equation in the general setting of fuzzy Banach spaces, allowing control functions that depend simultaneously on all variables, and we show that our fixed point approach yields stability results of Hyers–Ulam–Rassias type with rather mild conditions on the control function φ, thereby unifying and extending several previous fuzzy and non-fuzzy stability results for additive and Jensen-type equations [2, 20, 21, 23, 24]. Moreover, we establish hyperstability results for both even and odd values of m, showing that, under natural asymptotic assumptions, every approximate solution must be exactly additive in the fuzzy sense.

From a technical viewpoint, the main difficulties arise from combining the m-dimensional structure of equation (3.1) with the fuzziness of the underlying norm. In particular, one has to construct a suitable generalized metric on a space of mappings that reflects the behavior of the fuzzy norm and of the control function φ, and then verify that the associated operator is strictly contractive in the sense of the Diaz–Margolis fixed point alternative. Moreover, in the hyperstability part we must exploit delicate asymptotic conditions on φ to show that every approximate solution actually coincides with an exact additive mapping, which requires nontrivial substitutions and limiting arguments in the fuzzy context.

This paper is organized as follows. In Section 2, we provide definitions and preliminaries on fuzzy normed spaces and generalized metrics. Section 3 contains our main stability results using a fixed point framework. In Section 4, we present hyperstability results under natural assumptions.

2. Preliminaries

The notion of a fuzzy normed vector space was first introduced by Katsaras [17] in order to provide a framework for a topological vector structure enriched with fuzziness.

Since then, various definitions of fuzzy norms have been proposed from different perspectives. For instance, Krishna and Sarma developed separation properties and structural aspects of fuzzy normed linear spaces, highlighting how fuzzy norms interact with topological separation in this context [18].

Park, on the other hand, investigated fuzzy stability phenomena for functional equations associated with inner product spaces, showing how fuzzy norms can be used to derive stability results for such equations [22]. Among these approaches, the framework introduced by Bag and Samanta [4], building on the work of Cheng and Mordeson [11], has gained significant attention. Their formulation aligns the fuzzy metric with the well-known Karmosil Michalek type [16] and includes a decomposition theorem expressing the fuzzy norm as a family of crisp norms. Furthermore, several structural properties of fuzzy normed spaces have been investigated in this setting [5].

Following [4], we recall below some essential definitions and properties related to fuzzy normed spaces that will be used throughout this paper.

Definition 2.1. ([4]) Let χ be a real vector space. A function $\mathcal{N} : \chi \times \mathbb{R} \to [0,1]$ is called a fuzzy norm on χ if, for all $\xi, \gamma \in \chi$ and all $\alpha, \beta \in \mathbb{R}$, the following conditions hold:

(1) $\mathcal{N}(\xi, \alpha) = 0$ for $\alpha \leq 0$;
(2) $\xi = 0$ if and only if $\mathcal{N}(\xi, \alpha) = 1$ for all $\alpha > 0$;
(3) $\mathcal{N}(a\xi, \alpha) = \mathcal{N}\left(\xi, \frac{\alpha}{|a|}\right)$ if $a \neq 0$;
(4) $\mathcal{N}(\xi + \gamma, \alpha + \beta) \geq \min\{\mathcal{N}(\xi, \alpha), \mathcal{N}(\gamma, \beta)\}$;
(5) $\mathcal{N}(\xi, \cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{\alpha\to\infty} \mathcal{N}(\xi, \alpha) = 1$;
(6) for $\xi \neq 0$, $\mathcal{N}(\xi, \cdot)$ is continuous on $\mathbb{R}$.

The pair $(\chi, \mathcal{N})$ is called a fuzzy normed vector space.

Example 2.2. Let $(\chi, \|\cdot\|)$ be a normed linear space and let $a > 0$ be a real number. Then the function $\mathcal{N} : \chi \times \mathbb{R} \to [0,1]$ is defined as:

$$\mathcal{N}(\xi, \beta) = \begin{cases} e^{-\frac{a\|\xi\|}{\beta}} , & \text{if } \beta > 0; \\ 0 , & \text{if } \beta \leq 0 \end{cases}$$

is a fuzzy norm on χ.

Definition 2.3. ([4]) Let $(\chi, \mathcal{N})$ be a fuzzy normed vector space. A sequence $\{\xi_n\}$ in χ is called a convergent sequence if there exists an element $\xi \in \chi$ such that $\lim_{n\to\infty} \mathcal{N}(\xi_n - \xi, \beta) = 1$ for all $\beta > 0$. In this case, ξ is called the limit of the sequence $\{\xi_n\}$ in χ, and we denote it by $\mathcal{N} - \lim_{n\to\infty} \xi_n = \xi$.

Definition 2.4. ([4]) Let $(\chi, \mathcal{N})$ be a fuzzy normed vector space. A sequence $\{\xi_n\}$ in χ is called a Cauchy sequence if for each $\mu > 0$ and each $\beta > 0$ there exists an $k_0 \in \mathbb{N}$ such that for all $n \geq k_0$ and all $m > 0$, we have $\mathcal{N}(\xi_{n+m} - \xi_n, \beta) > 1 - \mu$.

