**Nonlinear Functional Analysis and Applications**
**Vol. 31, No. 1 (2026), pp. 303-316**
**ISSN: 1229-1595(print), 2466-0973(online)**

https://doi.org/10.22771/nfaa.2026.31.01.19
http://nfaa.kyungnam.ac.kr/journal-nfaa

# FUZZY ONE-POINT COMPACTIFICATION IN FUZZY SEMI-SPECTRAL SPACES

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**Abstract.** With an emphasis on fuzzy one-point compactification, the results of this work expand on the findings of [1] by offering a number of distinct results pertaining that Fuzzy Quasi-Spectral spaces are precisely those fuzzy topological spaces that become Fuzzy Spectral after fuzzy one-point compactification.

## 1. Introduction

The aim of this section is to highlight the study conducted on fuzzy quasi spectral and fuzzy semi spectral in relation with one-point compactification in fuzzy setting. Hochster [7] pioneered the concepts of spectral spaces and

[0]Received November 8, 2025. Revised December 11, 2025. Accepted December 13, 2025.
[0]2020 Mathematics Subject Classification: 06F30, 11F23, 11N45, 14F45.
[0]Keywords: Fuzzy irreducible closed sets, fuzzy generic sets, fuzzy sober spaces, fuzzy spectral spaces, fuzzy quasi-spectral spaces.
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spectral sets which are essential to domain theory, a field merges order and topology for theoretical computation applications. These spaces and sets also emerge organically in other mathematical research areas. Belaid, Echi and Gargouri [2] introduced the concept of A-spectral spaces. In their work [3], Bouacida, Echi and Salhi established a link between spectral spaces and foliation theory. Additionally, Hochster [7] introduced and examined quasi-spectral spaces to shed light on a conjecture about spectral sets. Echi and Yip [12] introduced quasi-homeomorphisms in spectral spaces. Further in [4], the study of quasi homeomorphisms, the Goldman prime spectrum and the Jacobson prime spectrum of a commutative ring was presented. Echi [5] dealt with the interactions between up-spectral spaces and A-spectral spaces.

Drawing inspiration from these ideas and using fuzzy set theory developed by Zadeh [13], this article presents new fuzzy concepts in spectral spaces. Madhuri and Amudhambigai [8] introduced the concept of fuzzy sober spaces and this concept was further developed as fuzzy spectral spaces and published in [1]. Motivated by the concept of spectral spaces in relation to diverse structres and a variety of topologies studied by Oprea1, et. al. [10] and Amartyagoswam [6], this article develops new concepts of fuzzy quasi-spectral spaces, fuzzy saturated sets and fuzzy special morphism along with several important properties and characterizations. Finally fuzzy one-point compactification is introduced and studied within fuzzy semi-spectral spaces including $\Gamma$-compact sets and $\xi$-sets. The rest of this work is organized as follows. In Section 2, we present some fundamentals of fuzzy sober and fuzzy spectral spaces. In Section 3, the notions of fuzzy spectral spaces, fuzzy-quasi-spectral paces, fuzzy saturated and fuzzy special morphisms with their properties are studied. In Section 4, the concepts of fuzzy semi-spectral spaces, fuzzy $\Gamma$-compact open sets, fuzzy $\Gamma$-compact closed sets, fuzzy $\Gamma$-compact clopen sets and fuzzy $\xi$-sets are initiated and some of their properties are presented. Finally, the conclusion of our work is given at the end.

## 2. Preliminaries

Some fundamentals of fuzzy sober and fuzzy spectral spaces are included in this section. In support of some needed results, propositions and important theorems are collected from different books and research articles. Also, this section includes almost all possible ground notions which are essential to make this paper self contained.

Throughout the paper, a nonempty set is denoted by $X$, the assortment of fuzzy sets in the interval $I = [0, 1]$, is denoted by $I^X$, $J$ is considered as an indexed set and the complement of $\lambda \in I^{\mathcal{X}}$ is $(1_{\mathcal{X}} - \lambda)$. Also, $\mathcal{FTS}(\mathcal{X}, \tau)$

denotes that $(\mathcal{X}, \tau)$ is a fuzzy topological space, fuzzy closed set is notated by $\mathcal{F}cl$ set and fuzzy open set is notated by $\mathcal{F}opp$ set.

**Definition 2.1.** ([8]) Any $\mu \in I^{\mathcal{X}}$ in $\mathcal{FTS}(\mathcal{X}, \tau)$ is defined to be fuzzy irreducible (denoted by, $\mathcal{FI}$) if $\mu \neq 0_{\mathcal{X}}$ and for all $\mathcal{F}cl$ sets $\gamma, \delta \in I^{\mathcal{X}}$ with $\mu \leq (\gamma \vee \delta)$ such that either $\mu \leq \gamma$ or $\mu \leq \delta$.

**Definition 2.2.** ([8]) Let $x_t \in \mathscr{FP}(\mathcal{X})$ in $\mathcal{FTS}(\mathcal{X}, \tau)$. Then $x_t$ is called a fuzzy generic point of $\gamma \in I^{\mathcal{X}}$ if $x_t \leq \gamma$ and $Fcl_\tau(x_t) = \gamma$.

**Definition 2.3.** ([8]) Any $\mathcal{FTS}(\mathcal{X}, \tau)$ is said to be a fuzzy sober space if for every $\mathcal{FI}$ closed set $\mu \in I^{\mathcal{X}}$, there exists a fuzzy generic point $x_t \in \mathscr{FP}(\mathcal{X})$ of $\mu$ such that $x_t \leq \mu$.

**Definition 2.4.** ([1]) Let $\mathcal{F}$ be the collection of fuzzy compact open sets in $\mathcal{FTS}(\mathcal{X}, \tau)$. Then $\mathcal{FTS}(\mathcal{X}, \tau)$ is known to be a fuzzy spectral space (denoted as $\mathcal{FSS}(\mathcal{X}, \tau)$) if it satisfies :

(i) $\mathcal{FTS}(\mathcal{X}, \tau)$ is a fuzzy $T_0$-space ;
(ii) $\mathcal{FTS}(\mathcal{X}, \tau)$ is fuzzy compact and has a base which is a assortment of fuzzy compact open sets ;
(iii) $\lambda_1 \wedge \lambda_2 \in \mathcal{F}$ if $\lambda_1, \lambda_2 \in \mathcal{F}$ ;
(iv) $\mathcal{FTS}(\mathcal{X}, \tau)$ is fuzzy sober.

**Example 2.5.** Let $X = \{a, b, c\}$ and $\mu_1, \mu_2 \in I^X$ be defined as follows $\mu_1(a) = 1, \mu_1(b) = 0.7, \mu_1(c) = 0.3, \mu_2(a) = 1, \mu_2(b) = 1, \mu_2(c) = 0.6,$. Thus $\tau = \{1_X, 0_X, \mu_1, \mu_2\}$ forms a fuzzy topology on X. Also $(X, \tau)$ is a fuzzy spectral space.

**Theorem 2.6.** ([11]) *Let $f$ be a function from $X$ to $Y$. If $a$ and $a_i$, $i \in I$, are fuzzy sets in $X$ and if $b$ and $b_j$, $j \in J$, are fuzzy sets in $Y$, then*

(i) $f(f^{-1}(b)) = b$ *when $f$ is onto $Y$.*
(ii) $f(\wedge a_i) \leqq \wedge f(a_i)$.
(iii) $f^{-1}(\wedge b_j) = \wedge f^{-1}(b_j)$.
(iv) $f(\vee a_i) = \vee f(a_i)$.
(v) $f^{-1}(\vee b_j) = \vee f^{-1}(b_j)$.
(vi) $f(f^{-1}(b) \wedge a) = b \wedge f(a)$.

**Definition 2.7.** ([1]) A $\mathcal{FTS}(\mathcal{X}, \tau)$ is defined to be fuzzy quasi-spectral (denoted as, $\mathcal{FQS}(\mathcal{X}, \tau)$) if it fulfills the axioms of (ii), (iii) and (iv) of the Definition 2.4.

