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FIXED POINT THEOREMS FOR GERAGHTY-REICH CONTRACTION MAPPINGS IN CONE METRIC SPACES OVER BANACH ALGEBRA

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Abstract. In this paper we prove the existence and uniqueness of fixed points of Geraghty-Reich type contraction mappings in generalized cone metric space over Banach algebra, which generalized some results in cone metric spaces. The results we establish in this work generalized some existing results in literature.

1. Introduction

It is almost impossible for a researcher to study the vast literature dealing with contraction mapping that either extended or generalized the Banach contraction mapping principle introduced in 1922 by Banach [3]. These extensions/generalizations in literature of metrical fixed point theory has been

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studied and recorded in the works of [4], [5], [6], [7], [8], [12], [13], [14], [16], [20], [21], [22], [23], [24], [25], [26] among others.

In 1972, Reich [29], gave a generalization of famous Banach fixed point theorem and proved that a fixed point exists. In the following year 1973, Geraghty [12] Extended Banach fixed point theorem by substituting the contractive constant in Banach contraction with the idea of class. He proved the existence of fixed point of the class-type contraction.

Malhotra et al [19] introduced the class of ordered weak φ-contractions in cone metric spaces over Banach algebras and proved some fixed point results for the mappings belonging to this new class. Ahmeda et al [1] proved an analogue of Banach and Kannan fixed point theorems by generalizing the Lipschitz constant k, in generalized Lipschitz mapping on cone metric spaces, this takes place over Banach algebra, while Fernandez et al [11] introduced the concept of extended cone b-metric-like spaces over Banach algebra as a generalization of cone b-metric-like spaces over Banach algebra. They also defined the notion of generalized Lipschitz mappings in the setup of extended cone b-metric-like spaces over Banach algebra and investigated some fixed point results and outcomes. They also examined the existence and uniqueness of solution of a Fredholm integral equation as an application.

Quan and Van [27] studied the existence of the solutions of a class of functional integral equations by using some fixed point results in cone b-metric spaces over Banach algebras. In order to obtain these results they introduced and proved some properties of generalized weak j-contractions, in which the j are nonlinear weak comparison functions. Semwal [31] obtained the coincidence and common fixed point theorems for two set of weakly commuting identity mappings in cone metric space over unital Banach algebra. Moreover, an example is given in support of the main result and they showed that their results are more general.

Alecsa [2] noticed that it is well known that fixed point problems of contractive type mappings defined on cone metric spaces over Banach algebra are not equal to those in usual metric spaces [10]. The novelty of their work represents the development of some fixed point results regarding sequences of contractions in the setting of cone metric spaces over Banach algebras. Also, based on the powerful notion of a cone metric space over a Banach algebra, they presented an important applications to systems of differential equations and coupled functional equations, respectively, that are linked to the concept of sequences of contractions.

Motivated by the works above, we pose the following question;

Question 1: Can the contractive constants in Riech contraction mapping be replace by more general class as defined by Geraghty?

The aim of this work is to provide an answer to the above problem.

2. Preliminaries

In this paper, we denote the set of fixed point set of a map $T : C \to C$ by $F(T) = \{x \in C : Tx = x\}$. we also denote the set of all positive Real numbers by $\mathbb{R}_+ \cup \{0\}$.

Next, the following definitions and lemmas will be useful in this paper.

Definition 2.1. ([3]) Suppose (X, d) is a metric space, a self mapping $T : X \to X$ is known as a Banach contraction mapping on X if we can get a positive real number $k \in [0, 1)$ such that $d(Tx, Ty) \leq kd(x, y)$ for all $x, y \in X$.

Theorem 2.2. ([3]) *Banach fixed point Theorem or contraction mapping principle (BCMP) states that Every contraction mapping $T : X \to X$ in a complete metric (X, d) has a unique fixed point. And the Picard iteration process $x_{n+1} = Tx_n$, where $x_1 \in X$ converges to the unique fixed point of T.*

Definition 2.3. ([29]) Let (X, d) be a complete metric space and a self mapping $T : X \to X$ is called a Reich contraction, if T satisfies the condition

$$d(Tx, Ty) \leq k_1 d(x, y) + k_2 d(x, Tx) + k_3 d(y, Ty),\ \forall x, y \in X,$$

where $k_i \geq 0$ for $i = 1, 2, 3$ and $\sum_{i=1}^{3} k_i < 1$.

We know that Reich contraction mapping T in a complete metric (X, d) has a unique fixed point and every sequence of converges to the fixed point in X.

Definition 2.4. ([12]) Let $\mathcal{F}_{Ger}$ be the family of all Geraghty functions, that is functions of the type $\alpha : [0, \infty) \to [0, 1)$ satisfying the condition: If $\alpha(t_n) \to 1$, then $t_n \to 0$ as $n \to \infty$.

Theorem 2.5. ([12]) *Let (X, ρ) be a complete metric space and $T : X \to X$ such that there is an $\alpha \in \mathcal{F}_{Ger}$ satisfying*

$$\rho(Tx, Ty) \leq \alpha(\rho(x, y))\rho(x, y) \tag{2.1}$$

for all $x, y \in X$. Then the sequence $\{x_k\}_{k\in\mathbb{N}}$ defined by $Tx_{k-1} = x_k$ converges to a unique fixed point of T in X for $k \geq 1$.

Definition 2.6. ([12]) A self mapping $T : X \to X$, where (X, d) is any metric space is called a Geraghty contraction, if there exist an auxiliary function $\beta : [0, +\infty) \to [0, 1)$ with $\lim_{n\to\infty} \gamma_n = 0$, whenever $\lim_{n\to\infty} \beta(\gamma_n) = 1$ such that

$$d(Tx, Ty) \leq \beta(d(x, y))d(x, y),\ \forall x, y \in X$$

and $\beta \in \mathcal{F}_{Ger}$, where $\mathcal{F}_{Ger}$ represent the Geraghty class of functions.

Definition 2.7. ([17]) Banach Algebra, $\mathbb{A}$ is a Banach space with an operation of multiplication defined on $\mathbb{A}$ satisfying the following axioms:

(i) $(xy)z = x(yz)$;
(ii) $x(y+z) = xy + xz$ and $(x+y)z = xz + yz$;
(iii) $\beta(xy) = x(\beta y)$;
(iv) $\|xy\| \leq \|x\|\|y\|$, $\forall x, y, z \in \mathbb{A}$ and $\beta \in \mathbb{R}$.

An identity element is unique if and only if there is a unit or identity element for multiplication e, in the Banach algebra $\mathbb{A}$ then $ex = xe = x$ for all $x \in \mathbb{A}$. An element with its inverse is equal to the identity element, hence $x \in \mathbb{A}$ is said to be invertible if there is an element $y \in \mathbb{A}$ such that $xy = yx = e$, then y is the inverse of x denoted by x^{-1}.

Definition 2.8. ([17]) A cone is a subset P of Banach algebra $\mathbb{A}$ which satisfy the following axioms:

(i) P is non-empty closed and $\{\theta, e\} \in P$; where e is an identity element and $\{\theta\}$ is the null of the Banach algebra $\mathbb{A}$;
(ii) $aP + bP \subset P$ for all $a, b \in \mathbb{R}_+$;
(iii) $P^2 = PP \subset P$;
(iv) $P \cap (-P) = \{\theta\}$.