It is a well-established fact that every convergent sequence within a fuzzy normed vector space is also a Cauchy sequence. When every Cauchy sequence in this context converges, the fuzzy norm is characterized as complete, thereby designating the fuzzy normed vector space as a fuzzy Banach space.

We say that a mapping $\psi : \chi \to Y$ between fuzzy normed vector spaces χ and Y is continuous at a point $\xi_0 \in \chi$ if, for every sequence $(\xi_n)_{n\in\mathbb{N}}$ in χ converging to ξ_0 in the sense of Definition 2.3, the sequence $(\psi(\xi_n))_{n\in\mathbb{N}}$ converges to $\psi(\xi_0)$ in Y (again in the sense of Definition 2.3 applied to the fuzzy norm on Y). If ψ is continuous at every point $\xi_0 \in \chi$, then ψ is said to be continuous on χ.

In 2003, Cădariu and Radu [9] highlighted the critical importance of a fixed point alternative method in addressing the Hyers-Ulam stability problem. This approach was subsequently utilized to examine the Hyers-Ulam-Rassias stability of the Jensen functional equation and the additive Cauchy functional equation [10], by employing a general control function $\varphi(\xi, \gamma)$ with suitable properties.

Definition 2.5. ([4]) Let χ be a set. A function $d : \chi \times \chi \to [0, \infty]$ is called a generalized metric on χ if d satisfies the following conditions:

(*i*) $d(\xi, \gamma) = 0$ if and only if $\xi = \gamma$ for all $\xi, \gamma \in \chi$;
(*ii*) $d(\xi, \gamma) = d(\gamma, \xi)$ for all $\xi, \gamma \in \chi$;
(*iii*) $d(\xi, \eta) \le d(\xi, \gamma) + d(\gamma, \eta)$ for all $\xi, \gamma, \eta \in \chi$.

Theorem 2.6. ([13]) *Let (χ, d) be a complete generalized metric space and $\mathcal{T} : \chi \to \chi$ be a strictly contractive mapping with Lipschitz constant $\mathcal{L} < 1$. Then, for all $\xi \in \chi$, either $d\big(\mathcal{T}^n\xi, \mathcal{T}^{n+1}\xi\big) = \infty$ for all nonnegative integers n or there exists a positive integer n_0 such that*

(1) $d\big(\mathcal{T}^n\xi, \mathcal{T}^{n+1}\xi\big) < \infty$ *for all* $n \ge n_0$*;*
(2) *the sequence* $\big\{\mathcal{T}^n\xi\big\}$ *converges to a fixed point* γ^* *of* $\mathcal{T}$*;*
(3) γ^* *is the unique fixed point of* $\mathcal{T}$ *in the set* $\mathcal{M} := \big\{\gamma \in \chi \;\big|\; d\big(\mathcal{T}^{n_0}\xi, \gamma\big) < \infty\big\}$*;*
(4) $d\big(\gamma, \gamma^*\big) \le \dfrac{1}{1-\mathcal{L}} d\big(\gamma, \mathcal{T}\gamma\big)$ *for all* $\gamma \in \chi$.

The next section will be devoted to applying these concepts to derive fixed point stability results for the additive functional equation in fuzzy Banach spaces.

3. Stability analysis

The problem of stability for functional equations, particularly in the sense of Hyers and Ulam, has been a subject of extensive investigation. The central question asks whether a mapping that approximately satisfies a functional equation must necessarily be close to an exact solution of that equation. This question has led to various notions of stability, including the classical Hyers-Ulam stability and its generalizations by Rassias and Găvruţă.

In the context of fuzzy normed spaces, the notion of stability acquires additional significance due to the presence of uncertainty and approximate structures. One of the most effective tools in this setting is the fixed point alternative, originally introduced by Diaz and Margolis [13], which has proven to be a powerful method for establishing stability results in both crisp and fuzzy frameworks.

In this section, we apply the fixed point approach to study the stability of the additive functional equation:

$$f\left(\sum_{i=1}^{m}\xi_i\right)=\sum_{i=1}^{m}f(\xi_i) \tag{3.1}$$

in fuzzy Banach spaces.

Recently, Bodaghi introduced the notion of d-dimensional multi-additive mappings in neutrosophic normed spaces, which provides a genuine multi-variable extension of the additive functional equation considered here [6]. From this point of view, our m-dimensional additive equation can be regarded as a basic building block for such multi-additive structures, but in the setting of fuzzy normed spaces rather than neutrosophic ones.

By constructing a suitable operator on an appropriately defined space of functions and equipping it with a generalized metric, we derive sufficient conditions under which a mapping that approximately satisfies the above equation admits a unique additive mapping close to it. The results obtained extend classical stability results to a more general fuzzy setting. Throughout this section, using the fixed point alternative approach, we prove the Hyers-Ulam stability of (3.1) in fuzzy Banach spaces.

We assume that χ is a vector space, $(\mathcal{Y},\mathcal{N})$ is a fuzzy Banach space, and $(\mathcal{Z},\mathcal{N}')$ is a fuzzy normed space. In the following two theorems, we investigate the stability of Eq. (3.1).