**Example 2.8.** Let $X = \{a, b, c\}$ and $\mu_1, \mu_2, \mu_3 \in I^X$ be defined as follows $\mu_1(a) = 1, \mu_1(b) = 0.6, \mu_1(c) = 0, \mu_2(a) = 0.8, \mu_2(b) = 0, \mu_2(c) = 0.7, \mu_3(a) = 1, \mu_3(b) = 0, \mu_3(c) = 0.4$. Thus $\tau = \{1_X, 0_X, \mu_1, \mu_2, \mu_3\}$ forms a fuzzy topology on X. Also $(X, \tau)$ is a fuzzy quasi-spectral space but not a fuzzy spectral space because failure of fuzzy generic point property.

**Theorem 2.9.** ([9]) *onto map. Then $f$ is fuzzy closed if and only if $f(\overline{a}) \leq \overline{f(a)}$ for all fuzzy set $a \in \mathcal{X}$.*

**Definition 2.10.** ([8]) Let $\varphi : \mathcal{FTS}(\mathcal{X}_1, \tau_1) \to \mathcal{FTS}(\mathcal{X}_2, \tau_2)$ be a fuzzy continuous function. Then $\varphi$ is fuzzy quasi homeomorphism if $\forall$ $\mathcal{F}op$ set (resp. $\mathcal{F}cl$ set) $\lambda \in I^{\mathcal{X}_1}$ in $\mathcal{FTS}(\mathcal{X}_1, \tau_1)$, there exists a unique $\mathcal{F}op$ set (resp. unique $\mathcal{F}cl$ set) $\mu \in I^{\mathcal{X}_2}$ in $\mathcal{FTS}(\mathcal{X}_2, \tau_2)$ such that $\lambda = \varphi^{-1}(\mu)$.

**Proposition 2.11.** ([1]) *Let $\phi : \mathcal{FTS}(\mathcal{X}_1, \tau_1) \to \mathcal{FTS}(\mathcal{X}_2, \tau_2)$ be a fuzzy quasi homeomorphism and let $\mathcal{FTS}(\mathcal{X}_2, \tau_2)$ be $\mathcal{FQS}(\mathcal{X}_2, \tau_2)$. Then the following statements hold :*

(i) *$\mathcal{FTS}(\mathcal{X}_1, \tau_1)$ is fuzzy compact and has a base of $\mathcal{FCO}$ sets ;*
(ii) *If $\mu, \sigma \in I^{\mathcal{X}_1}$ are any two $\mathcal{FCO}$ sets in $\mathcal{FTS}(\mathcal{X}_1, \tau_1)$, then $\mu \wedge \sigma$ is also $\mathcal{FC}$ in $\mathcal{FTS}(\mathcal{X}_1, \tau_1)$.*

**Proposition 2.12.** ([1]) *Let $\phi : \mathcal{FTS}(\mathcal{X}_1, \tau_1) \to \mathcal{FTS}(\mathcal{X}_2, \tau_2)$ be a fuzzy quasi homeomorphism. If $\mathcal{FTS}(\mathcal{X}_1, \tau_1)$ is $\mathcal{FQS}(\mathcal{X}_1, \tau_1)$ and $\mathcal{FTS}(\mathcal{X}_2, \tau_2)$ is $\mathcal{FSS}(\mathcal{X}_2, \tau_2)$, then $\phi$ is a surjective function.*

**Proposition 2.13.** ([1]) *Let $\varphi : \mathcal{FTS}(\mathcal{X}_1, \tau_1) \to \mathcal{FTS}(\mathcal{X}_2, \tau_2)$ be onto and fuzzy quasi homeomorphism. If $\mu \in I^{\mathcal{X}_2}$ in $\mathcal{FTS}(\mathcal{X}_2, \tau_2)$, then the given conditions are equivalent :*

(i) *$\mu$ is $\mathcal{FI}cl$ set in $\mathcal{FTS}(\mathcal{X}_2, \tau_2)$ ;*
(ii) *$\varphi^{-1}(\mu)$ is $\mathcal{FI}cl$ set in $\mathcal{FTS}(\mathcal{X}_1, \tau_1)$.*

**Definition 2.14.** ([3]) Let $p : X \to Y$ be a map on two sets $X$ and $Y$. Then $S \subseteq X$ is said to be saturated under the map $p$ if $p^{-1}(p(S)) = S$.

## 3. Fuzzy quasi-spectral spaces

The notions of fuzzy spectral spaces, fuzzy quasi-spectral spaces, fuzzy saturated and fuzzy special morphisms are introduced in this section and some properties are studied.

**Remark 3.1.** Let $\lambda \in I^{\mathcal{X}}$ be a fuzzy compact (denoted as, $\mathcal{FC}$) set in $\mathcal{FTS}(\mathcal{X}, \tau)$. $\lambda$ is said to be a fuzzy compact open (denoted as, $\mathcal{FC}o$) set if $\lambda \in \tau$. The complement of $\mathcal{FC}o$ set is a fuzzy compact closed (denoted as, $\mathcal{FC}cl$) set.

**Remark 3.2.** Let $\varphi : \mathcal{FTS}(\mathcal{X}_1, \tau_1) \to \mathcal{FTS}(\mathcal{X}_2, \tau_2)$ be a fuzzy quasi homeomorphism. For any $\lambda, \mu \in I^{\mathcal{X}_2}$, $\lambda = \mu$ if $\varphi^{-1}(\lambda) = \varphi^{-1}(\mu)$.

**Proposition 3.3.** *Let $\phi : \mathcal{FTS}(\mathcal{X}_1, \tau_1) \to \mathcal{FTS}(\mathcal{X}_2, \tau_2)$ be a onto fuzzy quasi homeomorphism. Let $\lambda \in I^{\mathcal{X}_2}$ be a $\mathcal{F}op$ set. Then the given conditions are equivalent :*

(i) *$\lambda$ is a $\mathcal{FC}$ set in $\mathcal{FTS}(\mathcal{X}_2, \tau_2)$ ;*
(ii) *$\phi^{-1}(\lambda)$ is a $\mathcal{FC}$ set in $\mathcal{FTS}(\mathcal{X}_1, \tau_1)$.*

*Proof.* (i) $\Rightarrow$ (ii): Let $\lambda \in I^{\mathcal{X}_2}$ be a $\mathcal{FCO}$ set in $\mathcal{FTS}(\mathcal{X}_2, \tau_2)$. Let $\{ \mu_i \in I^{\mathcal{X}_1}, i \in J \}$ be a fuzzy open covering of $\phi^{-1}(\lambda) \in I^{\mathcal{X}_1}$. Then $\phi^{-1}(\lambda) \leq \vee_{i\in J}\mu_i$. Since $\phi$ is a fuzzy quasi homeomorphism and by Definition 2.10, for each $\mu_i \in I^{\mathcal{X}_1}$ where $i \in J$, there exists a fuzzy open set $\sigma_i \in I^{\mathcal{X}_2}$ such that $\mu_i = \phi^{-1}(\sigma_i), i \in J$. Then $\phi^{-1}(\lambda) \leq \vee_{i\in J}\mu_i$. Thus $\phi^{-1}(\lambda) \leq \phi^{-1}(\vee_{i\in J}\sigma_i)$.

Since $\phi$ is a fuzzy quasi homeomorphism and by using Remark 3.2, $\lambda \leq \vee_{i\in J}\sigma_i$. Since $\lambda$ is a $\mathcal{FC}$ set, there exists a finite subset $J_0$ of $J$ such that $\lambda \leq \vee_{i\in J_0}\sigma_i$. Thus

$$\begin{aligned}\phi^{-1}(\lambda) &\leq \phi^{-1}(\vee_{i\in J_0}\sigma_i)\\ &= \vee_{i\in J_0}\phi^{-1}(\sigma_i), \text{ (as in (v) of Theorem 2.6)}\\ &= \vee_{i\in J_0}\mu_i, \text{ (since } \phi^{-1}(\sigma_i) = \mu_i).\end{aligned}$$

Therefore, $\phi^{-1}(\lambda) \leq \vee_{i\in J_0}\mu_i$. Hence $\phi^{-1}(\lambda)$ is a $\mathcal{FC}$ set in $\mathcal{FTS}(\mathcal{X}_1, \tau_1)$.