If the interior of P is not empty, then P is known as a solid cone. in this case, $\{\theta\}$ is the null of the Banach algebra $\mathbb{A}$. For a given cone $P \subset \mathbb{A}$, a partial ordering $\preceq$ with respect to P can be defined by $x \preceq y$ if and only if $y - x \in P$, and $x \neq y$, while $x \ll y$ stands for $y - x \in int(P)$, where $int(P)$ means the interior of P.

Definition 2.9. ([9, 17]) Suppose P is a cone and its interior is non empty in the Banach space $\mathbb{A}$. A sequence $\{x_k\} \subset P$ is called a $c-$sequence if for each $\theta \ll c$, there exist $N \in \mathbb{N}$ such that $x_k \ll c$ for all $k > N$.

Lemma 2.10. ([28]) *Suppose $\mathbb{A}$ is a Banach space and a cone P with nonempty interior, if $\theta \preceq \mu \ll c$ for each $\theta \ll c$, then $\mu = \theta$, where $\{\theta\}$ is the null of the Banach algebra $\mathbb{A}$.*

Lemma 2.11. ([28]) *Suppose $\mathbb{A}$ is a Banach space with a cone P where its interior is non void and $\{x_k\}_{k \in \mathbb{N}}$ is a sequence in the Banach Algebra $\mathbb{A}$, if $\|x_k\| \to 0$, then for each $\theta \ll c$, there exists $N \in \mathbb{N}$ such that $x_k \ll c$, where $k > N$.*

Definition 2.12. ([17]) The spectral radius of an element x of Banach Algebra $\mathbb{A}$ is the supremum of the absolute value, denoted by $r(x) = \sup|\lambda|$ such that $\lambda \in \sigma(x)$, where $\sigma(x)$ is the spectrum.

Lemma 2.13. ([30]) *Let $\mathbb{A}$ be a Banach algebra with a unit e and $x \in \mathbb{A}$. If the spectral radius $r(x)$ of x is less than 1, that is,*

$$r(x) = \lim_{k\to\infty} \left\|x^k\right\|^{\frac{1}{k}} = \inf_{k\geq 1} \left\|x^k\right\|^{\frac{1}{k}} < 1,$$

then $e - x$ is invertible. That is;

$$(e-x)^{-1} = \sum_{i=0}^{\infty} x^i.$$

Lemma 2.14. ([30]) *Suppose $\mathbb{A}$ is a Banach algebra and x, y are vectors in $\mathbb{A}$. If the commutativity law holds between x and y then we have the following:*

(i) $r(xy) \leq r(x)r(y)$;
(ii) $r(x+y) \leq r(x) + r(y)$;
(iii) $\left|r(x) - r(y)\right| \leq r(x-y)$.

Definition 2.15. ([18]) Suppose X is a nonempty set, then d is a cone metric on X if the following are satisfied:

(d_1) for every $(x, y) \in X \times X$, $d(x, y) \succeq \theta$ and $d(x, y) = \theta \iff x = y$;
(d_2) for every $(x, y) \in X \times X$, $d(x, y) = d(y, x)$;
(d_3) for every $(x, y, z) \in X \times X$, $d(x, y) \preceq d(x, z) + d(z, y)$.

The pair (X, d) is a cone metric space over the Banach algebra $\mathbb{A}$.

Let $\mathfrak{D} : X \times X \to \mathbb{A}$ be a given mapping. For every $x \in X$, write

$$c(\mathfrak{D}, X, x) = \{\{x_k\} \subset X : \lim_{k\to\infty} \mathfrak{D}(x, x_k) = \theta\}.$$

Definition 2.16. ([17]) A Generalized cone metric $\mathfrak{D}$ on X is a cone metric space which satisfies the following conditions:

($\mathfrak{D}_1$) for every $(x, y) \in X \times X$, $\mathfrak{D}(x, y) \succeq \theta$ and $\mathfrak{D}(x, y) = \theta$, then $x = y$;
($\mathfrak{D}_2$) for any $(x, y) \in X \times X$, $\mathfrak{D}(x, y) = \mathfrak{D}(y, x)$;
($\mathfrak{D}_3$) for $(x, y) \in X \times X$ and $\{x_k\} \in (\mathfrak{D}, X, x)$, then $\mathfrak{D}(x, y) \preceq c(\epsilon + \mathfrak{D}(x_k, y))$ if there exists $c \in P$ with $c \succ \theta$ satisfying $\epsilon \ll \theta$ and $N \in \mathbb{N}$, where $n > N$.

Definition 2.17. ([17]) Suppose $(X, \mathfrak{D})$ is a generalized cone metric space over the Banach algebra $\mathbb{A}$. Let $\{x_k\}$ be a sequence in X and $x \in X$. We say that $\{x_k\}$ $\mathfrak{D}$-converges to x, if $x_k \in c(\mathfrak{D}, X, x)$, where $c \in p$.

Definition 2.18. ([17]) Suppose $(X, \mathfrak{D})$ is a generalized cone metric space over Banach algebra $\mathbb{A}$. Let $\{x_k\}$ be a sequence in X. We say that $\{x_k\}$ is a $\mathfrak{D}$-Cauchy sequence if $\lim_{k,l\to\infty} \mathfrak{D}(x_k, x_{k+l}) = \theta$, where θ is the null of Banach Algebra.

Definition 2.19. ([17]) Suppose $(X, \mathfrak{D})$ is a generalized cone metric space over the same Banach algebra, then $(X, \mathfrak{D})$ is said to be $\mathfrak{D}$-complete if every $\mathfrak{D}$-Cauchy sequence in X is convergent to some element in X.

A cone metric space (X, d) over Banach Algebra $\mathbb{A}$ is a generalized cone metric space $(X, \mathfrak{D})$ over the same Banach Algebra $\mathbb{A}$.

Proposition 2.20. ([17]) *Suppose $(X, \mathfrak{D})$ is a generalized cone metric space over the Banach algebra $\mathbb{A}$. Let $\{x_k\}$ be a sequence in X and $(x, y) \in X \times X$. If $\{x_k\}$ $\mathfrak{D}$-converges to x and $\mathfrak{D}$-converges to y, then $x = y$.*

3. Main results

This work is motivated by the works of Leng and Yin [17] and Okeke and Francis [21]. Next, we prove the following results in generalized cone metric spaces.

Theorem 3.1. *Suppose $(X, \mathfrak{D})$ is a complete generalized cone metric space over the Banach algebra $\mathbb{A}$. If P is a solid cone and there is a mapping $T : X \to X$ for which the following conditions hold;*

(a)

$$\begin{aligned} \mathfrak{D}(Tx, Ty) \preceq\ & k_1(\mathfrak{D}(x, y))\mathfrak{D}(x, Tx) + k_2(\mathfrak{D}(x, y))\mathfrak{D}(y, Ty) \\ & + k_3(\mathfrak{D}(x, y))\mathfrak{D}(x, y) \end{aligned}$$

for any $x, y \in X$, where $k_1(\mathfrak{D}(x, y)), k_2(\mathfrak{D}(x, y)), k_3(\mathfrak{D}(x, y)) \in P$ with

$$\sup_{x,y\in X} \sum_{i=1}^{3} r(k_i(\mathfrak{D}(x, y))) \prec 1,$$

then there is a $c \in P$ with $c \succ \theta$ satisfying that for any $\epsilon \ll \theta$ there exists $N \in \mathbb{N}$ such that for any $n > N$, if $(x, y) \in X \times X$ and $\{x_k\} \in (\mathfrak{D}, X, x)$, so that $\mathfrak{D}(x, y) \preceq c(\epsilon + \mathfrak{D}(x_k, y))$ satisfying $\theta \prec r(ck_2(\mathfrak{D}(x, y))) \prec 1$ and $\theta \prec r(ck_1(\mathfrak{D}(x, y))) \prec 1$.

