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APPLICATION OF SOME COMMON FIXED POINT RESULTS FOR $\mathcal{F}-$CONTRACTION MAPPING IN $\mathtt{G}-$METRIC SPACES

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Abstract. This paper aims to establish the common fixed point theorem for $\mathcal{F}-$Kannan contraction mappings in complete $\mathtt{G}-$metric spaces and applies it to integral equations. The results extend recent fixed point findings in the literature and include examples and applications to nonlinear fractional differential equations.

1. Introduction

Fixed point theory is a well-established and extensive area of research in mathematical sciences. It combines elements of analysis, including topology, geometry, and algebra.

In 1922, Banach [2] introduced a foundational result in fixed point theory, known as the Banach Fixed Point Theorem. This theorem states that if Θ is a complete metric space and $\Gamma : \Theta \longrightarrow \Theta$ is a contraction mapping, satisfying $d(\Gamma\theta_1, \Gamma\theta_2) \leq kd(\theta_1, \theta_2)$ for all $\theta_1, \theta_2 \in \Theta$ with a constant $k \in [0, 1)$, then Γ has a unique fixed point, meaning $\Gamma\theta = \theta$ has a unique solution. Since

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then, researchers have been exploring methods to determine the fixed points of mappings by modifying one or more conditions, such as the contractive condition, the continuity of the mappings, or the completeness of the space, among others. An alternative contractive condition, which differed from the Banach contraction condition, was introduced by Kannan [7]. Some improvements to the Banach fixed point theorem address the contractive inequality, while others focus on generalizing the space.

A specific extension of metric spaces, known as the G-metric space, was introduced by Mustafa and Sims [10] in 2006. In their pioneering paper on G-metric spaces, Mustafa and Sims [10] introduced several properties of G-metric spaces and discussed topics such as its topology, compactness, completeness, product, as well as criteria for the convergence and continuity of sequences in G-metric spaces. They also proved some theorems related to these properties(See [14, 15, 16, 17]).

In 2012, the concept of $\mathcal{F}$-contraction was introduced by Wardowski [18], and a fixed point theorem for such mappings on a complete metric space was proved, which generalizes the Banach contraction principle in a different direction. The concept of $\mathcal{F}$-contractions has been expanded to include both single-valued and set-valued mappings.

2. Preliminaries

The proof of our results will rely on the following definitions and preliminary results.

Definition 2.1. ([10]) Let Λ be a non empty set and let $\mathtt{G} : \Lambda \times \Lambda \times \Lambda \to [0, \infty)$ be a function satisfying the following conditions:

(1) $\mathtt{G}(\varpi, \upsilon, \varrho) = 0$ if $\varpi = \upsilon = \varrho$,
(2) $\mathtt{G}(\varpi, \upsilon, \upsilon) > 0$; for all $\varpi, \upsilon \in \Lambda$ with $\varpi \neq \upsilon$,
(3) $\mathtt{G}(\varpi, \upsilon, \upsilon) \leq \mathtt{G}(\varpi, \upsilon, \varrho)$ for all $\varpi, \upsilon, \varrho \in \Lambda$ with $\upsilon \neq \varrho$,
(4) $\mathtt{G}(\varpi, \upsilon, \varrho) = \mathtt{G}(\varpi, \varrho, \upsilon) = \mathtt{G}(\varrho, \varpi, \upsilon) = ...$
(symmetry in all three variables),
(5) $\mathtt{G}(\varpi, \upsilon, \varrho) \leq \mathtt{G}(\varpi, \vartheta, \vartheta) + \mathtt{G}(\vartheta, \upsilon, \varrho)$ for all $\varpi, \upsilon, \varrho, \vartheta \in \Lambda$
(rectangle inequality).

Then the function $\mathtt{G}$ is called a generalized metric or more specially, a $\mathtt{G}$-metric on Λ and $(\Lambda, \mathtt{G})$ is called a $\mathtt{G}$-metric space.

Definition 2.2. ([10]) Let $(\Lambda, \mathtt{G})$ be a $\mathtt{G}$-metric space.

(i) The sequence $\{\varpi_\sigma\}$ $\mathtt{G}$-convergent to $\varpi \in \Lambda$ if and only if $\lim_{\sigma,\varrho\to\infty} \mathtt{G}(\varpi_\sigma, \varpi_\varrho, \varpi) = 0$.

(ii) The sequence $\{\varpi_\sigma\}$ $\mathtt{G}$-Cauchy sequence if and only if $\lim_{\varpi,\varrho,\zeta\to\infty} \mathtt{G}(\varpi_\sigma, \varpi_\varrho, \varpi_\zeta) = 0$.

(iii) $(\Lambda, \mathtt{G})$ is $\mathtt{G}$-complete if and only if every $\mathtt{G}$-Cauchy sequence in Λ is $\mathtt{G}$-convergent.

Definition 2.3. ([10]) A $\mathtt{G}-$metric space $(\Lambda, \mathtt{G})$ is called symmetric, if

$$\mathtt{G}(\varpi, \upsilon, \upsilon) = \mathtt{G}(\upsilon, \varpi, \varpi) \text{ for all } \varpi, \upsilon \in \Lambda. \tag{2.1}$$

An interesting generalization of the Banach fixed point theorem was given by Wardowski [18] using a different type of contraction called $\mathcal{F}$-contraction. Since then, new fixed point theorems have been constructed by many researchers following his approach, as seen in [6, 11] and the references therein.

The following definitions were provided by Wardowski [18].

Definition 2.4. Let $\mathcal{F}$ be a function defined as $\mathcal{F} : \mathbb{R}^+ \longrightarrow \mathbb{R}$, which satisfies the following conditions:

(F1) $\mathcal{F}$ is strictly increasing, that is, for all $\alpha, \beta \in \mathbb{R}^+$ such that $\alpha < \beta$, $\mathcal{F}(\alpha) < \mathcal{F}(\beta)$.

(F2) For each sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of positive numbers, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} \mathcal{F}(\alpha_n) = -\infty$.

(F3) There exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} (\alpha_n)^k \mathcal{F}(\alpha_n) = 0. \tag{2.2}$$

We denote the set of all functions satisfying properties $(F_1) - (F_3)$, by $\mathcal{F}$.

Based on the concept of $\mathcal{F}$-contraction introduced by Wardowski, Gupta et al. defined $\mathcal{F}$-contractions in G-metric space as follows:

Definition 2.5. Let (Θ, G) be a G-metric space. A mapping $\Gamma : \Theta \to \Theta$ is said to be an $\mathcal{F}$-contraction if there exists a number $\tau > 0$ such that

$$G(\Gamma\theta_1, \Gamma\theta_2, \Gamma\theta_3) > 0 \Rightarrow \tau + \mathcal{F}(G(\Gamma\theta_1, \Gamma\theta_2, \Gamma\theta_3)) \leq \mathcal{F}(G(\theta_1, \theta_2, \theta_3)) \tag{2.3}$$

for all $\theta_1, \theta_2, \theta_3 \in \Theta$.

Example 2.6. Let $\mathcal{F} : \mathbb{R}^+ \to \mathbb{R}$ be given by $\mathcal{F}(\beta) = \ln(\beta)$. It is clear that $\mathcal{F}$ satisfies $\mathcal{F}(1) - \mathcal{F}(3) < 0$ for any $k \in (0, 1)$. Each mapping $\Gamma : \mathcal{F} \to \mathcal{F}$ satisfying (2.3) is an $\mathcal{F}$-contraction such that

$$G(\Gamma\tau, \Gamma\sigma, \Gamma\upsilon) \leq e^{-\omega(G(\tau,\sigma,\upsilon))} \tag{2.4}$$

for all $\tau, \sigma, \upsilon \in \mathcal{F}$, $\Gamma\tau \neq \Gamma\sigma \neq \Gamma\upsilon$ [4]. It is clear that for $\tau, \sigma, \upsilon \in \mathcal{F}$ such that $\Gamma\tau = \Gamma\sigma = \Gamma\upsilon$, the inequality (2.4) also holds.

The theorem below was proven by Gupta et al. [5].

Theorem 2.7. ([5]) *Let (Θ, G) be a complete $G-$metric space and let $\Gamma : \Theta \to \Theta$ be an $\mathcal{F}-$contraction. Then Γ has a unique fixed point $\theta_1^* \in \Theta$ and for every $\theta_0 \in \Theta$ a sequence $\{\Gamma^n x_0\}_{n\in\mathbb{N}}$ is converges to θ_1^*.*

3. Main results

The main theorem of our paper aims to demonstrate that a map sequentially converges, given a pair of self-mappings in $\mathcal{F}$-Kannan mappings within a G-metric space.