Theorem 3.1. *Let $m\geq 2$ be an integer and let $\varphi:\chi^m\to\mathcal{Z}$ be a function such that there exists an $\mathcal{L}<1$ with*

$$\mathcal{N}'\Big(\varphi(\xi_1,\xi_2,\cdots,\xi_m),\alpha\Big)\geq\mathcal{N}'\left(m\mathcal{L}\varphi\left(\frac{\xi_1}{m},\frac{\xi_2}{m},\cdots,\frac{\xi_m}{m}\right),\alpha\right)$$

for all $\xi_1,\xi_2,\cdots,\xi_m\in\chi$ and all $\alpha>0$. Let $f:\chi\to\mathcal{Y}$ be a mapping such that

$$\mathcal{N}\left(f\left(\sum_{i=1}^{m}\xi_i\right)-\sum_{i=1}^{m}f(\xi_i),\alpha\right)\geq\mathcal{N}'\Big(\varphi(\xi_1,\xi_2,\cdots,\xi_m),\alpha\Big) \tag{3.2}$$

for all $\xi_1,\xi_2,\cdots,\xi_m\in\chi$ and all $\alpha>0$. Then the limit

$$\mathcal{A}(x):=\mathcal{N}-\lim_{n\longrightarrow\infty}\frac{f\left(m^n\xi\right)}{m^n}$$

exists for every $\xi\in\chi$ and defines a unique additive mapping $\mathcal{A}:\chi\to\mathcal{Y}$ such that

$$\mathcal{N}(f(\xi)-A(\xi),\alpha)\geq\mathcal{N}'\Big(\varphi(\xi,\cdots,\xi),m(1-\mathcal{L})\alpha\Big),\quad \xi\in\chi\quad and\quad \alpha>0. \tag{3.3}$$

Proof. For $i \in \{1, 2, \cdots, m\}$, putting $\xi_i = \xi$ in (3.2), we get

$$\mathcal{N}\left(\frac{1}{m}f(m\xi) - f(\xi), \frac{\alpha}{m}\right) \geq \mathcal{N}'\Big(\varphi(\xi, \cdots, \xi), \alpha\Big) \tag{3.4}$$

for all $\xi \in \chi$ and all $\alpha > 0$. Consider the set $\Omega := \{h : \chi \to \mathcal{Y}\}$ and we define the generalized metric d on Ω by

$$d(g,h) := \inf\left\{\eta \in [0,\infty) : \mathcal{N}\big(g(\xi) - h(\xi), \eta\alpha\big) \geq \mathcal{N}'\Big(\varphi(\xi, \cdots, \xi), \alpha\Big), \alpha > 0\right\},$$

where $\inf \phi = +\infty$. It is not hard to show that (Ω, d) is complete (see [19], proof of Lemma 2.1). For each $\xi \in \chi$, we define the linear mapping $\mathcal{T} : \Omega \to \Omega$ by $\mathcal{T} := \frac{1}{m}f(m\xi)$. Let $g, h \in \Omega$ be given such that $d(g,h) = \mu$. Therefore,

$$\mathcal{N}\big(g(\xi) - h(\xi), \mu\alpha\big) \geq \mathcal{N}'\big(\varphi(\xi, \cdots, \xi), \alpha\big)$$

for all $\xi \in \chi$ and all $\alpha > 0$. Hence, we have

$$\begin{aligned}\mathcal{N}\big(\mathcal{T}g(\xi) - \mathcal{T}h(\xi), \mathcal{L}\mu\alpha\big) &= \mathcal{N}\left(\frac{1}{m}g(m\xi) - \frac{1}{m}h(m\xi), \mathcal{L}\mu\alpha\right)\\ &= \mathcal{N}\big(g(m\xi) - h(m\xi), m\mathcal{L}\mu\alpha\big)\\ &\geq \mathcal{N}'\big(\varphi(m\xi, \cdots, m\xi), m\mathcal{L}\alpha\big)\\ &\geq \mathcal{N}'\big(m\mathcal{L}\varphi(\xi, \cdots, \xi), m\mathcal{L}\alpha\big)\\ &= \mathcal{N}'\big(\varphi(\xi, \cdots, \xi), \alpha\big)\end{aligned}$$

for all $\xi \in \chi$ and $\alpha > 0$. So, we deduce that $d(g,h) = \mu$ implies that $d\big(\mathcal{T}g, \mathcal{T}h\big) \leq \mathcal{L}\mu$, which means that $d\big(\mathcal{T}g, \mathcal{T}h\big) \leq \mathcal{L}d(g,h)$ for all $g, h \in \Omega$, that is, $\mathcal{T}$ is a strictly contractive operator. In view of (3.4), we get

$$d\big(\mathcal{T}f, f\big) \leq \frac{1}{m} < \infty.$$

According to Theorem 2.6, there exists a mapping $\mathcal{A} : \chi \to \mathcal{Y}$ that satisfies the following:

(1) The mapping $\mathcal{A}$ is a unique fixed point of $\mathcal{T}$ in the set $\mathcal{M} = \{h \in \Omega : d(g,h) < \infty\}$, that is, $(m\xi) = m\mathcal{A}(\xi)$ for all $\xi \in \chi$. This implies that there exists $\mu \in (0, \infty)$ such that

$$\mathcal{N}\big(f(\xi) - \mathcal{A}(\xi), \mu\alpha\big) \geq \mathcal{N}'\big(\varphi(\xi, \cdots, \xi), \alpha\big)$$

for all $\xi \in \chi$ and $\alpha > 0$.

