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ON COMMON COUPLED FIXED POINT THEOREMS WITHOUT COMPATIBILITY IN PARTIALLY ORDERED PARTIAL METRIC SPACES WITH APPLICATIONS

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Abstract. In this paper, we prove common coupled fixed point theorems for rational type contractive condition in the setting of partially ordered complete partial metric spaces without the assumption of (weak) compatibility. In many existing works in this direction, attempt have been made to prove the existence of common coupled points. However, only identity mappings can satisfy the conditions of the theorems. The method of proof presented in this work is powerful and it guarantees the existence of common coupled fixed points without using the (weak) compatibility conditions and identity mappings. In addition, we provide some consequences of the established results. To illustrate the results, an example is provided. Also, we give some applications of the result in terms of integral type contractions. Our results extend and generalize several previously published results from the existing literature (see, e.g., [41]).

1. Introduction

Bhashkar and Lakshmikantham [13] introduced the notion of a coupled fixed point and proved a coupled fixed point theorem in a metric space endowed with partial order (see, also [17]). The result has been generalised and extended Lakshmikantham and Ćirić [26] by introducing mixed g-monotone

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mapping and investigating coupled coincidence and coupled common fixed point theorems for such contractive mappings in partially ordered metric spaces. Several other researchers have generalized the results of [13]. Further, for more results on coupled fixed point theorems, interested readers can see [7, 16, 18, 25, 30, 31, 34, 35, 37, 40] and many others in the existing literature.

On the other hand, Matthews [28] introduced the notion of partial metric spaces to study the denotational semantics dataflow networks. In this space, the usual metric is replaced by partial metric with an interesting property that the self-distance of any point of space may not be zero. Later, Matthews proved the partial metric version of Banach fixed point theorem [9]. Oltra ans Valero [32] generalized the Matthews results in the sense of O'Neil [33] in complete partial metric space. Abdeljawad et al. [3] considered a general form of the weak ϕ-contraction and established some common fixed point results. Later, many authors focused on partial metric spaces and its topological properties(see, e.g., ([6, 7, 20, 21, 36, 39] and some others).

The problem of investigating common coupled fixed point theorems in the framework of partially ordered metric spaces using the mixed monotone property has been of immense research interest. Lakshmikantham and Ćirić [26] proved common coupled fixed point theorems by using mixed g-monotone property while one of the mappings is made a proper subset of the other. They proved the existence of coupled coincidence points while coupled fixed point can be proved only by assumption of identity on one of the mappings.

In a same fashion of argument, Abbas et al. [1] and Kim and Chandok [23] established coupled coincidence points for mappings using the mixed monotone property and making one of the mappings a proper subset of the other.

In addition, Abbas et al. [2] proved coupled coincidence point and common coupled fixed points for w-compatible mappings in partially ordered metric spaces. Also Kadelburg et al. [19] proved common coupled fixed points for compatible mappings in partially ordered metric spaces. Several other authors have proved common coupled fixed point theorems using the notions mentioned above (see, e.g., [4, 5, 15, 23, 24, 29]).

Recently, Tijani et al. [41] proved some common coupled fixed point results without the assumptions of (weak) compatibility condition in the setting of partially ordered metric spaces with mixed monotone property.

Motivated by the recent work of [41] and others, the aim of this paper is to study and establish some common coupled fixed point theorems in the setting of partially ordered complete partial metric space without using the compatible condition on the mappings but by enhancing the initial assumption of the theorem of Bhashkar and Lakshmikantham [13] together with the mixed monotone property. Also, we provide some consequences of the established result. Moreover, to illustrate the result, an example is provided and some applications of the established results in terms of integral type contractions are also included. Our results extend, generalize and enrich several results from the existing literature (see, e.g., [41] and some others).

2. Preliminaries

In this section, we need some definitions, lemmas and the basic results regarding partial metric space in the sequel.

Definition 2.1. ([28]) (PMS) Let $\Omega \neq \emptyset$ be a set. A partial metric on Ω is a function $p\colon \Omega \times \Omega \to [0, +\infty)$ such that for all $u, v, w \in \Omega$ the following axioms are satisfied:

(1) $u = v$ if and only if $p(u, u) = p(u, v) = p(v, v)$,
(2) $p(u, u) \leq p(u, v)$,
(3) $p(u, v) = p(v, u)$,
(4) $p(u, v) \leq p(u, w) + p(w, v) - p(w, w)$.

Then p is called a partial metric on Ω and the pair (Ω, p) is called a partial metric space (in short PMS).

It is clear that if $p(u, v) = 0$, then from axioms (1), (2), and (3), $u = v$. But if $u = v$, $p(u, v)$ may not be 0.

If p is a partial metric on Ω, then the function $p^s\colon \Omega \times \Omega \to [0, +\infty)$ given by

$$p^s(u, v) = 2p(u, v) - p(u, u) - p(v, v) \tag{2.1}$$

is a usual metric on Ω.

Each partial metric p on Ω generates a T_0 topology τ_p on Ω with the family of open p-balls $\{B_p(u, \varepsilon) : u \in \Omega, \varepsilon > 0\}$ where $B_p(u, \varepsilon) = \{v \in \Omega : p(u, v) < p(u, u) + \varepsilon\}$ for all $u \in \Omega$ and $\varepsilon > 0$. Similarly, closed p-ball is defined as $B_p[u, \varepsilon] = \{v \in \Omega : p(u, v) \leq p(u, u) + \varepsilon\}$ for all $u \in \Omega$ and $\varepsilon > 0$.

Example 2.2. ([8])
1. Let $\Omega = [0, +\infty)$ and $p\colon \Omega \times \Omega \to [0, +\infty)$ be given by $p(u, v) = \max\{u, v\}$ for all $u, v \in \Omega$. Then (Ω, p) is a partial metric space.
2. Let $I = \Omega$, where I denote the set of all intervals $[u_1, v_1]$ for any real numbers $u_1 \leq v_1$. Let $p\colon \Omega \times \Omega \to [0, \infty)$ be a function such that $p\big([u_1, v_1], [u_2, v_2]\big) = \max\{v_1, v_2\} - \min\{u_1, u_2\}$. Then (I, p) is a partial metric space.

