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OPTIMAL CONTROL POLICY APPLIED TO A PEST MANAGEMENT

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Abstract. This work aims to present the fractional pest management model. The fractional Caputo derivative is employed. Fundamental properties of the considered model involving its equilibria, as well as the existence and boundedness of the solutions, are investigated. Furthermore, optimal control theory is used to determine a cost function and the best economic costs. This is achieved through the maximum principle of Pontryagin, which characterizes the optimal solution analytically, and then computes the optimal solution numerically by employing a forward-backward method. Finally, some numerical simulations and their graphical representations are provided for a better insight into the results of the suggested model.

1. Introduction

Natural pest control is an important aspect of ecosystem management that preserves the stability of agriculture and has the potential to reduce or relieve pest control costs. The control methods, namely chemical and biological controls, are discussed in [33]. The chemical method can cause numerous negative impacts or effects, for example. The production of drug resistance, serious and

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significant environmental defilement or pollution, a defect in the ecological environment equilibrium, and putting human health a risk and future health problems. While the biological control can be corrupted and decomposed into native species. It can also affect the food web interaction. The management of pests is a compelling and challenging dilemma that humans have to face. So that an optimal management method is needed to reduce crop losses and the risk of pollution to gain the best influence and impact with lower and minimum cost.

There are many researchers and studies for optimal control management and harvesting, see [7, 9, 16, 29, 34]. In [33], the author studied and investigated the optimization of pesticides by applying impulsive differential equations to analyze and establish a mathematical model of spraying pesticides. He also considered the crops validation model. In [32], the authors studied a mathematical prey-predator system that excluded crop pests by applying natural enemies. They also analyzed and investigated the behavior of the equilibria of the considered model. Further, they modified the model in the presence of the control factor to obtain the optimal pest and examine the impact on the crop pest. This is done by using several chemical, physical, and biological controls.

Losses and damages to crop productivity due to pathogens, weeds, and animal pests can be intrinsic, and they can be reduced or prevented through crop protection and preservation measures. [23] gave an insight into several kinds of crop damages as well as different ways of pest control that have been improved during the last century. Ihza et al. [11] evaluated and studied the implementation of various controls. Their mathematical system described the fluctuations between insects and plants. They also discussed the spread of the pathogens. All optimal planning and designs achieved in controlling insects; their significant results are to reduce the potential loss from 65.36 % to 6.12% [30]. The author used and applied the optimal control theory to control pest problems. He controlled the net birth rate of the pests chemically by giving several numerical examples to allow the predators to operate. [34] investigated and studied the optimal economic policy of an insect pest through developing the spatial and dynamic of and dynamic bioeconomic models in the presence of natural enemies.

Scientists use living creatures to destroy or conceal the pest community. This is done by biological control. [6] et al. developed an optimal control to diminish the pest density simultaneously through optimal management effort. The biological control strategy is given in [19]. They implemented a combination strategy of susceptible and infective individuals in the model if the density of susceptible (pests) reaches the economic threshold.

The main important results are to apply and use an SI infectious disease system to integrate pest management, and several pest control policies are also stated, [13] and et al. studied and investigated a coffee berry borer model to optimize the application of the fungus. This is done through using a modelling approach. They also analyzed the stability of its equilibria, and established the conditions of optimality. They used the BOCOP software to get the numerical solution to the optimal control problem. In [33], the author used the optimal chemical control method to control agricultural pests.

Many papers use the optimal control theory to find and determine with chemical control method to achieve optimal pest management, see [7, 13]. They investigated a mathematical system of plants and studied two different control policies, namely, the removal and reduction policies. The results show that under a constant environment, if the system has a stable equilibrium, the optimal solution could be easier to employ. This is the case when the system presents chaotic dynamics under a non-constant environment. In [9], the authors investigated the mathematical model of the Jatropha curcas plantation, formulated, and investigated the application of integrated pesticides to control natural pests.

The model of Lotka-Volterra has been widely applied in biological study, the financial industry, resource development, the national economy, and other fields [15, 18, 27]. Therefore, many researchers modified and extended the Lotka-Volterra model to an optimal control problem and studied these topics widely with using ordinary differential equations and difference equations as well as fractional-order equations [2, 3, 4, 5, 20, 21]. Fractional derivatives have several different definitions; see [17, 24, 25, 26]. The Caputo definition of fractional derivatives is widely employed in real-world applications [27].

The structure of this paper is as follows: in Section two, a fractional predator-pest model is considered and investigated. We also prove that all its solutions are bounded. In addition, the behavior of its equilibria is studied. In section three, the model is extended to a discrete optimal control problem and solved by using the Pontryagin Maximum Principle. In Section four, the simulations are given to confirm the theoretical outcomes and solve the optimality problem. Finally, in Section five, concluding remarks are given.