(2) $d\big(\mathcal{T}^n f, \mathcal{A}\big) \longrightarrow 0$ as $n \longrightarrow \infty$, which implies that

$$\mathcal{N} - \lim_{n \longrightarrow \infty} \frac{1}{m^n} f\big(m^n\xi\big) = \mathcal{A}(\xi)$$

for all $\xi \in \chi$.

(3) For each $f \in \mathcal{M}$, we get

$$d(f, \mathcal{A}) \leq \frac{d(\mathcal{T}f, f)}{1 - \mathcal{L}},$$

which implies the inequality:

$$d(f, \mathcal{A}) \leq \frac{1}{m - m\mathcal{L}}.$$

Therefore, the inequality (3.3) holds. Furthermore, we have:

$$\begin{aligned}
&\mathcal{N}\left(\mathcal{A}\left(\sum_{i=1}^{m} \xi_i\right) - \sum_{i=1}^{m} \mathcal{A}(\xi_i), \alpha\right) \\
&\geq \mathcal{N} - \lim_{n\to\infty} \left(\frac{1}{m^n} f\left(\sum_{i=1}^{m} m^n \xi_i\right) - \frac{1}{m^n}\sum_{i=1}^{m} f\big(m^n \xi_i\big) \\
&= \mathcal{N} - \lim_{n\to\infty} \left(f\left(\sum_{i=1}^{m} m^n \xi_i\right) - \sum_{i=1}^{m} f\big(m^n \xi_i\big), m^n \alpha\right) \\
&\geq \mathcal{N}' - \lim_{n\to\infty} \big(\varphi(m^n\xi_1, \cdots, m^n\xi_m), m^n\alpha\big) \\
&\geq \mathcal{N}' - \lim_{n\to\infty} \big(m^n\mathcal{L}^n\varphi(\xi_1, \cdots, \xi_m), m^n\alpha\big) \to 1
\end{aligned}$$

for all $\xi \in \chi$ and all $\alpha > 0$. Thus,

$$\mathcal{N}\left(\mathcal{A}\left(\sum_{i=1}^{m} \xi_i\right) - \sum_{i=1}^{m} A(\xi_i), \alpha\right) = 1$$

for all $\xi_1, \cdots, \xi_m \in \chi$ and all $\alpha > 0$, which completes the proof. □

The following corollaries are direct consequences of Theorem 3.1.

Corollary 3.2. *Let $f : \chi \to \mathcal{Y}$ be a mapping such that*

$$\mathcal{N}\left(f\left(\sum_{i=1}^{m} \xi_i\right) - \sum_{i=1}^{m} f(\xi_i), \alpha\right) \geq e^{\left(\frac{-\theta\sum_{i=1}^{m}\|\xi_i\|^p}{\alpha}\right)}$$

for all $\xi_1, \xi_2, \cdots, \xi_m \in \chi$ and all $\alpha > 0$, where $0 < p < 2$ and $\theta \geq 0$ are two real numbers, and where $m \geq 2$ is an integer. Then, there exists a unique additive mapping $\mathcal{A} : \chi \to \mathcal{Y}$ such that

$$\mathcal{N}\big(f(\xi) - \mathcal{A}(\xi), \alpha\big) \geq e^{\left(\frac{-\theta\|\xi\|^p}{(1-2^{p-2})\alpha}\right)}$$

for all $\xi \in \chi$ and all $\alpha \in (0, \infty)$.

Proof. The proof follows from Theorem 3.1 by taking:

$$\mathcal{N}'\big(\varphi(\xi_1,\xi_2,\cdots,\xi_m),\alpha\big) := e^{\left(\dfrac{-\varphi(\xi_1,\xi_2,\cdots,\xi_m)}{\alpha}\right)}$$

and

$$\varphi\big(\xi_1,\xi_2,\cdots,\xi_m\big) := \theta\sum_{i=1}^{m}\|\xi_i\|^p$$

for all $\xi_1,\xi_2,\cdots,\xi_m \in \chi\setminus\{0\}$ and all $\alpha > 0$. Choosing $\mathcal{L} = 2^{p-2}$ in Theorem 3.1, we obtain the desired results. □

Corollary 3.3. *Let $f:\chi\to\mathcal{Y}$ be a mapping such that*

$$\mathcal{N}\left(f\left(\sum_{i=1}^{m}\xi_i\right)-\sum_{i=1}^{m}f(\xi_i),\alpha\right) \geq \frac{\alpha}{\alpha+\theta\prod_{i=1}^{m}\|\xi_i\|^p}$$

for all $\xi_1,\xi_2,\cdots,\xi_m \in \chi$ and all $\alpha > 0$, where $0 < p < \dfrac{2}{m}$ and $\theta \geq 0$ are two real numbers, and where $m \geq 2$ be an integer. Then, there exists a unique additive mapping $\mathcal{A}:\chi\to\mathcal{Y}$ such that

$$\mathcal{N}\big(f(\xi)-\mathcal{A}(\xi),\alpha\big) \geq \frac{m(1-2^{mp-2})\alpha}{m(1-2^{mp-2})\alpha+\theta\|\xi\|^{mp}}$$

for all $\xi\in\chi$ and $\alpha > 0$.