Definition 2.3. ([28]) Let (Ω, p) be a partial metric space.

(1) A sequence $\{u_n\}$ converges to a point $u \in \Omega$ if and only if $\lim_{n\to\infty} p(u, u_n) = p(u, u)$.
(2) A sequence $\{u_n\}$ in Ω is called a Cauchy sequence if and only if $\lim_{m,n\to\infty} p(u_m, u_n)$ exists (and finite).
(3) A partial metric space (Ω, p) is said to be complete if every Cauchy sequence $\{u_n\}$ in Ω converges, with respect to τ_p, to a point $u \in \Omega$, such that, $\lim_{m,n\to\infty} p(u_m, u_n) = p(u, u)$.
(4) A mapping $W\colon \Omega \to \Omega$ is said to be continuous at $u_0 \in \Omega$ if for every $\varepsilon > 0$, there exists $\beta > 0$ such that $W\big(B_p(u_0, \beta)\big) \subset B_p\big(W(u_0), \varepsilon\big)$.

Definition 2.4. ([13]) Let $(\Omega, \leq)$ be a partially ordered set. The mapping $F\colon \Omega \times \Omega \to \Omega$ is said to have the mixed monotone property if $F(u, v)$ is monotone non-decreasing in u and is monotone non-increasing in v, that is, for any $u, v \in \Omega$,

$$u_1, u_2 \in \Omega, \ \ u_1 \leq u_2 \ \Rightarrow \ F(u_1, v) \leq F(u_2, v)$$

and

$$v_1, v_2 \in \Omega, \ \ v_1 \leq v_2 \ \Rightarrow \ F(u, v_1) \geq F(u, v_2).$$

Definition 2.5. ([13, 26]) An element $(u, v) \in \Omega \times \Omega$ is said to be a coupled fixed point of the mapping $F\colon \Omega \times \Omega \to \Omega$ if $F(u, v) = u$ and $F(v, u) = v$.

Example 2.6. Let $\Omega = [0, +\infty)$ and $F\colon \Omega \times \Omega \to \Omega$ be defined by $F(u, v) = \frac{u+v}{3}$ for all $u, v \in \Omega$. Then F has a unique coupled fixed point $(0, 0)$.

Example 2.7. Let $\Omega = [0, +\infty)$ and $F\colon \Omega \times \Omega \to \Omega$ be defined by $F(u, v) = \frac{u+v}{2}$ for all $u, v \in \Omega$. Then F has two coupled fixed point $(0, 0)$ and $(1, 1)$, that is, the coupled fixed point is not unique.

Definition 2.8. ([10, 11, 12]) Consider a function $\psi\colon [0, +\infty) \to [0, +\infty)$ satisfying

(ψ_1) ψ is monotone increasing;
(ψ_2) $\psi^n(t) \to 0$, as $n \to \infty$;
(ψ_3) $\sum_{n=0}^{\infty} \psi^n(t)$ converges for all $t > 0$;
(ψ_4) $\psi(t) < t$ for all $t > 0$.

(A) A function ψ satisfying (ψ_1) and (ψ_2) above is called a comparison function.
(B) A function ψ satisfying (ψ_1) and (ψ_3) above is called a (c)-comparison function.

Remark 2.9. ([11, 12, 38])

(1) Any (c)-comparison function is a comparison function.

(2) Every comparison function satisfies $\psi(0) = 0$.
(3) (ψ_1), (ψ_2) and $(\psi_3) \Rightarrow (\psi_4)$.

Lemma 2.10. ([7, 27, 28])

(1) *A sequence $\{u_n\}$ is Cauchy in a partial metric space (Ω, p) if and only if $\{u_n\}$ is Cauchy in a metric space (Ω, p^s), where*
$$p^s(u,v) = 2p(u,v) - p(u,u) - p(v,v).$$
(2) *A partial metric space (Ω, p) is complete if a metric space (Ω, p^s) is complete, that is,*
$$\lim_{n\to\infty} p^s(u_n, u) = 0 \Leftrightarrow p(u,u) = \lim_{n\to\infty} p(u_n, u) = \lim_{n,m\to\infty} p(u_n, u_m).$$

Lemma 2.11. ([22]) *Let (Ω, p) be a partial metric space.*

(1) *If $u, v \in \Omega$, $p(u,v) = 0$, then $u = v$.*
(2) *If $u \neq v$, then $p(u,v) > 0$.*

Lemma 2.12. ([14]) *Let $u_n \to u$ as $n \to \infty$ in a partial metric space (Ω, p), where $p(u,u) = 0$. Then $\lim_{n\to\infty} p(u_n, z) = P(u, z)$ for all $z \in \Omega$.*

3. Main results

In this section, we prove common coupled fixed point theorems without using compatibility condition for rational type contraction condition in the setting of partially ordered complete partial metric spaces with mixed monotone property.

Theorem 3.1. *Let $(\Omega, p, \leq)$ be a partially ordered complete partial metric space. Let $F, G\colon \Omega \times \Omega \to \Omega$ be mappings having the mixed monotone property such that for some constants $\gamma_1, \gamma_2 \geq 0$, for all $u, v, r, s \in \Omega$, $p(r, F(r,s)) + p(u,r) > 0$ and ψ, a (c)-comparison function, we have*
$$\begin{aligned} p(F(u,v), G(r,s)) \leq \gamma_1 \Big(&\frac{p(u, F(u,v)) p(u, G(r,s)) p(r, F(u,v))}{1 + p(r, F(r,s)) + p(u,r)} \Big) \\ &+ \gamma_2 \Big(\frac{p(r, F(u,v))[1 + p(r, G(r,s))]}{1 + p(u,r)} \Big) + \psi(p(u,r)). \end{aligned} \tag{3.1}$$
Suppose that we endow the product space $\Omega \times \Omega$ with the following partial order:
$$\text{for } (u,v), (r,s) \in \Omega \times \Omega,\ (r,s) \leq (u,v) \Leftrightarrow u \geq r,\ v \leq s.$$
If there exist two elements $u_0, v_0 \in \Omega$ such that $u_0 \leq F(u_0, v_0) \leq G(u_0, v_0)$ and $v_0 \geq F(v_0, u_0) \geq G(v_0, u_0)$, then F and G have a common coupled fixed point and $p(a,a) = 0$ for some $a \in \Omega$.