Definition 3.1. Let $\mathcal{F}$ be a function that satisfies conditions (F1)-(F3). A pair of self-mappings, $\Gamma, \Lambda : \Theta \to \Theta$, is called an $\mathcal{F}$-Kannan mapping if the following condition holds:

$$\Gamma\Lambda\theta_1 \neq \Gamma\Lambda\theta_2 \neq \Gamma\Lambda\theta_3 \;\Rightarrow\; \Gamma\Lambda\theta_1 \neq \theta_1 \text{ or } \Gamma\Lambda\theta_2 \neq \theta_2 \text{ or } \Gamma\Lambda\theta_3 \neq \theta_3. \tag{3.1}$$

There is a $\tau > 0$ such that

$$\begin{aligned}\tau+&\mathcal{F}(G(\Gamma\Lambda\theta_1, \Gamma\Lambda\theta_2, \Gamma\Lambda\theta_3)) \\ &\leq \mathcal{F}\left[\frac{G(\Gamma\theta_1, \Gamma\Lambda\theta_1, \Gamma\Lambda\theta_1) + G(\Gamma\theta_2, \Gamma\Lambda\theta_2, \Gamma\Lambda\theta_2) + G(\Gamma\theta_3, \Gamma\Lambda\theta_3, \Gamma\Lambda\theta_3)}{3}\right]\end{aligned} \tag{3.2}$$

for all $\theta_1, \theta_2, \theta_3 \in \Theta$ with $\Gamma\Lambda\theta_1 \neq \Gamma\Lambda\theta_2 \neq \Gamma\Lambda\theta_3$.

Remark 3.2. Based on the properties of $\mathcal{F}$, each $\mathcal{F}$-Kannan mapping Γ on a G-metric space (Θ, G) satisfies the following necessary condition:

$$\begin{aligned}&G(\Gamma\Lambda\theta_1, \Gamma\Lambda\theta_2, \Gamma\Lambda\theta_3) \\ &\qquad\leq \frac{G(\Gamma\theta_1, \Gamma\Lambda\theta_1, \Gamma\Lambda\theta_1) + G(\Gamma\theta_2, \Gamma\Lambda\theta_2, \Gamma\Lambda\theta_2) + G(\Gamma\theta_3, \Gamma\Lambda\theta_3, \Gamma\Lambda\theta_3)}{3}\end{aligned} \tag{3.3}$$

for all $\theta_1, \theta_2, \theta_3 \in \Theta$.

Consider a pair of mappings, we present the subsequent examples.

Example 3.3. Define $F_1 : \mathbb{R}^+ \to \mathbb{R}$ by $F_1(\theta_3) = \ln(\theta_3)$. It is clear that $F_1(\theta_3)$ satisfies conditions (F1)-(F3). Specifically, (F3) is valid for all values of $k \in (0, 1)$. Furthermore, condition (3.2) is expressed as follows:

$$\begin{aligned}&G(\Gamma\Lambda\theta_1, \Gamma\Lambda\theta_2, \Gamma\Lambda\theta_3) \\ &\quad\leq e^{-\tau}\left[\frac{G(\Gamma\theta_1, \Gamma\Lambda\theta_1, \Gamma\Lambda\theta_1) + G(\Gamma\theta_2, \Gamma\Lambda\theta_2, \Gamma\Lambda\theta_2) + G(\Gamma\theta_3, \Gamma\Lambda\theta_3, \Gamma\Lambda\theta_3)}{3}\right]\end{aligned} \tag{3.4}$$

for all $\theta_1, \theta_2, \theta_3 \in \Theta$, with $\Gamma\Lambda\theta_1 \neq \Gamma\Lambda\theta_2 \neq \Gamma\Lambda\theta_3$. Therefore, if $\Gamma, \Lambda : \Theta \to \Theta$ represent a Kannan mapping with a constant $k \in (0,1)$ that satisfies the following condition:

$$
\begin{aligned}
&G(\Gamma\Lambda\theta_1, \Gamma\Lambda\theta_2, \Gamma\Lambda\theta_3) \\
&\quad \leq e^{-\tau}\left[\frac{G(\Gamma\theta_1, \Gamma\Lambda\theta_1, \Gamma\Lambda\theta_1) + G(\Gamma\theta_2, \Gamma\Lambda\theta_2, \Gamma\Lambda\theta_2) + G(\Gamma\theta_3, \Gamma\Lambda\theta_3, \Gamma\Lambda\theta_3)}{3}\right]
\end{aligned} \tag{3.5}
$$

for all $\theta_1, \theta_2, \theta_3 \in \Theta$, it follows that the mapping also satisfies conditions (3.4) and (3.2) with $\tau = \ln\left(\frac{1}{k}\right)$.

Example 3.4. Consider the complete G-metric space (Θ, G), where $\Theta = \mathbb{R}$ and

$$G(\theta_1, \theta_2, \theta_3) = |\theta_1 - \theta_2| + |\theta_2 - \theta_3| + |\theta_3 - \theta_1|.$$

Define the mappings $\Gamma, \Lambda : \Theta \to \Theta$ by

$$\Gamma(\theta) = \theta \quad \text{and} \quad \Lambda(\theta) = 2\theta.$$

Let $F_1 : \mathbb{R}^+ \to \mathbb{R}$ be defined by

$$F_1(t) = \ln(t).$$

Now, for $\theta_1, \theta_2, \theta_3 \in \Theta$, we have

$$G(\Gamma\Lambda\theta_1, \Gamma\Lambda\theta_2, \Gamma\Lambda\theta_3) = 2\,G(\theta_1, \theta_2, \theta_3).$$

Also,

$$
\begin{aligned}
&\frac{G(\Gamma\theta_1, \Gamma\Lambda\theta_1, \Gamma\Lambda\theta_1) + G(\Gamma\theta_2, \Gamma\Lambda\theta_2, \Gamma\Lambda\theta_2) + G(\Gamma\theta_3, \Gamma\Lambda\theta_3, \Gamma\Lambda\theta_3)}{3} \\
&= \frac{2(|\theta_1| + |\theta_2| + |\theta_3|)}{3}.
\end{aligned}
$$

Hence, the required $\mathcal{F}$-Kannan contractive inequality is not satisfied for all $\theta_1, \theta_2, \theta_3 \in \Theta$. Therefore, Γ and Λ fail to satisfy the assumptions of the main theorem.

Moreover, the mapping $\Lambda(\theta) = 2\theta$ does not admit a fixed point in $\mathbb{R}$ except $\theta = 0$, and the iterative sequence generated by the mapping is not necessarily convergent. Thus, the conclusion of the fixed point theorem cannot be guaranteed when the $\mathcal{F}$-Kannan condition is violated.

Lemma 3.5. *Consider* (Θ, G) *as a* G*-metric space and let* $\Gamma, \Lambda : \Theta \to \Theta$ *be an* $\mathcal{F}$*-Kannan mapping. Then*

$$G(\Gamma\Lambda^i\theta_0, \Gamma\Lambda^{i+1}\theta_0, \Gamma\Lambda^{i+1}\theta_0) \to 0 \text{ as } i \to \infty \tag{3.6}$$

for all $\theta_1 \in \Theta$.

Proof. Consider an arbitrary element $\theta_0 \in \Theta$. If $\Gamma\Lambda^i\theta_0 = \Gamma\Lambda^{i+1}\theta_0$ for some $i \in \mathbb{N}$, then it is guaranteed that the sequence $\{\theta_i\}_{i\in\mathbb{N}}$ converge in Θ and hence the sequence $G(\Gamma\Lambda^i\theta_0, \Gamma\Lambda^{i+1}\theta_0, \Gamma\Lambda^{i+1}\theta_0) \to 0$ as $i \to \infty$ for all $\theta_1 \in \Theta$.