Proof. The proof follows from Theorem 3.1 by taking:

$$\mathcal{N}'\big(\varphi(\xi_1,\xi_2,\cdots,\xi_m),\alpha\big) := \frac{\alpha}{\alpha+\varphi(\xi_1,\xi_2,\cdots,\xi_m)}$$

and

$$\varphi\big(\xi_1,\xi_2,\cdots,\xi_m\big) := \theta\prod_{i=1}^{m}\|\xi_i\|^p$$

for all $\xi_1,\xi_2,\cdots,\xi_m \in \chi\setminus\{0\}$ and all $\alpha > 0$. Choosing $\mathcal{L} = 2^{mp-2}$, in Theorem 3.1, we obtain the expected results. □

Corollary 3.4. *Let $f:\chi\to\mathcal{Y}$ be an even mapping such that*

$$\mathcal{N}\left(f\left(\sum_{i=1}^{m}\xi_i\right)-\sum_{i=1}^{m}f(\xi_i),\alpha\right) \geq \frac{\alpha}{\alpha+\theta\prod_{i=1}^{m}\|\xi_i\|^{p_i}}$$

for all $\xi_1, \xi_2, \cdots, \xi_m \in \chi$ and all $\alpha > 0$, where $\sum_{i=1}^m p_i < 2$ and $\theta \geq 0$ are two real numbers, and where $m \geq 2$ is an integer. Then, there exists a unique additive mapping $\mathcal{A} : \chi \to \mathcal{Y}$ such that:

$$\mathcal{N}\big(f(\xi) - \mathcal{A}(\xi), \alpha\big) \geq \frac{m(1 - 2^{(\sum_{i=1}^m p_i)-2})\alpha}{m(1 - 2^{(\sum_{i=1}^m p_i)-2})\alpha + \theta\|\xi\|^{(\sum_{i=1}^m p_i)}}$$

for all $\xi \in \chi$ and $\alpha > 0$.

Proof. Take

$$\mathcal{N}'\big(\varphi(\xi_1, \xi_2, \cdots, \xi_m), \alpha\big) := \frac{\alpha}{\alpha + \varphi(\xi_1, \xi_2, \cdots, \xi_m)}$$

and

$$\varphi\big(\xi_1, \xi_2, \cdots, \xi_m\big) := \theta \prod_{i=1}^m \|\xi_i\|^{p_i}$$

for all $\xi_1, \xi_2, \cdots, \xi_m \in \chi \setminus \{0\}$ and all $\alpha > 0$ in Theorem 3.1. Choosing $\mathcal{L} = 2^{(\sum_{i=1}^m p_i)-2}$ we obtain the desired results. $\square$

4. Hyperstability results

While the previous section established the stability of the additive functional equation under approximate conditions, we now turn to a more stringent phenomenon: hyperstability. Unlike standard stability, which ensures the existence of a nearby exact solution, hyperstability asserts that every approximate solution must, under suitable conditions, coincide with an exact one. This stronger form of rigidity becomes particularly significant when the control function exhibits specific asymptotic behavior.

In this section, we derive sufficient conditions under which the additive functional equation admits only exact solutions, even when initially perturbed. The fixed point framework developed earlier remains central to our approach, now refined to capture the limiting behavior of control functions. Our main results demonstrate that in the presence of vanishing perturbations, the space of approximate solutions collapses onto the set of additive mappingsthereby confirming hyperstability in the fuzzy normed context.

The following theorem concerns the hyperstability of Eq. (3.1) for even values of m, specifically when $m = 2k+2$ with $k \in \mathbb{N}$. Under this assumption, the equation takes the following form:

$$f\left(\sum_{i=1}^{2k+2} \xi_i\right) = \sum_{i=1}^{2k+2} f(\xi_i). \tag{4.1}$$

Theorem 4.1. *Let $f : \chi \to \mathcal{Y}$ and $\varphi : \chi \to \mathcal{Z}$ be two mappings that satisfy*

$$\mathcal{N}\left(f\left(\sum_{i=1}^{2k+2} \xi_i\right) - \sum_{i=1}^{2k+2} f(\xi_i), \alpha\right) \geq \mathcal{N}'\Big(\varphi(\xi_1, \xi_2, \cdots, \xi_{2k+2}), \alpha\Big), \tag{4.2}$$

$$\begin{aligned}\mathcal{N}' - \lim_{n\to\infty} \varphi\Big(&-n\xi, (-n-1)\xi, (-n+1)\xi, \cdots, (-n-k)\xi,\\ &(-n+k)\xi, ((2k+1)n+1)\xi\Big) = 1,\end{aligned} \tag{4.3}$$

$$\mathcal{N}' - \lim_{n\to\infty} \varphi\big(n\xi_1, n\xi_2, \cdots, n\xi_{2k+2}\big) = 1$$

and

$$\mathcal{N}' - \lim_{n\to\infty} \varphi\big(-n\xi_1, -n\xi_2, \cdots, -n\xi_{2k+2}\big) = 1$$

for all $\xi_1, \xi_2, \cdots, \xi_{2k+2} \in \chi$ and all $\alpha > 0$. Then f satisfies Eq. (4.1).