(ii) $\Rightarrow$ (i): Let $\phi^{-1}(\lambda) \in I^{\mathcal{X}_1}$ be a $\mathcal{FC}$ set in $\mathcal{FTS}(\mathcal{X}_1, \tau_1)$. Let $\{ \sigma_i \in I^{\mathcal{X}_2}, i \in J \}$ be a fuzzy open covering of $\lambda \in I^{\mathcal{X}_2}$. Then $\lambda \leq \vee_{i\in J}\sigma_i$. Therefore, $\phi^{-1}(\lambda) \leq \vee_{i\in J}\phi^{-1}(\sigma_i)$.

Since $\phi^{-1}(\lambda)$ is a $\mathcal{FC}$ set, there exists a finite subset $J_0$ of $J$ such that

$$\begin{aligned}\phi^{-1}(\lambda) &\leq \vee_{i\in J_0}\, \phi^{-1}(\sigma_i)\\ &= \phi^{-1}(\vee_{i\in J_0}\sigma_i), \text{ (as in (v) of Theorem 2.6).}\end{aligned}$$

Thus, $\phi^{-1}(\lambda) \leq \phi^{-1}(\vee_{i\in J_0}\sigma_i)$, which implies that

$$\phi(\phi^{-1}(\lambda)) \leq \phi(\phi^{-1}(\vee_{i\in J_0}\sigma_i)).$$

Therefore, $\lambda \leq \vee_{i\in J_0}\sigma_i$, since $\phi$ is onto. Hence $\lambda$ is a $\mathcal{FC}$ set in $\mathcal{FTS}(\mathcal{X}_2, \tau_2)$. $\square$

**Definition 3.4.** Let $\phi : \mathcal{FTS}(\mathcal{X}_1, \tau_1) \to \mathcal{FTS}(\mathcal{X}_2, \tau_2)$ be any function. Any $\lambda \in I^{\mathcal{X}_1}$ is defined to be fuzzy saturated under $\phi$ if $\phi^{-1}(\phi(\lambda)) = \lambda$.

**Remark 3.5.** Let $\phi : \mathcal{FTS}(\mathcal{X}_1, \tau_1) \to \mathcal{FTS}(\mathcal{X}_2, \tau_2)$ be any function. For a fuzzy saturated set $\lambda \in I^{\mathcal{X}_1}$ in $\mathcal{FTS}(\mathcal{X}_1, \tau_1)$ under $\phi$, $1_{\mathcal{X}_1} - \lambda$ is fuzzy saturated under $\phi$ and $\phi(1_{\mathcal{X}_1} - \lambda) = 1_{\mathcal{X}_2} - \phi(\lambda)$.

**Proposition 3.6.** *Let $\phi : \mathcal{FTS}(\mathcal{X}_1, \tau_1) \to \mathcal{FTS}(\mathcal{X}_2, \tau_2)$ be surjective. Then the given statements are equivalent :*

(i) *$\phi$ is a $\mathcal{F}op$ function and each $\mathcal{F}op$ set in $\mathcal{FTS}(\mathcal{X}_1, \tau_1)$ is fuzzy saturated under $\phi$;*
(ii) *$\phi$ is a $\mathcal{F}cl$ function and each $\mathcal{F}cl$ set in $\mathcal{FTS}(\mathcal{X}_1, \tau_1)$ is fuzzy saturated under $\phi$.*

*Proof.* (i) $\Rightarrow$ (ii): If $\lambda \in I^{\mathcal{X}_1}$ is a $\mathcal{F}cl$ set in $\mathcal{FTS}(\mathcal{X}_1, \tau_1)$, then $1_{\mathcal{X}_1} - \lambda$ is $\mathcal{F}op$ set in $\mathcal{FTS}(\mathcal{X}_1, \tau_1)$. Since each $\mathcal{F}op$ set in $\mathcal{FTS}(\mathcal{X}_1, \tau_1)$ is fuzzy saturated under $\phi$, $1_{\mathcal{X}_1} - \lambda$ is fuzzy saturated under $\phi$. By Remark 3.5, $1_{\mathcal{X}_1} - (1_{\mathcal{X}_1} - \lambda) = \lambda$ is also fuzzy saturated under $\phi$ and $\phi(1_{\mathcal{X}_1} - \lambda) = 1_{\mathcal{X}_2} - \phi(\lambda)$. Further, as $\phi$ is a $\mathcal{F}op$ function, $\phi(1_{\mathcal{X}_1} - \lambda) = 1_{\mathcal{X}_2} - \phi(\lambda)$ is a $\mathcal{F}op$ set in $\mathcal{FTS}(\mathcal{X}_2, \tau_2)$. Thus $\phi(\lambda)$ is a $\mathcal{F}cl$ set in $\mathcal{FTS}(\mathcal{X}_2, \tau_2)$. Therefore $\phi$ is a $\mathcal{F}cl$ function.

(ii) $\Rightarrow$ (i): The proof can be proved similarly. □

**Definition 3.7.** Let $\phi : \mathcal{FTS}(\mathcal{X}_1, \tau_1) \to \mathcal{FTS}(\mathcal{X}_2, \tau_2)$ be surjective and fuzzy continuous. Then $\phi$ is called as fuzzy special morphism if $\phi$ satisfies one of the following axioms :

(i) $\phi$ is a $\mathcal{F}op$ function and each $\mathcal{F}op$ set in $\mathcal{FTS}(\mathcal{X}_1, \tau_1)$ is fuzzy saturated under $\phi$;
(ii) $\phi$ is a $\mathcal{F}cl$ function and each $\mathcal{F}cl$ set in $\mathcal{FTS}(\mathcal{X}_1, \tau_1)$ is fuzzy saturated under $\phi$.

**Proposition 3.8.** *The composition of any two fuzzy special morphisms is also a fuzzy special morphism.*

*Proof.* Let $\phi : \mathcal{FTS}(\mathcal{X}_1, \tau_1) \to \mathcal{FTS}(\mathcal{X}_2, \tau_2)$ and $\varphi : \mathcal{FTS}(\mathcal{X}_2, \tau_2) \to \mathcal{FTS}(X_3, \tau_3)$ be any two fuzzy special morphisms. It is enough to show that $\varphi \circ \phi : \mathcal{FTS}(\mathcal{X}_1, \tau_1) \to \mathcal{FTS}(X_3, \tau_3)$ is also a fuzzy special morphism. Since $\phi$ and $\varphi$ are fuzzy special morphisms and by Definition 3.7, $\phi$ and $\varphi$ are $\mathcal{F}op$ functions (resp. $\mathcal{F}cl$ functions). Since composition of $\mathcal{F}op$ functions (resp. $\mathcal{F}cl$ functions) is $\mathcal{F}op$ (resp. $\mathcal{F}cl$), $\varphi \circ \phi$ is also a $\mathcal{F}op$ function (resp. $\mathcal{F}cl$ function).

Let $\lambda \in I^{\mathcal{X}_1}$ be a $\mathcal{F}op$ ( resp. $\mathcal{F}cl$ set ) in $\mathcal{FTS}(\mathcal{X}_1, \tau_1)$. Since $\lambda$ is fuzzy saturated under $\phi$, $\phi^{-1}(\phi(\lambda)) = \lambda$. Also, since $\lambda$ is fuzzy saturated under $\varphi$,

$\varphi^{-1}(\varphi(\lambda)) = \lambda$. Thus,

$$[\varphi \circ \phi]^{-1}((\varphi \circ \phi)(\lambda)) = (\phi^{-1}(\varphi^{-1}(\varphi(\phi(\lambda)))) \\ = \lambda.$$

Hence, we have $(\varphi \circ \phi)^{-1}((\varphi \circ \phi)(\lambda)) = \lambda$. Therefore $\lambda$ is fuzzy saturated under $\varphi \circ \phi$ and then $\varphi \circ \phi$ is fuzzy special morphism. □

**Note 3.9.** A $\mathcal{FTS}(\mathcal{X}, \tau)$ is a fuzzy $T_0$-space if and only if $\lambda, \mu \in I^{\mathcal{X}}$ are such that $Fcl_\tau(\lambda) = Fcl_\tau(\mu)$ where $Fcl_\tau$ is fuzzy closure under $\tau$. Thus $\lambda = \mu$.