Proof. Let $u_0, v_0 \in \Omega$ be arbitrary elements such that $u_0 \leq F(u_0, v_0) \leq G(u_0, v_0)$ and $v_0 \geq F(v_0, u_0) \geq G(v_0, u_0)$. Define $u_{2k+1} = F(u_{2k}, v_{2k})$, $v_{2k+1} = F(v_{2k}, u_{2k})$ and $u_{2k+2} = G(u_{2k+1}, v_{2k+1})$, $v_{2k+2} = G(v_{2k+1}, u_{2k+1})$ for all $k \geq 0$.

We have to prove that u_k is non-decreasing and v_k is non-increasing. That is, for all $k \geq 0$, $u_{2k} \leq u_{2k+1} \leq u_{2k+2}$ and $v_{2k} \geq v_{2k+1} \geq v_{2k+2}$.

Now, first we see that $u_0 \leq F(u_0, v_0) = u_1$ and $v_0 \geq F(v_0, u_0) = v_1$.

By the iterative process defined as above, we have

$$u_2 = G(u_1, v_1),\ v_2 = G(v_1, u_1).$$

Using the mixed monotone property of F and G, we have

$$u_1 = F(u_0, v_0) \leq G(u_0, v_0) \leq G(u_1, v_1) = u_2,$$

$$v_1 = F(v_0, u_0) \geq G(v_0, u_0) \geq G(v_1, u_1) = v_2.$$

Moreover,

$$u_2 = G(u_1, v_1) \leq G(u_2, v_2) = u_3,$$

$$v_2 = G(v_1, u_1) \geq G(v_2, u_2) = v_3$$

and

$$u_3 = F(u_2, v_2) \leq F(u_3, v_3) = u_4,$$

$$v_3 = F(v_2, u_2) \leq F(v_3, u_3) = v_4.$$

Therefore, for $k \geq 1$,

$$u_{2k+1} = F(u_{2k}, v_{2k}) \leq F(u_{2k+1}, v_{2k+1}) = u_{2k+2},$$

$$v_{2k+1} = F(v_{2k}, u_{2k}) \leq F(v_{2k+1}, u_{2k+1}) = v_{2k+2}$$

and

$$u_{2k+2} = G(u_{2k+1}, v_{2k+1}) \leq G(u_{2k+2}, v_{2k+2}) = u_{2k+3},$$

$$v_{2k+2} = G(v_{2k+1}, u_{2k+1}) \geq G(v_{2k+2}, u_{2k+2}) = v_{2k+3}.$$

Hence,

$$u_0 \leq u_1 \leq \cdots \leq u_{2k} \leq u_{2k+1} \leq \cdots,$$

$$v_0 \geq v_1 \geq \cdots \geq v_{2k} \geq v_{2k+1} \geq \cdots.$$

Now, using equation (3.1) for $u = u_{2k}$, $v = v_{2k}$, $r = u_{2k+1}$ and $s = v_{2k+1}$, we have

$$
\begin{aligned}
&p(u_{2k+1}, u_{2k+2}) \\
&\quad = p\big(F(u_{2k}, v_{2k}), G(u_{2k+1}, v_{2k+1})\big) \\
&\quad \le \gamma_1 \Big(\frac{p(u_{2k}, F(u_{2k}, v_{2k}))p(u_{2k}, G(u_{2k+1}, v_{2k+1}))p(u_{2k+1}, F(u_{2k}, v_{2k}))}{1 + p(u_{2k+1}, F(u_{2k+1}, v_{2k+1})) + p(u_{2k}, u_{2k+1})} \Big) \\
&\qquad + \gamma_2 \Big(\frac{p(u_{2k+1}, F(u_{2k}, v_{2k}))[1 + p(u_{2k+1}, G(u_{2k+1}, v_{2k+1}))]}{1 + p(u_{2k}, u_{2k+1})} \Big) \\
&\qquad + \psi(p(u_{2k}, u_{2k+1})) \\
&\quad = \gamma_1 \Big(\frac{p(u_{2k}, u_{2k+1})p(u_{2k}, u_{2k+2})p(u_{2k+1}, u_{2k+1})}{1 + p(u_{2k+1}, u_{2k+2}) + p(u_{2k}, u_{2k+1})} \Big) \\
&\qquad + \gamma_2 \Big(\frac{p(u_{2k+1}, u_{2k+1})[1 + p(u_{2k+1}, u_{2k+2})]}{1 + p(u_{2k}, u_{2k+1})} \Big) \\
&\qquad + \psi(p(u_{2k}, u_{2k+1})) \\
&\quad = \psi(p(u_{2k}, u_{2k+1})).
\end{aligned}
$$

Therefore,

$$
p(u_{2k+1}, u_{2k+2}) \le \psi(p(u_{2k}, u_{2k+1})) \tag{3.2}
$$

and

$$
p(v_{2k+1}, v_{2k+2}) \le \psi(p(v_{2k}, v_{2k+1})). \tag{3.3}
$$

Similarly, proceeding as above, we obtain

$$
p(u_{2k+2}, u_{2k+3}) \le \psi(p(u_{2k+1}, u_{2k+2})) \tag{3.4}
$$

and

$$
p(v_{2k+2}, v_{2k+3}) \le \psi(p(v_{2k+1}, v_{2k+2})). \tag{3.5}
$$

Hence, it can be deduced from equations (3.2)-(3.5) that

$$
\begin{aligned}
p(u_{2k+2}, u_{2k+3}) + p(v_{2k+2}, v_{2k+3}) &\le \psi(p(u_{2k+1}, u_{2k+2})) + \psi(p(v_{2k+1}, v_{2k+2})) \\
&\le \psi^2(p(u_{2k}, u_{2k+1})) + \psi^2(p(v_{2k}, v_{2k+1})) \\
&\le \cdots \le \psi^n(p(u_0, u_1)) + \psi^n(p(v_0, v_1)).
\end{aligned}
$$