2. Fractional pest model formulation and analysis

We consider and investigate a fractional predator-pest model. We start with the following model:

$$\begin{cases} \frac{dx_1}{d\tau} = r_1x_1(1-\frac{x_1}{k_1}) - vx_1 - a_1x_1x_2, \\ \frac{dx_2}{d\tau} = \ r_2x_2(1-\frac{x_2}{k_2}) - lvx_2 + b_1x_1x_2, \end{cases} \tag{2.1}$$

where $x_1(\tau), x_2(\tau), r_1,\ r_2, a_1, v, b_1$, and l represent the size population of the pest, the size population of the predator, the growth rate of the pest, the growth rate of the predator, the capture rate of the predator, the magnitude of the insecticide control effort and the fraction of conversion of the pest into predator, a comparative proportion of the effectiveness of the insecticide on predators and the pest, respectively. The parameters k_1 and k_2 are the maximum carrying capacity of the pest and predator, respectively. The following transformation is used for the dimensionless model (2.1) as follows: $x_1 = k_1 x,\ x_2 = k_2 y$ and $t = r_1\tau$, then the obtained system is as follows:

$$\begin{cases} \frac{dx}{dt} = x(1-x) - ux - axy, \\ \frac{dy}{dt} = ry(1-y) - luy + bxy, \end{cases} \tag{2.2}$$

where $r = \frac{r_2}{r_1}$, $u = \frac{v}{r_1}$, $a = \frac{a_1 k_2}{r_1}$ and $b = \frac{b_1 k_1}{r_1}$.

Now, we use and introduce the Caputo derivative type [12, 25] to modify system (2.2) to a fractional-order system. For $\alpha \in (0,1)$, the system (2.2) is given by:

$$\begin{cases} \mathcal{D}_t^{\alpha} x(t) = x(1-x) - ux - axy, \\ \mathcal{D}_t^{\alpha} y(t) = ry(1-y) - luy + bxy. \end{cases} \tag{2.3}$$

Theorem 2.1. *For each $\beta_0 = (x_0, y_0) \in \Omega = \left\{(x,y) \in \mathbb{R}^2 | \max\ \{|x|, |y|\}\ \le \Upsilon\right\}$, where Υ is sufficiently large, there is a unique solution $\beta = (x,y) \in \Omega$ of the system* (2.3) *with the initial condition β_0, that is defined for all $t \ge 0$.*

Proof. Assume that $\beta = (x,y)$, $\widetilde{\beta} = (\tilde{x}, \tilde{y})$ and a mapping $\mathcal{K}(\beta) = (\mathcal{K}_1(\beta), \mathcal{K}_2(\beta))$, where

$$\mathcal{K}_1(\beta) = x(1-x) - ux - axy$$

and

$$\mathcal{K}_2(\beta) = ry(1-y) - luy + bxy.$$

For any $\beta,\ \widetilde{\beta} \in \Omega$, we have

$$\begin{aligned}
\left\|\mathcal{K}(\beta)-\mathcal{K}(\widetilde{\beta})\right\| &= \left|\mathcal{K}_1(\beta)-\mathcal{K}_1\left(\widetilde{\beta}\right)\right| + \left|\mathcal{K}_2\left(\beta\right)-\mathcal{K}_2\left(\widetilde{\beta}\right)\right| \\
&= \left|\left[x\left(1-x\right)-ux-axy\right]-\left[\tilde{x}\left(1-\tilde{x}\right)-u\tilde{x}-a\tilde{x}\tilde{y}\right]\right| \\
&\quad + \left|\left[ry\left(1-y\right)-luy+bxy\right]-\left[r\tilde{y}\left(1-\tilde{y}\right)-lu\tilde{y}+b\tilde{x}\tilde{y}\right]\right| \\
&= \left|x-x^2-ux-axy-\tilde{x}+\tilde{x}^2+u\tilde{x}+a\tilde{x}\tilde{y}\right| \\
&\quad + \left|ry-ry^2-luy+bxy-r\tilde{y}+r\tilde{y}^2+lu\tilde{y}-b\tilde{x}\tilde{y}\right| \\
&= \left|\left(x-\tilde{x}\right)-\left(x^2-\tilde{x}^2\right)-u\left(x-\tilde{x}\right)-a\left(xy-\tilde{x}\tilde{y}\right)\right| \\
&\quad + \left|r\left(y-\tilde{y}\right)-r\left(y^2-\tilde{y}^2\right)-lu\left(y-\tilde{y}\right)+b(xy-\tilde{x}\tilde{y})\right|.
\end{aligned}$$