Suppose that $\Gamma\Lambda^i\theta_0 \neq \Gamma\Lambda^{i+1}\theta_0$ for every $i \in \mathbb{N}$. In this case, applying (3.2) with $\tau > 0$, we derive the following result:

$$\begin{aligned}
&\tau + \mathcal{F}(G(\Gamma\Lambda^i\theta_0, \Gamma\Lambda^{i+1}\theta_0, \Gamma\Lambda^{i+1}\theta_0)) \\
&\quad\leq \mathcal{F}\left[\left\{\frac{G(\Gamma\Lambda^{i-1}\theta_0, \Gamma\Lambda^i\theta_0, \Gamma\Lambda^i\theta_0)}{3}\right\}\right. \\
&\qquad\left. + \left\{\frac{G(\Gamma\Lambda^i\theta_0, \Gamma\Lambda^{i+1}\theta_0, \Gamma\Lambda^{i+1}\theta_0) + G(\Gamma\Lambda^i\theta_0, \Gamma\Lambda^{i+1}\theta_0, \Gamma\Lambda^{i+1}\theta_0)}{3}\right\}\right].
\end{aligned} \tag{3.7}$$

From Remark 3.2, we have

$$\begin{aligned}
&G(\Gamma\Lambda^i\theta_0, \Gamma\Lambda^{i+1}\theta_0, \Gamma\Lambda^{i+1}\theta_0) \\
&\quad\leq \left\{\frac{G(\Gamma\Lambda^{i-1}\theta_0, \Gamma\Lambda^i\theta_0, \Gamma\Lambda^i\theta_0)}{3}\right\} \\
&\qquad + \left\{\frac{G(\Gamma\Lambda^i\theta_0, \Gamma\Lambda^{i+1}\theta_0, \Gamma\Lambda^{i+1}\theta_0) + G(\Gamma\Lambda^i\theta_0, \Gamma\Lambda^{i+1}\theta_0, \Gamma\Lambda^{i+1}\theta_0)}{3}\right\}.
\end{aligned} \tag{3.8}$$

Using (3.8) in (3.7), as results yield to

$$\tau + G(\Gamma\Lambda^i\theta_0, \Gamma\Lambda^{i+1}\theta_0, \Gamma\Lambda^{i+1}\theta_0) \leq G(\Gamma\Lambda^i\theta_0, \Gamma\Lambda^{i+1}\theta_0, \Gamma\Lambda^{i+1}\theta_0). \tag{3.9}$$

Taking the limit as $i \to \infty$ in (3.9), we obtain

$$\tau \leq 0, \tag{3.10}$$

which leads to a contradiction. Hence, $G(\Gamma\Lambda^i\theta_0, \Gamma\Lambda^{i+1}\theta_0, \Gamma\Lambda^{i+1}\theta_0) \to 0$ as $i \to \infty$. □

Theorem 3.6. *Let (Θ, G) be a complete G-metric space, and let $\Gamma, \Lambda : \Theta \to \Theta$ be an $\mathcal{F}$-Kannan mapping, where Γ is continuous, injective, and subsequentially convergent. It is also assumed that for $\lambda \in [0, \frac{1}{3})$ and $\tau > 0$,*

$$\begin{aligned}
&\tau + \mathcal{F}(G(\Gamma\Lambda\theta_1, \Gamma\Lambda\theta_2, \Gamma\Lambda\theta_3)) \\
&\quad\leq \mathcal{F}\left[\frac{G(\Gamma\theta_1, \Gamma\Lambda\theta_1, \Gamma\Lambda\theta_1) + G(\Gamma\theta_2, \Gamma\Lambda\theta_2, \Gamma\Lambda\theta_2) + G(\Gamma\theta_3, \Gamma\Lambda\theta_3, \Gamma\Lambda\theta_3)}{3}\right],
\end{aligned} \tag{3.11}$$

then there exists a unique fixed point for Λ. Furthermore, if Γ is subsequentially convergent, then the sequence of iterates $\{\Lambda^i\theta_0\}$ converge to this fixed point for every $\theta_0 \in \Theta$.

Proof. Let $\theta_0 \in \Theta$ be an arbitrary element of Θ. Define the sequence $\{\theta_i\}_{i\geq 1}$ by the recurrence relation $\theta_{i+1} = \Lambda\theta_i$, where each term θ_i is determined by applying Λ to the initial value θ_0 for all $i = 1, 2, 3, \ldots$.

From inequality (3.11), we get:

$$G(\Gamma\theta_i, \Gamma\theta_{i+1}, \Gamma\theta_{i+1}) = G(\Gamma\Lambda\theta_{i-1}, \Gamma\Lambda\theta_i, \Gamma\Lambda\theta_i),$$

$$\begin{aligned}&\mathcal{F}(G(\Gamma\theta_i, \Gamma\theta_{i+1}, \Gamma\theta_{i+1}))\\ &\quad \leq \mathcal{F}\left[\frac{G(\Gamma\theta_{i-1}, \Gamma\Lambda\theta_{i-1}, \Gamma\Lambda\theta_{i-1}) + 2G(\Gamma\theta_i, \Gamma\Lambda\theta_i, \Gamma\Lambda\theta_i)}{3}\right] - \tau. \end{aligned} \tag{3.12}$$

Since $\mathcal{F}$ is strictly increasing, it follows from Remark 3.2 that we can deduce

$$G(\Gamma\theta_i, \Gamma\theta_{i+1}, \Gamma\theta_{i+1}) \leq \frac{G(\Gamma\theta_{i-1}, \Gamma\Lambda\theta_{i-1}, \Gamma\Lambda\theta_{i-1}) + 2G(\Gamma\theta_i, \Gamma\Lambda\theta_i, \Gamma\Lambda\theta_i)}{3},$$

$$G(\Gamma\theta_i, \Gamma\theta_{i+1}, \Gamma\theta_{i+1}) \leq \frac{G(\Gamma\theta_{i-1}, \Gamma\theta_i, \Gamma\theta_i) + 2G(\Gamma\theta_i, \Gamma\theta_{i+1}, \Gamma\theta_{i+1})}{3}.$$

Hence, we have

$$3G(\Gamma\theta_i, \Gamma\theta_{i+1}, \Gamma\theta_{i+1}) - 2G(\Gamma\theta_i, \Gamma\theta_{i+1}, \Gamma\theta_{i+1}) < G(\Gamma\theta_{i-1}, \Gamma\theta_i, \Gamma\theta_i),$$

this implies that

$$G(\Gamma\theta_i, \Gamma\theta_{i+1}, \Gamma\theta_{i+1}) < G(\Gamma\theta_{i-1}, \Gamma\theta_i, \Gamma\theta_i).$$

According to $(F1)$, this leads to

$$\mathcal{F}(G(\Gamma\theta_i, \Gamma\theta_{i+1}, \Gamma\theta_{i+1})) < \mathcal{F}(G(\Gamma\theta_{i-1}, \Gamma\theta_i, \Gamma\theta_i)). \tag{3.13}$$

Consequently, we get

$$\mathcal{F}(G(\Gamma\theta_i, \Gamma\theta_{i+1}, \Gamma\theta_{i+1})) \leq \mathcal{F}(G(\Gamma\theta_{i-1}, \Gamma\theta_i, \Gamma\theta_i)) - \tau. \tag{3.14}$$

Similarly, we obtain

$$\mathcal{F}(G(\Gamma\theta_{i+1}, \Gamma\theta_{i+2}, \Gamma\theta_{i+2})) \leq \mathcal{F}(G(\Gamma\theta_{i-1}, \Gamma\theta_i, \Gamma\theta_i)) - 2\tau. \tag{3.15}$$

Applying induction and (3.14), we conclude:

$$\mathcal{F}(G(\Gamma\theta_i, \Gamma\theta_{i+1}, \Gamma\theta_{i+1})) \leq \mathcal{F}(G(\Gamma\theta_{i-1}, \Gamma\theta_i, \Gamma\theta_i)) - i\tau. \tag{3.16}$$

Taking the limit as $i \to \infty$ in (17) and applying property (F2) of $\mathcal{F}$ yields

$$\lim_{i\to\infty} G(\Gamma\theta_i, \Gamma\theta_{i+1}, \Gamma\theta_{i+1}) = 0. \tag{3.17}$$

By Lemma (3.5), we have $G(\Gamma\theta_i, \Gamma\theta_{i+1}, \Gamma\theta_{i+1}) \to 0$ as $i \to \infty$.