(b) *there exists $x_0 \in X$ such that for any $n \in \mathbb{N}$, $\mathfrak{D}(x_0, Tx_0)$ is bounded and $\mathfrak{D}(x_0, Tx_0) \succ \theta$.*

Then T has a unique fixed point and $\{T^n x_0\}_{n\in\mathbb{N}}$ $\mathfrak{D}$-converges to fixed point of T.

Proof. First observe that $X \neq \emptyset$. If on a contrary, we have nothing to prove. Now let $x_0 \in X$ such that for any $n \in \mathbb{N}$, $\mathfrak{D}(x_0, Tx_0)$ is bounded and $\mathfrak{D}(x_0, Tx_0) \succ \theta$. To show that $\{T^n x_0\}_{n\in\mathbb{N}}$ $\mathfrak{D}$-converges to fixed point of T, then we assume that $T^n x_0 \neq T^{n+1}x_0$ which implies that $\mathfrak{D}(T^n x_0, T^{n+1}x_0) \succ \theta$. Thus, this means that $\mathfrak{D}(T^n x_0, T^{n+1}x_0) = \mathfrak{D}(TT^{n-1}x_0, TT^n x_0)$. From condition (a) we get

$$\begin{aligned}
\mathfrak{D}(T^{n+1}x_0, T^n x_0) \preceq\, & k_1(\mathfrak{D}(T^n x_0, T^{n-1}x_0))\mathfrak{D}(T^n x_0, TT^n x_0) \\
& + k_2(\mathfrak{D}(T^n x_0, T^{n-1}x_0))\mathfrak{D}(T^{n-1}x_0, TT^{n-1}x_0) \\
& + k_3(\mathfrak{D}(T^n x_0, T^{n-1}x_0))\mathfrak{D}(T^n x_0, T^{n-1}x_0) \\
=\, & k_1(\mathfrak{D}(T^n x_0, T^{n-1}x_0))\mathfrak{D}(T^n x_0, T^{n+1}x_0) \\
& + k_2(\mathfrak{D}(T^n x_0, T^{n-1}x_0))\mathfrak{D}(T^{n-1}x_0, T^n x_0) \\
& + k_3(\mathfrak{D}(T^n x_0, T^{n-1}x_0))\mathfrak{D}(T^n x_0, T^{n-1}x_0). \qquad (3.1)
\end{aligned}$$

Therefore, from inequality (3.1), we see that

$$\begin{aligned}
&(e - k_1(\mathfrak{D}(T^n x_0, T^{n-1}x_0)))\mathfrak{D}(T^n x_0, T^{(n+1)}x_0) \\
&\quad \preceq (k_2(\mathfrak{D}(T^n x_0, T^{n-1}x_0))) + k_3(\mathfrak{D}(T^n x_0, T^{n-1}x_0)))) \times \mathfrak{D}(T^n x_0, T^{n-1}x_0). \qquad (3.2)
\end{aligned}$$

Thus

$$\begin{aligned}
&\mathfrak{D}(T^n x_0, T^{n+1}x_0) \\
&\quad \preceq \frac{k_2(\mathfrak{D}(T^n x_0, T^{n-1}x_0))) + k_3(\mathfrak{D}(T^n x_0, T^{n-1}x_0))))}{(e - k_1(\mathfrak{D}(T^n x_0, T^{n-1}x_0)))}\mathfrak{D}(T^n x_0, T^{n-1}x_0) \\
&\quad \preceq (k_2(\mathfrak{D}(T^n x_0, T^{n-1}x_0)) + k_3(\mathfrak{D}(T^n x_0, T^{n-1}x_0))))\mathfrak{D}(T^n x_0, T^{n-1}x_0) \\
&\quad \preceq (k_2(\mathfrak{D}(T^n x_0, T^{n-1}x_0)) + k_3(\mathfrak{D}(T^n x_0, T^{n-1}x_0))))^2\mathfrak{D}(T^n x_0, T^{n-1}x_0) \\
&\quad \preceq (k_2(\mathfrak{D}(T^n x_0, T^{n-1}x_0)) + k_3(\mathfrak{D}(T^n x_0, T^{n-1}x_0))))^3\mathfrak{D}(x_0, T^{n-1}x_0) \\
&\quad \vdots \\
&\quad \preceq \mathfrak{D}(x_0, Tx_0). \qquad (3.3)
\end{aligned}$$

Hence, $\{\mathfrak{D}(T^n x_0, T^{n+1}x_0)\}$ is non-increasing sequence that is bounded below, so the $\mathfrak{D}$-sequence converges to some real number, $\eta \geq \theta$. Suppose by contradiction that $\eta > \theta$, then observe that

$$\begin{aligned}
&\mathfrak{D}(T^n x_0, T^{n+1}x_0) \\
&\quad \preceq (k_2(\mathfrak{D}(T^n x_0, T^{n-1}x_0)) + k_3(\mathfrak{D}(T^n x_0, T^{n-1}x_0)))\mathfrak{D}(T^n x_0, T^{n-1}x_0), \qquad (3.4)
\end{aligned}$$

so that

$$\frac{\mathfrak{D}(T^nx_0, T^{n+1}x_0)}{\mathfrak{D}(T^nx_0, T^{n-1}x_0)} \preceq k_2(\mathfrak{D}(T^nx_0, T^{n-1}x_0)) + k_3(\mathfrak{D}(T^nx_0, T^{n-1}x_0)). \tag{3.5}$$

Therefore, from inequality (3.5), by the existence of limit and after taking the limit on both sides, we have

$$1 \preceq \lim_{n\to\infty}(k_2(\mathfrak{D}(T^nx_0, T^{n-1}x_0)) + k_3(\mathfrak{D}(T^nx_0, T^{n-1}x_0)))). \tag{3.6}$$

Inequality (3.6) shows that

$$\lim_{n\to\infty}\mathfrak{D}(T^nx_0, T^{n-1}x_0) = \theta. \tag{3.7}$$

This is a contradiction. Therefore,

$$\lim_{n\to\infty}\mathfrak{D}(T^nx_0, T^{n+1}x_0) = \theta. \tag{3.8}$$

We now show that $\{T^nx_0\}_{n\in\mathbb{N}}$ is a $\mathfrak{D}$-Cauchy sequence in X. By contradiction, we assume that there exists two subsequences $\{T^{m_k}x_0\}_{k\in\mathbb{N}}$ and $\{T^{n_k}x_0\}_{k\in\mathbb{N}}$ of $\{T^nx_0\}_{n\in\mathbb{N}}$ and ϵ^* such that for any $m_k > n_k > k$,

$$\mathfrak{D}(T^{m_k}x_0, T^{n_k}x_0) \succeq \epsilon^*,$$

$$\mathfrak{D}(T^{m_k}x_0, T^{n_k-1}x_0) \prec \epsilon^*$$

and

$$\mathfrak{D}(T^{m_k}x_0, T^{m_k-1}x_0) = \theta$$

as $k \to \infty$. Now, we can rewrite $\mathfrak{D}(T^{m_k}x_0, T^{n_k}x_0) \succeq \epsilon^*$ as $\epsilon^* \preceq \mathfrak{D}(T^{m_k}x_0, T^{n_k}x_0)$, so we have from condition (a) that;