**Proposition 3.10.** *Let $\phi : \mathcal{FTS}(\mathcal{X}_1, \tau_1) \to \mathcal{FTS}(\mathcal{X}_2, \tau_2)$ be a fuzzy special morphism. If $\mathcal{FTS}(\mathcal{X}_1, \tau_1)$ is a fuzzy $T_0$-space, then $\phi$ is fuzzy homeomorphism.*

*Proof.* As $\phi$ is a fuzzy special morphism and by Definition 3.7, $\phi$ is a fuzzy closed and fuzzy continuous function. Hence it is sufficient to prove that $\phi$ is injective. Let $\lambda, \mu \in I^{\mathcal{X}_1}$ be such that $\phi(\lambda) = \phi(\mu)$. Then $Fcl_{\tau_2}(\phi(\lambda)) = Fcl_{\tau_2}(\phi(\mu))$. Because $\phi$ is $\mathcal{F}cl$ function and by Theorem 2.9,

$$Fcl_{\tau_2}(\phi(\lambda)) \leq \phi(Fcl_{\tau_1}(\lambda)) \text{ and } Fcl_{\tau_2}(\phi(\mu)) \leq \phi(Fcl_{\tau_1}(\mu)). \tag{3.1}$$

As $\phi$ is fuzzy continuous,

$$Fcl_{\tau_2}(\phi(\lambda)) \geq \phi(Fcl_{\tau_1}(\lambda)) \text{ and } Fcl_{\tau_2}(\phi(\mu)) \geq \phi(Fcl_{\tau_1}(\mu)). \tag{3.2}$$

Thus, from Equations (3.1) and (3.2), $Fcl_{\tau_2}(\phi(\lambda)) = \phi(Fcl_{\tau_1}(\lambda))$ and $Fcl_{\tau_2}(\phi(\mu)) = \phi(Fcl_{\tau_1}(\mu))$. Therefore $Fcl_{\tau_2}(\phi(\lambda)) = Fcl_{\tau_2}(\phi(\mu))$ implies that $\phi(Fcl_{\tau_1}(\lambda)) = \phi(Fcl_{\tau_1}(\mu))$. From Proposition 3.6, it is known that every $\mathcal{F}cl$ set in $(\mathcal{X}_1, \tau_1)$ is fuzzy saturated under $\phi$. Since $\phi$ is fuzzy saturated, $\phi^{-1}(\phi(Fcl_{\tau_1}(\lambda))) = Fcl_{\tau_1}(\lambda)$ and $\phi^{-1}(\phi(Fcl_{\tau_1}(\mu))) = Fcl_{\tau_1}(\mu)$. Therefore, $Fcl_{\tau_1}(\lambda) = Fcl_{\tau_1}(\mu)$. Since $(\mathcal{X}_1, \tau_1)$ is a fuzzy $T_0$-space and by Note 3.9, $\lambda = \mu$. Thus, $\phi$ is injective. Therefore, $\phi$ is a fuzzy homeomorphism. □

**Example 3.11.** Let $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2\}$ and $\mu \in I^X$, $\lambda, \gamma \in I^Y$ be defined by $\mu(x_1) = 1, \mu(x_2) = 0.4, \lambda(y_1) = 1, \lambda(y_2) = 0.7, \gamma(y_1) = 1, \gamma(y_2) = 0.2$. Clearly $\tau_X = \{0_X, 1_X, \mu\}$ and $\tau_Y = \{0_Y, 1_Y, \lambda, \gamma\}$ be fuzzy topologies on X and Y respectively. Let $f$ be function form X to Y be defined by $f(x_1) = y_1, f(x_2) = y_2$. Here $f$ is fuzzy special morphism but not a fuzzy homeomorphism because all inverse images of a fuzzy open set does not lie in the fuzzy topology of the domain.

**Theorem 3.12.** *Let $\phi : \mathcal{FTS}(\mathcal{X}_1, \tau_1) \to \mathcal{FTS}(\mathcal{X}_2, \tau_2)$ be fuzzy quasi homeomorphism. Suppose that $\mathcal{FTS}(\mathcal{X}_1, \tau_1)$ is fuzzy quasi-spectral, in that case $\mathcal{FTS}(\mathcal{X}_2, \tau_2)$ is also fuzzy quasi-spectral.*

*Proof.* Consider $\mathcal{FTS}(\mathcal{X}_1, \tau_1)$ is fuzzy quasi-spectral. Let $\{ \mu_i \in I^{\mathcal{X}_1}, i \in J \}$ be a base which is the assortment of $\mathcal{FCO}$ sets in $\mathcal{FTS}(\mathcal{X}_1, \tau_1)$. Since $\phi$ is fuzzy quasi homeomorphism, there exists $\mathcal{F}op$ set $\sigma_i \in I^{\mathcal{X}_2}$ where $i \in J$ such that $\mu_i = \phi^{-1}(\sigma_i)$, where each $\mu_i \in I^{\mathcal{X}_1}$ is $\mathcal{FCO}$ in $\mathcal{FTS}(\mathcal{X}_1, \tau_1)$. Since $\mu_i$ is $\mathcal{FC}$ and $\mu_i = \phi^{-1}(\sigma_i)$, $\phi^{-1}(\sigma_i)$ is also $\mathcal{FC}$ in $\mathcal{FTS}(\mathcal{X}_1, \tau_1)$. By Proposition 3.3, $\sigma_i$ is $\mathcal{FC}$ in $\mathcal{FTS}(\mathcal{X}_2, \tau_2)$. Let $\sigma \in I^{\mathcal{X}_2}$ be a $\mathcal{F}op$ set in $\mathcal{FTS}(\mathcal{X}_2, \tau_2)$. As $\phi$ is fuzzy continuous, $\phi^{-1}(\sigma)$ is $\mathcal{F}op$ set in $\mathcal{FTS}(\mathcal{X}_1, \tau_1)$. Therefore, there is a subset $J_0$ of $J \ni \phi^{-1}(\sigma) \leq \vee_{i \in J_0} \mu_i \leq \vee_{i \in J_0} \phi^{-1}(\sigma_i) \leq \phi^{-1}(\vee_{i \in J_0} \sigma_i)$, as in (iii) of Theorem 2.6. Since $\phi$ is fuzzy quasi homeomorphism and by Remark 3.2, $\sigma \leq \vee_{i \in J_0} \sigma_i$. Thus $\{ \sigma_i \in I^{\mathcal{X}_2}, i \in J \}$ is a base which is the assortment of $\mathcal{FCO}$ sets in $\mathcal{FTS}(\mathcal{X}_2, \tau_2)$.

Let $\mu, \sigma \in I^{\mathcal{X}_2}$ be any two $\mathcal{FCO}$ sets in $\mathcal{FTS}(\mathcal{X}_2, \tau_2)$. Then by Proposition 3.3, $\phi^{-1}(\mu)$ and $\phi^{-1}(\sigma)$ are two $\mathcal{FCO}$ sets of $\mathcal{FTS}(\mathcal{X}_1, \tau_1)$. Also, $\phi^{-1}(\mu) \wedge \phi^{-1}(\sigma) = \phi^{-1}(\mu \wedge \sigma)$. Since $\mathcal{FTS}(\mathcal{X}_1, \tau_1)$ is fuzzy quasi-spectral, $\phi^{-1}(\mu \wedge \sigma)$ is $\mathcal{FC}$ in $\mathcal{FTS}(\mathcal{X}_1, \tau_1)$. Therefore, $\mu \wedge \sigma$ is $\mathcal{FC}$ in $\mathcal{FTS}(\mathcal{X}_2, \tau_2)$, by Proposition 3.3.

