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CONTEMPORARY METHODS OF COMPUTATIONAL NONLINEAR COUPLED VARIATIONAL-HEMIVARIATIONAL INEQUALITY PROBLEMS

Jong Kyu Kim[1] **and Salahuddin**[2]

[1]Department of Mathematics Education, Kyungnam University,
Changwon Gyeongnam 51767, Korea
e-mail: `jongkyuk@kyungnam.ac.ac.kr`

[2]Department of Mathematics, College of Science, Jazan University,
Jazan-45142, P.O. Box 114, Kingdom of Saudi Arabia
e-mail: `dr_salah12@yahoo.com; smohammad@jazanu.edu.sa`

Abstract. In this paper, we introduce nonlinear coupled variational-hemivariational inequality problems on Banach spaces. These problems are defined by hemivariational and variational inequality equations that are mildly nonlinear. We then apply the Tychonoff fixed point theorem to demonstrate that solutions to these nonlinear coupled variational-hemivariational inequality problems that are mildly nonlinear are both nonempty and compact.

1. Historical background

We are studying a new class of variational-hemivariational inequality problems and their possible solutions. These inequalities involve a nonlinear operator, a convex potential, the generalized directional derivative of a locally Lipschitz function, and a constraint set. They also have nonlinear and mildly coupled constraints. Principal properties of nonlinear operators with mildly

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[0]Corresponding author: J. K. Kim(`jongkyuk@kyungnam.ac.kr`).

coupling nonlinearity and generalized directional derivatives are the main subjects of our study. Our results address the existence and compactness of the solution set for these equations through an alternative method, building upon recent results in [21].

It is interesting and significant to study the generalization of variational inequality proposed by Hassouni and Moudafi [13]. It gives us a unified, organic, novel, and general method to examine a broad range of problems that come up in various branches of engineering sciences, economics, and mathematics (see, for instance, [1, 3, 5, 15]). The theory of variational inequality recognizes the creation of an effective and workable algorithm for resolving a variety of variational inequalities and inclusions as one of its major and fascinating challenges. Numerous numerical techniques have been developed recently for solving different classes of variational inequalities and inclusions in Hilbert spaces or Euclidean spaces. These techniques include Newton's method, descent method, linear approximation, projection methods and their varient forms, Newton's method, and methods based on auxiliary principle techniques. Specifically, the approach based on the resolvent operator technique, a generalization of the projection method, has been applied extensively to the solution of variational inclusions.

A new class of intriguing problems known as the system of variational inequality problems was introduced and examined. Pang [24] examined a system of scalar variational inequalities and demonstrated that variational inequality modelling can be used to solve traffic equilibrium, spatial equilibrium, Nash equilibrium, and general equilibrium programming problems. He conducted a study on the convergence of such methods and broke down the original variational inequality into a system of easily solvable variations. The fixed point theorem was utilized by Chang et al. [6] to examine the solvability of system of nonlinear set-valued variational inclusions. In their analysis of a system of generalized variational inclusions, Salahuddin [25] proved the approximation salvability for the system of nonlinear variational type inclusions in Banach spaces. Kim et al. [17] introduced a system of parametric general regularized nonconvex variational inequalities in Banach spaces and studied the sensitivity of solutions for this system of general regularized nonconvex variational inequalities in Banach spaces.

In-depth research has been done on a number of variational-hemivariational inequalities in the literature. For instance, noncoercive hemivariational inequalities are examined in the context of equilibrium problems and discusses their application to problems in contact mechanics, *see* [19]. Furthermore, a variety of classes of variational-hemivariational and hemivariational inequalities pertinent to contact mechanics have been studied by [16, 22]. The work of [12] focuses on the analysis of singular perturbations in inequality problems, whereas [8] explores nonconvex star-shaped constraint sets within the context of evolution hemivariational inequalities. In this field, [18, 20] primarily addresses elliptic variational-hemivariational inequalities, while [4, 9, 23] investigate differential hemivariational inequalities.

Throughout the paper, we will utilize the functional framework delineated below to introduce the problem.

Let $(\Omega, \|\cdot\|_\Omega)$ and $(\mho, \|\cdot\|_\mho)$ be real reflexive Banach spaces, and let $\emptyset \neq \mathcal{U} \subset \Omega$ and $\emptyset \neq \mathcal{V} \subset \mho$ be closed and convex sets. Let $\mathscr{M}\colon \mho \times \Omega \to \Omega^*$ and $\mathscr{N}\colon \Omega \times \mho \to \mho^*$ be the nonlinear operators, and the convex functionals be $\varphi\colon \Omega \times \Omega \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ and $\phi\colon \mho \times \mho \to \bar{\mathbb{R}}$. Let us assume that $J\colon W_1 \times X \to \mathbb{R}$ and $I\colon W_2 \times Y \to \mathbb{R}$ are nonlinear function, and that there exist four linear operators $\zeta_1\colon \Omega \to X$, $\zeta_2\colon \mho \to Y$, $\lambda_1\colon \mho \to W_1$, and $\lambda_2\colon \Omega \to W_2$.

The nonlinear variational-hemivariational inequality problems are given below:

Problem 1.1. Find $u \in \mathcal{U}$ and $v \in \mathcal{V}$ that satisfy the inequality equations below:

$$\begin{aligned} &\langle \mathscr{M}(v,u), x-u\rangle_\Omega + J^0(\lambda_1 v, \zeta_1 u; \zeta_1(x-u)) + \varphi(x,u) - \varphi(u,u) \\ &\qquad \geq \langle f, x-u\rangle_\Omega, \quad \forall x \in \mathcal{U}, f \in \Omega^*, \end{aligned} \tag{1.1}$$

$$\begin{aligned} &\langle \mathscr{N}(u,v), y-v\rangle_\mho + I^0(\lambda_2 u, \zeta_2 v; \zeta_2(y-v)) + \phi(y,v) - \phi(v,v) \\ &\qquad \geq \langle g, y-v\rangle_\mho, \quad \forall y \in \mathcal{V}, g \in \mho^*. \end{aligned} \tag{1.2}$$

We present interesting specific cases of Problem 1.1.

(i) If $\varphi(u,u) \equiv \varphi(u)$ and $\phi(v,v) \equiv \phi(v)$, Problem 1.1 can be reduced to the following inequality problems: Find $(u,v) \in \mathcal{U} \times \mathcal{V}$ such that

$$\left\{ \begin{aligned} &\langle \mathscr{M}(v,u), x-u\rangle_\Omega + J^0(\lambda_1 v, \zeta_1 u; \zeta_1(x-u)) + \varphi(x) - \varphi(u) \\ &\qquad \geq \langle f, x-u\rangle_\Omega, \ \ \forall x \in \mathcal{U}, f \in \Omega^*, \\ &\langle \mathcal{N}(u,v), y-v\rangle_\mho + I^0(\lambda_2 u; \zeta_2 v; \zeta_2(y-v)) + \phi(y) - \phi(v) \\ &\qquad \geq \langle g, y-u\rangle_\mho, \ \ \forall y \in \mathcal{V}, g \in \mho^*. \end{aligned} \right. \tag{1.3}$$

This system was analyzed in [2].

(ii) If $J \equiv 0$ and $I \equiv 0$, Problem 1.1 transforms into the following system of variational inequality problems: Find $(u, v) \in \mathcal{U} \times \mathcal{V}$ such that

$$\begin{cases} \langle \mathscr{M}(v,u), x-u\rangle_\Omega + \varphi(x,u) - \varphi(u,u) \\ \qquad \geq \langle f, x-u\rangle_\Omega, \ \ \forall x \in \mathcal{U}, f \in \Omega^*, \\ \langle \mathscr{N}(u,v), y-v\rangle_V + \phi(y,v) - \phi(v,v) \\ \qquad \geq \langle g, y-v\rangle_\mho, \ \ \forall y \in \mathcal{V}, g \in \mho^*. \end{cases} \tag{1.4}$$

This system was investigated in [10].