By applying the triangle inequality and noticing that max $\{|x|,|y|\} \leq \varUpsilon$, we can show that

$$\begin{aligned}
\left\|\mathcal{K}\left(\beta\right)-\mathcal{K}\left(\widetilde{\beta}\right)\right\| &\leq |x-\tilde{x}|+2\varUpsilon\,|x-\tilde{x}|+u\,|x-\tilde{x}|+a\,|xy-\tilde{x}\tilde{y}| \\
&\quad + r\,|y-\tilde{y}|+2r\varUpsilon\,|y-\tilde{y}|+lu\,|y-\tilde{y}|+b\,|xy-\tilde{x}\tilde{y}| \\
&= |x-\tilde{x}|+2\varUpsilon\,|x-\tilde{x}|+u\,|x-\tilde{x}| \\
&\quad + a\,|xy-\tilde{x}\tilde{y}+x\tilde{y}-x\tilde{y}|+r\,|y-\tilde{y}| \\
&\quad + 2r\varUpsilon\,|y-\tilde{y}|+lu\,|y-\tilde{y}|+b\,|xy-\tilde{x}\tilde{y}+x\tilde{y}-x\tilde{y}| \\
&\leq |x-\tilde{x}|+2\varUpsilon\,|x-\tilde{x}|+u\,|x-\tilde{x}| \\
&\quad + a\varUpsilon\,|y-\tilde{y}|+a\varUpsilon\,|x-\tilde{x}|+r\,|y-\tilde{y}| \\
&\quad + 2r\varUpsilon\,|y-\tilde{y}|+lu\,|y-\tilde{y}|+b\varUpsilon\,|y-\tilde{y}|+b\varUpsilon\,|x-\tilde{x}| \\
&= \left(1+2\varUpsilon+u+a\varUpsilon+b\varUpsilon\right)|x-\tilde{x}| \\
&\quad + \left(r+2r\varUpsilon+lu+a\varUpsilon+b\varUpsilon\right)|y-\tilde{y}| \\
&= L\ \left\|\beta-\widetilde{\beta}\right\|,
\end{aligned}$$

where $L = \max\ \left\{\left(1+2\varUpsilon+u+a\varUpsilon+b\varUpsilon\right),\ \left(r+2r\varUpsilon+lu+a\varUpsilon+b\varUpsilon\right)\right\}$. Therefore, $\mathcal{K}\left(\beta\right)$ has the Lipschitz condition. Then, by Lemma in [14], the model (2.3) has a unique solution. □

Theorem 2.2. *Assume that $B_+ = \{\ (x,y)\,|x \geq 0,\ y \geq 0\}$ represents all non-negative real number in $\mathbb{R}^2$. Then for each solution to the system* (2.3) *such that $x_0 \geq 0, y_0 \geq 0$ is non-negative and uniformly bounded.*

Proof. First to show that $x\left(t\right) \geq 0$ for all $t \geq 0$, let $x\left(0\right) > 0$ with $t = 0$. Suppose that $x\left(t\right) \geq 0$ for all $t \geq 0$ is not true. Hence, there is a constant

$t_1 > 0$ with

$$\begin{aligned} x(t) &> 0 \qquad 0 \le t < t_1, \\ x(t) &= 0 \qquad t = t_1, \\ x(t) &< 0 \qquad t > t_1. \end{aligned}$$

So, we have

$$\mathcal{D}_{t_1}^{\alpha} x\,(t)\,|_{t=t_1} = 0.$$

By Lemma in [22], we have $x\left(t_1^+\right) = 0$, that contradicts with the fact that $x\left(t_1^+\right) < 0$, that is, $x\,(t) < 0, \quad t > t_1$. It follows that $x\,(t) \ge 0$ for all $t \ge 0$. We can apply the same procedure to show that $y\,(t) \ge 0$ for all $t \ge 0$.

Now, let $\omega(t) = x(t) + \frac{a}{b}\, y(t)$, Then, for all $d > 0$ it follows