Let $\delta \in I^{\mathcal{X}_2}$ be a $\mathcal{FI}cl$ set in $\mathcal{FTS}(\mathcal{X}_2, \tau_2)$. Then $\phi^{-1}(\delta)$ is $\mathcal{FI}cl$ in $\mathcal{FTS}(\mathcal{X}_1, \tau_1)$, by Proposition 2.13. Since $\mathcal{FTS}(\mathcal{X}_1, \tau_1)$ is a fuzzy quasi-spectral space, $(\mathcal{X}_1, \tau_1)$ is a Fuzzy Sober Space. Since $\phi^{-1}(\delta)$ is $\mathcal{FI}cl$ in $\mathcal{FTS}(\mathcal{X}_1, \tau_1)$, $\exists$ a fuzzy generic point $x_t \in \mathscr{F}\mathscr{P}(\mathcal{X}_1)$ of $\phi^{-1}(\delta)$ such that $x_t \leq \phi^{-1}(\delta)$. Since $x_t$ is a fuzzy generic point of $\phi^{-1}(\delta)$,

$$\phi^{-1}(\delta) = Fcl_{\tau_1}(x_t).$$

Then $\phi^{-1}(\delta) \leq \phi^{-1}(\phi(Fcl_{\tau_1}(x_t)))$. So $\phi^{-1}(\delta) \leq \phi^{-1}(Fcl_{\tau_2}(\phi(x_t)))$. Thus by Remark 3.2,

$$\delta \leq Fcl_{\tau_2}(\phi(x_t)). \tag{3.3}$$

Also, since $x_t \leq \phi^{-1}(\delta)$, it follows that

$$\begin{aligned} \phi(x_t) &\leq \phi(\phi^{-1}(\delta)) \\ &\leq \delta. \end{aligned}$$

Therefore, $\phi(x_t) \leq \delta$ and so $Fcl_{\tau_2}(\phi(x_t)) \leq Fcl_{\tau_2}(\delta)$. Since $\delta$ is $\mathcal{F}cl$ in $(\mathcal{X}_2, \tau_2)$,

$$Fcl_{\tau_2}(\phi(x_t)) \leq \delta. \tag{3.4}$$

Thus, from Equations (3.3) and (3.4), $Fcl_{\tau_2}(\phi(x_t)) = \delta$. Therefore, for any $\mathcal{FI}cl$ set $\delta \in I^{\mathcal{X}_2}$, there exists a fuzzy generic point $\phi(x_t) \in I^{\mathcal{X}_2}$ of $\delta$ such that $\phi(x_t) \leq \delta$. As $\phi(x_t)$ is a fuzzy generic point of $\delta$, $Fcl_{\tau_2}(\phi(x_t)) = \delta$. Hence, $\mathcal{FTS}(\mathcal{X}_2, \tau_2)$ is a fuzzy sober space. Therefore, $\mathcal{FTS}(\mathcal{X}_2, \tau_2)$ is fuzzy quasi-spectral. □

**Proposition 3.13.** *Let $\phi : \mathcal{FTS}(\mathcal{X}_1, \tau_1) \to \mathcal{FTS}(\mathcal{X}_2, \tau_2)$ be onto fuzzy continuous. Then the assertions that follow are equivalent :*

(i) *$\phi$ is a fuzzy quasi homeomorphism ;*

(ii) *If $\phi$ is a $\mathcal{F}cl$ function and $\lambda \in I^{\mathcal{X}_1}$ is a $\mathcal{F}cl$ set, then $\phi^{-1}(\phi(\lambda)) = \lambda$.*

*Proof.* (i) $\Rightarrow$ (ii): By hypothesis, as $\lambda \in I^{\mathcal{X}_1}$ is $\mathcal{F}cl$ set and $\phi$ is $\mathcal{F}cl$ function, $\phi(\lambda)$ is $\mathcal{F}cl$ function in $\mathcal{FTS}(\mathcal{X}_2, \tau_2)$. Since $\phi$ is fuzzy quasi homeomorphism, for each $\mathcal{F}cl$ set $\lambda \in I^{\mathcal{X}_1}$ in $(\mathcal{X}_1, \tau_1)$, there exists a unique $\mathcal{F}cl$ set $\mu \in I^{\mathcal{X}_2}$ in $\mathcal{FTS}(\mathcal{X}_2, \tau_2)$ such that $\phi^{-1}(\mu) = \lambda$. Thus $\phi(\lambda) = \mu$. Therefore $\phi^{-1}(\phi(\lambda)) = \phi^{-1}(\mu) = \lambda$. Hence $\phi^{-1}(\phi(\lambda)) = \lambda$.

(ii) $\Rightarrow$ (i): Since $\phi$ is $\mathcal{F}cl$ function and $\lambda \in I^{\mathcal{X}_1}$ is $\mathcal{F}cl$ set, $\phi(\lambda)$ is $\mathcal{F}cl$ in $\mathcal{FTS}(\mathcal{X}_2, \tau_2)$. Thus for each $\mathcal{F}cl$ set $\lambda \in I^{\mathcal{X}_1}$ in $\mathcal{FTS}(\mathcal{X}_1, \tau_1)$, there exists a unique $\mathcal{F}cl$ set $\phi(\lambda) \in I^{\mathcal{X}_2}$ such that $\phi^{-1}(\phi(\lambda)) = \lambda$. Thus $\phi$ is fuzzy quasi homeomorphism. □

**Proposition 3.14.** *Let $\phi : \mathcal{FTS}(\mathcal{X}_1, \tau_1) \to \mathcal{FTS}(\mathcal{X}_2, \tau_2)$ be a $\mathcal{F}cl$ function and fuzzy quasi homeomorphism. If $\mathcal{FTS}(\mathcal{X}_2, \tau_2)$ is Fuzzy spectral and $\phi$ is surjective, then $\mathcal{FTS}(\mathcal{X}_1, \tau_1)$ is fuzzy sober.*

*Proof.* Let $\mu \in I^{\mathcal{X}_1}$ be $\mathcal{FI}cl$ set in $\mathcal{FTS}(\mathcal{X}_1, \tau_1)$. As $\phi$ is fuzzy quasi homeomorphism, for each $\mathcal{F}cl$ set $\mu \in I^{\mathcal{X}_1}$ in $(\mathcal{X}_1, \tau_1)$, there exists a unique $\mathcal{F}cl$ set $\lambda \in I^{\mathcal{X}_2}$ in $\mathcal{FTS}(\mathcal{X}_2, \tau_2)$ such that $\phi^{-1}(\lambda) = \mu$. Since $\phi^{-1}(\lambda) = \mu$ and $\mu$ is $\mathcal{FI}cl$ set in $\mathcal{FTS}(\mathcal{X}_1, \tau_1)$, $\phi^{-1}(\lambda)$ is $\mathcal{FI}cl$ set in $\mathcal{FTS}(\mathcal{X}_1, \tau_1)$.

Thus, $\lambda \in I^{\mathcal{X}_2}$ is $\mathcal{FI}cl$ in $\mathcal{FTS}(\mathcal{X}_2, \tau_2)$, by Proposition 2.13. Since $\mathcal{FTS}(\mathcal{X}_2, \tau_2)$ is fuzzy spectral, it is fuzzy sober. Since $\lambda$ is $\mathcal{FI}cl$ in $\mathcal{FTS}(\mathcal{X}_2, \tau_2)$, there exists a fuzzy generic point $y_t \in \mathscr{FP}(\mathcal{X}_2)$ of $\lambda$ such that $\lambda = Fcl_{\tau_2}(y_t)$. Since $\phi$ is surjective, for every $y_t \in \mathscr{FP}(\mathcal{X}_2)$, there exists a $x_t \in \mathscr{FP}(\mathcal{X}_1)$ such that $y_t = \phi(x_t)$. Thus

$$\begin{aligned}\mu &= \phi^{-1}(\lambda)\\ &= \phi^{-1}(Fcl_{\tau_2}(y_t))\\ &= \phi^{-1}(Fcl_{\tau_2}(\phi(x_t)))\\ &= \phi^{-1}(\phi(Fcl_{\tau_1}(x_t))).\end{aligned}$$

Hence $\mu = Fcl_{\tau_1}(x_t)$. This implies that there exists a fuzzy generic point $x_t$ of a $\mathcal{FI}cl$ set $\mu$ such that $\mu = Fcl_{\tau_1}(x_t)$. Hence $\mathcal{FTS}(\mathcal{X}_1, \tau_1)$ is fuzzy sober. □

## 4. Fuzzy one-point compactification in $\mathcal{FSS}(\mathcal{X}, \tau)$

The concepts of fuzzy semi-spectral spaces, fuzzy $\Gamma$-compact open sets, fuzzy $\Gamma$-compact closed sets, fuzzy $\Gamma$-compact clopen sets and fuzzy $\xi$-sets are initiated and few of their properties are studied in this section.

**Note 4.1.** Fuzzy quasi-spectral spaces are precisely those fuzzy topological spaces that become fuzzy spectral after fuzzy one-point compactification.