In our study, we used a set-valued extension of the Tychonoff fixed-point principle in a Banach space with concepts from nonsmooth analysis, generalized monotonicity, the Minty lemma to addressed the convergence of solution for Problem 1.1.

2. Comprehensive background

In this section, we review the foundational concepts and results that will be utilized throughout this paper. First, we introduce the set-valued version of the Tychonoff fixed point principle in a Banach space, which is an essential tool for our analysis. We will then explore the fundamentals of nonsmooth analysis theory and its integration with generalized monotonicity and the Minty lemma. The combination of these ideas offers the mathematical structure required to solve the current problems.

Let $(\Omega, \|\cdot\|_\Omega)$ be a Banach space, Ω^* its dual space, and $\langle\cdot,\cdot\rangle_\Omega$ the duality pairing between Ω^* and Ω. Using the symbols $\rightharpoonup$ and $\to$, we denote the weak convergence and the strong convergence in distinct spaces, respectively.

First, we review the definitions and fundamental characteristics of upper semicontinuous set-valued operators, which are essential to our analysis.

Definition 2.1. ([9]) Let Y and W be topological spaces, let $\mathcal{V} \subset Y$ be a nonempty set, and let $\mathscr{G}\colon Y \to 2^W$ be a set-valued map.

(i) The map $\mathcal{G}$ is referred to as upper semicontinuous at $y \in Y$, if there is a neighborhood $N(y)$ of y that satisfies the following condition for each open set $\mathcal{O} \subset W$ such that $\mathcal{G}(y) \subset \mathcal{O}$, and

$$\mathcal{G}(N(y)) = \bigcup_{w \in N(y)} \mathcal{G}(w) \subset \mathcal{O}.$$

It is said that $\mathcal{G}$ is upper semicontinuous in $\mathcal{V}$ if it holds for each $y \in \mathcal{V}$.

(ii) The map $\mathcal{G}$ is closed at $y \in Y$, if for every sequence $\{(y_n, w_n)\} \subset Gr(\mathcal{G})$ satisfying

$$(y_n, w_n) \longrightarrow (y, w) \in Y \times W,$$

it holds

$$(y, w) \in Gr(\mathcal{G}),$$

where $Gr(\mathcal{G})$ is the graph of the map $\mathcal{G}$ dened by

$$Gr(\mathcal{G}) = \{(y, w) \in Y \times W \mid w \in \mathcal{G}(y)\}.$$

It is said that $\mathcal{G}$ is closed (or $\mathcal{G}$ has a closed graph) if it holds for each $y \in Y$.

Definition 2.2. ([10]) Let X_1 and X_2 be two Banach spaces. A set-valued map $\mathcal{F}\colon X_1 \to 2^{X_2}$ is called sequentially weak weakly-closed, if it is sequentially closed from X_1 endowed with the weak topology to the subsets of X_2 with the weak topology.

Proposition 2.3. ([2]) *The topological spaces X and Y should be associated with $\mathcal{F}\colon X \to 2^Y$. These conditions are equivalent:*

(i) *$\mathcal{F}$ is an upper semicontinuous.*
(ii) *Every closed set $\mathcal{U} \subset Y, \mathcal{F}^-(\mathcal{U}) = \{x \in X | \mathcal{F}(x) \cap \mathcal{U} \neq \emptyset\}$ is closed in X.*
(iii) *Every open set $\mathcal{O} \subset Y, \mathcal{F}^+(\mathcal{O}) = \{x \in X \mid \mathcal{F}(x) \subset \mathcal{O}\}$ is open in X.*

Definition 2.4. ([10]) Suppose Ω is a reexive Banach space, and $\varphi\colon \Omega \to \bar{\mathbb{R}}$ is a proper, convex and lower semi-continuous function. Let $\mathcal{M}\colon \Omega \to 2^{\Omega^*}$ be a set-valued operator. The operator $\mathcal{M}$ is called:

(i) φ-pseudomonotone, if $u, x \in \Omega$ are fixed, there exists a $u^* \in \mathcal{M}u$ such that

$$\langle u^*, x - u\rangle_X + \varphi(x) - \varphi(u) \geq 0.$$

Then we have

$$\langle x^*, x - u\rangle_X + \varphi(x) - \varphi(u) \geq 0, \ \ \forall x^* \in \mathcal{M}(x).$$

(ii) φ-stable-pseudomonotone with respect to the set $Z \subset \Omega^*$, if $\mathcal{M}$ and $\Omega \ni u \rightarrowtail \mathcal{M}u - v \subset \Omega^*$ are ψ-pseudomonotone for each $v \in Z$.

Definition 2.5. ([7]) Assume X is a Banach space with the dual space X^* and norm $\|\cdot\|_X$. A function $J\colon X \to \mathbb{R}$ is said to be a locally Lipschitz continuous at $u \in X$, if there is a neighborhood $N(u)$ of u and a constant $\measuredangle_u > 0$ with

$$|J(v) - J(x)| \leq \measuredangle_u \|v - x\|_X, \ \forall x, v \in N(u).$$

The generalized (Clarke) directional derivative of a locally Lipschitz function $J\colon X \to \mathbb{R}$ at a point $u \in X$ in the direction $x \in X$, denoted by $J^0(u;x)$, is defined as:

$$J^0(u;x) = \lim_{\rho\to 0^+} \sup_{v\to x} \frac{J(v+\rho x) - J(v)}{\rho}.$$

Additionally, the generalized subgradient of $J\colon X \to \mathbb{R}$ at $u \in X$ is given by

$$\partial J(u) = \{\xi \in X^* \mid J^0(u;x) \geq \langle \xi, x\rangle_{X^*\times X},\ \forall x \in X\}.$$

Proposition 2.6. ([22]) *Let $J\colon X \to \mathbb{R}$ be a locally Lipschitz function. Then*

(i) *the function $X \ni x \rightarrowtail J^0(x;v) \in \mathbb{R}$ is positively homogeneous and subadditive for every $x \in X$, i.e.,*

$$J^0(x;\rho v) = \rho J^0(x;v),\ \forall \rho \geq 0, v \in X,$$

$$J^0(x;v_1+v_2) = J^0(x;v_1) + J^0(x;v_2),\ \forall v_1, v_2 \in X, \textit{respectively.}$$

(ii) *the set $\partial J(x)$ for each $x \in X$ is bounded by the Lipschitz constant $\curlywedge_x > 0$ of J near x. It is a nonempty, convex, and weakly* compact subset of X^*.*

(iii) *the graph of ∂J of J is closed in $X \times (v^* - X^*)$ topology, that is, if $\{x_n\} \subset X$ and $\{\xi_n\} \subset X^*$ are sequences that*

$$\xi_n \in \partial J(x_n) \text{ and } x_n \to x \in X,\ \xi_n \rightharpoonup \xi \in X^*,$$

then

$$\xi \in \partial J(x),$$

where $(v^* - X^*)$ denotes the space X^* equipped with weak* topology.

Theorem 2.7. ([11]) *Let $\mathcal{U}$ be a bounded, closed and convex subset of a reflexive Banach space Ω, and $T\colon \mathcal{U} \to 2^{\mathcal{U}}$ be a bounded, closed, convex and weak weakly-upper semicontinuous. Then T has a fixed point in $\mathcal{U}$.*

3. Main results

This section summarizes the key findings of the paper, focusing on the nonemptiness and compactness of the solution set to Problem 1.1. To state and prove these results, we make the following assumptions about Problem 1.1.

(A) $\emptyset \neq \mathcal{U}$ and $\emptyset \neq \mathcal{V}$ are closed and convex subsets of Ω and $\mho$, *respectively.*

(B) $f \in \Omega^*$ and $g \in \mho^*$.