Proof. First, let $2 \leq i \leq 2k$ be even, and let $1 \leq j \leq 2k+1$ be odd. Replacing ξ_i with $\left(-n - \frac{i}{2}\right)\xi$, ξ_j with $\left(-n + \frac{j-1}{2}\right)\xi$, and ξ_{2k+2} with $\big((2k+1)n+1\big)\xi$ in (4.2), we obtain

$$\begin{aligned}\mathcal{N}\Bigg(&f(\xi) - f\bigg(\big((2k+1)n+1\big)\xi\bigg) - f(-n\xi)\\ &- \sum_{\ell=1}^{k} f\big((-n-\ell)\xi\big) - \sum_{\ell=1}^{k} f\big((-n+\ell)\xi\big), \alpha\Bigg)\\ \geq \mathcal{N}'\Bigg(&\varphi\bigg(-n\xi, (-n-1)\xi, (-n+1)\xi, \cdots, (-n-k)\xi,\\ &(-n+k)\xi, \big((2k+1)n+1\big)\xi\bigg), \alpha\Bigg)\end{aligned}$$

for all $\xi \in \chi$, all $n \in \mathbb{N}$, and all $\alpha > 0$. In view of (4.3), we deduce that

$$\begin{aligned}f(\xi) = \mathcal{N} - \lim_{n\to\infty} \Bigg\{&f(-n\xi) + f\bigg(\big((2k+1)n+1\big)\xi\bigg)\\ &+ \sum_{\ell=1}^{k} f\big((-n-\ell)\xi\big) + \sum_{\ell=1}^{k} f\big((-n+\ell)\xi\big)\Bigg\}\end{aligned}$$

for all $\xi \in \chi$. Therefore,

$$f(\xi_i) = \mathcal{N} - \lim_{n\to\infty} \left\{ f(-n\xi_i) + f\Big(\big((2k+1)n+1\big)\xi_i\Big) + \sum_{\ell=1}^{k} f\big((-n-\ell)\xi_i\big) + \sum_{\ell=1}^{k} f\big((-n+\ell)\xi_i\big) \right\}$$

for $i = 1, 2, 3, \cdots, (2k+2)$, and

$$f\left(\sum_{i=1}^{2k+2} \xi_i\right) = \mathcal{N} - \lim_{n\to\infty} \left\{ f\left(-n\sum_{i=1}^{2k+2} \xi_i\right) + f\left(\big((2k+1)n+1\big)\sum_{i=1}^{2k+2} \xi_i\right) + \sum_{\ell=1}^{k} f\left((-n-\ell)\sum_{i=1}^{2k+2} \xi_i\right) + \sum_{\ell=1}^{k} f\left((-n+\ell)\sum_{i=1}^{2k+2} \xi_i\right) \right\}$$

for all $\xi_1, \xi_2, \cdots, \xi_{2k+2} \in \chi$.

Now, we come back to (4.2) and we acquire:

$$\begin{aligned}
&\mathcal{N}\left(f\left(\sum_{i=1}^{2k+2} \xi_i\right) - \sum_{i=1}^{2k+2} f(\xi_i), \alpha\right) \\
&= \mathcal{N}\Bigg(\lim_{n\to\infty} \left\{ f\left(-n\sum_{i=1}^{2k+2} \xi_i\right) + f\left(\big((2k+1)n+1\big)\sum_{i=1}^{2k+2} \xi_i\right) \right. \\
&\quad + \sum_{\ell=1}^{k} f\left((-n-\ell)\sum_{i=1}^{2k+2} \xi_i\right) + \sum_{\ell=1}^{k} f\left((-n+\ell)\sum_{i=1}^{2k+2} \xi_i\right) - \sum_{i=1}^{2k+2} f(-n\xi_i) \\
&\quad - \sum_{i=1}^{2k+2} f\Big(\big((2k+1)n+1\big)\xi_i\Big) - \sum_{i=1}^{2k+2}\sum_{\ell=1}^{k} f\big((-n-\ell)\xi_i\big) \\
&\quad - \sum_{i=1}^{2k+2}\sum_{\ell=1}^{k} f\big((-n+\ell)\xi_i\big), \alpha\Bigg)
\end{aligned}$$