**Proposition 4.2.** *Let $Y \subseteq X$ and $\mathcal{FTS}(Y, \tau_Y)$ be a subspace of $\mathcal{FTS}(\mathcal{X}, \tau)$. If $\sigma \in I^Y$ is a $\mathcal{FI}cl$ set in $\mathcal{FTS}(Y, \tau_Y)$, then $Fcl_\tau(\sigma)$ is a $\mathcal{FI}cl$ set in $\mathcal{FTS}(\mathcal{X}, \tau)$.*

*Proof.* Let $\mathcal{FTS}(Y, \tau_Y)$ be a subspace of $\mathcal{FTS}(\mathcal{X}, \tau)$. Since $\sigma \in I^Y$ is a $\mathcal{FI}cl$ set in $\mathcal{FTS}(Y, \tau_Y)$, for all $\mathcal{F}cl$ sets $\gamma, \delta \in I^Y$ in $\mathcal{FTS}(Y, \tau_Y)$ such that $\sigma \leq (\gamma \vee \delta)$, then either $\sigma \leq \gamma$ or $\sigma \leq \delta$. Thus, $\sigma \leq (\gamma \vee \delta)$ implies that $Fcl_\tau \sigma \leq Fcl_\tau(\gamma \vee \delta)$ and so $Fcl_\tau \sigma \leq Fcl_\tau(\gamma) \vee Fcl_\tau(\delta)$.

It gives that either $Fcl_\tau \sigma \leq Fcl_\tau(\gamma)$ or $Fcl_\tau \sigma \leq Fcl_\tau(\delta)$. Hence, $Fcl_\tau(\sigma)$ is $\mathcal{FI}cl$ in $\mathcal{FTS}(\mathcal{X}, \tau)$. □

**Definition 4.3.** In a $\mathcal{FTS}(\mathcal{X}, \tau)$, consider a point $w \notin X$ and $X^* = X \cup \{ w \}$. A fuzzy set $\lambda \in I^{\mathcal{X}^*}$ in $\mathcal{FTS}(\mathcal{X}^*, \tau^*)$ where $\tau^*$ is a fuzzy topology on $X^*$, is called $\mathcal{F}op$ in $\mathcal{FTS}(\mathcal{X}^*, \tau^*)$ if either $\lambda(w) = 0$ and $\lambda|_X$ is $\mathcal{F}op$ in $\mathcal{FTS}(\mathcal{X}, \tau)$ or $(1_{\mathcal{X}^*} - \lambda)|_X$ is $\mathcal{F}cl$ in $\mathcal{FTS}(\mathcal{X}, \tau)$ and $\{a \in X^* : \lambda(a) \neq 1\} \subseteq A$, where $A$ is the collection of supports of a fuzzy compact subspace of $\mathcal{FTS}(\mathcal{X}, \tau)$. Then $\mathcal{FTS}(\mathcal{X}^*, \tau^*)$ is called the fuzzy one-point compactification of $\mathcal{FTS}(\mathcal{X}, \tau)$.

**Remark 4.4.** A $\mathcal{FTS}(A \cup B, \tau_{A \cup B})$ is fuzzy compact subspace of $\mathcal{FTS}(\mathcal{X}, \tau)$ if $\mathcal{FTS}(A, \tau_A)$ and $\mathcal{FTS}(B, \tau_B)$ are the fuzzy compact subspaces of $\mathcal{FTS}(\mathcal{X}, \tau)$, where $A, B \subseteq X$.

**Proposition 4.5.** *Let $X^* = X \cup \{w\}$. The fuzzy one-point compactification $\mathcal{FTS}(\mathcal{X}^*, \tau^*)$ of $\mathcal{FTS}(\mathcal{X}, \tau)$ is fuzzy compact and $\mathcal{FTS}(\mathcal{X}, \tau)$ is a fuzzy subspace of $\mathcal{FTS}(\mathcal{X}^*, \tau^*)$.*

*Proof.* Let $\lambda, \mu \in I^{\mathcal{X}^*}$ be $\mathcal{F}op$ sets in $\mathcal{FTS}(\mathcal{X}^*, \tau^*)$. Then the following cases arise :

Case (i): $\lambda(w) = \mu(w) = 0$. Then $(\lambda \wedge \mu)|_{\mathcal{X}} = (\lambda|_{\mathcal{X}}) \wedge (\mu|_{\mathcal{X}})$ is $\mathcal{F}op$ in $\mathcal{FTS}(\mathcal{X}, \tau)$ and thus $\lambda \wedge \mu$ is $\mathcal{F}op$ in $\mathcal{FTS}(\mathcal{X}^*, \tau^*)$.

Case (ii): $\lambda(w) = 1$, $\mu(w) = 0$. This implies that $(1_{\mathcal{X}^*} - \lambda)|_{\mathcal{X}}$ is $\mathcal{F}cl$ in $\mathcal{FTS}(\mathcal{X}, \tau)$. Then $\lambda|_{\mathcal{X}}$ is $\mathcal{F}op$ in $\mathcal{FTS}(\mathcal{X}, \tau)$. Now, $(\lambda \wedge \mu)(w) = 0$ and $\lambda|_{\mathcal{X}}$, $\mu|_{\mathcal{X}}$ are $\mathcal{F}op$ in $\mathcal{FTS}(\mathcal{X}, \tau)$. Then $(\lambda \wedge \mu)|_{\mathcal{X}} = (\lambda|_{\mathcal{X}}) \wedge (\mu|_{\mathcal{X}})$ is $\mathcal{F}op$ in $\mathcal{FTS}(\mathcal{X}, \tau)$ and therefore $\lambda \wedge \mu$ is $\mathcal{F}op$ set in $\mathcal{FTS}(\mathcal{X}^*, \tau^*)$.

Case (iii): $\lambda(w) = \mu(w) = 1$. Suppose $\{a \in X^* : \lambda(a) \neq 1\} \subseteq A$ and $\{a \in X^* : \mu(a) \neq 1\} \subseteq B$ where $A$ and $B$ are the two families of supports of fuzzy compact subspaces of $\mathcal{FTS}(\mathcal{X}, \tau)$. Now, $(1_{\mathcal{X}^*} - (\lambda \wedge \mu))|_{\mathcal{X}} = ((1_{\mathcal{X}^*} - \lambda) \vee (1_{\mathcal{X}^*} - \mu))|_{\mathcal{X}} = (1_{\mathcal{X}^*} - \lambda)|_{\mathcal{X}} \vee (1_{\mathcal{X}^*} - \mu)|_{\mathcal{X}}$ is $\mathcal{F}cl$ in $\mathcal{FTS}(\mathcal{X}, \tau)$ and $\{a \in X^* : (\lambda \wedge \mu)(a) \neq 1\} \subseteq A \cup B$. By Remark 4.4, $A \cup B$ is the collection of supports of a fuzzy compact subspace of $\mathcal{FTS}(\mathcal{X}, \tau)$.

Let $\mathfrak{C} = \{ \lambda_i \in I^{\mathcal{X}^*} : i \in J \}$ be a collection of $\mathcal{F}op$ sets of $\mathcal{FTS}(\mathcal{X}^*, \tau^*)$. Then there are two possibilities.

Case (1) If $\lambda_i(w) = 0$ whenever $\lambda_i \in \mathfrak{C}$, then $\vee_{i\in J}\{\lambda_i : \lambda_i \in \mathfrak{C}\}$ is $\mathcal{F}op$ in $\mathcal{FTS}(\mathcal{X}^*, \tau^*)$.

Case (2) Let $\lambda_i \in \mathfrak{C}$, where $i \in J$ be such that $\lambda_i(w) = 1$ and $\{a \in X^* : \lambda_i(a) \neq 1\} \subseteq A$, where $A$ is the family of support of a fuzzy compact subspace of $\mathcal{FTS}(\mathcal{X}, \tau)$. Therefore, for $\lambda_i \in \mathfrak{C}$, $(\vee_{i\in J}\lambda_i)(w) = 1$ and $(1_{\mathcal{X}^*} - \lambda_i)|_X$ is a $\mathcal{F}cl$ set in $\mathcal{FTS}(\mathcal{X}, \tau)$ for each $\lambda_i \in \mathfrak{C}$. Moreover, $(1_{\mathcal{X}^*} - (\wedge_{i\in J}\lambda_i))|_{\mathcal{X}} = \vee_{i\in j}\{(1_{\mathcal{X}^*} - \lambda_i)|_{\mathcal{X}} : \lambda_i \in \mathfrak{C}\}$ is $\mathcal{F}cl$ set in $\mathcal{FTS}(\mathcal{X}, \tau)$ and $\{a \in X^* : \vee_{i\in J}\lambda_i(a) \neq 1$, where $\lambda_i \in \mathfrak{C}\} \subseteq A$. Thus, $\vee_{i\in J}\{\lambda_i : \lambda_i \in \mathfrak{C}\}$ is $\mathcal{F}op$ in $\mathcal{FTS}(\mathcal{X}^*, \tau^*)$.