(C) $\zeta_1\colon \Omega \to X$, $\zeta_2\colon \mho \to Y$, $\lambda_1\colon \mho \to W_1$ and $\lambda_2\colon \Omega \to W_2$ are bounded, linear and compact.

(D) $\varphi\colon \Omega\times\Omega \to \overline{\mathbb{R}}$ is such that

(i) $\varphi(\tau,\cdot)\colon \Omega \to \overline{\mathbb{R}}$ is a proper, convex and lower semicontinuous function,

(ii) there exists $\eta_\varphi \geq 0$ such that

$$\varphi(\tau_1, u_2) - \varphi(\tau_1, u_1) + \varphi(\tau_2, u_1) - \varphi(\tau_2, u_2) \\ \leq \eta_\varphi \|\tau_1 - \tau_2\|_\Omega \|u_1 - u_2\|_\Omega, \ \forall \tau_1, \tau_2, u_1, u_2 \in \Omega,$$

(iii) for each $\tau \in \Omega$, there exists $\eta_\varphi(\tau) > 0$ such that

$$\varphi(\tau, u_1) - \varphi(\tau, u_2) \leq \eta_\varphi(\tau) \|u_1 - u_2\|_\Omega, \ \forall u_1, u_2 \in \Omega.$$

(E) $\phi\colon \mho \times \mho \to \overline{\mathbb{R}}$ is such that

(i) $\phi(\tau, \cdot)\colon \mho \to \overline{\mathbb{R}}$ is a proper, convex and lower semicontinuous function,

(ii) there exists $\eta_\phi \geq 0$ such that

$$\phi(\tau_1, v_2) - \phi(\tau_1, v_1) + \phi(\tau_2, v_1) - \phi(\tau_2, v_2) \\ \leq \eta_\phi \|\tau_1 - \tau_2\|_\mho \|v_1 - v_2\|_\mho, \ \forall \tau_1, \tau_2, v_1, v_2 \in \mho, \tag{3.1}$$

(iii) for each $\tau \in \mho$, there exists $\eta_\phi(\tau) > 0$ such that

$$\phi(\tau, v_1) - \phi(\tau, v_2) \leq \eta_\phi(\tau) \|v_1 - v_2\|_\mho, \ \forall v_1, v_2 \in \mho.$$

(F) $J\colon W_1 \times X \to \mathbb{R}$ is such that

(i) for every $v \in W_1, X \ni u \rightarrowtail J(v, u) \in \mathbb{R}$ is locally Lipschitz continuous.

(ii) there exists $\eta_J \geq 0$ such that

$$\|\mu\|_{X^*} \leq \eta_J(1 + \|u\|_X + \|v\|_{W_1}), \ \forall \mu \in \partial J(v, u), u \in X \text{ and } v \in W_1.$$

(iii) this inequality is true

$$\limsup_{n\to\infty} J^0(v_n, u; x) \leq J^0(v, u; x),$$

if sequence $\{v_n\} \subset W_1$ and u, x are such that

$$v_n \to v \in W_1 \text{ as } n \to \infty.$$

(G) $I\colon W_2 \times Y \to \mathbb{R}$ is such that

(i) for every $u \in W_2, Y \ni v \rightarrowtail I(u, v) \in \mathbb{R}$ is locally Lipschitz continuous.

(ii) there exists $\eta_I \geq 0$ such that

$$\|\vartheta\|_{Y^*} \leq \eta_I(1 + \|u\|_{W_2} + \|v\|_Y), \ \forall \vartheta \in \partial I(u, v), u \in W_2 \text{ and } v \in Y,$$

(iii) this inequality is true

$$\limsup_{n\to\infty} I^0(u_n, v; y) \leq I^0(u, v; y),$$

if the sequence $\{v_n\} \subset W_2$ and $v, y \in Y$ are such that

$$v_n \to v \in W_2 \text{ as } n \to \infty.$$

(H) $\mathcal{M}\colon \mho \times \Omega \to \Omega^*$ fulfills the subsequent requirements:

(i) This inequality holds true for any $v \in \mho$ and $x, u \in \Omega$

$$\limsup_{\rho \to 0} \langle \mathcal{M}(v, \rho x + (1-\rho)u), x - u \rangle_\Omega \leq \langle \mathcal{M}(v, u), x - u \rangle_\Omega.$$

(ii) for any $v \in \mho$ fixed, the set-valued mapping $\Omega \ni u \rightarrowtail \mathcal{M}(v, u) + \zeta_1^* \partial J(\lambda_1 v, \zeta_1 u) \subset \Omega^*$ is φ-stable-pseudomonotone with $\{f\}$.

(iii) if $\{v_n\} \subset \mho$ and $\{u_n\} \subset \Omega$ are such that

$$v_n \rightharpoonup v \in \mho \text{ and } u_n \rightharpoonup u \in \Omega \text{ as } n \to \infty \text{ for } (v, u) \in \mho \times \Omega,$$

implies that

$$\limsup_{n \to \infty} \langle \mathcal{M}(v_n, x), x - u_n \rangle_\Omega \leq \langle \mathcal{M}(v, x), x - u \rangle_\Omega.$$

(iv) the growth assumption is met

$$\|\mathcal{M}(v, u)\|_{\Omega^*} \leq \alpha_{\mathcal{M}}(1 + \|u\|_\Omega + \|v\|_\mho, \ \forall (v, u) \in \Omega \times \mho \text{ with } \alpha_{\mathcal{M}} > 0,$$

(v) there is a function $\curlywedge_{\mathcal{M}} \colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ such that

$$\begin{aligned} &\langle \mathcal{M}(v, u), u \rangle_\Omega - J^0(\lambda_1 v, \zeta_1 u; -\zeta_1 u) \\ &\quad \geq \curlywedge_{\mathcal{M}}(\|u\|_\Omega, \|v\|_\mho)\|u\|_\Omega, \ \forall u \in \Omega, v \in \mho, \end{aligned}$$

and

* considering each bounded and nonempty set $\mathcal{O} \subset \mathbb{R}^+$, we have

$$\curlywedge_{\mathcal{M}}(t, s) \to +\infty \text{ as } t \to +\infty, \ \forall s \in \mathcal{O},$$

* it holds for any constants $\pi_1, \pi_2 \geq 0$,

$$\curlywedge_{\mathcal{M}}(t, \pi_1 t + \pi_2) \to +\infty \text{ as } t \to +\infty,$$

* for sequences $\{s_n\} \subset \mathbb{R}^+$ and $\{t_n\} \subset \mathbb{R}^+$ such that $s_n \to +\infty$, $t_n \to +\infty$ and

$$\frac{t_n}{s_n} \to 0 \text{ as } n \to \infty,$$

hence, $\curlywedge_{\mathcal{M}}(s_n, t_n) \to +\infty$ as $n \to \infty$.

(I) $\mathcal{N} \colon \Omega \times \mho \to \mho^*$ fulfills the subsequent requirements:

(i) This inequality holds true for any $u \in \Omega$ and $y, v \in \mho$

$$\limsup_{\rho \to 0} \langle \mathcal{N}(u, \rho y + (1-\rho)v), y - v \rangle_\mho \leq \langle \mathcal{N}(u, v), y - v \rangle_\mho.$$

(ii) for any $u \in \Omega$ fixed, the set-valued mapping $\mho \ni v \rightarrowtail \mathcal{N}(u, v) + \zeta_2^* \partial I(\lambda_2 u, \zeta_2 v) \subset \mho^*$ is ϕ-stable-pseudomonotone with $\{g\}$.