$$\geq \lim_{n\to\infty} \min_{1\leq \ell\leq k} \Bigg\{\mathcal{N}\left(f\left((-n-\ell)\sum_{i=1}^{2k+2}\xi_i\right)-\sum_{i=1}^{2k+2} f\big((-n-\ell)\xi_i\big),\alpha_\ell\right),$$
$$\mathcal{N}\left(f\left((-n+\ell)\sum_{i=1}^{2k+2}\xi_i\right)-\sum_{i=1}^{2k+2} f\big((-n+\ell)\xi_i\big),\alpha_\ell\right),$$
$$\mathcal{N}\left(f\left(-n\sum_{i=1}^{2k+2}\xi_i\right)-\sum_{i=1}^{2k+2} f(-n\xi_i),\alpha_{2k+1}\right),$$
$$\mathcal{N}\left(f\left(\big((2k+1)n+1\big)\sum_{i=1}^{2k+2}\xi_i\right)-\sum_{i=1}^{2k+2} f\Big(\big((2k+1)n+1\big)\xi_i\Big),\alpha_{2k+2}\right)\Bigg\}$$
$$\geq \lim_{n\to\infty} \min_{1\leq \ell\leq k} \Bigg\{\mathcal{N}'\left(\varphi\big((-n-\ell)\xi_1,\cdots,(-n-\ell)\xi_{2k+2}\big),\alpha_\ell\right),$$
$$\mathcal{N}'\left(\varphi\big((-n+\ell)\xi_1,\cdots,(-n+\ell)\xi_{2k+2}\big),\alpha_\ell\right),$$
$$\mathcal{N}'\left(\varphi\big(-n\xi_1,\cdots,-n\xi_{2k+2}\big),\alpha_{2k+1}\right),$$
$$\mathcal{N}'\left(\varphi\Big(\big((2k+1)n+1\big)\xi_1,\cdots,\big((2k+1)n+1\big)\xi_{2k+2}\Big),\alpha_{2k+2}\right)\Bigg\}$$
$$=1$$

for all $\xi_1,\xi_2,\cdots,\xi_{2k+2}\in\chi$ and all $\alpha,\alpha_1,\cdots,\alpha_{2k+2}\in(0,\infty)$ such that $\alpha=\sum_{i=1}^{2k+2}\alpha_i>0$. This is yields that f satisfies Eq. (4.1). □

By the same method, we can prove the following theorem, which examines the stability of Eq. (4.1) in the case where m is an odd integer. Specifically when $m=2k+1$ with $k\in\mathbb{N}$. Under this assumption, the equation takes the following form:

$$f\left(\sum_{i=1}^{2k+1}\xi_i\right)=\sum_{i=1}^{2k+1} f(\xi_i).$$

Theorem 4.2. *Let $f:\chi\to\mathcal{Y}$ and $\varphi:\chi\to\mathcal{Z}$ be two mappings that satisfy*

$$\mathcal{N}\left(f\left(\sum_{i=1}^{2k+1}\xi_i\right)-\sum_{i=1}^{2k+1} f(\xi_i),\alpha\right)\geq \mathcal{N}'\Big(\varphi(\xi_1,\xi_2,\cdots,\xi_{2k+1}),\alpha\Big), \tag{4.4}$$

$$\mathcal{N}'-\lim_{n\to\infty}\varphi\Big(-n\xi,(-n-1)\xi,(-n+1)\xi,\cdots,(-n-k)\xi,$$
$$(-n+k)\xi,(2kn+1)\xi\Big)=1,$$

$$\mathcal{N}' - \lim_{n\to\infty} \varphi\big(n\xi_1, n\xi_2, \cdots, n\xi_{2k+1}\big) = 1$$

and

$$\mathcal{N}' - \lim_{n\to\infty} \varphi\big(-n\xi_1, -n\xi_2, \cdots, -n\xi_{2k+1}\big) = 1$$

for all $\xi_1, \xi_2, \cdots, \xi_{2k+1} \in \chi$ *and all* $\alpha > 0$. *Then* f *satisfies Eq.* (4.1).

Proof. Let $2 \le i \le 2k$ be even, and let $3 \le j \le 2k-1$ be odd. Replacing ξ_i with $\left(-n - \dfrac{i}{2}\right)\xi$, ξ_j with $\left(-n + \dfrac{j+1}{2}\right)\xi$, ξ_1 with $-n\xi$, and ξ_{2k+1} with $\big(2kn+1\big)\xi$ in (4.4), we obtain

$$\begin{aligned}\mathcal{N}\Bigg(f(\xi) - f\Big(\big(2kn+1\big)\xi\Big) - f(-n\xi)&\\ - \sum_{\ell=1}^{k} f\big((-n-\ell)\xi\big) - \sum_{\ell=1}^{k} f\big((-n+\ell)\xi\big), \alpha\Bigg)&\\ \ge \mathcal{N}'\Bigg(\varphi\Big(-n\xi, (-n-1)\xi, (-n+1)\xi, \cdots, (-n-k)\xi,&\\ (-n+k)\xi, \big(2kn+1\big)\xi\Big), \alpha\Bigg)&\end{aligned}$$

for all $\xi \in \chi$, all $n \in \mathbb{N}$, and all $\alpha > 0$. The remainder of the proof is analogous to that of Theorem 4.1. □

Corollary 4.3. *Let* $\theta \ge 0$ *and* $p_1, p_2, \cdots, p_m$ *be real numbers such that*

$$\sum_{i=1}^{m} p_i < 0$$

and assume that $f : \chi \to \mathcal{Y}$ *is a mapping satisfying*

$$\mathcal{N}\left(f\left(\sum_{i=1}^{m} \xi_i\right) - \sum_{i=1}^{m} f(\xi_i), \alpha\right) \ge \frac{t}{t + \theta \prod_{i=1}^{m} \|x_i\|^{p_i}}$$

for all $\xi_1, \xi_2, \cdots, \xi_m \in \chi$ *and all* $\alpha > 0$. *Then the mapping* f *satisfies Eq.* (3.1).