$$\begin{aligned} \mathcal{D}_t^{\alpha}\omega\,(t) + d\omega\,(t) =&\ \mathcal{D}_t^{\alpha} x\,(t) + \frac{a}{b}\,\mathcal{D}_t^{\alpha} y\,(t) + dx\,(t) + \frac{ad}{b}\, y\,(t) \\ &= x\,(1-x) - ux - axy + \frac{a}{b}\, ry\,(1-y) \\ &\quad - \frac{a}{b} luy + \frac{a}{b}\, bxy + dx + \frac{ad}{b}\, y \\ &= -x^2 + x - ux + dx - \frac{ar}{b}\, y^2 + \frac{ar}{b}\, y - \frac{alu}{b}\, y + \frac{ad}{b}\, y \\ &= -x^2 + (1-u+d)\,x - \frac{ar}{b}\, y^2 + \frac{ar}{b}\,\left(1 - \frac{lu}{r} + \frac{d}{r}\right) y \\ &= -x^2 + (1-u+d)\,x - \frac{(1-u+d)^2}{4} + \frac{(1-u+d)^2}{4} \\ &\quad - \frac{ar}{b}\left[y^2 - \left(1 - \frac{lu}{r} + \frac{d}{r}\right) y - \frac{\left(1 - \frac{lu}{r} + \frac{d}{r}\right)^2}{4} + \frac{\left(1 - \frac{lu}{r} + \frac{d}{r}\right)^2}{4}\right] \\ &= -\left[x^2 - (1-u+d)\,x + \frac{(1-u+d)^2}{4}\right] \\ &\quad - \frac{ar}{b}\,\left[y^2 - \left(1 - \frac{lu}{r} + \frac{d}{r}\right) y + \frac{\left(1 - \frac{lu}{r} + \frac{d}{r}\right)^2}{4}\right] \\ &\quad + \frac{ar\left(1 - \frac{lu}{r} + \frac{d}{r}\right)^2}{4b} + \frac{(1-u+d)^2}{4} \\ &= -\left[x - \frac{(1-u+d)}{2}\right]^2 - \frac{ar}{b}\,\left[y - \frac{\left(1 - \frac{lu}{r} + \frac{d}{r}\right)}{2}\right]^2 \\ &\quad + \frac{b(1-u+d)^2 + ar\left(1 - \frac{lu}{r} + \frac{d}{r}\right)^2}{4b}. \end{aligned}$$

Therefore,

$$\mathcal{D}_t^{\alpha}\omega\,(t)+d\omega(t)\leq\frac{b(1-u+d)^2+ar\big(1-\frac{lu}{r}+\frac{d}{r}\big)^2}{4b}=\xi.$$

Now, using Lemma in [15], we have

$$\omega(t)\leq(\omega(0)-\ \frac{\xi}{d})E_{\alpha}(-dt^{\alpha})+\frac{\xi}{d},$$

where, E_α represents the Mittag-Leffler function. Now, $E_{\alpha}\left(-dt^{\alpha}\right)\longrightarrow 0$ as $t\longrightarrow\infty$, (see lemma and corollary in [8]) we have $\omega\,(t)\leq\frac{\xi}{d}$, $t\longrightarrow\infty$. It follows that every solution to the system (2.3) starts from G is confined in

$$\mathrm{G}=\left\{(x,y)\in\mathbb{R}^2\mid\omega\,(t)\leq\frac{\xi}{d}\ +\varepsilon\ \text{ for any }\ \varepsilon>0\right\}.$$

□

Let $\mathcal{D}_t^{\beta}x\,(t)=g(x\,(t))$, $x\,(0)=x_0$, be any fractional-order differential system such that $\beta\in(0,1)$ and $x\in D\subset\mathbb{R}^n$. Any solution to the equation $g\,(x)=0$ is called an equilibrium point E^*. E^* is called locally asymptotically stable, if $|\arg{(\lambda_i)}|>\frac{\beta\pi}{2}$ for all λ_i, where λ_i are the eigenvalues of the Jacobian matrix $J(E^*)$. Otherwise, it is unstable, see [17].

After solving the following equations:

$$\mathcal{D}_t^{\alpha}x\,(t)=0,$$
$$\mathcal{D}_t^{\alpha}y\,(t)=0.$$

We get all possible equilibrium points of the system (2.3) as follows:

(1) The point $E_0=(0,0)$ is always feasible.
(2) The point $E_1=(1-u,\ 0),\ 0<u<1$ always exists.
(3) The point $E_2=(0,\ \frac{r-lu}{r})$, $0<u<1$, $0<l<1$ always exists.
(4) The interior point $\overline{E}=(\overline{x},\ \overline{y})$, where $\overline{x}=\frac{r+alu-ar-ru}{ab+r}$ positive if $r+alu>ar+ru$ and $\overline{y}=\frac{1-u-\overline{x}}{a}$, $\overline{E}$ positive if $\overline{x}<(1-u)$.

The local stability of system 2.3 around its equilibria will be discussed by computing the general Jacobian matrix $\mathcal{J}$ of model (2.3),

$$\mathcal{J}\,(x,\ y)=\begin{bmatrix}1-u-2x-ay & -ax\\ by & r-lu-2ry+bx\end{bmatrix}$$

the polynomial $P(\lambda)=\lambda^2+\mathrm{T}\lambda+D=0$ represents the characteristic polynomial of the Jacobian $\mathcal{J}\,(x,\ y)$, where

$$\mathrm{T}=\ u-1-r+2x+lu+2ry-bx+ay$$

and

$$\begin{aligned} D =& r - lu - 2ry + bx - ur + lu^2 + 2ruy - bux - 2rx \\ &+ 2lux + 4rxy - 2bx^2 - ary + aluy + 2ray^2. \end{aligned}$$

Theorem 2.3. *For the system* (2.3)*, we have the followings:*

(1) *The equilibrium point* $E_0 = (0,\ 0)$ *is unstable point.*

(2) *The equilibrium point* $E_1 = (1-u, 0)$ *is locally stable point if* $r + b < lu + bu$.