(iii) if $\{v_n\} \subset \mho$ and $\{u_n\} \subset \Omega$ are such that

$$v_n \rightharpoonup v \in \mho \text{ and } u_n \rightharpoonup u \in \Omega \text{ as } n \to \infty, \text{ for`}(v,u) \in \mho \times \Omega,$$

implies that

$$\limsup_{n\to\infty} \langle \mathcal{N}(u_n, y), y - v_n \rangle_{\mho} \leq \langle \mathcal{N}(u, y), y - v \rangle_{\mho},$$

(iv) the growth assumption is met

$$\|\mathcal{N}(u,v)\|_{\mho^*} \leq \alpha_{\mathcal{N}}(1+\|u\|_{\Omega}+\|v\|_{\mho}), \ \forall (v,u) \in \mho\times\Omega \text{ with } \alpha_{\mathcal{N}} > 0,$$

(v) there is a function $\curlywedge_{\mathcal{N}} \colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ such that

$$\begin{aligned}&\langle \mathcal{N}(u,v), v \rangle_{\mho} - I^0(\lambda_2 u, \zeta_2 v; -\zeta_2 v)\\ &\qquad \geq \curlywedge_{\mathcal{N}}(\|v\|_{\mho}, \|u\|_{\Omega})\|v\|_{\mho}, \ \forall u \in \Omega, v \in \mho,\end{aligned}$$

and

- ⋆ considering each bounded and nonempty set $\mathcal{O} \subset \mathbb{R}^+$, we have

$$\curlywedge_{\mathcal{N}}(t,s) \to +\infty \text{ as } t \to +\infty, \forall s \in \mathcal{O},$$

- ⋆ it holds for any constants $\pi_1, \pi_2 \geq 0$,

$$\curlywedge_{\mathcal{N}}(t, \pi_1 t + \pi_2) \to +\infty \text{ as } t \to +\infty,$$

- ⋆ for sequences $\{s_n\} \subset \mathbb{R}^+$ and $\{t_n\} \subset \mathbb{R}^+$ such that

$$s_n \to +\infty, t_n \to +\infty \text{ and } \frac{t_n}{s_n} \to 0 \text{ as } n \to \infty,$$

hence, $\curlywedge_{\mathcal{N}}(s_n, t_n) \to +\infty$ as $n \to \infty$.

Theorem 3.1. *Given **(A)**-**(I)**, assume that they hold. Thus, $\mathscr{S}(f,g)$, the set of solutions to Problem* 1.1*, is nonempty and weakly compact in* $\Omega \times \mho$.

Proof. The proof of this theorem consists of these five steps.

Step 1: If the set of solutions to Problem 1.1 is not empty, then $\mathscr{S}(f,g)$ is bounded. Assume that

$$\mathscr{S}(f,g) \neq \emptyset.$$

Let $(u,v) \in \mathscr{S}(f,g)$, and $u_0 \in dom\varphi \cap \mathcal{U}$ and $v_0 \in dom\phi \cap \mathcal{V}$ be an arbitrary fixed. Then, we have

$$\begin{cases} \langle \mathcal{M}(v,u), u_0 - u \rangle_{\Omega} + J^0(\lambda_1 v, \zeta_1 u; \zeta_1(u_0 - u)) + \varphi(u_0, u) - \varphi(u,u) \\ \qquad \geq \langle f, u_0 - u \rangle_{\Omega}, \\ \langle \mathcal{N}(u,v), v_0 - v \rangle_{\mho} + I^0(\lambda_2 u, \zeta_2 v; \zeta_2(v_0 - v)) + \phi(v_0, v) - \phi(v,v) \\ \qquad \geq \langle g, v_0 - v \rangle_{\mho}. \end{cases} \tag{3.2}$$

With the help of $z \to J^0(v,u;z)$ and $z \to I^0(u,v;z)$, subadditivity, we obtain

$$\begin{cases} \langle \mathcal{M}(v,u),u\rangle_\Omega - J^0(\lambda_1 v,\zeta_1 u;-\zeta_1 u) & \leq \langle \mathcal{M}(v,u),u_0\rangle_\Omega + J^0(\lambda_1 v,\zeta_1 u;\zeta_1 u_0) \\ & \quad +\varphi(u_0,u) - \varphi(u,u) - \langle f, u_0 - u\rangle_\Omega, \\ \langle \mathcal{N}(u,v),v\rangle_\mho - I^0(\lambda_2 u,\zeta_2 v;-\zeta_2 v) & \leq \langle \mathcal{N}(u,v),v_0\rangle_\mho + I^0(\lambda_2 u,\zeta_2 v;\zeta_2 v_0) \\ & \quad +\phi(v_0,v) - \phi(v,v) - \langle g, v_0 - v\rangle_\mho. \end{cases} \tag{3.3}$$

Let $\mu \in X^*$ and $\vartheta \in Y^*$ be such that

$$\begin{cases} \mu \in \partial J(\lambda_1 v,\zeta_1 u) \text{ and } \langle \mu, \zeta_1 u_0\rangle_X = J^0(\lambda_1 v,\zeta_1 u;\zeta_1 u_0), \\ \vartheta \in \partial I(\lambda_2 u,\zeta_2 v) \text{ and } \langle \vartheta, \zeta_2 v_0\rangle_Y = I^0(\lambda_2 u,\zeta_2 v;\zeta_1 v_0). \end{cases} \tag{3.4}$$

By utilizing **H**(iv)-(v) and **I**(iv)-(v) in conjunction with (3.3)-(3.4), we can ascertain that

$$\begin{aligned} \curlywedge_{\mathcal{M}}(\|u\|_\mho,\|v\|_\mho)\|u\|_\Omega &\leq \langle \mathcal{M}(v,u),u\rangle_\mho - J^0(\lambda_1 v,\zeta_1 u,-\zeta_1 u) \\ &\leq \langle \mathcal{M}(v,u),u_0\rangle_\Omega + \langle \mu,\zeta_1 u_0\rangle_X + \varphi(u_0,u) \\ &\quad - \varphi(u,u) - \langle f,u_0-u\rangle_\mho \\ &\leq \alpha_{\mathcal{M}}(1+\|u\|_\Omega+\|v\|_\mho)\|u_0\|_\Omega \\ &\quad + \varrho_J(1+\|\zeta_1 u\|_X+\|\lambda_1 v\|_{W_1})\|\zeta_1 u_0\|_X \\ &\quad + \varphi(u_0,u)-\varphi(u,u)+\|f\|_{\Omega^*}(\|u_0\|_\Omega+\|u\|_\Omega) \\ &\leq \alpha_{\mathcal{M}}(1+\|u\|_\Omega+\|v\|_\mho)\|u_0\|_\Omega \\ &\quad + \varrho_J(1+\|\zeta_1\|\|u\|_\Omega+\|\lambda_1\|\|v\|_\mho)\|\zeta_1\|\|u_0\|_\Omega \\ &\quad + \varrho_\varphi(u)\|u_0-u\|_\Omega+\|f\|_{\Omega^*}(\|u_0\|_\Omega+\|u\|_\Omega) \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} \curlywedge_{\mathcal{N}}(\|v\|_\mho,\|u\|_\Omega)\|v\|_\mho &\leq \langle \mathcal{N}(u,v),v\rangle_\mho - I^0(\lambda_2 u,\zeta_2 v;-\zeta_2 v) \\ &\leq \langle \mathcal{N}(u,v),v_0\rangle_\mho + I^0(\lambda_2 u,\zeta_2 v;\zeta_2 v_0) \\ &\quad + \phi(v_0,v)-\phi(v,v)-\langle g,v_0-v\rangle_\mho \\ &\leq \alpha_{\mathcal{N}}(1+\|u\|_\Omega+\|v\|_\mho)\|v_0\|_\mho \\ &\quad + \varrho_I(1+\|\lambda_2\|\|u\|_\Omega+\|\zeta_2\|\|v\|_\mho)\|\zeta_2\|\|v_0\|_\mho \\ &\quad + \varrho_\phi(v)\|v_0-v\|_\mho+\|g\|_{\mho^*}(\|v_0\|_\mho+\|v\|_\mho). \end{aligned} \tag{3.6}$$