$$\begin{aligned}
\epsilon^* &\preceq \mathfrak{D}(T^{m_k}x_0, T^{n_k}x_0)\\
&= \mathfrak{D}(TT^{m_k-1}x_0, TT^{n_k-1}x_0)\\
&\preceq k_1(\mathfrak{D}(T^{m_k-1}x_0, T^{n_k-1}x_0))\mathfrak{D}(T^{m_k-1}x_0, TT^{m_k-1}x_0)\\
&\quad + k_2(\mathfrak{D}(T^{m_k-1}x_0, T^{n_k-1}x_0))\mathfrak{D}(T^{n_k-1}x_0, TT^{n_k-1}x_0)\\
&\quad + k_3(\mathfrak{D}(T^{m_k-1}x_0, T^{n_k-1}x_0))\mathfrak{D}(T^{m_k-1}x_0, T^{n_k-1}x_0)\\
&= k_1(\mathfrak{D}(T^{m_k-1}x_0, T^{n_k-1}x_0))\mathfrak{D}(T^{m_k-1}x_0, T^{m_k}x_0)\\
&\quad + k_2(\mathfrak{D}(T^{m_k-1}x_0, T^{n_k-1}x_0))\mathfrak{D}(T^{n_k-1}x_0, T^{n_k}x_0)\\
&\quad + k_3(\mathfrak{D}(T^{m_k-1}x_0, T^{n_k-1}x_0))\mathfrak{D}(T^{m_k-1}x_0, T^{n_k-1}x_0).
\end{aligned} \tag{3.9}$$

However, from inequality (3.9), we can see that

$$\begin{aligned}
\epsilon^* &\preceq \mathfrak{D}(T^{m_k}x_0, T^{n_k}x_0) \\
&\preceq \mathfrak{D}(T^{m_k-1}x_0, T^{m_k}x_0) \\
&\quad + \mathfrak{D}(T^{n_k-1}x_0, T^{n_k}x_0) \\
&\quad + \mathfrak{D}(T^{m_k-1}x_0, T^{n_k-1}x_0) \\
&\preceq \mathfrak{D}(T^{m_k-1}x_0, T^{m_k}x_0) \\
&\quad + \mathfrak{D}(T^{n_k-1}x_0, T^{n_k}x_0) \\
&\quad + \mathfrak{D}(T^{m_k-1}x_0, T^{m_k}x_0) \\
&\quad + \mathfrak{D}(T^{m_k}x_0, T^{n_k-1}x_0) \\
&\prec 2\mathfrak{D}(T^{m_k-1}x_0, T^{m_k}x_0) \\
&\quad + \mathfrak{D}(T^{n_k-1}x_0, T^{n_k}x_0) + \epsilon^*. \qquad (3.10)
\end{aligned}$$

Inequality (3.10) implies that

$$\begin{aligned}
\epsilon^* &\preceq \mathfrak{D}(T^{m_k}x_0, T^{n_k}x_0) \\
&\prec 2\mathfrak{D}(T^{m_k-1}x_0, T^{m_k}x_0) + \mathfrak{D}(T^{n_k-1}x_0, T^{n_k}x_0) + \epsilon^*. \qquad (3.11)
\end{aligned}$$

Hence,

$$\begin{aligned}
\epsilon^* &\preceq \lim_{k\to\infty} \mathfrak{D}(T^{m_k}x_0, T^{n_k}x_0) \\
&\prec 2\lim_{k\to\infty} \mathfrak{D}(T^{m_k-1}x_0, T^{m_k}x_0) + \lim_{k\to\infty} \mathfrak{D}(T^{n_k-1}x_0, T^{n_k}x_0) + \epsilon^*, \qquad (3.12)
\end{aligned}$$

which implies that

$$\begin{aligned}
\epsilon^* &\preceq \lim_{k\to\infty} \mathfrak{D}(T^{m_k}x_0, T^{n_k}x_0) \\
&\preceq 3\theta + \epsilon^* \qquad (3.13)
\end{aligned}$$

and

$$\begin{aligned}
\epsilon^* - \theta &\preceq \lim_{k\to\infty} \mathfrak{D}(T^{m_k}x_0, T^{n_k}x_0) \\
&\preceq \theta + \epsilon^*. \qquad (3.14)
\end{aligned}$$

$$\lim_{k\to\infty} \mathfrak{D}(T^{m_k}x_0, T^{n_k}x_0) = \theta + \epsilon^*. \qquad (3.15)$$

Again,

$$\begin{aligned}\mathfrak{D}(T^{m_k}x_0, T^{n_k}x_0) &\preceq \mathfrak{D}(T^{m_k-1}x_0, T^{m_k}x_0)\\ &\quad + \mathfrak{D}(T^{n_k-1}x_0, T^{n_k}x_0)\\ &\quad + \mathfrak{D}(T^{m_k-1}x_0, T^{n_k-1}x_0)\\ &\preceq \mathfrak{D}(T^{m_k-1}x_0, T^{m_k}x_0)\\ &\quad + \mathfrak{D}(T^{m_k}x_0, T^{n_k}x_0)\\ &\quad + \mathfrak{D}(T^{n_k}x_0, T^{n_k-1}x_0)\\ &\quad + \mathfrak{D}(T^{m_k-1}x_0, T^{m_k}x_0)\\ &\quad + \mathfrak{D}(T^{n_k-1}x_0, T^{n_k}x_0),\end{aligned} \tag{3.16}$$

which implies that

$$\begin{aligned}\lim_{k\to\infty} \mathfrak{D}(T^{m_k}x_0, T^{n_k}x_0) &\preceq \lim_{k\to\infty} \mathfrak{D}(T^{m_k-1}x_0, T^{n_k-1}x_0)\\ &\preceq \lim_{k\to\infty} \mathfrak{D}(T^{m_k}x_0, T^{n_k}x_0) + 4\theta.\end{aligned} \tag{3.17}$$

So that from Equation (3.15),

$$\theta + \epsilon^* \preceq \lim_{k\to\infty} \mathfrak{D}(T^{m_k-1}x_0, T^{n_k-1}x_0) \preceq \theta + \epsilon^* + 4\theta,$$

$$\lim_{k\to\infty} \mathfrak{D}(T^{m_k-1}x_0, T^{n_k-1}x_0) = \theta + \epsilon^*. \tag{3.18}$$

Therefore,

$$1 \preceq \lim \inf_{k\to\infty} k_3(\mathfrak{D}(T^{m_k-1}x_0, T^{n_k-1}x_0))$$

for any $c \in \mathbb{A}$ with $\theta \ll c$, there exists $m_k > n_k > k$ such that

$$\mathfrak{D}(T^{m_k}x_0, T^{n_k}x_0) \ll c + \epsilon^*.$$

Therefore, from Lemma 2.10, $\lim_{k\to\infty} \mathfrak{D}(T^{m_k}x_0, T^{n_k}x_0) = \theta + \epsilon^*$, as ϵ^* is arbitrary, hence, $\{T^n x_0\}_{n\in\mathbb{N}}$ is a $\mathfrak{D}$-Cauchy sequence in X. The $\mathfrak{D}$-completeness of $(X, \mathfrak{D})$ shows that there exists $u \in X$ such that $\{T^n x_0\}$ $\mathfrak{D}$-converges to u.

According to the condition (3) of Definition 2.16, for any $\theta \ll \epsilon$, there exist $c \in P$ with $c \succ \theta$ and $n_0 \in \mathbb{N}$, where $n > n_0$ such that

$$\mathfrak{D}(u, Tu) \preceq c(\epsilon + \mathfrak{D}(T^n x_0, Tu)), \tag{3.19}$$

where $\mathfrak{D}(T^n x_0, u) = \epsilon$.