(3) *The equilibrium point* $E_2 = \left(0,\ \frac{r-lu}{r}\right)$ *is locally stable point if* $1+\frac{alu}{r} < u + a$ *and* $lu < r$.

Proof. (1): The Jacobian $\mathcal{J}$ at E_0 is:

$$\mathcal{J}\left(E_0\right) = \left[\begin{array}{cc} 1-u & 0 \\ 0 & r-lu \end{array}\right].$$

The eigenvalues of $J\left(E_0\right)$ are $\lambda_1 = 1-u$ and $\lambda_2 = r - lu$. Hence, $|arg\lambda_i| < \frac{\alpha\pi}{2},\ i = 1,\ 2$. Then the point $E_0 = (0,\ 0)$ is unstable point.

(2): Clearly, the Jacobian $\mathcal{J}$ at E_1 is:

$$\mathcal{J}\left(E_1\right) = \left[\begin{array}{cc} -(1-u) & -a(1-u) \\ 0 & r-lu+b-bu \end{array}\right]$$

and the eigenvalues of $\mathcal{J}\left(E_1\right)$ are $\lambda_1 = -(1-u)$ and $\lambda_2 = r - lu + b - bu$. Since $0 < u < 1$ then λ_1 will be negative, so $|arg\lambda_1| = \pi > \frac{\alpha\pi}{2}$. Now, if $r + b < lu + bu$, then $|arg\lambda_2| = \pi > \frac{\alpha\pi}{2}$. Then the point E_1 is locally asymptotically stable.

(3): The Jacobian $\mathcal{J}$ at E_2 is:

$$\mathcal{J}\left(E_2\right) = \left[\begin{array}{cc} 1-u-\frac{a(r-lu)}{r} & 0 \\ \frac{b(r-lu)}{r} & lu-r \end{array}\right].$$

The eigenvalues of $\mathcal{J}\left(E_2\right)$ are $\lambda_1 = 1-u-\frac{a(r-lu)}{r}$ and $\lambda_2 = lu - r$. Now, if $1+\frac{alu}{r} < u+a$ and $lu < r$. Then λ_1 and λ_2 are negative values, respectively, and $|arg\lambda_i| > \frac{\alpha\pi}{2},\ i = 1,\ 2$. Then the point E_2 is locally asymptotically stable. ☐

Theorem 2.4. *The equilibrium point* $\overline{E} = \left(\overline{x},\ \overline{y}\right),\ \overline{x} = \frac{r+alu-ar-ru}{ab+r},\ \overline{y} = \frac{1-u-\overline{x}}{a}$, *of the system* (2.3) *is locally asymptotically stable point if one of the following conditions satisfies:*

(1) *If* $B > 0$,

(2) *If* $B < 0,\ 4C > B^2,\ \left|tan^{-1}\frac{\sqrt{4C-B^2}}{B}\right| > \frac{\alpha\pi}{2}$,

where $B =\ lu - r + 2r\overline{y} - b\overline{x} + \overline{x}$ *and* $C = \overline{x}B - \overline{x}^2 + ab\overline{x}\overline{y}$.

Proof. The Jacobian $\mathcal{J}$ at $\overline{E}$ is:

$$\mathcal{J}\left(\overline{E}\right)=\left[\begin{array}{cc} 1-u-2\overline{x}-a\overline{y} & -a\overline{x} \\ b\overline{y} & r-lu-2r\overline{y}+b\overline{x} \end{array}\right].$$

The characteristic equation of the Jacobian $\mathcal{J}\left(\overline{E}\right)$ is

$$\lambda^2+B\lambda+C=0,$$

where $B=\ lu-r+2r\overline{y}-b\overline{x}+\overline{x}$, $C=\overline{x}B-\overline{x}^2+ab\overline{xy}.$

Now, the eigenvalues of $\mathcal{J}\left(\overline{E}\right)$ are: $\lambda_{1,2}=\frac{-B\mp\sqrt{B^2-4C}}{2}$. If $B>0$, then $C>0$ and the Routh-Hurwitz conditions [1] are satisfied, then $|arg\lambda_i|=\pi>\frac{\alpha\pi}{2}$, $\alpha\in(0,1)$. Therefore, the point $\overline{E}$ is locally asymptotically stable. Also, if $B<0$, $4C>B^2$ and if $|\ arg\lambda_{1,2}|=\left|tan^{-1}\frac{\sqrt{4C-B^2}}{B}\right|>\frac{\alpha\pi}{2}$ holds, then $\overline{E}$ is locally asymptotically stable. □