Consequently, we have

$$\begin{aligned}\curlywedge_{\mathcal{M}}(\|u\|_\Omega, \|v\|_\mho) \le{}& \frac{\alpha_{\mathcal{M}}(1+\|u\|_\Omega+\|v\|_\mho)\|u_0\|_\Omega}{\|u\|_\Omega}\\ &+\frac{\varrho_J(1+\|\zeta_1\|\|u\|_\Omega+\|\lambda_1\|\|v\|_\mho)\|\zeta_1\|\|u_0\|_\Omega}{\|u\|_\Omega}\\ &+\frac{\varrho_\varphi(u)\|u_0-u\|_\Omega+\|f\|_{\Omega^*}(\|u_0\|_\Omega+\|u\|_\Omega)}{\|u\|_\Omega}\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}\curlywedge_{\mathcal{N}}(\|v\|_\mho, \|u\|_\Omega) \le{}& \frac{\alpha_{\mathcal{N}}(1+\|u\|_\Omega+\|v\|_\mho)\|v_0\|_\mho}{\|v\|_\mho}\\ &+\frac{\varrho_I(1+\|\lambda_2\|\|u\|_\Omega+\|\zeta_2\|\|v\|_\mho)\|\zeta_2\|\|v_0\|_\mho}{\|v\|_\mho}\\ &+\frac{\varrho_\phi(v)\|v_0-v\|+\|g\|_{\mho^*}(\|v_0\|_\mho+\|v\|_\mho)}{\|v\|_\mho}.\end{aligned} \tag{3.8}$$

Let $\mathscr{S}(f,g)$ be an unbounded set. We can assume that there exists a sequence $\{(u_n,v_n)\}\subset\mathscr{S}(f,g)$ that satisfies

$$\|u_n\|_\Omega+\|v_n\|_\mho\to\infty \text{ as } n\to\infty,$$

if one of the following conditions is met:

$$\|u_n\|_\Omega\uparrow+\infty \text{ as } n\to\infty \text{ and } \{v_n\} \text{ is bounded in } \mho \tag{3.9}$$

or

$$\|v_n\|_\mho\uparrow+\infty \text{ as } n\to\infty \text{ and } \{u_n\} \text{ is bounded in } \Omega \tag{3.10}$$

or

$$\begin{cases}\|v_n\|_\mho\uparrow+\infty,\\ \|u_n\|_\Omega\uparrow+\infty \text{ as } n\to\infty.\end{cases} \tag{3.11}$$

If (3.9) is true, then (3.7) gives us

$$\begin{aligned}\curlywedge_{\mathcal{M}}(\|u_n\|_\Omega, \|v_n\|_\mho) \le{}& \frac{\alpha_{\mathcal{M}}(1+\|u_n\|_\Omega+\|v_n\|_\mho)\|u_0\|_\Omega}{\|u_n\|_\Omega}\\ &+\frac{\varrho_J(1+\|\zeta_1\|\|u_n\|_\Omega+\|\lambda_1\|\|v_n\|_\mho)\|\zeta_1\|\|u_0\|_\Omega}{\|u_n\|_\Omega}\\ &+\frac{\varrho_\varphi(u)\|u_0-u\|_\Omega+\|f\|_{\Omega^*}(\|u_0\|_\Omega+\|u_n\|_\Omega)}{\|u_n\|_\Omega}.\end{aligned}$$

Based on **(H)**(v) and taking the limit as $n \to \infty$ in the inequality above, it yields

$$\begin{aligned} +\infty &= \lim_{n\to\infty} \curlywedge_{\mathcal{M}}(\|u_n\|_\Omega, \|v_n\|_\mho) \\ &\le \lim_{n\to\infty}\left[\frac{\alpha_{\mathcal{M}}(1+\|u_n\|_\Omega + \|v_n\|_\mho)\|u_0\|_\Omega}{\|u_n\|_\Omega}\right] \\ &\quad + \lim_{n\to\infty}\left[\frac{\varrho_J(1+\|\zeta_1\|\|u_n\|_\Omega + \|\lambda_1\|\|v_n\|_\mho)\|\zeta_1\|\|u_0\|_\Omega}{\|u_n\|_\Omega}\right] \\ &\quad + \lim_{n\to\infty}\left[\frac{\varrho_\varphi(u)\|u_0-u\| + \|f\|_{\Omega^*}(\|u_0\|_\Omega + \|u_n\|_\Omega)}{\|u_n\|_\Omega}\right] \\ &= \alpha_{\mathcal{M}}\|u_0\|_\Omega + \varrho_J\|\zeta_1\|^2\|u_0\|_\Omega + \|f\|_{\Omega^*}. \end{aligned}$$

This brings about a paradox. Thus, $\mathscr{S}(f,g)$ is bounded.

Similarly, we can use the same reasoning to arrive at a contradiction in the case of (3.10). If (3.11) is true, we further differentiate between the subsequent scenarios:

(i) $$\frac{\|u_n\|_\Omega}{\|v_n\|_\mho} \to +\infty \text{ or } \frac{\|v_n\|_\mho}{\|u_n\|_\Omega} \to +\infty \text{ as } n \to \infty.$$

(ii) there exists $n_0 \in \mathbb{N}$ such that

$$0 < \varrho_0 \le \frac{\|u_n\|_\Omega}{\|v_n\|_\mho} \le \varrho_1,\ \forall n \ge n_0 \text{ for some } \varrho_0, \varrho_1 > 0.$$

In case (i), if $\frac{\|u_n\|_\Omega}{\|v_n\|_\mho} \to +\infty$, we obtain a similar proof when $\frac{\|v_n\|_\mho}{\|u_n\|_\Omega} \to +\infty$ as $n \to \infty$.

Using (3.7), we have

$$\begin{aligned} \curlywedge_{\mathcal{M}}(\|u_n\|_\Omega, \|v_n\|_\mho) \le{}& \frac{\alpha_{\mathcal{M}}(1+\|u_n\|_\Omega + \|v_n\|_\mho)\|u_0\|_\Omega}{\|u_n\|_\Omega} \\ &+ \frac{\varrho_J(1+\|\zeta_1\|\|u_n\|_\Omega + \|\lambda_1\|\|v_n\|_\mho)\|\zeta_1\|\|u_0\|_\Omega}{\|u_n\|_\Omega} \\ &+ \frac{\varrho_\varphi(u)\|u_0-u\| + \|f\|_{\Omega^*}(\|u_0\|_\Omega + \|u_n\|_\Omega)}{\|u_n\|_\Omega}. \end{aligned}$$

Using (3.11) and condition (i), we can conclude that

$$
\begin{aligned}
+\infty &= \lim_{n\to\infty} \curlywedge_{\mathcal{M}}(\|u_n\|_\Omega, \|v_n\|_\mho) \\
&\le \lim_{n\to\infty}\left[\frac{\alpha_{\mathcal{M}}(1+\|u_n\|_\Omega+\|v_n\|_\mho)\|u_0\|_\Omega}{\|u_n\|_\Omega}\right] \\
&+ \lim_{n\to\infty}\left[\frac{\varrho_J(1+\|\zeta_1\|\|u_n\|_\Omega+\|\lambda_1\|\|v_n\|_\mho)\|\zeta_1\|\|u_0\|_\Omega}{\|u_n\|_\Omega}\right] \\
&+ \lim_{n\to\infty}\left[\frac{\varrho_\varphi(u)\|u_0-u_n\|+\|f\|_{\Omega^*}(\|u_0\|_\Omega+\|u_n\|_\Omega)}{\|u_n\|_\Omega}\right] \\
&= \alpha_{\mathcal{M}}\|u_0\|_\Omega+\varrho_J\|\zeta_1\|^2\|u_0\|_\Omega+\alpha_\varphi(u)+\|f\|_{\Omega^*}.
\end{aligned}
$$

This also results in an inconsistency. This suggests a boundedness of $\mathscr{S}(f,g)$.