Now consider $\mathfrak{D}(T^n x_0, Tu)$. From Condition (a) of the theorem, we get

$$\begin{aligned}\mathfrak{D}(T^n x_0, Tu) \preceq & k_1(\mathfrak{D}(T^{n-1}x_0, u))\mathfrak{D}(T^{n-1}x_0, TT^{n-1}x_0) \\ & + k_2(\mathfrak{D}(T^{n-1}x_0, u))\mathfrak{D}(u, Tu) \\ & + k_3(\mathfrak{D}(T^{n-1}x_0, u))\mathfrak{D}(T^{n-1}x_0, u) \\ = & k_1(\mathfrak{D}(T^{n-1}x_0, u))\mathfrak{D}(T^{n-1}x_0, T^n x_0) \\ & + k_2(\mathfrak{D}(T^{n-1}x_0, u))\mathfrak{D}(u, Tu) \\ & + k_3(\mathfrak{D}(T^{n-1}x_0, u))\mathfrak{D}(T^{n-1}x_0, u). \end{aligned} \tag{3.20}$$

Putting Equation (3.20) into (3.19), we have

$$\begin{aligned}(e - ck_2(\mathfrak{D}(T^{n-1}x_0, u)))\mathfrak{D}(u, Tu) \preceq & ck_1(\mathfrak{D}(T^{n-1}x_0, u))\mathfrak{D}(T^{n-1}x_0, T^n x_0) \\ & + ck_3(\mathfrak{D}(T^{n-1}x_0, u))\mathfrak{D}(T^{n-1}x_0, u) + c\epsilon. \end{aligned} \tag{3.21}$$

Since $\{T^n x_0\}$ is $\mathfrak{D}$-convergent to u and $\epsilon \in \mathbb{A}$, from Equations (3.20) and (3.7) we have

$$\begin{aligned}(e - ck_2(\epsilon)\mathfrak{D}(u, Tu) & \preceq ck_1(\epsilon)\theta + ck_3(\epsilon)\epsilon + c\epsilon. \\ & \preceq ck_1(\epsilon)\theta. \end{aligned} \tag{3.22}$$

Therefore, $\mathfrak{D}(u, Tu) = \theta$, that is, $Tu = u$. Hence u is a fixed point of T.

For uniqueness, assume that there exists another fixed point of T say u^*. Now we claim that the two fixed points of T are not unique. That is $u \neq u^*$. Therefore, $\mathfrak{D}(u, u^*) = \mathfrak{D}(Tu, Tu^*)$. From condition (a) above we have

$$\begin{aligned}\mathfrak{D}(u, u^*) & = \mathfrak{D}(Tu, Tu^*) \\ & \preceq k_1(\mathfrak{D}(u, u^*))\mathfrak{D}(u, Tu) + k_2(\mathfrak{D}(u, u^*))\mathfrak{D}(u^*, Tu^*) \\ & \quad + k_3(\mathfrak{D}(u, u^*))\mathfrak{D}(u, u^*) \\ & = k_1(\mathfrak{D}(u, u^*))\mathfrak{D}(u, u) + k_2(\mathfrak{D}(u, u^*))\mathfrak{D}(u^*, u^*) \\ & \quad + k_3(\mathfrak{D}(u, u^*))\mathfrak{D}(u, u^*). \end{aligned} \tag{3.23}$$

We can now see that $\mathfrak{D}(u, u^*) = \theta$. This shows that the fixed points of T is indeed unique. □

Remark 3.2. If $k_i(\mathfrak{D}(x, y)) = k_i$ for $i = 1, 2, 3$, then we get Theorem 4.1 in Leng and Yin [17]. Hence Theorem 3.1 is a generalization of the results of [17] among others.

Theorem 3.3. *Suppose $(X, \mathfrak{D})$ is a complete and generalized cone metric space over the Banach algebra $\mathbb{A}$. If P is a solid cone and a self mapping $T : X \to X$ for which the following conditions hold;*

(a)

$$\mathfrak{D}(Tx,Ty) \preceq k_1(\mathfrak{D}(Tx,y))\mathfrak{D}(x,Tx) \\ + k_2(\mathfrak{D}(Tx,y))\mathfrak{D}(y,Ty) + k_3(\mathfrak{D}(Tx,y))\mathfrak{D}(x,y)$$

for any $x, y \in X$, where $k_1(\mathfrak{D}(Tx,y)), k_2(\mathfrak{D}(Tx,y)), k_3(\mathfrak{D}(Tx,y)) \in P$ with

$$\sup_{x,y\in X} \sum_{i=1}^{3} r(k_i(\mathfrak{D}(Tx,y))) \prec 1,$$

then there is a $c \in P$ with $c \succ \theta$, if for any $\epsilon \ll \theta$ there exists $N \in \mathbb{N}$ such that $\mathfrak{D}(x,y) \preceq c(\epsilon + \mathfrak{D}(x_k,y))$, where $n > N$, if $(x,y) \in X \times X$ and $\{x_k\} \in (\mathfrak{D}, X, x)$ satisfying $\theta \prec r(ck_2(\mathfrak{D}(x,y))) \prec 1$ and $\theta \prec r(ck_1(\mathfrak{D}(x,y))) \prec 1$,

(b) *there exists $x_0 \in X$ such that for any $n \in \mathbb{N}$, $\mathfrak{D}(x_0, Tx_0)$ is bounded and $\mathfrak{D}(x_0, Tx_0) \succ \theta$. Then T has a unique fixed point and $\{T^n x_0\}_{n\in\mathbb{N}}$ $\mathfrak{D}$-converges to fixed point of T.*

Proof. First observe that $X \neq \emptyset$. If on a contrary, we have nothing to prove.

Now let $x_0 \in X$ such that for any $n \in \mathbb{N}$, $\mathfrak{D}(x_0, Tx_0) \succ \theta$, since $\mathfrak{D}(x_0, Tx_0)$ is bounded. To show that $\{T^n x_0\}_{n\in\mathbb{N}}$ $\mathfrak{D}$-converges to fixed point of T, then we assume that $T^n x_0 \neq T^{n+1}x_0$, which implies that $\mathfrak{D}(T^n x_0, T^{n+1}x_0) \succ \theta$. Thus, this means that

$$\mathfrak{D}(T^n x_0, T^{n+1}x_0) = \mathfrak{D}(TT^{n-1}x_0, TT^n x_0).$$

From condition (a) of Theorem 3.3,

$$\mathfrak{D}(Tx,Ty) \preceq k_1(\mathfrak{D}(Tx,y))\mathfrak{D}(x,Tx) \\ + k_2(\mathfrak{D}(Tx,y))\mathfrak{D}(Ty,y) + k_3(\mathfrak{D}(Tx,y))\mathfrak{D}(x,y)$$

for all $x, y \in X$, put $x = T^{n-1}x_0$ and $y = T^n x_0$, we get

$$\begin{aligned} \mathfrak{D}(T^{n+1}x_0, T^n x_0) &\preceq k_1(\mathfrak{D}(TT^n x_0, T^{n-1}x_0))\mathfrak{D}(T^n x_0, TT^n x_0) \\ &\quad + k_2(\mathfrak{D}(TT^n x_0, Tn-1))\mathfrak{D}(T^{n-1}x_0, TT^{n-1}x_0) \\ &\quad + k_3(\mathfrak{D}(TT^n x_0, T^{n-1}x_0))\mathfrak{D}(T^n x_0, T^{n-1}x_0) \\ &= k_1(\mathfrak{D}(T^{n+1}x_0, T^{n-1}x_0))\mathfrak{D}(T^n x_0, T^{n+1}x_0) \\ &\quad + k_2(\mathfrak{D}(T^{n+1}x_0, T^{n-1}x_0))\mathfrak{D}(T^{n-1}x_0, T^n x_0) \\ &\quad + k_3(\mathfrak{D}(T^{n+1}x_0, T^{n-1}x_0))\mathfrak{D}(T^n, T^{n-1}x_0). \end{aligned} \tag{3.24}$$