From condition (F3) of the function $\mathcal{F}$, it follows that there is a $k \in (0, 1)$ such that

$$\lim_{i\to\infty} (G(\Gamma\theta_i, \Gamma\theta_{i+1}, \Gamma\theta_{i+1}))^k \mathcal{F}(G(\Gamma\theta_i, \Gamma\theta_{i+1}, \Gamma\theta_{i+1})) = 0. \tag{3.18}$$

According to (3.16), for all $i \in \mathbb{N}$, we get

$$\begin{aligned}
&(G(\Gamma\theta_i, \Gamma\theta_{i+1}, \Gamma\theta_{i+1}))^k \mathcal{F}(G(\Gamma\theta_i, \Gamma\theta_{i+1}, \Gamma\theta_{i+1})) \\
&\quad \leq (G(\Gamma\theta_i, \Gamma\theta_{i+1}, \Gamma\theta_{i+1}))^k \mathcal{F}(G(\Gamma\theta_i, \Gamma\theta_{i+1}, \Gamma\theta_{i+1})) \\
&\qquad - i\tau(G(\Gamma\theta_i, \Gamma\theta_{i+1}, \Gamma\theta_{i+1}))^k. \qquad (3.19)
\end{aligned}$$

This implies,

$$\begin{aligned}
&(G(\Gamma\theta_i, \Gamma\theta_{i+1}, \Gamma\theta_{i+1}))^k \mathcal{F}(G(\Gamma\theta_i, \Gamma\theta_{i+1}, \Gamma\theta_{i+1})) \\
&- (G(\Gamma\theta_i, \Gamma\theta_{i+1}, \Gamma\theta_{i+1}))^k \mathcal{F}(G(\Gamma\theta_i, \Gamma\theta_{i+1}, \Gamma\theta_{i+1})) \leq -i\tau(G(\Gamma\theta_i, \Gamma\theta_{i+1}, \Gamma\theta_{i+1}))^k.
\end{aligned}$$

That is,

$$\begin{aligned}
&(G(\Gamma\theta_i, \Gamma\theta_{i+1}, \Gamma\theta_{i+1}))^k [\mathcal{F}(G(\Gamma\theta_i, \Gamma\theta_{i+1}, \Gamma\theta_{i+1})) - \mathcal{F}(G(\Gamma\theta_i, \Gamma\theta_{i+1}, \Gamma\theta_{i+1}))] \\
&\quad \leq -i\tau(G(\Gamma\theta_i, \Gamma\theta_{i+1}, \Gamma\theta_{i+1}))^k \leq 0.
\end{aligned}$$

By letting $i \to \infty$ in (3.19), we derive

$$\lim_{i\to\infty} i(G(\Gamma\theta_i, \Gamma\theta_{i+1}, \Gamma\theta_{i+1}))^k = 0. \qquad (3.20)$$

By (3.20), we can find $i_1 \in \mathbb{N}$ such that $i(G(\Gamma\theta_i, \Gamma\theta_{i+1}, \Gamma\theta_{i+1}))^k \leq 1$ for all $i \geq i_1$, which leads to

$$G(\Gamma\theta_i, \Gamma\theta_{i+1}, \Gamma\theta_{i+1}) \leq i^{\frac{-1}{k}}, \ \forall i \geq i_1. \qquad (3.21)$$

Therefore, $\displaystyle\sum_{i=0}^{\infty} G(\Gamma\theta_i, \Gamma\theta_{i+1}, \Gamma\theta_{i+1})$ converges.

$$\begin{aligned}
G(\Gamma\theta_i, \Gamma\theta_j, \Gamma\theta_j) &\leq G(\Gamma\theta_i, \Gamma\theta_{i+1}, \Gamma\theta_{i+1}) + G(\Gamma\theta_{i+1}, \Gamma\theta_{i+2}, \Gamma\theta_{i+2}) \\
&\quad + ... + G(\Gamma\theta_{j-1}, \Gamma\theta_j, \Gamma\theta_j) \\
&\leq G(\Gamma\theta_i, \Gamma\theta_{i+1}, \Gamma\theta_{i+1}) + G(\Gamma\theta_{i+1}, \Gamma\theta_{i+2}, \Gamma\theta_{i+2}) \\
&\quad + G(\Gamma\theta_{i+2}, \Gamma\theta_{i+3}, \Gamma\theta_{i+3}) + ... + G(\Gamma\theta_j, \Gamma\theta_{j+1}, \Gamma\theta_{j+1}) \\
&= \sum_{i}^{j-1} G(\Gamma\theta_i, \Gamma\theta_{i+1}, \Gamma\theta_{i+1}) \\
&\leq \sum_{j=1}^{\infty} G(\Gamma\theta_i, \Gamma\theta_{i+1}, \Gamma\theta_{i+1}) \\
&\leq \sum_{i=1}^{\infty} i^{\frac{-1}{k}}.
\end{aligned}$$

This demonstrates that the series $\sum_{i=1}^{\infty} i^{\frac{-1}{k}}$ G-converges, which results in

$$\lim_{i,j\to\infty} G(\Gamma\theta_i, \Gamma\theta_j, \Gamma\theta_j) = 0. \tag{3.22}$$

By the result in (3.22), we establish that $\{\Gamma\theta_i\}$ is a G-Cauchy sequence, as (Θ, G) is G-complete. Suppose $i, j, k \in \mathbb{N}$ such that

$$\lim_{i,j,k\to\infty} G(\Gamma\theta_i, \Gamma\theta_j, \Gamma\theta_k) \leq \lim_{i,j\to\infty} G(\Gamma\theta_i, \Gamma\theta_j, \Gamma\theta_j) + \lim_{j,k\to\infty} G(\Gamma\theta_j, \Gamma\theta_k, \Gamma\theta_k).$$

Hence, we have

$$\lim_{i,j,k\to\infty} G(\Gamma\theta_i, \Gamma\theta_j, \Gamma\theta_k) = 0. \tag{3.23}$$

Consequently, $\Gamma\theta_i$ forms a G-Cauchy sequence in Θ. Since Completeness of $(G, \Theta))$ ensures the existence of $\theta_1^* \in \Theta$ such that

$$G(\theta_1^*, \Gamma\theta_1^*, \Gamma\theta_1^*) = \lim_{i,j,k\to\infty} G(\Gamma\theta_i, \Gamma\theta_j, \Gamma\theta_k) = 0 = \lim_{j,k\to\infty} G(\theta_1^*, \Gamma\theta_j, \Gamma\theta_k) = 0. \tag{3.24}$$

From (3.24), it follows that $\theta_{i+1} \to \theta_1^*$ as $i \to \infty$. Given the continuity of Λ and Γ, we obtain

$$\begin{aligned}
\theta_1^* &= \lim_{i\to\infty} \theta_i = \lim_{i\to\infty} \theta_{i+1} = \lim_{i\to\infty} \Lambda\theta_i = \Lambda\theta_1^*,\\
\theta_1^* &= \lim_{i\to\infty} \theta_i = \lim_{i\to\infty} \theta_{i+1} = \lim_{i\to\infty} \Gamma\theta_i = \Gamma\theta_1^*.
\end{aligned}$$

Since Θ is a complete $G-$metric space, it follows that there exists $\theta_1^* \in \Theta$ such that

$$\lim_{i\to\infty} \Gamma\theta_i = \theta_1^*. \tag{3.25}$$

We proceed to prove that θ_1^* is a fixed point of Γ. Hence, by (v) of Definition 2.1, we derive

$$G(\theta_1^*, \Gamma\theta_1^*, \Gamma\theta_1^*) \leq G(\theta_1^*, \Gamma\theta_i, \Gamma\theta_i) + G(\Gamma\theta_i, \Gamma\theta_{i+1}, \Gamma\theta_{i+1}) + G(\Gamma\theta_{i+1}, \Gamma\theta_1^*, \Gamma\theta_1^*). \tag{3.26}$$

Based on Remark 3.2, it can be concluded that

$$\begin{aligned}
&G(\Gamma\theta_{i+1}, \Gamma\theta_1^*, \Gamma\theta_1^*)\\
&\quad \leq \frac{G(\Gamma\theta_1^*, \Gamma\theta_i, \Gamma\theta_i) + G(\Gamma\theta_i, \Gamma\theta_{i+1}, \Gamma\theta_{i+1}) + G(\Gamma\theta_{i+1}, \Gamma\theta_{i+1}, \Gamma\theta_1^*)}{3}.
\end{aligned} \tag{3.27}$$