3. THE OPTIMAL CONTROL STRATEGY

Let $x(0)=x_0$ and $y(0)=y_0$ be the initial conditions of the system (2.3). Then, its discretization with piecewise constant arguments, see [10], is as follows:

$$\begin{cases} x_{n+1}=x_n+\delta\ (x_n(1-x_n)-ux_n-ax_ny_n), \\ y_{n+1}=y_n+\delta(ry_n(1-y_n)-luy_n+bx_ny_n), \end{cases} \tag{3.1}$$

where, $\delta=\ \frac{s^\alpha}{\alpha\Gamma(\alpha)}$. We introduce the cost function, which is given by $J(u_t)=\sum_{t=0}^{T-1}(cx_t+du_t^2)$. That relates to the presence of prey on the crop; the cost relates to using insecticide, including the current economic cost of employing pesticides. T is the time of application of the control. The u_t refers to the control variable at time t. Limiting aspect is taken of insecticide control effort such that $0<u_t\leq u_{Max}<1$. The parameters c and d are positive constants.

The state equations of the optimal control problem are:

$$\begin{cases} x_{t+1}=x_t+\delta\ (x_t(1-x_t)-u_tx_t-ax_ty_t), \\ y_{t+1}=y_t+\delta(ry_t(1-y_t)-lu_ty_t+bx_ty_t), \end{cases} \tag{3.2}$$

where, the parameters δ,r,a,b,l and u_t are defined as before. It is clear that if $l\ =0$ matches a pesticide harmless to the predator, otherwise the predator effects at a rate lu_ty_t.

The discrete version of the maximum principle of Pontryagin is used for this problem [28, 31]. The Hamiltonian function H_t for $t=0,1,\ldots,T-1$, which is written by:

$$\begin{aligned} H_t\ &=cx_t+du_t^2+\lambda_{t+1}\left(x_t+\delta\ \ \left(x_t\left(1-x_t\right)-u_tx_t-ax_ty_t\right)\right) \\ &\quad+\mu_{t+1}\left(y_t+\delta\left(ry_t\left(1-y_t\right)-lu_ty_t+bx_ty_t\right)\right), \end{aligned}$$

where, λ_t and μ_t are the adjoint variables.

Now, by the maximum principle of Pontryagin procedure. The necessary conditions are:

$$\lambda_t = \frac{\partial H_t}{\partial x_t} = c + \lambda_{t+1}\left(1 + \delta\left[1 - 2x_t - u_t - ay_t\right]\right) + \mu_{t+1}\left(b\delta y_t\right),$$
$$\mu_t = \frac{\partial H_t}{\partial y_t} = \lambda_{t+1}\left(-a\delta x_t\right) + \mu_{t+1}\left(1 + \delta\left[r - 2ry_t - lu_t + bx_t\right]\right)$$

and $\frac{\partial H_t}{\partial u_t} = 0$, that is, $2du_t - \lambda_{t+1}\delta x_t - \mu_{t+1}\delta l y_t = 0$, which implies that

$$u_t = \frac{\lambda_{t+1}\delta x_t + \mu_{t+1}\delta l y_t}{2d}.$$

After applying the boundary conditions for control harvesting, we get:

$$u_t^* = \begin{cases} 0 & \frac{\lambda_{t+1}\delta x_t + \mu_{t+1}\delta l y_t}{2d} < 0, \\ \frac{\lambda_{t+1}\delta x_t + \mu_{t+1}\delta l y_t}{2d} & 0 < \frac{\lambda_{t+1}\delta x_t + \mu_{t+1}\delta l y_t}{2d} \leq u_{max}, \\ u_{max} & u_{max} < \frac{\lambda_{t+1}\delta x_t + \mu_{t+1}\delta l y_t}{2d}. \end{cases}$$

4. NUMERICAL OUTCOMES

In this section, we present a numerical study to confirm the theoretical results and to investigate the behaviors of the solution to the fractional pest-predator model. We give example to study the stability of equilibria. We also solve the optimality problem.

Example 4.1. We give different values of parameters in Table 1 that are used to illustrate and show the local stability of equilibrium points E_1, E_2 and $\overline{E}$ of the fractional predator-pest model (2.3). The initial conditions $(x_0, y_0) =$ $(0.7, 0.5)$, $(0.5, 0.9)$ and $(5.8, 0.6)$ respectively. These are done according to Theorem 2.3. Figure 1 illustrates the stability of the points E_1, E_2, and $\overline{E}$, respectively.