Additionally, if condition (ii) is true, we can apply **(I)**(v) to obtain

$$
\begin{aligned}
+\infty &= \lim \curlywedge_{\mathcal{N}}(\|v_n\|_\mho, \|u_n\|_\Omega) \\
&\le \frac{\alpha_{\mathcal{N}}(1+\|u_n\|_\Omega+\|v_n\|_\mho)\|v_0\|_\mho}{\|v_n\|_\mho} \\
&\quad + \frac{\varrho_I(1+\|\lambda_2\|\|u_n\|_\Omega+\|\zeta_2\|\|v_n\|_\mho)\|\zeta_2\|\|v_0\|_\mho}{\|v_n\|_\mho} \\
&\quad + \frac{\varrho_\phi(v)\|v_0-v_n\|_\mho+\|g\|_{\mho^*}(\|v_0\|_\mho+\|v_n\|_\mho)}{\|v_n\|_\mho} \\
&\le \alpha_{\mathcal{N}}(\varrho_1+1)\|v_0\|_\mho+\varrho_I(\|\lambda_2\|\varrho_1+\|\zeta_2\|)\|\zeta_2\|\|v_0\|_\mho+\varrho_\phi(v)+\|g\|_{\mho^*}.
\end{aligned}
$$

This suggests that there is a bounded set $\mathscr{S}(f,g)$.

Step 2: For every $v \in \mho$ (resp. $u \in \Omega$) fixed, there exists a nonempty, bounded, and closed solution set for the inequality problem (1.1) (resp. (1.2)). Let $v \in \mho$ be an arbitrary fixed variable. Based on the generalised subgradient, we possess

$$
\begin{aligned}
\langle \mathcal{M}(v,u)+\zeta_1^*\mu, u\rangle_\Omega &= \langle \mathcal{M}(v,u), u\rangle_\Omega - \langle \mu, -\zeta_1 u\rangle_X \\
&\ge \langle \mathcal{M}(v,u), u\rangle_\Omega - J^0(\lambda_1 y, \zeta_1 u; -\zeta_1 u),\ \forall \mu \in \partial J(\lambda_1 v, \zeta_1 u).
\end{aligned}
$$

Given **(H)**(v) and the previously mentioned inequality, it produces

$$
\begin{aligned}
\frac{\inf_{\mu\in\partial J(\lambda_1 v,\zeta_1 u)}\langle \mathcal{M}(v,u)+\zeta^*\mu, u\rangle_\Omega}{\|u\|_\Omega} &\ge \frac{\langle \mathcal{M}(v,u), u\rangle_\Omega - J^0(\lambda_1 y, \zeta_1 u; -\zeta_1 u)}{\|u\|_\Omega} \\
&\ge \curlywedge_{\mathcal{M}}(\|u\|_\Omega, \|v\|_\mho),\ \forall u \in \Omega.
\end{aligned}
$$

Therefore, the set-valued operator

$$
\Omega \ni u \to \mathcal{M}(v,u)+\zeta_1^*\partial J(\lambda_1 v, \zeta_1 u) \subset \Omega^*
$$

is coercive. In the same way, we can verify that the set-valued operator

$$\mho \ni v \to \mathcal{N}(u,v) + \zeta_2^* \partial I(\lambda_2 u, \zeta_2 v) \subset \mho^*, \ u \in \Omega,$$

is coercive too. Thus, the set of solutions to (1.1) (resp. (1.2)) is closed, bounded, and nonempty.

We now present the set-valued map $\daleth \colon \mathcal{U} \times \mathcal{V} \to 2^{\mathcal{U}\times\mathcal{V}}$ defined by

$$\daleth(u,v) = (\mathcal{A}(u), \mathcal{B}(v)), \ \forall (u,v) \in \mathcal{U} \times \mathcal{V}, \tag{3.12}$$

where $\mathcal{A} \colon \Omega \to 2^{\mathcal{V}}$ and $\mathcal{B} \colon \mho \to 2^{\mathcal{U}}$ are solution mappings of (1.1) and (1.2) and denoted by $\mathcal{A}(u)$ for $u \in \Omega$ and $\mathcal{B}(v)$ for $v \in \mho$. It is easy to demonstrate from the definition of $\daleth$ that $(u,v) \in \mathcal{U} \times \mathcal{V}$ is a fixed point of $\daleth$ if and only if it is a solution of Problem 1.1. We will confirm, using this property, that $\daleth$ has at least one fixed point in $\mathcal{U} \times \mathcal{V}$.

Step 3: A convex, bounded, and closed subset X of $\mathcal{U} \times \mathcal{V}$ has $\daleth \colon X \to X$. It is enough to prove that there is a constant $\ell_0 > 0$ such that

$$\daleth(\mathcal{O}(\ell_0)) \subset \mathcal{O}(\ell_0), \tag{3.13}$$

where $\mathcal{O}(\ell_0)$ is given by

$$\mathcal{O}(\ell_0) = \{(u,v) \in \mathcal{U} \times \mathcal{V} \mid \|u\|_\Omega \leq \ell_0 \text{ and } \|v\|_\mho \leq \ell_0\}.$$

According to contradiction, there is $\ell_0 > 0$ such that (3.13) holds. For each $n \in \mathbb{N}$, there are sequences $\{(u_n, v_n)\}, \{(x_n, y_n)\} \subset \mathcal{U} \times \mathcal{V}$ that satisfy

$$\begin{cases} (u_n, v_n) \in \mathcal{O}(n), \\ (x_n, y_n) \in \daleth(u_n, v_n) \end{cases}$$

and

$$\begin{cases} \|x_n\|_\Omega > n, \ \text{or} \\ \|y_n\|_\mho > n. \end{cases}$$

Let us assume that $\|x_n\|_\Omega > n$. Equation (3.5) states that

$$\begin{aligned} \text{⅄}_{\mathcal{M}}(\|x_n\|_\Omega, \|v_n\|_\mho) \leq{}& \frac{\alpha_{\mathcal{M}}(1 + \|v_n\|_\mho + \|x_n\|_\Omega)\|u_0\|_\Omega}{\|x_n\|_\Omega} \\ &+ \frac{\varrho_J(1 + \|\zeta_1\|\|x_n\|_\Omega + \|\lambda_1\|\|v_n\|_\mho)\|\zeta_1\|\|u_0\|_\Omega}{\|x_n\|_\Omega} \\ &+ \frac{\varrho_\varphi(x)\|u_0 - x\|_\Omega + \|f\|_{\Omega^*}(\|u_0\|_\Omega + \|x_n\|_\Omega)}{\|x_n\|_\Omega}. \end{aligned}$$

Applying the assumption $\|v_n\|_{\mho} \leq n < \|x_n\|_\Omega$ and **(H)**(v) to find

$$\begin{aligned}
+\infty &= \lim_{n\to\infty} \curlywedge_{\mathcal{M}}(\|x_n\|_\Omega, \|v_n\|_{\mho}) \\
&\leq \lim_{n\to\infty}\left[\frac{\alpha_{\mathcal{M}}(1+\|x_n\|_\Omega+\|v_n\|_{\mho})\|u_0\|_\Omega}{\|x_n\|_\Omega}\right] \\
&\quad + \lim_{n\to\infty}\left[\frac{+\varrho_J(1+\|\zeta_1\|\|x_n\|_\Omega+\|\lambda_1\|\|v_n\|_{\mho})\|\zeta_1\|\|u_0\|_\Omega}{\|x_n\|_\Omega}\right] \\
&\quad + \lim_{n\to\infty}\left[\frac{\varrho_\varphi(x)\|u_0-x\|_\Omega+\|f\|_{\Omega^*}(\|u_0\|_\Omega+\|x_n\|_\Omega)}{\|x_n\|_\Omega}\right] \\
&\leq 2\alpha_{\mathcal{M}}\|u_0\|_\Omega+\varrho_J(\|\zeta_1\|+\|\lambda_1\|)\|\zeta_1\|\|u_0\|_\Omega+\varrho_\varphi+\|f\|_{\Omega^*}.
\end{aligned}$$

Thus we have a contradiction. As previously discussed, it can also lead to a contraction when

$$\|y_n\|_{\mho} > n.$$

This implies the existence of a constant $\ell_0 > 0$ required for the validity of (3.13). We conclude that there is a bounded, closed and convex subset X of $\mathcal{U}\times\mathcal{V}$, and that $\urcorner\colon X \to X$.