Applying (3.28) in (3.27), we obtain

$$\begin{aligned}
&G(\theta_1^*, \Gamma\theta_1^*, \Gamma\theta_1^*)\\
&\quad\leq G(\theta_1^*, \Gamma\theta_i, \Gamma\theta_i) + G(\Gamma\theta_i, \Gamma\theta_{i+1}, \Gamma\theta_{i+1})\\
&\qquad + \frac{G(\Gamma\theta_1^*, \Gamma\theta_i, \Gamma\theta_i) + G(\Gamma\theta_i, \Gamma\theta_{i+1}, \Gamma\theta_{i+1}) + G(\Gamma\theta_{i+1}, \Gamma\theta_{i+1}, \Gamma\theta_1^*)}{3}. \quad (3.28)
\end{aligned}$$

As $i \to \infty$ and utilizing Lemma 3.5 in the inequality above, we get

$$\begin{aligned}
G(\theta_1^*, \Gamma\theta_1^*, \Gamma\theta_1^*) &\leq G(\theta_1^*, \theta_1^*, \theta_1^*) + G(\theta_1^*, \theta_1^*, \theta_1^*)\\
&\quad + \frac{G(\Gamma\theta_1^*, \theta_1^*, \theta_1^*) + G(\theta_1^*, \theta_1^*, \theta_1^*) + G(\theta_1^*, \theta_1^*, \Gamma\theta_1^*)}{3},
\end{aligned}$$

this implies that

$$G(\theta_1^*, \Gamma\theta_1^*, \Gamma\theta_1^*) \leq 0.$$

Hence, $\Gamma\theta_1^* = \theta_1^*$.

We now demonstrate that θ_1^* is the unique fixed point of Γ. Assume the opposite, that is, let $\mu^* \in \mathrm{Card}\{\mathrm{Fix}\ \Gamma\}$ such that $\theta_1^* \neq \mu^*$.

Suppose $\Gamma\theta_i \to \mu^*$, and that μ^* is a fixed point of Γ. By Remark 3.2 and Lemma 3.5, we obtain $\theta_1^* = \mu^*$, which contradicts our assumption.

Thus, Γ has a unique fixed point in Θ.

In addition, Γ being subsequentially convergent implies that the sequence $\{\theta_i\}$ has a convergent subsequence, and there exists $\mu^* \in \Theta$ along with a subsequence $\{\theta_{i(k)}\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} \theta_{i(k)} = \mu^*$.

Since Γ is continuous and

$$\lim_{k\to\infty} \theta_{i(k)} = \mu^*. \quad (3.29)$$

As a result of the continuity of Γ, we deduce that

$$\lim_{k\to\infty} \Gamma\theta_{i(k)} = \Gamma\mu^*. \quad (3.30)$$

By (3.26), we conclude that

$$\Gamma\mu^* = \theta_1^*. \quad (3.31)$$

Using Remark 3.2 and (v) of Definition 2.1, we get

$$\begin{aligned}
G(\Gamma\theta_i, \Gamma\theta_{i+1}, \Gamma\theta_{i+1}) &= G(\Gamma\Lambda\theta_{i-1}, \Gamma\Lambda\theta_i, \Gamma\Lambda\theta_i)\\
&\leq \lambda(G(\Gamma\theta_{i-1}, \Gamma\Lambda\theta_{i-1}, \Gamma\Lambda\theta_{i-1}) + 2G(\Gamma\theta_i, \Gamma\Lambda\theta_i, \Gamma\Lambda\theta_i))\\
&= \lambda(G(\Gamma\theta_{i-1}, \Gamma\Lambda\theta_{i-1}, \Gamma\Lambda\theta_{i-1}) + 2G(\Gamma\theta_i, \Gamma\theta_{i+1}, \Gamma\theta_{i+1})).
\end{aligned}$$

Hence, we have

$$G(\Gamma\theta_i, \Gamma\theta_{i+1}, \Gamma\theta_{i+1}) \leq \frac{\lambda}{1-2\lambda} G(\Gamma\theta_{i-1}, \Gamma\Lambda\theta_{i-1}, \Gamma\Lambda\theta_{i-1}). \quad (3.32)$$

Therefore, by applying equation (3.2) and (v) of Definition 2.1, we obtain

$$\tau + \mathcal{F}(G(\Gamma\theta_{i+1}, \Gamma\theta_{i+2}, \Gamma\theta_{i+2})) = \tau + G(\Gamma\Lambda\theta_i, \Gamma\Lambda\theta_{i+1}, \Gamma\Lambda\theta_{i+1}).$$

Hence, we have

$$\begin{aligned}&\mathcal{F}(G(\Gamma\Lambda\mu^*, \Gamma\mu^*, \Gamma\mu^*))\\ &\quad \leq \mathcal{F}(G(\Gamma\Lambda\mu^*, \Gamma\theta_i, \Gamma\theta_i) + G(\Gamma\theta_i, \Gamma\theta_{i+1}, \Gamma\theta_{i+1}) + G(\Gamma\theta_{i+1}, \Gamma\mu^*, \Gamma\mu^*)).\end{aligned} \tag{3.33}$$

Since $\mathcal{F}$ is sequentially increasing, it follows that

$$\begin{aligned}&G(\Gamma\Lambda\mu^*, \Gamma\mu^*, \Gamma\mu^*)\\ &\quad \leq G(\Gamma\Lambda\mu^*, \Gamma\theta_i, \Gamma\theta_i) + G(\Gamma\theta_i, \Gamma\theta_{i+1}, \Gamma\theta_{i+1}) + G(\Gamma\theta_{i+1}, \Gamma\mu^*, \Gamma\mu^*)\\ &\quad \leq G(\Gamma\Lambda\mu^*, \Gamma\Lambda^{i(k)}\theta_0, \Gamma\Lambda^{i(k)}\theta_0) + G(\Gamma\Lambda^{i(k)}\theta_0, \Gamma\Lambda^{i(k)+1}\theta_0, \Gamma\Lambda^{i(k)+1}\theta_0)\\ &\qquad + G(\Gamma\Lambda^{i(k)+1}\theta_0, \Gamma\mu^*, \Gamma\mu^*).\end{aligned} \tag{3.34}$$

By applying Lemma 3.5, when $\Gamma\Lambda^i\theta_0 \neq \Gamma\Lambda^{i+1}\theta_0$ for all $i \in \mathbb{N}$ and using (3.12), we obtain

$$\begin{aligned}&G(\Gamma\Lambda\mu^*, \Gamma\Lambda^{i(k)}\theta_0, \Gamma\Lambda^{i(k)}\theta_0)\\ &\quad \leq \lambda[G(\Gamma\mu^*, \Gamma\Lambda\mu^*, \Gamma\Lambda\mu^*) + 2G(\Gamma^{i(k)-1}\theta_0, \Gamma\Lambda^{i(k)}\theta_0, \Gamma\Lambda^{i(k)}\theta_0)]\\ &\quad \leq \lambda G(\Gamma\mu^*, \Gamma\Lambda\mu^*, \Gamma\Lambda\mu^*) + \lambda\left(\frac{\lambda}{1-2\lambda}\right)^{i(k)-1} G(\Gamma\theta_1, \Gamma\theta_0, \Gamma\theta_0).\end{aligned} \tag{3.35}$$

Therefore, we have

$$G(\Gamma\Lambda^{i(k)+1}\theta_0, \Gamma\Lambda^{i(k)}\theta_0, \Gamma\Lambda^{i(k)}\theta_0) \leq \left(\frac{\lambda}{1-2\lambda}\right)^{i(k)} G(\Gamma\theta_1, \Gamma\theta_0, \Gamma\theta_0). \tag{3.36}$$

Using (3.35) and (3.36) in (3.34), we obtain

$$\begin{aligned}&G(\Gamma\Lambda\mu^*, \Gamma\mu^*, \Gamma\mu^*)\\ &\quad \leq \lambda G(\Gamma\mu^*, \Gamma\Lambda\mu^*, \Gamma\Lambda\mu^*) + \lambda\left(\frac{\lambda}{1-2\lambda}\right)^{i(k)-1} G(\Gamma\theta_1, \Gamma\theta_0, \Gamma\theta_0)\\ &\qquad + \left(\frac{\lambda}{1-2\lambda}\right)^{i(k)} G(\Gamma\theta_1, \Gamma\theta_0, \Gamma\theta_0) + G(\Gamma\theta_{i(k)+1}, \Gamma\mu^*, \Gamma\mu^*).\end{aligned} \tag{3.37}$$

As $k \to \infty$ in (3.37), we derive

$$G(\Gamma\Lambda\mu^*, \Gamma\mu^*, \Gamma\mu^*) = 0. \tag{3.38}$$

Since Γ is injection, $\Lambda\mu^* = \mu^*$. Thus, a fixed point of Λ exists.