If $\lambda$ is $\mathcal{F}op$ in $\mathcal{FTS}(\mathcal{X}, \tau)$, then $\lambda = \lambda^*|_{\mathcal{X}}$, where $\lambda^*$ is $\mathcal{F}op$ set in $\mathcal{FTS}(\mathcal{X}^*, \tau^*)$ such that $\lambda^*(a) = 0$ if $a = w$ and $\lambda^*(a) = \lambda(a)$ if $a \neq w$. On the other hand, if $\lambda$ is relatively $\mathcal{F}op$, that is $\lambda = \mu|_{\mathcal{X}}$, where $\mu$ is $\mathcal{F}op$ set in $\mathcal{FTS}(\mathcal{X}^*, \tau^*)$, then $\lambda$ is $\mathcal{F}op$ set in $\mathcal{FTS}(\mathcal{X}, \tau)$. If $\mu(w) = 0$, then $\mu(a) = 0$ if $a = w$ and $\mu(a) = \lambda(a)$ if $a \neq w$, where $\lambda$ is $\mathcal{F}op$ in $\mathcal{FTS}(\mathcal{X}, \tau)$. If $\mu(w) = 1$, then $(1_{\mathcal{X}^*} - \mu)|_{\mathcal{X}}$ is $\mathcal{F}cl$ in $\mathcal{FTS}(\mathcal{X}, \tau)$ and $\lambda$ is $\mathcal{F}op$ in $\mathcal{FTS}(\mathcal{X}, \tau)$. Clearly, $\mathcal{FTS}(\mathcal{X}, \tau)$ is a fuzzy subspace of $\mathcal{FTS}(\mathcal{X}^*, \tau^*)$.

Let $\mathfrak{C} = \{\ \lambda_i \in I^{\mathcal{X}^*} : i \in J$, where each $\lambda_i$ is $\mathcal{F}op$ in $\mathcal{FTS}(\mathcal{X}^*, \tau^*)\ \}$ such that $\vee_{i\in J}\lambda_i = 1_{\mathcal{X}^*}$. Then $\lambda_i(w) = 1$, where $\lambda_i \in \mathfrak{C}$ for some $i \in J$ and $\{a \in X^* : \lambda_i(a) \neq 1\} \subseteq A$, where $A$ is the family of support of fuzzy compact subspaces of $\mathcal{FTS}(\mathcal{X}, \tau)$. Then there is a finite subset $J_0$ of $J$ such that $(\vee_{i\in J_0}\lambda_i) \vee \lambda = 1_{\mathcal{X}^*}$. Therefore, $\mathcal{FTS}(\mathcal{X}^*, \tau^*)$ is fuzzy compact. □

**Proposition 4.6.** *Let $w \notin X$ and $X^* = X \cup \{w\}$. Consider the fuzzy one-point compactification of $\mathcal{FTS}(\mathcal{X}, \tau)$ as $\mathcal{FTS}(\mathcal{X}^*, \tau^*)$. Then $\mathcal{FTS}(\mathcal{X}, \tau)$ is fuzzy sober, if the fuzzy one-point compactification $\mathcal{FTS}(\mathcal{X}^*, \tau^*)$ is fuzzy sober.*

*Proof.* Suppose $\mathcal{FTS}(\mathcal{X}^*, \tau^*)$ is fuzzy sober and $\gamma \in I^{\mathcal{X}}$ is $\mathcal{FI}cl$ in $\mathcal{FTS}(\mathcal{X}, \tau)$. Then $\mathcal{FTS}(\mathcal{X}, \tau)$ is fuzzy subspace of $\mathcal{FTS}(\mathcal{X}^*, \tau^*)$, by Proposition 4.5. Thus, $Fcl_{\tau^*}(\gamma)$ is also $\mathcal{FI}cl$ set in $\mathcal{FTS}(\mathcal{X}^*, \tau^*)$, by Proposition 4.2. As $(\mathcal{X}^*, \tau^*)$ is fuzzy sober and $Fcl_{\tau^*}(\gamma)$ is $\mathcal{FI}cl$ in $\mathcal{FTS}(\mathcal{X}^*, \tau^*)$, there exists a fuzzy generic point $x_t \in \mathscr{F}\mathscr{P}(X)$ of $Fcl_{\tau^*}(\gamma)$ such thst $x_t \leq Fcl_{\tau^*}(\gamma)$. Since $x_t$ is a fuzzy generic point of $Fcl_{\tau^*}(\gamma)$ and by Definition 2.2, $Fcl_{\tau^*}(x_t) = Fcl_{\tau^*}(\gamma)$. Hence $\gamma = Fcl_{\tau^*}(\gamma)|_X \wedge 1_{\mathcal{X}} = Fcl_{\tau^*}(x_t)|_X \wedge 1_{\mathcal{X}} = Fcl_\tau(x_t)$. Then there is a fuzzy generic point $x_t$ of $\gamma$ such that $x_t \leq \gamma$. Since $x_t$ is fuzzy generic point of a $\mathcal{FI}cl$ set $\gamma$ in $\mathcal{FTS}(\mathcal{X}, \tau)$, $\gamma = Fcl_\tau(x_t)$. Hence $\mathcal{FTS}(\mathcal{X}, \tau)$ is fuzzy sober. □

**Example 4.7.** Let $X = (0, 1)$ be a base space with fuzzy topology. Let fuzzy open sets be $\mu_\alpha(x) = \alpha.x$ for $\alpha \in (0.1]$ They are locally fuzzy compact near any point. So X is fuzzy quasi-spectral (not fuzzy compact). Add a point $\infty$ with membership defined by point $\mu(\infty) = 1 - \sup_{x\in X}\mu(x)$ The new space $X \cup \{\infty\}$ is fuzzy compact and satisfies spectral conditions. It becomes a fuzzy spectral space.

**Definition 4.8.** In a $\mathcal{FTS}(\mathcal{X},\tau)$, assume that $\delta \in I^{\mathcal{X}}$. Then

(i) $\delta$ is defined to be fuzzy $\Gamma$-compact open (briefly, $\mathcal{F}\Gamma\mathcal{C}o$) if for all $\mathcal{FC}o$ set $\mu \in I^{\mathcal{X}}$, $\delta \wedge \mu$ is $\mathcal{FC}$.

(ii) $\delta$ is defined to be fuzzy $\Gamma$-compact closed (briefly, $\mathcal{F}\Gamma\mathcal{C}cl$) if for all $\mathcal{FC}cl$ set $\mu \in I^{\mathcal{X}}$, $\delta \wedge \mu$ is $\mathcal{FC}$.

(iii) $\delta$ is defined to be fuzzy $\Gamma$-compact clopen (briefly, $\mathcal{F}\Gamma\mathcal{C}ocl$) if it is both $\mathcal{F}\Gamma\mathcal{C}o$ and $\mathcal{F}\Gamma\mathcal{C}cl$.

(iv) $\delta$ is defined to be fuzzy co-compact (fuzzy co-$\Gamma$-compact open, fuzzy co-$\Gamma$-compact closed, fuzzy co-$\Gamma$-compact clopen) if $1_{\mathcal{X}} - \delta$ is fuzzy compact ($\mathcal{F}\Gamma\mathcal{C}o$, $\mathcal{F}\Gamma\mathcal{C}cl$, $\mathcal{F}\Gamma\mathcal{C}ocl$ respectively).

**Definition 4.9.** Any $\mathcal{FTS}(\mathcal{X},\tau)$ is a fuzzy semi-spectral space if $\lambda_1, \lambda_2 \in I^{\mathcal{X}}$ are any two $\mathcal{FC}o$ sets, then $\lambda_1 \wedge \lambda_2$ is $\mathcal{FC}$ in $\mathcal{FTS}(\mathcal{X},\tau)$.