Proof. According to the parity of m, we can define $\varphi : \chi^m \to \mathcal{Y}$ in Theorems 4.1 or 4.2 as $\varphi(\xi_1, \xi_2, \cdots, \xi_m) := \theta \prod_{i=1}^{m} \|\xi_i\|^{p_i}$, for all $\xi_1, \xi_2, \cdots, \xi_m \in \chi$. It follows that all the conditions are fulfilled, implying that the mapping f satisfies Eq. (3.1). □

5. Conclusion

This paper established the fuzzy stability of a m-dimensional additive functional equation in the sense of Găvruţă and examined its stability under Rassias-type conditions. It also showed that, under suitable assumptions, the equation is hyperstable.

References

[1] A.M. Alanazi, G. Muhiuddin, K. Tamilvanan, E.N. Alenze, A. Ebaid and K. Loganathan, *Fuzzy Stability Results of Finite Variable Additive Functional Equation: Direct and Fixed Point Methods,* Mathematics, **8**(7) (2020), 1050. DOI: 10.3390/math8071050.

[2] G. Alsahli and A. Bodaghi, *A System of Generalized Quadratic Functional Equations and Fuzzy Difference Results,* Symmetry, **17** (2025), 866

[3] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan, **2** (1950), 64–66.

[4] T. Bag and S.K. Samanta, *Finite dimensional fuzzy normed linear spaces*, J. Fuzzy Math., **11** (2003), 687–705.

[5] T. Bag and S.K. Samanta, *Fuzzy bounded linear operators*, Fuzzy Sets Syst., **151** (2005), 513–547.

[6] A. Bodaghi, *Almost d–dimensional multi–additive mappings in neutrosophic normed spaces,* Commun. Korean Math. Soc., **40**(3) (2025), 673–685.

[7] D.G. Bourgin, *Classes of transformations and bordering transformations*, Bull. Amer. Math. Soc., **57** (1951), 223–237.

[8] J. Brzdęk and K. Ciepliński, *A fixed point approach to the stability of functional equations in non-Archimedean metric spaces*, Nonlinear Anal., **74**(18) (2011), 6861–6867.

[9] L. Cădariu and V. Radu, *Fixed points and the stability of Jensens functional equation*, J. Inequ. Pure Appl. Math., **4**(1) (2003), Article 4.

[10] L. Cădariu and V. Radu, *On the stability of the Cauchy functional equation: a fixed point approach*, Grazer Math. Berichte, **346** (2004), 43–52.

[11] S.C. Cheng and J.N. Mordeson, *Fuzzy linear operators and fuzzy normed linear spaces*, Bull. Calcutta Math. Soc., **86** (1994), 429–436.

[12] P.W. Cholewa, *Remarks on the stability of functional equations*, Aequationes Math., **27** (1984), 76–86.

[13] J. Diaz and B. Margolis, *A fixed point theorem of the alternative for contractions on a generalized complete metric space*, Bull. Amer. Math. Soc., **74** (1968), 305–309, doi:10.1090/S0002-9904-1968-11933-0

[14] P. Găvruţă, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl., **184** (1994), 431–436.

[15] D.H. Hyers, *On the stability of the linear functional equation*, Proc. National Acad. Sci. USA., **27** (1941), 222–224.

[16] I. Karmosil and J. Michalek, *Fuzzy metric and statistical metric spaces*, Kybernetica, **11** (1975), 326–334.

[17] A.K. Katsaras, *Fuzzy topological vector spaces*, Fuzzy Sets Syst., **12** (1984), 143–154.

[18] S.V. Krishna and K.K.M. Sarma, *Separation of fuzzy normed linear spaces*, Fuzzy Set Sys., **63** (1994), 207–217.

[19] D. Miheţ and V. Radu, *On the stability of the additive Cauchy functional equation in random normed spaces*, J. Math. Anal. Appl., **343** (2008), 567–572.
[20] M. Mirmostafaee, M. Mirzavaziri and M.S. Moslehian, *Fuzzy stability of the Jensen functional equation,* Fuzzy Sets Syst. **159** (2008), 730–738.
[21] A. Mirmostafaee and M. Moslehian, *Fuzzy versions of HyersUlamRassias theorem,* Fuzzy Sets Syst., **159** (2008), 720–729.
[22] C. Park, *Fuzzy stability of a functional equation associated with inner product spaces*, Fuzzy Sets Syst., **160** (2009), 1632–1642.
[23] C. Park, *Stability of the CauchyJensen functional equation in fuzzy Banach algebras,* Appl. Math. Lett., **24** (2011), 2024–2029.
[24] C. Park, A. Bodaghi, and M. Donganont, *Fuzzy stability of multi-additive mappings,* Demonstratio Math., **58** (2025), Art. 20250159.
[25] Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc., **72**(1978), 297–300.
[26] S.M. Ulam, *Problems in Modern Mathematics*, Science Editions, Wiley, New York, 1964.