Thus, it follows that

$$
p(u_n, u_{n+1}) + p(v_n, v_{n+1}) \le \psi^n(p(u_0, u_1)) + \psi^n(p(v_0, v_1)). \tag{3.6}
$$

Again, for $n, r \in \mathbb{N}$, using equation (3.6) inductively and using condition (4) of PMS repeatedly, we obtain

$$\begin{aligned}
&p(u_n, u_{n+r}) + p(v_n, v_{n+r}) \\
&\quad \le [p(u_n, u_{n+1}) + p(v_n, v_{n+1})] + [p(u_{n+1}, u_{n+2}) \\
&\qquad + p(v_{n+1}, v_{n+2})] + \cdots + [p(u_{n+r-1}, u_{n+r}) + p(v_{n+r-1}, v_{n+r})] \\
&\qquad - [p(u_{n+1}, u_{n+1}) + p(v_{n+1}, v_{n+1})] - [p(u_{n+2}, u_{n+2}) + p(v_{n+2}, v_{n+2})] \\
&\qquad - \cdots - [p(u_{n+r-1}, u_{n+r-1}) + p(v_{n+r-1}, v_{n+r-1})] \\
&\quad \le [p(u_n, u_{n+1}) + p(v_n, v_{n+1})] + [p(u_{n+1}, u_{n+2}) + p(v_{n+1}, v_{n+2})] \\
&\qquad + \cdots + [p(u_{n+r-1}, u_{n+r}) + p(v_{n+r-1}, v_{n+r})] \\
&\quad \le \psi^n(p(u_0, u_1)) + \psi^n(p(v_0, v_1)) + \psi^{n+1}(p(u_0, u_1)) \\
&\qquad + \psi^{n+1}(p(v_0, v_1)) + \cdots + \psi^{n+r-1}(p(u_0, u_1)) + \psi^{n+r-1}(p(v_0, v_1)) \\
&\quad = \sum_{\kappa=n}^{n+r-1} \psi^\kappa(p(u_0, u_1)) + \sum_{\kappa=n}^{n+r-1} \psi^\kappa(p(v_0, v_1)) \\
&\quad = \sum_{\kappa=0}^{n+r-1} \psi^\kappa(p(u_0, u_1)) - \sum_{\kappa=0}^{n-1} \psi^\kappa(p(u_0, u_1)) \\
&\qquad + \sum_{\kappa=0}^{n+r-1} \psi^\kappa(p(v_0, v_1)) - \sum_{\kappa=0}^{n-1} \psi^\kappa(p(v_0, v_1)) \\
&\quad \to 0 \text{ as } \kappa \to \infty,
\end{aligned}$$

since ψ is a (c)-comparison function. Hence, $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences in a partial metric space (Ω, p) such that $\lim_{m,n\to\infty} p(u_m, u_n) = 0$ and $\lim_{m,n\to\infty} p(v_m, v_n) = 0$, where $m = n + r$ with $m > n \in \mathbb{N}$. Again, since (Ω, p) is a complete partial metric space, there exist $u^*, v^* \in \Omega$ such that (by Lemma 2.10(2)),

$$p(u^*, u^*) = \lim_{n\to\infty} p(u_n, u^*) = \lim_{n\to\infty} p(u_n, u_n) = 0 \tag{3.7}$$

and

$$p(v^*, v^*) = \lim_{n\to\infty} p(v_n, v^*) = \lim_{n\to\infty} p(v_n, v_n) = 0. \tag{3.8}$$

Now, we have to show that (u^*, v^*) is a coupled fixed point of F and G. Using contractive condition (3.1) again and noting that $\psi(0) = 0$, we have

$$\begin{aligned}
p(G(u^*, v^*), u^*) &\leq p(G(u^*, v^*), u_{2n+1}) + p(u_{2n+1}, u^*) - p(u_{2n+1}, u_{2n+1}) \\
&\leq p(G(u^*, v^*), u_{2n+1}) + p(u_{2n+1}, u^*) \\
&= p(G(u^*, v^*), F(u_{2n}, v_{2n})) + p(u_{2n+1}, u^*) \\
&= p(F(u_{2n}, v_{2n}), G(u^*, v^*)) + p(u_{2n+1}, u^*) \\
&\leq \gamma_1 \Big(\frac{p(u_{2n}, F(u_{2n}, v_{2n}))p(u_{2n}, G(u_*, v_*))p(u_*, F(u_{2n}, v_{2n}))}{1 + p(u_*, F(u_*, v_*)) + p(u_{2n}, u^*)}\Big) \\
&\quad + \gamma_2 \Big(\frac{p(u^*, F(u_{2n}, v_{2n}))[1 + p(u^*, G(u^*, v^*))]}{1 + p(u_{2n}, u^*)}\Big) \\
&\quad + \psi(p(u_{2n}, u^*)) + p(u_{2n+1}, u^*) \\
&= \gamma_1 \Big(\frac{p(u_{2n}, u_{2n+1})p(u_{2n}, G(u_*, v_*))p(u_*, u_{2n+1})}{1 + p(u_*, F(u_*, v_*)) + p(u_{2n}, u^*)}\Big) \\
&\quad + \gamma_2 \Big(\frac{p(u^*, u_{2n+1})[1 + p(u^*, G(u^*, v^*))]}{1 + p(u_{2n}, u^*)}\Big) \\
&\quad + \psi(p(u_{2n}, u^*)) + p(u_{2n+1}, u^*). \qquad (3.9)
\end{aligned}$$