**Remark 4.10.** Every fuzzy semi-spectral space need not be a fuzzy spectral space.

**Example 4.11.** Let $X = \{a, b, c\}$ and $\mu \in I^X$ be defined as follows $\mu(a) = 1$, $\mu(b) = 0.5$, $\mu(c) = 0$. Thus $\tau = \{1_X, 0_X, \mu\}$ forms a fuzzy topology on X. Clearly $(X, \tau)$ is a fuzzy T-0 and fuzzy semi-spectral space. Since $\mu$ is not fuzzy compact, $(X, \tau)$ is not a fuzzy spectral space.

**Proposition 4.12.** Let $\mathcal{FTS}(\mathcal{X}^*,\tau^*)$ be fuzzy one-point compactification of $\mathcal{FTS}(\mathcal{X},\tau)$. If $\mathcal{FTS}(\mathcal{X},\tau)$ is fuzzy semi-spectral, then the assertions that follow are equivalent :

(i) $\delta \in I^{\mathcal{X}^*}$ is $\mathcal{F}op$ set in $\mathcal{FTS}(\mathcal{X}^*,\tau^*)$.

(ii) If $\delta|_{\mathcal{X}}$ is $\mathcal{F}op$ set in $\mathcal{FTS}(\mathcal{X},\tau)$, then $(1_{\mathcal{X}} - \delta|_{\mathcal{X}}) \in I^{\mathcal{X}}$ is $\mathcal{FC}$ and $\delta|_{\mathcal{X}}$ is fuzzy co-$\Gamma$-compact closed in $\mathcal{FTS}(\mathcal{X},\tau)$.

*Proof.* (i) $\Rightarrow$ (ii): Let $\delta \in I^{\mathcal{X}^*}$ be a $\mathcal{F}op$ set in $\mathcal{FTS}(\mathcal{X}^*,\tau^*)$. Then by Definition 4.3, $(1_{\mathcal{X}} - \delta|_{\mathcal{X}}) \in I^{\mathcal{X}}$ is $\mathcal{FC}$ in $\mathcal{FTS}(\mathcal{X},\tau)$ and $\{a \in X^* : \lambda(a) \neq 1\} \subseteq A$, where $A$ is the collection of supports of a fuzzy compact subspace of $\mathcal{FTS}(\mathcal{X},\tau)$. Let $\mu \in I^{\mathcal{X}}$ be a $\mathcal{FCC}$ set in $\mathcal{FTS}(\mathcal{X},\tau)$. As $\mathcal{FTS}(\mathcal{X},\tau)$ is a fuzzy semi-spectral Space and $\mu, (1_{\mathcal{X}} - \delta|_{\mathcal{X}})$ are $\mathcal{FC}$ in $\mathcal{FTS}(\mathcal{X},\tau)$, $\mu \wedge (1_{\mathcal{X}} - \delta|_{\mathcal{X}})$ is also $\mathcal{FC}$ in $\mathcal{FTS}(\mathcal{X},\tau)$. Thus $(1_{\mathcal{X}} - \delta|_{\mathcal{X}})$ is fuzzy co-$\Gamma$-compact closed. Therefore $\delta|_{\mathcal{X}}$ is fuzzy co-$\Gamma$-compact closed.

(ii) $\Rightarrow$ (i): Let $\delta|_{\mathcal{X}} \in I^{\mathcal{X}}$ be a fuzzy open set in $\mathcal{FTS}(\mathcal{X},\tau)$. Then $(1_{\mathcal{X}} - \delta|_{\mathcal{X}})$ is fuzzy closed in $\mathcal{FTS}(\mathcal{X},\tau)$. As $(1_{\mathcal{X}} - \delta|_{\mathcal{X}})$ is $\mathcal{FC}$ in $\mathcal{FTS}(\mathcal{X},\tau)$ and by Definition 4.3, $\delta$ is a fuzzy open set in $\mathcal{FTS}(\mathcal{X}^*,\tau^*)$. □

**Definition 4.13.** Any $\lambda \in I^{\mathcal{X}}$ is said to be a fuzzy $\xi$-set if $\lambda$ is fuzzy closed, $\mathcal{FC}$ and $\mathcal{F}\Gamma\mathcal{C}\wr$ in $\mathcal{FTS}(\mathcal{X},\tau)$.

**Proposition 4.14.** Let $\mathcal{FTS}(\mathcal{X},\tau)$ be fuzzy semi-spectral with a base of $\mathcal{FCO}$ sets and $\lambda \in I^{\mathcal{X}}$. Then the assertions that follow are equivalent:

(i) $\lambda$ is $\mathcal{FC}$ and $\mathcal{F}cl$ set.

(ii) $\lambda$ is a fuzzy $\xi$-set in $\mathcal{FTS}(\mathcal{X},\tau)$.

(iii) $\lambda$ is $\mathcal{F}cl$ set and there exist two $\mathcal{FCO}$ sets $\alpha,(1_{\mathcal{X}}-\beta)\in I^{\mathcal{X}}$ such that $\lambda=\alpha\wedge(1_{\mathcal{X}}-\beta)$.

*Proof.* (i) $\Rightarrow$ (ii): Let $\lambda \in I^{\mathcal{X}}$ be $\mathcal{FC}$. Since $\mathcal{FTS}(\mathcal{X},\tau)$ has a base of $\mathcal{FCO}$ sets, $\mu \in I^{\mathcal{X}}$ is also $\mathcal{FCO}$. Since $\lambda$ and $\mu$ are $\mathcal{FC}$ and $\mathcal{FTS}(\mathcal{X},\tau)$ is a Fuzzy semi-spectral space, $\lambda\wedge\mu$ is $\mathcal{FC}$. Since $\mu$ is $\mathcal{FCO}$ and $\lambda\wedge\mu$ is $\mathcal{FC}$, by (i) of Definition 4.8, $\lambda$ is $\mathcal{F}\Gamma\mathcal{C}o$. Therefore, $\lambda$ is a fuzzy $\xi$-set in $\mathcal{FTS}(\mathcal{X},\tau)$.

(ii) $\Rightarrow$ (iii): Since $\lambda$ is a fuzzy $\xi$-set, $\lambda$ is fuzzy closed and fuzzy co-$\Gamma$-compact open. Thus $1_{\mathcal{X}}-\lambda$ is fuzzy $\Gamma$-compact open. By hypothesis on $\mathcal{FTS}(\mathcal{X},\tau)$, there is a fuzzy compact open set $\alpha \in I^{\mathcal{X}}$. Thus $\alpha\wedge(1_{\mathcal{X}}-\lambda)=\beta$ (say). Since $1_{\mathcal{X}}-\lambda$ is $\mathcal{F}\Gamma\mathcal{C}o$ and by (i) of Definition 4.8, $\beta$ is $\mathcal{FC}$. Therefore, $\alpha\wedge(1_{\mathcal{X}}-\lambda)=\beta$ which implies that $\alpha-\lambda=\beta$, so $\alpha-\beta=\lambda$. Thus we have $\alpha\wedge(1_{\mathcal{X}}-\beta)=\lambda$.

(iii) $\Rightarrow$ (i): Suppose $\lambda$ is $\mathcal{F}cl$ set and also there exist two $\mathcal{FCO}$ sets $\alpha,(1_{\mathcal{X}}-\beta)\in I^{\mathcal{X}}$ such that $\lambda=\alpha\wedge(1_{\mathcal{X}}-\beta)$. Since $\mathcal{FTS}(\mathcal{X},\tau)$ is fuzzy semi-spectral and $\alpha,(1_{\mathcal{X}}-\beta)$ is $\mathcal{FC}$, $\lambda$ is also $\mathcal{FC}$. Hence $\lambda$ is $\mathcal{FC}$ and $\mathcal{F}cl$ set in $\mathcal{FTS}(\mathcal{X},\tau)$. $\square$

## 5. Conclusion

The concepts of Fuzzy Spectral spaces, Fuzzy Semi-Spectral Spaces, Fuzzy Quasi-Spectral Spaces, fuzzy special morphisms and fuzzy saturated sets are introduced. Further, the concept of fuzzy one-point compactification is discussed in Fuzzy Semi-Spectral Spaces. Some equivalent statements in Fuzzy Semi-Spectral Spaces are also given. Furthermore, this concept can be extended to many areas in fuzzy topology.

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