Therefore, from (3.24), we get

$$\begin{aligned}(e-k_1(\mathfrak{D}(T^{n+1}x_0,T^{n-1}x_0))\mathfrak{D}(T^nx_0,T^{n+1}x_0)\\ \preceq [k_2(\mathfrak{D}(T^{n+1}x_0,T^{n-1}x_0))+k_3(\mathfrak{D}(T^{n+1}x_0,T^{n-1}x_0))]\mathfrak{D}(T^nx_0,TT^{n-1}x_0).\end{aligned} \tag{3.25}$$

Hence,

$$\begin{aligned}&\mathfrak{D}(T^nx_0,T^{n+1}x_0)\\ &\quad\preceq \frac{k_2(\mathfrak{D}(T^{n+1}x_0,T^{n-1}x_0))+k_3\mathfrak{D}(T^{n+1}x_0,T^{n-1}x_0)\mathfrak{D}(T^nx_0,T^{n-1}x_0)}{(e-K_1(\mathfrak{D}(T^{n+1}x_0,T^{n-1}x_0))}\\ &\quad\preceq k_2(\mathfrak{D}(T^{n+1}x_0,T^{n-1}x_0))+k_3(\mathfrak{D}(T^{n+1}x_0,T^{n-1}x_0))\mathfrak{D}(T^nx_0,T^{n-1}x_0)\\ &\quad\preceq (k_2(\mathfrak{D}(T^{n+1}x_0,T^{n-1}x_0))+k_3(\mathfrak{D}(T^{n+1}x_0,T^{n-1}x_0)))^2\mathfrak{D}(T^nx_0,T^{n-1}x_0)\\ &\quad\preceq (k_2(\mathfrak{D}(T^{n+1}x_0,T^{n-1}x_0))+k_3(\mathfrak{D}(T^{n+1}x_0,T^{n-1}x_0)))^n\mathfrak{D}(x_0,T^{n-1}x_0)\\ &\quad\vdots\\ &\quad\preceq \mathfrak{D}(x_0,Tx_0).\end{aligned} \tag{3.26}$$

Hence, $\{\mathfrak{D}(T^nx_0,T^{n+1}x_0)\}_{n\in\mathbb{N}}$ is non-increasing sequence that is bounded below. Hence the $\mathfrak{D}$-sequence converges to some real number $a\geq 0$.

Now suppose that $a>0$. Then we observe that

$$\begin{aligned}&\mathfrak{D}(T^nx_0,T^{n+1}x_0)\\ &\quad\preceq ((k_2(\mathfrak{D}(T^{n+1}x_0,T^{n-1}x_0))+k_3(\mathfrak{D}(T^{n+1}x_0,T^{n-1}x_0)))\mathfrak{D}(T^nx_0,T^{n-1}x_0).\end{aligned}$$

From this inequality, we get

$$\frac{\mathfrak{D}(T^nx_0,T^{n+1}x_0}{\mathfrak{D}(T^nx_0,T^{n-1}x_0)}\preceq\sum_{i=2}^{3}k_i(\mathfrak{D}(T^{n+1}x_0,T^{n-1}x_0)). \tag{3.27}$$

By the existence of limit and taking limit as $n\to\infty$ on both sides, we get

$$1\preceq\lim_{n\to\infty}\sum_{i=2}^{3}k_i(\mathfrak{D}(T^{n+1}x_0,T^{n-1}x_0)). \tag{3.28}$$

In fact, (3.28) is the Geraghty condition, thus

$$\lim_{n\to\infty}\mathfrak{D}(T^{n+1}x_0,T^{n-1}x_0)=\theta,$$

which is a contradiction to our initial claim, therefore,

$$\lim_{n\to\infty}\mathfrak{D}(T^nx_0,T^{n+1}x_0)=\theta.$$

Next, we demonstrate that the sequence $\{T^nx_0\}_{n\in\mathbb{N}}$ is a $\mathfrak{D}$-Cauchy sequence in X. By contradiction, assume that there are two sub-sequence $\{T^{m_k}x_0\}_{k\in\mathbb{N}}$ and $\{T^{n_k}x_0\}_{k\in\mathbb{N}}$ of $\{T^nx_0\}_{n\in\mathbb{N}}$ and $\in^*>0$ such that for any $m_k>n_k>k$,

$\mathfrak{D}(T^{m_k}x_0, T^{n_k}x_0) \succeq \in^*$, $\mathfrak{D}(T^{m_k}x_0, T^{n_{k-1}}x_0) \preceq \in^*$ and $\mathfrak{D}(T^{m_k}x_0, T^{m_{k-1}}x_0) = \theta$ as $k \to \infty$.

Now we write

$$\mathfrak{D}(T^{m_k}x_0, T^{n_k}x_0) \succeq \in^*$$

as $\in^* \preceq \mathfrak{D}(T^{m_k}x_0, T^{n_k}x_0)$, from condition (a) above, we have

$$\begin{aligned}
\epsilon^* &\preceq \mathfrak{D}(T^{m_k}x_0, T^{n_k}x_0) \\
&= \mathfrak{D}(TT^{m_k-1}x_0, TT^{n_k-1}x_0) \\
&\preceq k_1(\mathfrak{D}(TT^{m_k-1}x_0, T^{n_k-1}x_0))\mathfrak{D}(T^{m_k-1}x_0, TT^{m_k-1}x_0) \\
&\quad + k_2(\mathfrak{D}(TT^{m_k-1}x_0, T^{n_k-1}x_0))\mathfrak{D}(T^{n_k-1}x_0, TT^{n_k-1}x_0) \\
&\quad + k_3(\mathfrak{D}(TT^{m_k-1}x_0, T^{n_k-1}x_0))\mathfrak{D}(T^{m_k-1}x_0, T^{n_k-1}x_0) \\
&= k_1(\mathfrak{D}(T^{m_k}x_0, T^{n_k-1}x_0))\mathfrak{D}(T^{m_k-1}x_0, TT^{m_k}x_0) \\
&\quad + k_2(\mathfrak{D}(T^{m_k}x_0, T^{n_k-1}x_0))\mathfrak{D}(T^{n_k-1}x_0, TT^{n_k}x_0) \\
&\quad + k_3(\mathfrak{D}(TT^{m_k}x_0, T^{n_k-1}x_0))\mathfrak{D}(T^{m_k-1}x_0, T^{n_k-1}x_0).
\end{aligned} \tag{3.29}$$