Letting $n \to \infty$ in equation (3.9) and using equation (3.7), we obtain

$$p(G(u^*, v^*), u^*) \to 0 \text{ as } n \to \infty.$$

Thus, we conclude that $p(G(u^*, v^*), u^*) = 0$ and so by Lemma 2.11(1)

$$G(u^*, v^*) = u^*.$$

Again by using inequality (3.1), we have

$$\begin{aligned}
p(G(v^*, u^*), v^*) &\leq p(G(v^*, u^*), v_{2n+1}) + p(v_{2n+1}, v^*) - p(v_{2n+1}, v_{2n+1}) \\
&\leq p(G(v^*, u^*), v_{2n+1}) + p(v_{2n+1}, v^*) \\
&= p(G(v^*, u^*), F(v_{2n}, u_{2n})) + p(v_{2n+1}, v^*) \\
&= p(F(v_{2n}, u_{2n}), G(v^*, u^*)) + p(v_{2n+1}, v^*)
\end{aligned}$$

$$
\begin{aligned}
&\leq \gamma_1 \Big(\frac{p(v_{2n}, F(v_{2n}, u_{2n}))p(v_{2n}, G(v^*, u^*))p(v^*, F(v_{2n}, u_{2n}))}{1 + p(v^*, F(v^*, u^*)) + p(v_{2n}, v^*)}\Big) \\
&\quad + \gamma_2 \Big(\frac{p(v^*, F(v_{2n}, u_{2n}))[1 + p(v^*, G(v^*, u^*))]}{1 + p(v_{2n}, v^*)}\Big) \\
&\quad + \psi(p(v_{2n}, v^*)) + p(v_{2n+1}, v^*) \\
&= \gamma_1 \Big(\frac{p(v_{2n}, v_{2n+1})p(v_{2n}, G(v^*, u^*))p(v^*, v_{2n+1})}{1 + p(v^*, F(v^*, u^*)) + p(v_{2n}, v^*)}\Big) \\
&\quad + \gamma_2 \Big(\frac{p(v^*, v_{2n+1})[1 + p(v^*, G(v^*, u^*))]}{1 + p(v_{2n}, v^*)}\Big) \\
&\quad + \psi(p(v_{2n}, v^*)) + p(v_{2n+1}, v^*). \qquad (3.10)
\end{aligned}
$$

Letting $n \to \infty$ in equation (3.10) and using equation (3.7) and condition $\psi(0) = 0$, we obtain

$$p(G(v^*, u^*), v^*) \to 0 \text{ as } n \to \infty.$$

Thus, we get

$$p(G(v^*, u^*), v^*) = 0 \;\Rightarrow\; G(v^*, u^*) = v^*,$$

by Lemma 2.11(1). Hence (u^*, v^*) is a coupled fixed point of mapping G.

Next, we have to show that (u^*, v^*) is also a coupled fixed point of mapping F. By inequality (3.1), we have

$$
\begin{aligned}
p(F(u^*, v^*), u^*) &= p(F(u^*, v^*), G(u^*, v^*)) \\
&\leq \gamma_1 \Big(\frac{p(u^*, F(u^*, v^*))p(u^*, G(u^*, v^*))p(u^*, F(u^*, v^*))}{1 + p(u^*, F(u^*, v^*) + p(u^*, u^*)}\Big) \\
&\quad + \gamma_2 \Big(\frac{p(u^*, F(u^*, v^*))[1 + p(u^*, G(u^*, v^*))]}{1 + p(u^*, u^*)}\Big) + \psi(p(u^*, u^*)) \\
&= \gamma_1 \Big(\frac{p(u^*, F(u^*, v^*))p(u^*, u^*)p(u^*, F(u^*, v^*))}{1 + p(u^*, F(u^*, v^*) + p(u^*, u^*)}\Big) \\
&\quad + \gamma_2 \Big(\frac{p(u^*, F(u^*, v^*))[1 + p(u^*, u^*)]}{1 + p(u^*, u^*)}\Big) + \psi(0) \\
&= \gamma_2 \, p(F(u^*, v^*), u^*),
\end{aligned}
$$

which implies that $p(F(u^*, v^*), u^*) = 0$, since $\gamma_2 \geq 0$ and so by Lemma 2.11(1), $F(u^*, v^*) = u^*$. By a similar fashion, using inequality (3.1), we can prove that $F(v^*, u^*) = v^*$. Thus, (u^*, v^*) is a common coupled fixed point of F and G. This completes the proof. □

Example 3.2. Let $\Omega = [0, 4]$ be endowed with the usual partial metric given by $p(u, v) = \max\{u, v\}$ for all $u, v \in \Omega$. Let $\psi(t) = \frac{1}{2}t$ for all $t \in \Omega$. Clearly,

$\psi(t)$ is a (c)-comparison function. Define $F, G\colon [0,4]\times[0,4]\to[0,4]$ by

$$F(u,v) = 3u - 2v$$

and

$$G(u,v) = \begin{cases} u - v + 1, & \text{if } u \geq v, \\ \frac{1}{7}, & \text{if } u < v, \end{cases}$$

for all $u, v \in [0,4]$. Clearly, F and G have the mixed monotone property.

Let $u_0 = \frac{2}{3}$, $v_0 = \frac{1}{2} \in \Omega$.

$$F(u_0,v_0) = F\Big(\frac{2}{3},\frac{1}{2}\Big) = 1,$$

$$F(v_0,u_0) = F\Big(\frac{1}{2},\frac{2}{3}\Big) = \frac{1}{6},$$

$$G(u_0,v_0) = G\Big(\frac{2}{3},\frac{1}{2}\Big) = \Big(\frac{2}{3}\Big) - \frac{1}{2} + 1 = \frac{7}{6},$$

$$G(v_0,u_0) = G\Big(\frac{1}{2},\frac{2}{3}\Big) = \frac{1}{7}.$$

Thus,

$$\frac{2}{3} < 1 < \frac{7}{6} \text{ and } \frac{1}{2} > \frac{1}{6} > \frac{1}{7}.$$

Hence, $u_0 \leq F(u_0,v_0) \leq G(u_0,v_0)$ and $v_0 \geq F(v_0,u_0) \geq G(v_0,u_0)$.