TABLE 1. The parameter's value for the points E_1, E_2, and $\overline{E}$ with different values of α

The value of Parameter	E_1	E_2	$\overline{E}$
a	0.8	0.8	0.6
b	0.2	0.5	0.5
u	0.7	0.4	0.2
l	0.3	0.1	0.1
r	0.1	0.9	1.1

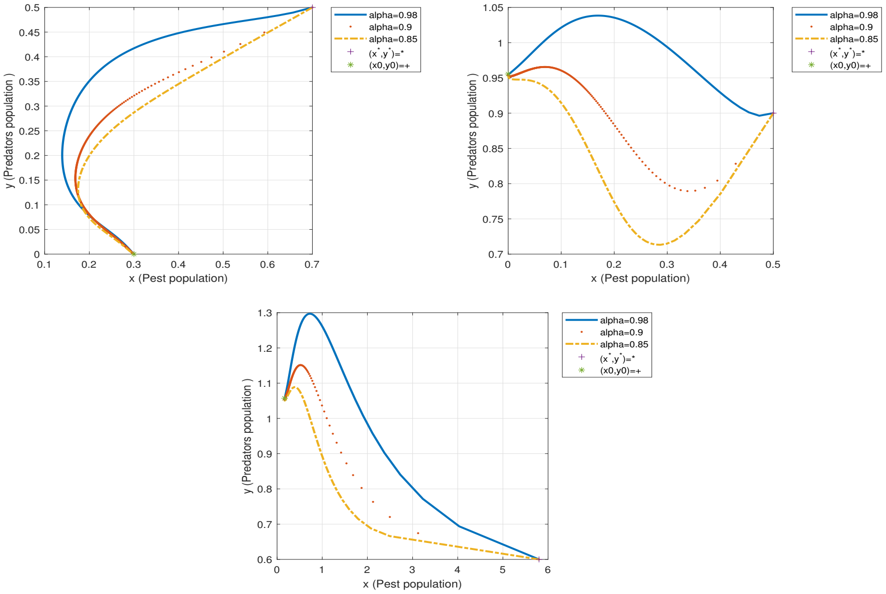

FIGURE 1. The local stability of the points E_1, E_2 and $\overline{E}$ at different values of α, respectively.

Now, we solve the optimality problem, so we will discuss three cases. In case 1, we illustrate that pesticide control is vital to the pest (prey) population, which means using insecticide control $l = 0$, but does not kill the predator population. In case 2, we assume that the insecticide control affects to kill the pest and predator population at the same rate by using the insecticide control l=1. In the last case, we assume that the parameter l is between 0 and 1. In our simulation, we chose l=0.1. which means that the pesticide control affects or kills the predator by a factor l. The values of the parameters that are employed in our simulation are set in Table 2.

TABLE 2. Parameters' values for the optimal control problem.

Parameters	Values
a	0.6
b	0.5
l	$0, 0.1, 1$
u_t	u^*
r	1.1

The results in cases 1, 2, and 3 reveal that the total cost functional (J) is 0.17103, 0.182917, and 0.273713 at the optimal control path, respectively. In

Table 3, we give the total cost functional (J) at other strategies for the three cases.

Table 3. The optimal cost functional is compared with other strategies.

Different strategies	The cost function (J) with l =0	The cost function (J) with l =0.1	The cost function (J) with l =1
$u_t = u^*$	$J_{opt} = 0.171034$	$J_{opt} = 0.182917$	$J = 0.273713$
$u_t = 0.1$	J =0.240057	J =0.243968	$J = 0.279350$
$u_t = 0.13$	J =0.225099	$J = 0.230147$	$J = 0.275941$
$u_t = 0.15$	$J = 0.216212$	$J = 0.222003$	$J = 0.274666$
$u_t = 0.2$	J =0.197897	$J = 0.205468$	$J = 0.274972$
$u_t = 0.31$	$J = 0.179453$	J =0.1898975	J =0.293255
$u_t = 0.35$	J =0.182362	$J = 0.192752$	$J = 0.305924$
$u_t = 0.4$	J =0.196916	$J = 0.2053759$	$J = 0.326295$
$u_t = 0.5$	J =0.265414	J =0.2684055	J =0.382250