As a result, $\Lambda\theta_1 \subset \Theta$ and $\Gamma\theta_1 \subset \Theta$, leading to the existence of a point $\mu^* \subset \Theta$ such that $\mu^* \in \Lambda\mu^* \cap \Gamma\mu^*$, which implies that μ^* is a common fixed point for both Λ and Γ. □

Example 3.7. Let $\Theta = \mathbb{R}$ and define the G-metric

$$G(\theta_1, \theta_2, \theta_3) = |\theta_1 - \theta_2| + |\theta_2 - \theta_3| + |\theta_3 - \theta_1|$$

for all $\theta_1, \theta_2, \theta_3 \in \Theta$. Then (Θ, G) is a complete G-metric space.

Define the mappings $\Gamma, \Lambda : \Theta \to \Theta$ by

$$\Gamma(\theta) = \theta \quad \text{and} \quad \Lambda(\theta) = \frac{\theta}{4}.$$

Also define $\mathcal{F} : \mathbb{R}^+ \to \mathbb{R}$ by

$$\mathcal{F}(t) = \ln(t),$$

which satisfies conditions $(F1)$-$(F3)$. Clearly, Γ is continuous, injective, and subsequentially convergent. Let $\tau = \ln 2 > 0$.

Now, for any $\theta_1, \theta_2, \theta_3 \in \Theta$, we obtain

$$G(\Gamma\Lambda\theta_1, \Gamma\Lambda\theta_2, \Gamma\Lambda\theta_3) = \frac{1}{4}G(\theta_1, \theta_2, \theta_3).$$

Further,

$$G(\Gamma\theta_i, \Gamma\Lambda\theta_i, \Gamma\Lambda\theta_i) = 2\left|\theta_i - \frac{\theta_i}{4}\right| = \frac{3}{2}|\theta_i|, \quad i = 1, 2, 3.$$

Hence,

$$\begin{aligned}
\tau + &\mathcal{F}(G(\Gamma\Lambda\theta_1, \Gamma\Lambda\theta_2, \Gamma\Lambda\theta_3)) \\
&= \ln 2 + \ln\left(\frac{1}{4}G(\theta_1, \theta_2, \theta_3)\right) \\
&= \ln\left(\frac{1}{2}G(\theta_1, \theta_2, \theta_3)\right) \\
&\le \mathcal{F}\left[\frac{G(\Gamma\theta_1, \Gamma\Lambda\theta_1, \Gamma\Lambda\theta_1) + G(\Gamma\theta_2, \Gamma\Lambda\theta_2, \Gamma\Lambda\theta_2) + G(\Gamma\theta_3, \Gamma\Lambda\theta_3, \Gamma\Lambda\theta_3)}{3}\right].
\end{aligned}$$

Therefore, all conditions of Theorem 3.6 are satisfied. Consequently, Λ has a unique fixed point in Θ. Since

$$\Lambda(\theta) = \frac{\theta}{4},$$

the unique fixed point is $\theta = 0$. Moreover, for every $\theta_0 \in \Theta$,

$$\Lambda^i(\theta_0) = \frac{\theta_0}{4^i} \to 0 \quad \text{as } i \to \infty.$$

Thus, the sequence of iterates converges to the unique fixed point.

4. Applications

In the following section, we will explore some applications of the theorem demonstrated in the previous section.

4.1. **Existence of a Solution for Nonlinear Fractional Differential Equation:** The objective of this section is to illustrate the application of Theorem (3.6) to derive a common solution for a nonlinear fractional differential equation, employing $\mathcal{F}-$Kannan mappings in $G-$metric spaces.

This part of the paper investigates the Caputo derivative of fractional order in the context of the nonlinear fractional differential equation. The fractional derivative for a continuous function $\Upsilon : [0, \infty) \to \mathbb{R}$ is expressed as

$$({}^{c}D_t^{\varkappa})\Upsilon(\xi) = \frac{1}{\Gamma(n-\varkappa)}\int_0^{\xi}(\xi-\zeta)^{n-\varkappa-1}\Upsilon^{n}(\zeta)d\zeta, (n-1<\varkappa, n=[\varkappa]+1), \tag{4.1}$$

where $[\varkappa]$ represents the greatest integer less than or equal to the real number $\varkappa$. Additionally, the Riemann-Liouville fractional integral of order $\varkappa$ is defined as

$$({}^{c}I_t^{\varkappa})\Upsilon(\xi) = \frac{1}{\Gamma(\varkappa)-1}\int_0^{\xi}(\xi-\zeta)^{\varkappa}\Upsilon(\zeta)d\zeta,\ (\varkappa>0), \tag{4.2}$$

$$\begin{cases} {}^{c}D_s^{\varkappa}x(\xi) = \Upsilon(\xi, \theta_1(\xi)), \xi \in (0,1), 1<\varkappa\leq 2, \\ \\ \theta_1(0) = 0, \theta_1(1) = \displaystyle\int_0^{v}\theta_1(\zeta)d\zeta(0<v<1), \end{cases} \tag{4.3}$$

where $\Upsilon : [0,1] \to \Theta$ is a continuous function and ${}^{c}D_s^{\varkappa}$ represents the Caputo fractional derivative of order $\varkappa$.

Let $\Theta = C(I)$, where $I = [0,1]$, be the space of continuous functions defined on the interval I. Assume that $(\Theta, ||\cdot||)$ is a Banach space, and let $I := [0,\Gamma]$, where $\Gamma > 0$.

Consider the Banach space $C(I,\Theta)$, which consists of all continuous functions from I into Θ. The norm on this space is defined by

$$||\theta_1|| := \sup_{\xi\in I}|\theta_1(\xi)| = L$$

for $\theta_1 \in C(I,\Theta)$. This space defines the $G-$metric by

$$\begin{aligned} G(\theta_1,\theta_2,\theta_3) = &\sup_{\xi\in[0,\Gamma]}\{|\theta_1(\xi)-\theta_2(\xi)|\} + \sup_{\xi\in[0,\Gamma]}\{|\theta_2(\xi)-\theta_3(\xi)|\} \\ &+ \sup_{\xi\in[0,\Gamma]}\{|\theta_3(\xi)-\theta_1(\xi)|\}, \end{aligned} \tag{4.4}$$

for all $\theta_1,\theta_2,\theta_3 \in \Theta$. This is a complete $G-$metric space.

Equation (4.4) represents the nonlinear fractional equation, which can be written as

$$\begin{aligned} \theta_1(\xi) = & \frac{1}{\Gamma(\varkappa)} \int_0^{\xi} (\xi - \zeta)^{\varkappa-1} \Upsilon(\zeta, \theta_1(\zeta)) d\zeta \\ & - \frac{2t}{(2-v^2)\Gamma(\varkappa)} \int_0^1 (1-\zeta)^{\varkappa-1} \Upsilon(\zeta, \theta_1(\zeta)) d\zeta \\ & + \frac{2t}{(2-v^2)\Gamma(\varkappa)} \int_0^v \left[\int_0^{\zeta} (\zeta - \theta_3)^{\varkappa-1} \Upsilon(\theta_3, \theta_1(\theta_3)) dz \right] d\zeta. \end{aligned} \tag{4.5}$$

A solution to the fractional differential integral equation (4.5) is equivalent to a solution to the nonlinear fractional differential equation (4.3) for a function $\theta_1 \in C(I, \Theta)$.

Let us proceed to prove the theorem below.