Now,

$$\begin{aligned}
\in^* &\preceq \mathfrak{D}(T^{m_k}x_0, T^{n_k}x_0) \\
&\preceq \mathfrak{D}(T^{m_k-1}x_0, T^{mk}x_0) + \mathfrak{D}(T^{n_k-1}x_0, T^{n_k}x_0) \\
&\quad + \mathfrak{D}((T^{mk-1}x_0, T^{nk-1}x_0)) \\
&\preceq \mathfrak{D}(T^{m_k-1}x_0, T^{m_k}x_0) + \mathfrak{D}(T^{n_k-1}x_0, T^{n_k}x_0) \\
&\quad + \mathfrak{D}(T^{m_k-1}x_0, T^{m_k}x_0) + \mathfrak{D}((T^{m_k}x_0, T^{n_k-1}x_0)) \\
&< 2\mathfrak{D}(T^{m_k-1}x_0, T^{n_k}x_0) + \mathfrak{D}(T^{n_k-1}x_0, T^{n_k}x_0) + \in^* \\
&\preceq \liminf_{k\to\infty} \mathfrak{D}(T^{m_k}x_0, T^{n_k}x_0) \\
&\preceq 2\liminf_{k\to\infty} \mathfrak{D}(T^{m_k-1}x_0, T^{m_k}x_0) + \liminf_{k\to\infty} \mathfrak{D}(T^{n_k-1}x_0, T^{n_k}x_0) \\
&\preceq \in^* \liminf_{k\to\infty} \mathfrak{D}(T^{m_k}x_0, T^{n_k}x_0) \\
&\preceq \in^* \liminf_{k\to\infty} \mathfrak{D}(T^{m_k}x_0, T^{n_k}x_0) = \in^* + \theta
\end{aligned}$$

and

$$\lim_{k\to\infty} \mathfrak{D}(T^{m_k}x_0, T^{n_k}x_0) = \in^* + \theta. \tag{3.30}$$

Again,

$$\begin{aligned}&\mathfrak{D}(T^{m_k}x_0, T^{n_k}x_0)\\ &\quad\preceq \mathfrak{D}(T^{m_k-1}x_0, T^{m_k}x_0) + \mathfrak{D}(T^{n_k-1}x_0, T^{n_k}x_0) + \mathfrak{D}(T^{m_k-1}x_0, T^{n_k-1}x_0).\end{aligned} \tag{3.31}$$

$$\begin{aligned}&\mathfrak{D}(T^{m_k}x_0, T^{n_k}x_0)\\ &\quad\preceq \mathfrak{D}(T^{m_k-1}x_0, T^{m_k}x_0) + \mathfrak{D}(T^{m_k}x_0, T^{n_k}x_0) + \mathfrak{D}(T^{n_k}x_0, T^{n_k-1}x_0)\\ &\qquad + \mathfrak{D}(T^{m_k-1}x_0, T^{m_k}x_0) + \mathfrak{D}(T^{n_k-1}x_0, T^{n_k}x_0).\end{aligned} \tag{3.32}$$

This implies

$$\begin{aligned}\liminf_{k\to\infty} \mathfrak{D}(T^{m_k}x_0, T^{n_k}x_0) &\preceq \liminf_{k\to\infty} \mathfrak{D}(T^{m_k-1}x_0, T^{n_k-1}x_0)\\ &\preceq \liminf_{k\to\infty} \mathfrak{D}(T^{m_k}x_0, T^{n_k}x_0) + \theta.\end{aligned} \tag{3.33}$$

After some algebraic simplification, we get

$$\lim_{k\to\infty} \mathfrak{D}(T^{m_k-1}x_0, T^{n_k-1}x_0) = \in^* + \theta.$$

Thus, it follows that

$$1 \preceq \lim_{k\to\infty} k_3(\mathfrak{D}(T^{m_k}x_0, T^{n_k-1}x_0)).$$

$$\lim_{k\to\infty} \mathfrak{D}(T^{m_k}x_0, T^{n_k}x_0) = \lim_{k\to\infty} \mathfrak{D}(T^{m_k-1}x_0, T^{n_k-1}x_0) = \in^* + \theta,\ x_0 \in X,$$

therefore, $\lim_{k\to\infty} \mathfrak{D}(T^{m_k}x_0, T^{n_k}x_0) = \theta + \epsilon^*$, which is a contradiction. Hence $\{T^n x_0\}_{n\in\mathbb{N}}$ is a $\mathfrak{D}$-Cauchy sequence in X. The $\mathfrak{D}$-completeness of $(X, \mathfrak{D})$ implies that there exists $u^* \in X$ such that $\{T^n x_0\}_{n\in\mathbb{N}}$ $\mathfrak{D}$-converges to $u^* \in X$.

From condition (3) of definition 2.13, for any $\theta \ll \epsilon$, there exist $c \in P$ with $c \succ \theta$ and $n_0 \in \mathbb{N}$ such that for any $n > n_0$

$$\mathfrak{D}(u^*, Tu^*) \preceq c(\epsilon + \mathfrak{D}(T^n x_0, Tu^*)), \tag{3.34}$$

where $\mathfrak{D}(u^*, Tu^*) = \epsilon$. Now consider $\mathfrak{D}(T^n x_0, Tu^*)$ from condition (a) we get

$$\begin{aligned}\mathfrak{D}(T^n x_0, Tu^*) &\preceq k_1(\mathfrak{D}(TT^{n-1}x_0, u^*))\mathfrak{D}(T^{n-1}x_0, TT^{n-1}x_0)\\ &\quad + k_2(\mathfrak{D}(TT^{n-1}x_0, u^*))\mathfrak{D}(u^*, Tu^*)\\ &\quad + k_3(\mathfrak{D}(TT^{n-1}x_0, u^*)\mathfrak{D}(T^{n-1}x_0, u^*)\\ &= k_1(\mathfrak{D}(T^n x_0, u^*))\mathfrak{D}(T^{n-1}x_0, T^n x_0)\\ &\quad + k_2(\mathfrak{D}(T^n x_0, u^*))\mathfrak{D}(u^*, Tu^*)\\ &\quad + k_3(\mathfrak{D}(T^n x_0, u^*)\mathfrak{D}(T^{n-1}x_0, u^*).\end{aligned} \tag{3.35}$$

Using inequalities (3.34) and (3.35) we get

$$(e - ck_2(\mathfrak{D}(T^n x_0, u^*)))\mathfrak{D}(u^*, Tu^*) \preceq ck_1(\mathfrak{D}(T^n x_0, u^*))\mathfrak{D}(T^n x_0, T^n x_0) + ck_3(\mathfrak{D}(T^n x_0, u^*))\mathfrak{D}(T^n x_0, u^*) + c\epsilon. \tag{3.36}$$

Since $\{T^n x_0\} \to u^*$ as $n \to \infty$, thus

$$(e - ck_2)(\epsilon))\mathfrak{D}(u^*, Tu^*) \preceq ck_1(\epsilon)\theta.$$

Therefore, $\mathfrak{D}(u^*, Tu^*) = \theta$. Hence u^* is the fixed point of T. The uniqueness is clear as it follows from Theorem 3.1 above. □

If $k_1(\mathfrak{D}(x, y)) \in P$, in Theorem 3.3 then we get the following corollary;

Corollary 3.4. *Let $(X, \mathfrak{D})$ be a complete generalized cone metric space over the Banach algebra $\mathbb{A}$. If P is a solid cone and a self mapping $T : X \to X$ for which the following conditions hold;*

(a)

$$\mathfrak{D}(Tx, Ty) \preceq k_1(\mathfrak{D}(Tx, y))\mathfrak{D}(x, Tx)$$

for any $x, y \in X$, where $k_1\mathfrak{D}(Tx, y) \prec 1$.