Now, consider the contractive condition (3.1) in Theorem 3.1. Let $u = \frac{4}{3}$, $v = \frac{1}{3}$, $\gamma_1 = \gamma_2 = 1$, $r = \frac{4}{5}$, $s = \frac{1}{5}$ and assume that $u \geq v \geq r \geq s$. Then

$$F(u,v) = F\Big(\frac{4}{3},\frac{1}{3}\Big) = 3\Big(\frac{4}{3}\Big) - 2\Big(\frac{1}{3}\Big) = 4 - \frac{2}{3} = \frac{10}{3},$$

$$G(r,s) = G\Big(\frac{4}{5},\frac{1}{5}\Big) = \Big(\frac{4}{5}\Big) - \frac{1}{5} + 1 = \frac{8}{5},$$

$$F(r,s) = F\Big(\frac{4}{5},\frac{1}{5}\Big) = 3\Big(\frac{4}{5}\Big) - 2\Big(\frac{1}{5}\Big) = \frac{12}{5} - \frac{2}{5} = 2,$$

$$p(u,F(u,v)) = p\Big(\frac{4}{3},\frac{10}{3}\Big) = \max\Big\{\frac{4}{3},\frac{10}{3}\Big\} = \frac{10}{3},$$

$$p(r,F(r,s)) = p\Big(\frac{4}{5},2\Big) = \max\Big\{\frac{4}{5},2\Big\} = 2,$$

$$p(r,F(u,v)) = p\Big(\frac{4}{5},\frac{10}{3}\Big) = \max\Big\{\frac{4}{5},\frac{10}{3}\Big\} = \frac{10}{3},$$

$$p(u,F(r,s)) = p\Big(\frac{4}{3},2\Big) = \max\Big\{\frac{4}{3},2\Big\} = 2,$$

$$p(u,r) = p\Big(\frac{4}{3},\frac{4}{5}\Big) = \max\Big\{\frac{4}{3},\frac{4}{5}\Big\} = \frac{4}{3},$$

$$p(u,G(r,s)) = p\Big(\frac{4}{3},\frac{8}{5}\Big) = \max\Big\{\frac{4}{3},\frac{8}{5}\Big\} = \frac{8}{5},$$

$$p(r,G(r,s)) = p\Big(\frac{4}{5},\frac{8}{5}\Big) = \max\Big\{\frac{4}{5},\frac{8}{5}\Big\} = \frac{8}{5}.$$

Now, using contractive condition (3.1), we have

$$p(F(u,v),G(r,s)) \leq \gamma_1 \Big(\frac{p(u,F(u,v))p(u,G(r,s))p(r,F(u,v))}{1+p(r,F(r,s))+p(u,r)}\Big) + \gamma_2 \Big(\frac{p(r,F(u,v))[1+p(r,G(r,s))]}{1+p(u,r)}\Big) + \psi(p(u,r)),$$

implies that,

$$\begin{aligned} p\Big(\frac{10}{3},\frac{8}{5}\Big) &= \max\Big\{\frac{10}{3},\frac{8}{5}\Big\} = \frac{10}{3} = 3.333 \\ &\leq 1.\Big(\frac{\frac{10}{3}.\frac{8}{5}.\frac{10}{3}}{1+2+\frac{4}{3}}\Big) + 1.\Big(\frac{\frac{10}{3}[1+\frac{8}{5}]}{1+\frac{4}{3}}\Big) + \frac{1}{2}.\frac{4}{3} \\ &= \frac{160}{39} + \frac{26}{7} + \frac{2}{3} = 8.483, \end{aligned}$$

that is, $3.333 \leq 8.483$. Since all the assumptions of Theorem 3.1 are satisfied, so F and G have a common coupled fixed point in $\Omega = [0,4]$. Here $F(1,1) = 1 = G(1,1)$, that is, 1 is a common coupled fixed point of F and G in $\Omega = [0,4]$.

If we take $\psi(t) = \gamma_3\, t$ for all $t > 0$ where $\gamma_3 \in [0,1)$ in Theorem 3.1, then we have the following result.

Corollary 3.3. *Let $(\Omega, p, \leq)$ be a partially ordered complete partial metric space. Let $F, G \colon \Omega \times \Omega \to \Omega$ be mappings having the mixed monotone property such that for some constants $\gamma_1, \gamma_2 \geq 0$, $\gamma_3 \in [0,1)$ and for all $u, v, r, s \in \Omega$, where $p(r,F(r,s)) + p(u,r) > 0$, we have*

$$p(F(u,v),G(r,s)) \leq \gamma_1 \Big(\frac{p(u,F(u,v))p(u,G(r,s))p(r,F(u,v))}{1+p(r,F(r,s))+p(u,r)}\Big) + \gamma_2 \Big(\frac{p(r,F(u,v))[1+p(r,G(r,s))]}{1+p(u,r)}\Big) + \gamma_3\, p(u,r). \tag{3.11}$$

Suppose that we endow the product space $\Omega \times \Omega$ with the following partial order:

for $(u,v),(r,s) \in \Omega\times\Omega$, $(r,s) \leq (u,v) \Leftrightarrow u \geq r,\ v \leq s$.

If there exist two elements $u_0, v_0 \in \Omega$ such that $u_0 \leq F(u_0,v_0) \leq G(u_0,v_0)$ and $v_0 \geq F(v_0,u_0) \geq G(v_0,u_0)$, then F and G have a common coupled fixed point and $p(a,a) = 0$ for some $a \in \Omega$.