Theorem 4.1. *Assume the following condition is satisfied:*

(i) *The function $\Upsilon \in C(I \times \Theta, \Theta)$ is sequentially continuous.*
(ii) *A continuous function $\Upsilon : [0,1] \times \mathbb{R} \to \mathbb{R}^+$ exists, satisfying*

$$|\Upsilon(\xi, \theta_1(\zeta)) - \Upsilon(\xi, \theta_2(\zeta))| \leq e^{-\tau} L |\theta_1(\zeta) - \theta_2(\zeta)|, \tag{4.6}$$

$$|\Upsilon(\xi, \theta_2(\zeta)) - \Upsilon(\xi, \theta_3(\zeta))| \leq e^{-\tau} L |\theta_2(\zeta) - \theta_3(\zeta)|, \tag{4.7}$$

$$|\Upsilon(\xi, \theta_3(\zeta)) - \Upsilon(\xi, \theta_1(\zeta))| \leq e^{-\tau} L |\theta_3(\zeta) - \theta_1(\zeta)| \tag{4.8}$$

for every $\xi \in [0,1]$ and for all $\theta_1, \theta_2, \theta_3 \in \Theta$ such that

$$G(\theta_1(\xi), \theta_2(\xi), \theta_3(\xi)) \geq 0,$$

there exists a constant L and a constant k such that

$$Lk \leq 1,$$

$$k = \left[\frac{e^{-\tau} L}{\Gamma(\varkappa)} + \frac{2te^{-\tau} L}{(2-v^2)\Gamma(\varkappa)} + \frac{2te^{-\tau} L}{(2-v^2)\Gamma(\varkappa)} \right]. \tag{4.9}$$

Then the fractional differential equation (4.3) has a common solution, which is a fixed point $\theta_1^* \in C(I, \Theta)$.

Proof. We define $\Gamma, \Lambda : C(I) \to C(I)$ as follows

$$\begin{aligned}\Gamma\Lambda\theta_1(\xi) = & \frac{1}{\Gamma(\varkappa)}\int_0^{\xi}(\xi-\zeta)^{\varkappa-1}\Upsilon(\zeta,\theta_1(\zeta))d\zeta \\ & - \frac{2t}{(2-v^2)\Gamma(\varkappa)}\int_0^1(1-\zeta)^{\varkappa-1}\Upsilon(\zeta,\theta_1(\zeta))d\zeta \\ & + \frac{2t}{(2-v^2)\Gamma(\varkappa)}\int_0^v\left[\int_0^{\zeta}(\zeta-\theta_3)^{\varkappa-1}\Upsilon(\theta_3,\theta_1(\theta_3))dz\right]d\zeta \end{aligned} \tag{4.10}$$

for $\xi \in [0,1]$, then $\Gamma\Lambda$ is sequentially continuous.

Assume that

$$\Lambda\theta_1(\xi) = \int_0^{\zeta}(\zeta-\theta_3)^{\varkappa-1}\Upsilon(\theta_3,\theta_1(\theta_3))dz. \tag{4.11}$$

This leads to the conclusion that Λ is contained within $\Gamma\Lambda$ and possesses a fixed point, denoted by $\theta_1^* \in \Gamma\Lambda$.

To establish the existence of a fixed point for $\Gamma\Lambda$, we need to demonstrate that $\Gamma\Lambda$ is both sequentially continuous and a contraction.

To prove sequential continuity, assume that $\Gamma\Lambda\theta_1 \neq \Gamma\Lambda\theta_2$ for any $\theta_1, \theta_2 \in [0,\Gamma]$. According to condition (ii), we obtain the following results

$$\begin{aligned} & G(\Gamma\Lambda\theta_1,\Gamma\Lambda\theta_2,\Gamma\Lambda\theta_3) \\ & = ||\Gamma\Lambda\theta_1-\Gamma\Lambda\theta_2||_\infty + ||\Gamma\Lambda\theta_2-\Gamma\Lambda\theta_3||_\infty + ||\Gamma\Lambda\theta_3-\Gamma\Lambda\theta_1||_\infty \\ & = \frac{1}{\Gamma(\varkappa)}\int_0^{\xi}(\xi-\zeta)^{\varkappa-1}|\Upsilon(\zeta,\theta_1(\zeta))-\Upsilon(\zeta,\theta_2(\zeta))|d\zeta \\ & \quad + \frac{2t}{(2-v^2)\Gamma(\varkappa)}\int_0^1(1-\zeta)^{\varkappa-1}|\Upsilon(\zeta,\theta_1(\zeta))-\Upsilon(\zeta,\theta_2(\zeta))|d\zeta \\ & \quad + \frac{2t}{(2-v^2)\Gamma(\varkappa)}\int_0^v\left[\int_0^{\zeta}(\zeta-\theta_3)^{\varkappa-1}|\Upsilon(\theta_3,\theta_1(\theta_3))-\Upsilon(\theta_3,\theta_2(\theta_3))|dz\right]d\zeta \\ & \quad + \frac{1}{\Gamma(\varkappa)}\int_0^{\xi}(\xi-\zeta)^{\varkappa-1}|\Upsilon(\zeta,\theta_2(\zeta))-\Upsilon(\zeta,\theta_3(\zeta))|d\zeta \\ & \quad + \frac{2t}{(2-v^2)\Gamma(\varkappa)}\int_0^1(1-\zeta)^{\varkappa-1}|\Upsilon(\zeta,\theta_2(\zeta))-\Upsilon(\zeta,\theta_3(\zeta))|d\zeta \\ & \quad + \frac{2t}{(2-v^2)\Gamma(\varkappa)}\int_0^v\left[\int_0^{\zeta}(\zeta-\theta_3)^{\varkappa-1}|\Upsilon(\theta_3,\theta_2(\theta_3))-\Upsilon(\theta_3,\theta_3(\theta_3))|dz\right]d\zeta \\ & \quad + \frac{1}{\Gamma(\varkappa)}\int_0^{\xi}(\xi-\zeta)^{\varkappa-1}|\Upsilon(\zeta,\theta_3(\zeta))-\Upsilon(\zeta,\theta_1(\zeta))|d\zeta \\ & \quad + \frac{2t}{(2-v^2)\Gamma(\varkappa)}\int_0^1(1-\zeta)^{\varkappa-1}|\Upsilon(\zeta,\theta_3(\zeta))-\Upsilon(\zeta,\theta_1(\zeta))|d\zeta \end{aligned}$$

$$+\frac{2t}{(2-v^2)\Gamma(\varkappa)}\int_0^v\left[\int_0^\zeta(\zeta-\theta_3)^{\varkappa-1}|\Upsilon(\theta_3,\theta_3(\theta_3))-\Upsilon(\theta_3,\theta_1(\theta_3))|dz\right]d\zeta$$