Step 4: The upper semicontinuous function $\urcorner$ is weakly-weak.

Let $\mathscr{D} \subset \mathcal{U}\times\mathcal{V}$ be a nonempty weakly closed set such that

$$\urcorner^-(\mathscr{D}) \neq \emptyset.$$

Proposition 2.3 requires proving that $\urcorner^-(\mathscr{D})$ is weakly closed in $\Omega\times\mho$. Suppose $\{(x_n, y_n)\} \subset \urcorner^-(\mathscr{D})$ is chosen so that

$$(u_n, v_n) \to (u, v) \in \Omega\times\mho \text{ as } n\to\infty. \tag{3.14}$$

Therefore, for each $n\in\mathbb{N}$, we can find $(u_n, v_n)\in\Omega\times\mho$ satisfying

$$(x_n, y_n) \in \urcorner(u_n, v_n)\cap\mathscr{D}.$$

Applying (3.7) and (3.8), we have

$$\begin{aligned}
\curlywedge_{\mathcal{M}}(\|x_n\|_\Omega, \|v_n\|_{\mho}) \leq{}& \frac{\alpha_{\mathcal{M}}(1+\|x_n\|_\Omega+\|v_n\|_{\mho})\|u_0\|_\Omega}{\|x_n\|_\Omega} \\
&+ \frac{\varrho_J(1+\|\zeta_1\|\|x_n\|_\Omega+\|\lambda_1\|\|v_n\|_{\mho})\|\zeta_1\|\|u_0\|_\Omega}{\|x_n\|_\Omega} \\
&+ \frac{\varrho_\varphi(u)\|u_0-x_n\|_\Omega+\|f\|_{\Omega^*}(\|u_0\|_\Omega+\|x_n\|_\Omega)}{\|x_n\|_\Omega}
\end{aligned}$$

and

$$\curlywedge_{\mathcal{N}}(\|y_n\|_{\mho}, \|u_n\|_{\Omega}) \leq \frac{\alpha_{\mathcal{N}}(1+\|u_n\|_{\Omega}+\|y_n\|_{\mho})\|v_0\|_{\mho}}{\|y_n\|_{\mho}} + \frac{\varrho_I(1+\|\lambda_2\|\|u_n\|_{\mho}+\|\zeta_2\|\|y_n\|_{\mho})\|\zeta_2\|\|v_0\|_{\mho}}{\|y_n\|_{\mho}} + \frac{\varrho_{\phi}(v)\|v_0-y_n\|+\|g\|_{\mho^*}(\|v_0\|_{\mho}+\|y_n\|_{\mho})}{\|y_n\|_{\mho}}.$$

Combining this with **(H)**(v) and **(I)**(v) indicates that the sequence $\{(x_n, y_n)\}$ is bounded in $\Omega \times \mho$. If needed, we can go to a relabeled subsequence and assume that

$$(x_n, y_n) \to (x, y) \in \Omega \times \mho \text{ as } n \to \infty. \tag{3.15}$$

Next, for each $n \in N$, let $\mu_n \in X^*$ and $\vartheta_n \in Y^*$ be such that

$$\begin{cases} \langle \mu_n, \zeta_1(w_1 - x_n)\rangle_X = J^0(\lambda_1 v_n, \zeta_1 x_n; \zeta_1(w_1 - x_n)), \\ \langle \vartheta_n, \zeta_2(w_2 - y_n)\rangle_Y = I^0(\lambda_2 u_n, \zeta_2 y_n; \zeta_2(w_2 - y_n)). \end{cases}$$

Then, we have

$$\begin{cases} \langle \mathcal{M}(v_n, x_n) + \zeta_1^* \mu_n, w_1 - x_n\rangle_\Omega + \varphi(w_1, x_n) - \varphi(x_n, x_n) \\ \qquad \geq \langle f, w_1 - x_n\rangle_\Omega, \ \forall w_1 \in \mathcal{U}, \\ \langle \mathcal{N}(u_n, y_n) + \zeta_2^* \vartheta_n, w_2 - y_n\rangle_\mho + \phi(w_2, y_n) - \phi(y_n, y_n) \\ \qquad \geq \langle g, w_2 - y_n\rangle_\mho, \ \forall w_2 \in \mathcal{V}. \end{cases}$$

Next, we employ **(H)**(ii) and **(I)**(ii) to discover that

$$\begin{aligned} \langle f, w_1 - x_n\rangle_\Omega &\leq \langle \mathcal{M}(v_n, x) + \zeta_1^* \Upsilon_n, w_1 - x_n\rangle_\Omega + \varphi(w_1, x_n) - \varphi(x_n, x_n) \\ &\leq \langle \mathcal{M}(v_n, w_1), w_1 - x_n\rangle_\Omega + J^0(\lambda_1 v_n, \zeta_1 w_1; \zeta_1(w_1 - x_n)) \\ &\quad + \varphi(w_1, x_n) - \varphi(x_n, x_n), \ \forall \Upsilon_n \in \partial J(\lambda_1 v_n, \zeta_1 w_1) \text{ and } w_1 \in \mathcal{U} \end{aligned}$$

and

$$\begin{aligned} \langle g, w_2 - y\rangle_\mho &\leq \langle \mathcal{N}(u_n, w_2) + \zeta_2^* \Upsilon_n, w_2 - y_n\rangle_\mho + \phi(w_2, y_n) - \phi(y_n, y_n) \\ &\leq \langle \mathcal{N}(u_n, w_2), w_2 - y_n\rangle_\mho + I^0(\lambda_2 u_n, \zeta_2 w_2; \zeta_2(w_2 - y_n)) \\ &\quad + \phi(w_2, y_n) - \phi(y_n, y_n), \ \forall \Upsilon_n \in \partial I(\lambda_2 u_n, \zeta_2 w_2) \text{ and } w_2 \in \mathcal{V}. \end{aligned}$$

Using the hypotheses **(J)**(iii), **(F)**(iii), **(H)**(iii), and **(I)**(iii), and passing to the upper limit as $n \to \infty$ in the aforementioned inequalities, we obtain

$$\begin{aligned}
\langle f, w_1 - x\rangle_\Omega &= \limsup_{n\to\infty}\langle f, w_1 - x_n\rangle_\Omega \\
&\le \limsup_{n\to\infty} \big[\langle \mathcal{M}(v, w_1), w_1 - x_n\rangle_\Omega + J^0(\lambda v_n, \zeta w_1; \zeta(w_1 - x_n)) \\
&\quad + \varphi(w_1, x_n) - \varphi(x_n, x_n)\big] \\
&\le \limsup_{n\to\infty}\langle \mathcal{M}(v, w_1), w_1 - x_n\rangle_\Omega + \limsup_{n\to\infty} J^0(\lambda v_n, \zeta w_1; \zeta(w_1 - x_n)) \\
&\quad + \liminf_{n\to\infty} \varphi(w_1, x_n) - \liminf_{n\to\infty} \varphi(x_n, x_n) \\
&\le \langle \mathcal{M}(v, w_1), w_1 - x_n\rangle_\Omega + J^0(\lambda_1 v_n, \zeta_1 w_1; \zeta_1(w_1 - x_n)) \\
&\quad + \varphi(w_1, x_n) - \varphi(x_n, x_n)
\end{aligned}$$