Proof. Follows from Theorem 3.1 by taking $\psi(t) = \gamma_3\, t$ for all $t > 0$ where $\gamma_3 \in [0,1)$. □

If we take $\gamma_3 = 0$ in Corollary 3.3, then we have the following result.

Corollary 3.4. *Let $(\Omega, p, \leq)$ be a partially ordered complete partial metric space. Let $F, G\colon \Omega\times\Omega \to \Omega$ be mappings having the mixed monotone property such that for some constants $\gamma_1, \gamma_2 \geq 0$ and for all $u, v, r, s \in \Omega$, where $p(r, F(r, s)) + p(u, r) > 0$, we have*

$$\begin{aligned} p(F(u,v), G(r,s)) \leq \gamma_1 \Big(&\frac{p(u, F(u,v))p(u, G(r,s))p(r, F(u,v))}{1 + p(r, F(r,s)) + p(u,r)}\Big) \\ &+ \gamma_2 \Big(\frac{p(r, F(u,v))[1 + p(r, G(r,s))]}{1 + p(u,r)}\Big). \end{aligned} \tag{3.12}$$

Suppose that we endow the product space $\Omega\times\Omega$ with the following partial order:

$$\text{for } (u,v), (r,s) \in \Omega\times\Omega,\ (r,s) \leq (u,v) \Leftrightarrow u \geq r,\ v \leq s.$$

If there exist two elements $u_0, v_0 \in \Omega$ such that $u_0 \leq F(u_0, v_0) \leq G(u_0, v_0)$ and $v_0 \geq F(v_0, u_0) \geq G(v_0, u_0)$, then F and G have a common coupled fixed point and $p(a, a) = 0$ for some $a \in \Omega$.

Proof. Follows from Corollary 3.3 by taking $\gamma_3 = 0$. □

If we take $\gamma_1 = \gamma_2 = 0$ and $\gamma_3 = \mu$ where $\mu \in [0, 1)$ in Corollary 3.3, then we have the following result.

Corollary 3.5. *Let $(\Omega, p, \leq)$ be a partially ordered complete partial metric space. Let $F, G\colon \Omega\times\Omega \to \Omega$ be mappings having the mixed monotone property such that for all $u, v, r, s \in \Omega$,*

$$p(F(u,v), G(r,s)) \leq \mu\, p(u,r), \tag{3.13}$$

where $\mu \in [0, 1)$ is a constant. Suppose that we endow the product space $\Omega\times\Omega$ with the following partial order:

$$\text{for } (u,v), (r,s) \in \Omega\times\Omega,\ (r,s) \leq (u,v) \Leftrightarrow u \geq r,\ v \leq s.$$

If there exist two elements $u_0, v_0 \in \Omega$ such that $u_0 \leq F(u_0, v_0) \leq G(u_0, v_0)$ and $v_0 \geq F(v_0, u_0) \geq G(v_0, u_0)$, then F and G have a common coupled fixed point.

Now, we prove a common coupled fixed point theorem for contractive condition of non-rational-type.

Theorem 3.6. *Let $(\Omega, p, \leq)$ be a partially ordered complete partial metric space (POCPMS). Suppose that $\{S_\alpha\}_{\alpha\in\Delta}$ with $S_\alpha\colon \Omega\times\Omega \to \Omega$ is a family of mappings having the mixed monotone property and Δ is an index set. If there exists a $\beta \in \Delta$ such that for each $\alpha \in \Delta$,*

$$\begin{aligned} p(S_\alpha(u,v), S_\beta(r,s)) \leq h_1\, p(u,r) + h_2\, &[p(u, S_\alpha(u,v)) + p(r, S_\beta(r,s))] \\ &+ h_3\, [p(u, S_\beta(r,s)) + p(r, S_\alpha(u,v))] \end{aligned} \tag{3.14}$$

for all $u, v, r, s \in \Omega$ and for some constants $h_1, h_2, h_3 \geq 0$ with $h_1+2h_2+2h_3 < 1$. Suppose that we endow the product space $\Omega \times \Omega$ with the following partial order:

$$\text{for } (u,v),(r,s) \in \Omega \times \Omega,\ (r,s) \leq (u,v) \Leftrightarrow u \geq r,\ v \leq s.$$

If for all $\alpha \in \Delta$, there exist $u_0, v_0 \in \Omega$ such that $u_0 \leq S_\alpha(u_0,v_0) \leq S_\beta(u_0,v_0)$ and $v_0 \geq S_\alpha(v_0,u_0) \geq S_\beta(v_0,u_0)$, then all S_α have a common coupled fixed point and $p(a,a) = 0$ for some $a \in \Omega$.

Proof. Choose $u_0, v_0 \in \Omega$ such that $u_0 \leq S_\alpha(u_0,v_0) \leq S_\beta(u_0,v_0)$ and $v_0 \geq S_\alpha(v_0,u_0) \geq S_\beta(v_0,u_0)$ and define sequences $\{u_n\}$ and $\{v_n\}$ in Ω in the following way: $u_{2n+1} = S_\alpha(u_{2n}, v_{2n})$, $v_{2n+1} = S_\alpha(v_{2n}, u_{2n})$ and $u_{2n+2} = S_\beta(u_{2n+1}, v_{2n+1})$, $v_{2n+2} = S_\beta(v_{2n+1}, u_{2n+1})$ for all $n \geq 0$.