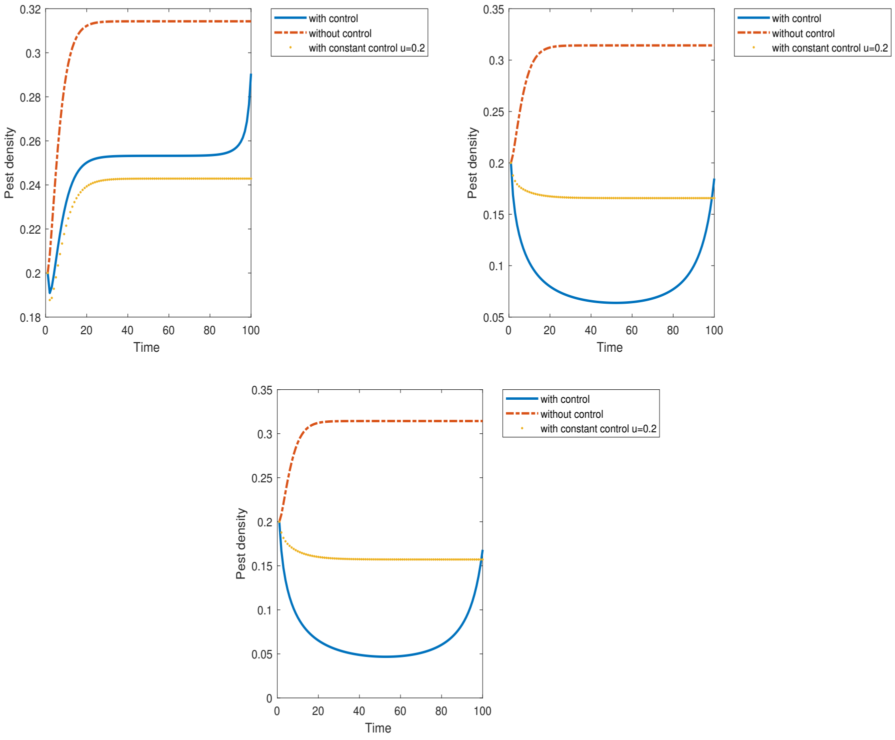

Figure 2. The pest population with different values of l, $l = 0$, $l = 0.1$ and $l = 1$, respectively.

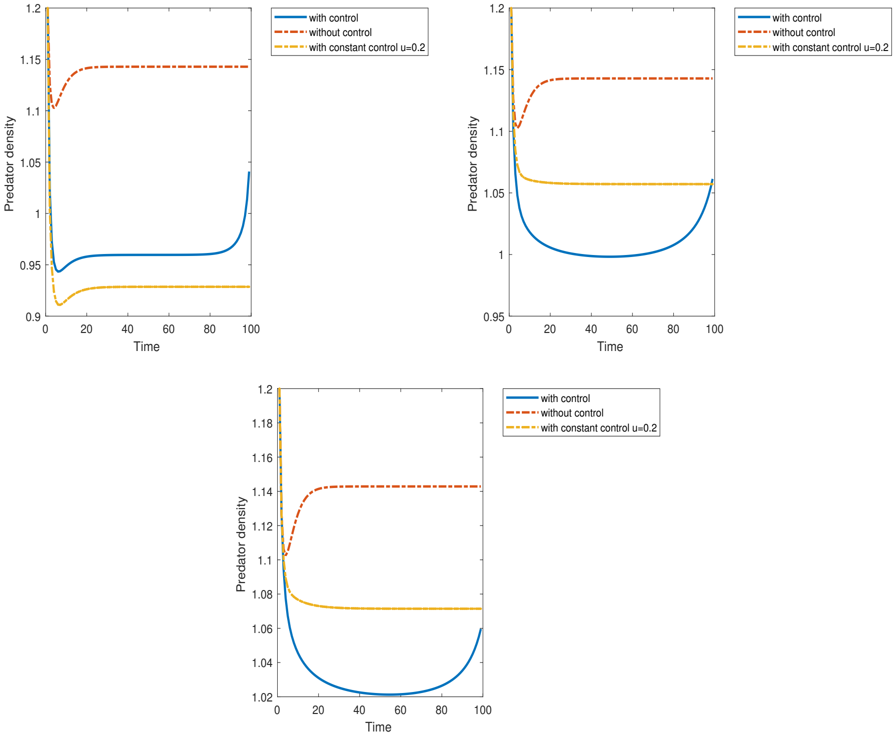

FIGURE 3. The predator population with different values of l, $l = 0$, $l = 0.1$ and $l = 1$, respectively.

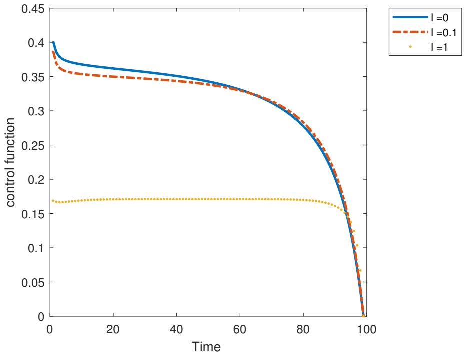

FIGURE 4. The control function with different values of l, namely, $l = 0$, $l = 0.1$ and $l = 1$.

5. Conclusions

In this article, a fractional order pest management model is discussed and investigated. All possible equilibrium points of our proposed fractional-order system have been determined. Moreover, we have extended the model to the optimal control problem using the discrete-time Pontryagin Maximum Principle. It has been observed that when l=0, the effect of the pesticide used is harmless to predators, and when l = 1, the effect on both remains the same. This is based on the numerical results of the model considered.

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