and

$$\begin{aligned}
\langle g, w_2 - y_n\rangle_\mho &= \limsup_{n\to\infty}\langle g, w_2 - y_n\rangle_\mho \\
&\le \limsup_{n\to\infty} \big[\langle \mathcal{N}(u_n, w_2), w_2 - y_n\rangle_\mho + I^0(\lambda_2 u_n, \zeta_2 w_2; \zeta_2(w_2 - y_n)) \\
&\quad + \phi(w_2, y_n) - \phi(y_n, y_n)\big] \\
&\le \limsup_{n\to\infty}\langle \mathcal{N}(u_n, w_2), w_2 - y_n\rangle_\mho + \limsup_{n\to\infty} I^0(\lambda u_n, \zeta w_2; \zeta(w_2 - y_n)) \\
&\quad + \liminf_{n\to\infty} \phi(w_2, y_n) - \liminf_{n\to\infty} \phi(y_n, y_n) \\
&\le \langle \mathcal{N}(u, w_2), w_2 - y\rangle_\mho + I^0(\lambda_2 u, \zeta_2 w_2; \zeta_2(w_2 - y)) \\
&\quad + \phi(w_2, y) - \phi(y, y).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\langle \mathcal{M}(v, w_1), w_1 - x\rangle_\Omega + J^0(\lambda_1 v, \zeta_1 w_1; \zeta_1(w_1 - x)) + \varphi(w_1, x) - \varphi(x, x) \\
&\qquad \ge \langle f, w_1 - x\rangle_\Omega, \ \forall w_1 \in \mathcal{U} \qquad (3.16)
\end{aligned}$$

and

$$\begin{aligned}
&\langle \mathcal{N}(u, w_2), w_2 - y\rangle_\mho + I^0(\lambda_2 u, \zeta_2 w_2; \zeta_2(w_2 - y)) + \phi(w_2, y) - \phi(y, y) \\
&\qquad \ge \langle g, w_2 - y\rangle_\mho, \ \forall w_2 \in \mathcal{V}. \qquad (3.17)
\end{aligned}$$

Assume that $\curlywedge \in \mathcal{U}$ and $\rho \in (0, 1)$ are arbitrary. Applying the convexity of φ and the positive homogeneity of $x \to J^0(v, u; x)$, we insert $w = w_\rho = \rho \curlywedge + (1 - \rho)x$ in (3.16) to obtain

$$\begin{aligned}
&\rho\langle \mathcal{M}(v, w_\rho), \curlywedge - x\rangle_\Omega + J^0(\lambda_1 v, \zeta_1 w_\rho; \zeta_1(\curlywedge - x)) + \psi(w_\rho, x) - \psi(x, x) \\
&\quad \ge \rho\langle \mathcal{M}(v, w_\rho), \curlywedge - x\rangle_\Omega + J^0(\lambda_1 v, \zeta_1 w_\rho; \rho\zeta_1(\curlywedge - x)) + \varphi(w_\rho, x) - \varphi(x, x) \\
&\quad \ge \rho\langle f, \curlywedge - x\rangle_\Omega.
\end{aligned}$$

Thus, we have

$$\langle \mathcal{M}(v, w_\rho), \curlywedge - x\rangle_\Omega + J^0(\lambda_1 v, \zeta_1 w_\rho; \zeta_1(\curlywedge - x)) + \varphi(\curlywedge, x) - \varphi(x, x) \geq \langle f, \curlywedge - x\rangle_\Omega.$$

Utilizing **(H)**(i), the upper semicontinuity of $(u, x) \to J^0(v, u; x)$, and passing the upper limit in the inequality above as $\rho \downarrow 0$, we have

$$\begin{aligned}
&\langle \mathcal{M}(v, x), \curlywedge - x\rangle_\Omega + J^0(\lambda_1 v, \zeta_1 x; \zeta_1(\curlywedge - x)) + \varphi(\curlywedge, x) - \varphi(x, x) \\
&\quad \geq \limsup_{\rho\downarrow 0} \langle \mathcal{M}(v, w_\rho), \curlywedge - x\rangle_\Omega + J^0(\lambda_1 v, \zeta_1 w_\rho; \zeta_1(\curlywedge - x)) + \varphi(\curlywedge, x) - \varphi(x, x) \\
&\quad \geq \langle f, \curlywedge - x\rangle_\Omega.
\end{aligned}$$

Given the arbitrary nature of $\curlywedge \in \mathcal{U}$, we can conclude that $x \in \mathcal{U}$ represents a solution to problem (1.1), which corresponds to $v \in \mathcal{V}$. In the same way, we can also determine that $y \in \mathcal{V}$ is a solution to problem (1.2), which corresponds to $u \in \mathcal{U}$. According to this,

$$(x, y) \in \daleth(u, v).$$

Due to the weak closedness of $\mathscr{D}$, we derive the result that

$$(x, y) \in \mathscr{D}, \text{ or } (u, v) \in \daleth^-(\mathscr{D}).$$

Based on Proposition 2.3, we consequently conclude that $\daleth$ is weakly-weak upper semi continuous. Thus, it is evident that every requirement outlined in Theorem 3.1 and Theorem 2.7 has been satisfied. This theorem leads us to the conclusion that $\daleth$ has at least one fixed point

$$(u^*, v^*) \in \mathcal{U} \times \mathcal{V} \text{ in } X.$$

This suggests that another solution to Problem 1.1 is

$$(u^*, v^*) \in \mathcal{U} \times \mathcal{V}.$$

Step 5: Weak compactness of the set $\mathscr{S}(f, g)$.

It is evident from Step 1 that the set $\mathscr{S}(f, g)$ is bounded. Since $\Omega \times \mho$ is reflexive, we can show that $\mathscr{S}(f, g)$ is weakly closed. Consider a sequence $\{(u_n, v_n)\} \subset \mathscr{S}(f, g)$ such that

$$(u_n, v_n) \to (u, v) \in \Omega \times \mho \text{ as } n \to \infty. \tag{3.18}$$

The definition of $\daleth$ suggests that

$$(u_n, v_n) \in \daleth(u_n, v_n).$$

Since $\daleth$ is weakly-weak upper semi-continuous with nonempty, bounded, closed and convex values. [[14], Theorem 1.1.4] implies that it is weakly-weak closed. This, combined with the convergence of (3.18), implies that

$$(u, v) \in \daleth(u, v).$$

Based on the definition of $\daleth$, we infer that

$$(u, v) \in \mathscr{S}(f, g).$$

In other words, $\mathscr{S}(f, g)$ is weakly closed. Thus, we conclude that $\mathscr{S}(f, g)$ is weakly compact in $\Omega \times \mho$. This wraps up the proof. $\square$

4. Concluding remarks

In our study of mild nonlinearities and constraints within Banach spaces, we analyzed a system of variational-hemivariational inequality problems. By employing a set-valued version of the Tychonoff fixed point principle in a Banach space, along with the Minty technique, generalized monotonicity arguments, and nonsmooth analysis, we established a general existence theorem for this system. Our findings expand upon the results presented in [11, Theorem 7]. Many engineering challenges can be formulated as systems of coupled variational-hemivariational inequalities. We intend to leverage the theoretical results from this paper to explore various real-world engineering problems in the near future.

Additionally, we aim to incorporate concepts such as homogenization, stability analysis, optimal control, and sensitivity into the mathematical framework for systems of variational-hemivariational inequalities. Numerous researchers have indicated that a wide array of intricate physical phenomena and engineering applications, including multi-player Nash equilibrium problems, contact mechanics, transport optimization, and related areas can be modeled using the complex systems generated by variational inequalities.

Our future endeavors include applying the theoretical outcomes established herein to address a multi-player Nash equilibrium problem and investigating a system of generalized mixed quasi-variational inequality problems. Furthermore, we plan to propose our abstract results in a novel context, specifically targeting a system of elliptic mixed boundary value problems with nonlocal effects and set-value boundary conditions, as well as a feedback control issue focused on the least energy condition relative to the control variable.

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