And

$$\begin{aligned}
&G(\Gamma\Lambda\theta_1,\Gamma\Lambda\theta_2,\Gamma\Lambda\theta_3)\\
&\quad\leq\frac{1}{\Gamma(\varkappa)}\int_0^\xi(\xi-\zeta)^{\varkappa-1}|\theta_1(\zeta)-\theta_2(\zeta)|d\zeta\\
&\qquad+\frac{2t}{(2-v^2)\Gamma(\varkappa)}\int_0^1(1-\zeta)^{\varkappa-1}|\theta_1(\zeta)-\theta_2(\zeta)|d\zeta\\
&\qquad+\frac{2t}{(2-v^2)\Gamma(\varkappa)}\int_0^v\left[\int_0^\zeta(\zeta-\theta_3)^{\varkappa-1}|\theta_1(\theta_3)-\theta_2(\theta_3)|dz\right]d\zeta\\
&\qquad+\frac{1}{\Gamma(\varkappa)}\int_0^\xi(\xi-\zeta)^{\varkappa-1}|\theta_2(\zeta)-\theta_3(\zeta)|d\zeta\\
&\qquad+\frac{2t}{(2-v^2)\Gamma(\varkappa)}\int_0^1(1-\zeta)^{\varkappa-1}|\theta_2(\zeta)-\theta_3(\zeta)|d\zeta\\
&\qquad+\frac{2t}{(2-v^2)\Gamma(\varkappa)}\int_0^v\left[\int_0^\zeta(\zeta-\theta_3)^{\varkappa-1}|\theta_2(\theta_3)-\theta_3(\theta_3)|dz\right]d\zeta\\
&\qquad+\frac{1}{\Gamma(\varkappa)}\int_0^\xi(\xi-\zeta)^{\varkappa-1}|\theta_3(\zeta)-\theta_1(\zeta)|d\zeta\\
&\qquad+\frac{2t}{(2-v^2)\Gamma(\varkappa)}\int_0^1(1-\zeta)^{\varkappa-1}|\theta_3(\zeta)-\theta_1(\zeta)|d\zeta\\
&\qquad+\frac{2t}{(2-v^2)\Gamma(\varkappa)}\int_0^v\left[\int_0^\zeta(\zeta-\theta_3)^{\varkappa-1}|\theta_3(\theta_3)-\theta_1(\theta_3)|dz\right]d\zeta\\
&\quad=\frac{e^{-\tau}L}{\Gamma(\varkappa)}||\theta_1-\theta_2||_\infty\int_0^\xi(\xi-\zeta)^{\varkappa-1}d\zeta\\
&\qquad+\frac{2te^{-\tau}L}{(2-v^2)\Gamma(\varkappa)}||\theta_1-\theta_2||_\infty\int_0^1(1-\zeta)^{\varkappa-1}d\zeta\\
&\qquad+\frac{2te^{-\tau}L}{(2-v^2)\Gamma(\varkappa)}||\theta_1-\theta_2||_\infty\int_0^v\left[\int_0^\zeta(\zeta-\theta_3)^{\varkappa-1}dz\right]d\zeta\\
&\qquad+\frac{e^{-\tau}L}{\Gamma(\varkappa)}||\theta_2-\theta_3||_\infty\int_0^\xi(\xi-\zeta)^{\varkappa-1}d\zeta\\
&\qquad+\frac{2te^{-\tau}L}{(2-v^2)\Gamma(\varkappa)}||\theta_2-\theta_3||_\infty\int_0^1(1-\zeta)^{\varkappa-1}d\zeta
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{2te^{-\tau}L}{(2-v^2)\Gamma(\varkappa)}||\theta_2-\theta_3||_\infty \int_0^v \left[\int_0^\zeta (\zeta-\theta_3)^{\varkappa-1}dz\right]d\zeta \\
&+ \frac{e^{-\tau}L}{\Gamma(\varkappa)}||\theta_3-\theta_1||_\infty \int_0^\xi (\xi-\zeta)^{\varkappa-1}d\zeta \\
&+ \frac{2te^{-\tau}L}{(2-v^2)\Gamma(\varkappa)}||\theta_3-\theta_1||_\infty \int_0^1 (1-\zeta)^{\varkappa-1}d\zeta \\
&+ \frac{2te^{-\tau}L}{(2-v^2)\Gamma(\varkappa)}||\theta_3-\theta_1||_\infty \int_0^v \left[\int_0^\zeta (\zeta-\theta_3)^{\varkappa-1}dz\right]d\zeta.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
G(\Gamma\Lambda\theta_1,\Gamma\Lambda\theta_2,\Gamma\Lambda\theta_3) &\le e^{-\tau}L\left[\frac{e^{-\tau}L}{\Gamma(\varkappa)}+\frac{2te^{-\tau}L}{(2-v^2)\Gamma(\varkappa)}+\frac{2te^{-\tau}L}{(2-v^2)\Gamma(\varkappa)}\right] \\
&\quad \times \Big(||\theta_1-\theta_2||_\infty+||\theta_2-\theta_3||_\infty+||\theta_3-\theta_1||_\infty\Big) \\
&\le e^{-\tau}Lk\Big(||\theta_1-\theta_2||_\infty+||\theta_2-\theta_3||_\infty+||\theta_3-\theta_1||_\infty\Big).
\end{aligned}
$$

Since $Lk \le 1$, we have

$$G(\Gamma\Lambda\theta_1,\Gamma\Lambda\theta_2,\Gamma\Lambda\theta_3) \le e^{-\tau}\Big(||\theta_1-\theta_2||_\infty+||\theta_2-\theta_3||_\infty+||\theta_3-\theta_1||_\infty\Big).$$

Thus, for each $\theta_1,\theta_2\in\Theta$, we have

$$G(\Gamma\Lambda\theta_1,\Gamma\Lambda\theta_2,\Gamma\Lambda\theta_3) \le e^{-\tau}\mathcal{M}(\theta_1,\theta_2,\theta_3). \tag{4.12}$$

By applying the logarithm to both sides of (4.12), using $F_1(\theta_3)=\ln(\theta_3)$ and the properties of $\mathcal{F}$, we obtain

$$\ln(G(\Gamma\Lambda\theta_1,\Gamma\Lambda\theta_2,\Gamma\Lambda\theta_3)) \le \ln(e^{-\tau}\mathcal{M}(\theta_1,\theta_2,\theta_3)). \tag{4.13}$$

Equivalently to

$$\tau+\mathcal{F}(G(\Gamma\Lambda\theta_1,\Gamma\Lambda\theta_2,\Gamma\Lambda\theta_3)) \le \mathcal{F}(\mathcal{M}(\theta_1,\theta_2,\theta_3)) \tag{4.14}$$

for $Lk\in[0,1)$, $\tau>0$, and suppose that $\mathcal{M}(\theta_1,\theta_2,\theta_3)$ is an $\mathcal{F}$-Kannan mapping.

As a result, $\Gamma\Lambda$ is a contraction mapping on Θ. Given that all the conditions of Theorem (4.1) are met, it follows that there exists a common fixed point $\theta_1^*\in C(I)$ for both Γ and Λ. Hence, θ_1^* is a solution to the fractional nonlinear differential equation (4.3). □

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References

[1] H. Aydi, E. Karapinar and H. Yazidi, *Modified $\mathcal{F}-$contractions via $\varkappa$-admissible mappings and application to integral equations*, Filomat, **31** (2017), 1141–1148.

[2] S. Banach, *Sure operations dans tes ensembles abstraits et leur application aux equations integrals*, Fund. Math., **3** (1922), 133–181.

[3] R. Batra and S. Vashistha, *Fixed points of an $\mathcal{F}-$contraction on metric spaces with a graph*, Int. J. Comput. Math., **91** (2014), 2483–2490.

[4] N. Goswami, N. Haokip and V.N. Mishra, *$\mathcal{F}-$contractive type mappings in b-metric spaces and some related fixed point results*, Fixed Point Theory Appl., **2019** (2019), Article ID 13.

[5] A. Gupta and P.S. Kaurav, *Fixed points of a new type of contractive mappings in G-metric spaces*, J. Math. Its Appl., **7** (2019), 95–100.

[6] R. Jain, G.S. Saluja and H.G. Hyun, *Results for Coupled Fixed Points in Partial metric Spaces Under A Contractive Type Condition*, Nonlinear Funct. Anal. Appl., **29**(3) (2024), 929–945.

[7] R. Kannan, *Some results on fixed points*, Bull. Calcutta Math. Soc., **60** (1968), 71-76.

[8] T. Kanwal, A. Hussain, H. Aghani and M. de la Sen, *New fixed point theorems in orthogonal $\mathcal{F}-$metric spaces with application to fractional differential equation*, Symmetry, **12** (2020), 832.

[9] Z. Mustafa and H. Obiedat, *A fixed point theorem of Reich in G-metric spaces*, CUBO, **12** (2010), 83–93.

[10] Z. Mustafa and B. Sims, *A new approach to generalized metric spaces*, J. Nonlinear Convex Anal., **7** (2006), 289–297.

[11] A.Z. Rezazgui, W. Shatanawi and A.A. Tallafha, *Common fixed point theorems in the setting of extended quasi b-metric spaces under extended A-contraction Mappings*, Nonlinear Funct. Anal., Appl., **28**(1) (2023), 251–263.

[12] S.S. Sepet and C. Aydin, *Common Fixed Point Theorems for $\mathcal{F}-$contractions in $\mathtt{G}-$metric Spaces using Compatible Mappings*, Konuralp J. Math., **6** (2018), 92–97.

[13] W. Shatanawi, *Fixed point theory for contractive mappings satisfying Φ-maps in G-metric spaces*, Fixed Point Theory Appl., **2010** (2010), 181650.

[14] R.A. Thirumalai and S. Thalapathiraj, *A Novel Approach for Digital Image Compression in Some Fixed Point Results on Complete G-metric Space Using Comparison Function*, Int. J. Anal. Appl., **21** (2023), 110–110.

[15] R.A. Thirumalai and S. Thalapathiraj, *A Novel Approach for Some Fixed Point Results in Complete G-Metric Space by Using Asymptotically Regular Mapping in Digital Image Compression and Decompression Applications*, Int. J. Appl. Math. **37** (2024), 311–332.

[16] R.A. Thirumalai and S. Thalapathiraj, *Solution of the Boundary Value Problems via Fixed Point Theorem on G-metric Space*, IAENG Int. J. Appl. Math., **53** (2023), 1357–1369.

[17] R.A. Thirumalai and S. Thalapathiraj, *Some common fixed point results in complete G-Metric space by using four mappings*, Amer. Insti. Physics Conference Series, **2852** (2023).

[18] D. Wardowski, *Fixed points of a new type of contractive mappings in complete metric spaces*, Fixed Point Theory Appl., **2012** (2012), Article ID 94.