(b) *there exists $x_0 \in X$ such that $\mathfrak{D}(x_0, Tx_0)$ is bounded for $n \in \mathbb{N}$.*

Then T has a unique fixed point and $\{T^n x_0\}_{n\in\mathbb{N}}$ $\mathfrak{D}$-converges to fixed point of T.

Proof. Observe that $X \neq \emptyset$. If on a contrary, we have nothing to prove. Now let $x_0 \in X$ such that for any $n \in \mathbb{N}$, $\mathfrak{D}(x_0, Tx_0)$ is bounded and $\mathfrak{D}(x_0, Tx_0) \succ \theta$. Since $\{T^n x_0\}_{n\in\mathbb{N}}$ $\mathfrak{D}$-converges to fixed point of T, then we assume that $T^n x_0 \neq T^{n+1}x_0$ which implies that $\mathfrak{D}(T^n x_0, T^{n+1}x_0) \succ \theta$. Thus, this means that

$$\mathfrak{D}(T^n x_0, T^{n+1}x_0) = \mathfrak{D}(TT^{n-1}x_0, TT^n x_0).$$

By proof of Theorem 3.1, we conclude that T has a unique fixed point and $\{T^n x_0\}_{n\in\mathbb{N}}$ $\mathfrak{D}$-converges to fixed point of T. □

If $k_3(\mathfrak{D}(x, y)) \in P$, in Theorem 3.3, then we get the following corollary;

Corollary 3.5. *Let $(X, \mathfrak{D})$ be a complete generalized cone metric space over the Banach algebra $\mathbb{A}$. Suppose that P is a cone and its interior is nonempty, consider a self map $T : X \to X$ for which these conditions hold;*

(a)

$$\mathfrak{D}(Tx, Ty) \preceq k_3(\mathfrak{D}(Tx, y))\mathfrak{D}(x, y),$$

for any $x, y \in X$, where $k_1(\mathfrak{D}(x,y)) \in P$ with

$$\sup_{x,y\in X} r(k_3(\mathfrak{D}(Tx,y))) \prec 1.$$

(b) *there exists $x_0 \in X$ such that $\mathfrak{D}(x_0, Tx_0)$ is bounded.*

Then T has a unique fixed point and $\{T^n x_0\}_{n\in\mathbb{N}}$ $\mathfrak{D}$-converges to fixed point of T.

Proof. Observe that $X \neq \emptyset$. If on a contrary, we have nothing to prove. Now let $x_0 \in X$ such that for any $n \in \mathbb{N}$, $\mathfrak{D}(x_0, Tx_0)$ is bounded and $\mathfrak{D}(x_0, Tx_0) \succ \theta$. Since $\{T^n x_0\}_{n\in\mathbb{N}}$ $\mathfrak{D}$-converges to fixed point of T, then we assume that $T^n x_0 \neq T^{n+1}x_0$ which implies that $\mathfrak{D}(T^n x_0, T^{n+1}x_0) \succ \theta$. Thus, this means that

$$\mathfrak{D}(T^n x_0, T^{n+1}x_0) = \mathfrak{D}(TT^{n-1}x_0, TT^n x_0).$$

By proof of Theorem 3.1, we conclude that T has a unique fixed point and $\{T^n x_0\}_{n\in\mathbb{N}}$ $\mathfrak{D}$-converges to fixed point of T. □

Corollary 3.6. *Let $(X, \mathfrak{D})$ be a complete generalized cone metric space over the Banach algebra $\mathbb{A}$. Suppose that P is a cone with its interior not empty and a self mapping $T : X \to X$ for which the following conditions hold;*

(a)

$$\mathfrak{D}(Tx, Ty) \preceq \frac{e + k_2(\mathfrak{D}(x,y))\mathfrak{D}(y, T^2y)}{e + k_2(\mathfrak{D}(x,y))\mathfrak{D}(y, Tx)} k_2(\mathfrak{D}(x,y))\mathfrak{D}(x,y)$$

for any $x, y \in X$, where $k_2(\mathfrak{D}(x,y)) \in P$ with

$$\sup_{x,y\in X} r(k_2(\mathfrak{D}(Tx,y))) \prec 1$$

and $e \in \mathbb{A}$.

(b) *there exists $x_0 \in X$ such that $\mathfrak{D}(x_0, Tx_0)$ is bounded.*

Then the fixed point of T is unique and $\{T^n x_0\}_{n\in\mathbb{N}}$ $\mathfrak{D}$-converges to fixed point of T.

Proof. Observe that $X \neq \emptyset$. If on a contrary, we have nothing to prove. Now, let $x_0 \in X$ such that for any $n \in \mathbb{N}$, $\mathfrak{D}(x_0, Tx_0)$ is bounded and $\mathfrak{D}(x_0, Tx_0) \succ \theta$. Since $\{T^n x_0\}_{n\in\mathbb{N}}$ $\mathfrak{D}$-converges to fixed point of T, then we assume that $T^n x_0 \neq T^{n+1}x_0$ which implies that $\mathfrak{D}(T^n x_0, T^{n+1}x_0) \succ \theta$. Thus, this means that

$$\mathfrak{D}(T^n x_0, T^{n+1}x_0) = \mathfrak{D}(TT^{n-1}x_0, TT^n x_0).$$

By proof of Theorem 3.1, we conclude that T has a unique fixed point. □

Example 3.7. ([17]) Let $\mathbb{A} = \mathbb{R}^2$, for each $(x_1, x_2) \in \mathbb{A}$, the norm of (x_1, x_2) is defined by $\|(x_1, x_2)\| = |x_1| + |x_2|$. A multiplication is defined by $xy = (x_1, x_2)(y_1, y_2) = (x_1y_1, x_1y_2 + x_2y_1)$ for any pair $(x_1, x_1), (y_1, y_2) \in \mathbb{A}$, then $\mathbb{A}$ is a Banach Algebra with unite $e = (1, 0)$. Therefore, P is a cone in $\mathbb{A}$ if $p = (x_1, x_2) \in \mathbb{R}^2$ such that $x_1, x_2 \geq 0$.

Example 3.8. ([17]) Let $\mathbb{A} = \mathbb{R}^2$, for each $(x_1, x_2) \in \mathbb{A}$, the norm of (x_1, x_2) is defined by $\|(x_1, x_2)\| = |x_1| + |x_2|$. Then $\|.\|$ is a norm in $\mathbb{A}$. A multiplication in $\mathbb{A}$ is defined by $xy = (x_1, x_2)(y_1, y_2) = (x_1y_1, x_1y_2 + x_2y_1)$ for any pair $(x_1, x_1), (y_1, y_2) \in \mathbb{A}$, then $\mathbb{A}$ is a Banach Algebra with unite $e = (1, 0)$. Therefore, P is a cone in $\mathbb{A}$ if $p = (x_1, x_2) \in \mathbb{R}^2$ such that $x_1, x_2 \geq 0$. Put $X = \{-1, 0, 1\}$. Define $\mathfrak{D} : X \times X \to \mathbb{A}$ by $\mathfrak{D}(x, y)= \mathfrak{D}(y, x)$ for all $x, y \in X$, then $\mathfrak{D}(-1, -1)= \mathfrak{D}(1, 1) = (0, 0)$ and $\mathfrak{D}(-1, 0)= \mathfrak{D}(3, 3)$, again $\mathfrak{D}(-1, 1)= \mathfrak{D}(0, 1) = \mathfrak{D}(0, 0) = (1, 1)$. Then $(X, \mathfrak{D})$ is a generalized cone metric space over the Banach Algebra $\mathbb{A}$.

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