We have to prove that the sequence $\{u_n\}$ is non-decreasing and the sequence $\{v_n\}$ is non-increasing, that is, for all $n \geq 0$,

$$u_{2n} \leq u_{2n+1} \quad \text{and} \quad v_{2n} \geq v_{2n+1}.$$

Now, first we set, $u_0 \leq S_\alpha(u_0,v_0) = u_1$ and $v_0 \geq S_\alpha(v_0,u_0) = v_1$. By the iterative process defined as above, we have

$$u_2 = S_\beta(u_1,v_1),\ v_2 = S_\beta(v_1,u_1).$$

Using the mixed monotone property of S_α and S_β, we have

$$u_1 = S_\alpha(u_0,v_0) \leq S_\beta(u_0,v_0) \leq S_\beta(u_1,v_1) = u_2,$$

$$v_1 = S_\alpha(v_0,u_0) \geq S_\beta(v_0,u_0) \geq S_\beta(v_1,u_1) = v_2.$$

Moreover,

$$u_2 = S_\beta(u_1,v_1) \leq S_\beta(u_2,v_2) = u_3,$$
$$v_2 = S_\beta(v_1,u_1) \geq S_\beta(v_2,u_2) = v_3$$

and

$$u_3 = S_\alpha(u_2,v_2) \leq S_\alpha(u_3,v_3) = u_4,$$
$$v_3 = S_\alpha(v_2,u_2) \leq S_\alpha(v_3,u_3) = v_4.$$

Therefore, for $k \geq 1$,

$$u_{2n+1} = S_\alpha(u_{2n},v_{2n}) \leq S_\alpha(u_{2n+1},v_{2n+1}) = u_{2n+2},$$

$$v_{2n+1} = S_\alpha(v_{2n},u_{2n}) \leq S_\alpha(v_{2n+1},u_{2n+1}) = v_{2n+2}$$

and

$$u_{2n+2} = S_\beta(u_{2n+1},v_{2n+1}) \leq S_\beta(u_{2n+2},v_{2n+2}) = u_{2n+3},$$
$$v_{2n+2} = S_\beta(v_{2n+1},u_{2n+1}) \geq S_\beta(v_{2n+2},u_{2n+2}) = v_{2n+3}.$$

Hence,

$$u_0 \leq u_1 \leq \cdots \leq u_{2n} \leq u_{2n+1} \leq \cdots$$

and

$$v_0 \geq v_1 \geq \cdots \geq v_{2n} \geq v_{2n+1} \geq \cdots .$$

Now, using equation (3.14) for $u = u_{2n}$, $v = v_{2n}$, $r = u_{2n+1}$, $s = v_{2n+1}$ and using PMS condition (4), we have

$$\begin{aligned}
p(u_{2n+1}, u_{2n+2}) &= p\big(S_\alpha(u_{2n}, v_{2n}), S_\beta(u_{2n+1}, v_{2n+1})\big) \\
&\leq h_1\, p(u_{2n}, u_{2n+1}) + h_2\, [p(u_{2n}, S_\alpha(u_{2n}, v_{2n})) \\
&\quad + p(u_{2n+1}, S_\beta(u_{2n+1}, v_{2n+1}))] \\
&\quad + h_3\, [p(u_{2n}, S_\beta(u_{2n+1}, v_{2n+1})) + p(u_{2n+1}, S_\alpha(u_{2n}, v_{2n}))] \\
&= h_1\, p(u_{2n}, u_{2n+1}) + h_2\, [p(u_{2n}, u_{2n+1}) + p(u_{2n+1}, u_{2n+2})] \\
&\quad + h_3\, [p(u_{2n}, u_{2n+2}) + p(u_{2n+1}, u_{2n+1})] \\
&\leq h_1\, p(u_{2n}, u_{2n+1}) + h_2\, [p(u_{2n}, u_{2n+1}) + p(u_{2n+1}, u_{2n+2})] \\
&\quad + h_3\, [p(u_{2n}, u_{2n+1}) + p(u_{2n+1}, u_{2n+2}) - p(u_{2n+1}, u_{2n+1}) \\
&\quad + p(u_{2n+1}, u_{2n+1})] \\
&\leq h_1\, p(u_{2n}, u_{2n+1}) + h_2\, [p(u_{2n}, u_{2n+1}) + p(u_{2n+1}, u_{2n+2})] \\
&\quad + h_3\, [p(u_{2n}, u_{2n+1}) + p(u_{2n+1}, u_{2n+2})] \\
&= (h_1 + h_2 + h_3) p(u_{2n}, u_{2n+1}) + (h_2 + h_3) p(u_{2n+1}, u_{2n+2}).
\end{aligned}$$

This implies that

$$\begin{aligned}
p(u_{2n+1}, u_{2n+2}) &\leq \Big(\frac{h_1 + h_2 + h_3}{1 - h_2 - h_3}\Big) p(u_{2n}, u_{2n+1}) \\
&= \sigma\, p(u_{2n}, u_{2n+1}),
\end{aligned} \tag{3.15}$$

similarly by using inequality (3.14) again, we obtain

$$p(v_{2n+1}, v_{2n+2}) \leq \sigma\, p(v_{2n}, v_{2n+1}), \tag{3.16}$$

where $\sigma = \Big(\frac{h_1+h_2+h_3}{1-h_2-h_3}\Big) < 1$, since by assumption $h_1 + 2h_2 + 2h_3 < 1$.

Now, by adding equations (3.15) and (3.16), we have

$$p(u_{2n+1}, u_{2n+2}) + p(v_{2n+1}, v_{2n+2}) \leq \sigma\, [p(u_{2n}, u_{2n+1}) + p(v_{2n}, v_{2n+1})]. \tag{3.17}$$

Likewise by equation (3.14), we obtain

$$\begin{aligned}
p(u_{2n}, u_{2n+1}) &= p\big(S_\alpha(u_{2n-1}, v_{2n-1}), S_\beta(u_{2n}, v_{2n})\big) \\
&\leq h_1\, p(u_{2n-1}, u_{2n}) + h_2\, [p(u_{2n-1}, S_\alpha(u_{2n-1}, v_{2n-1})) \\
&\quad + p(u_{2n}, S_\beta(u_{2n}, v_{2n}))] \\
&\quad + h_3\, [p(u_{2n-1}, S_\beta(u_{2n}, v_{2n})) + p(u_{2n}, S_\alpha(u_{2n-1}, v_{2n-1}))] \\
&= h_1\, p(u_{2n-1}, u_{2n}) + h_2\, [p(u_{2n-1}, u_{2n}) + p(u_{2n}, u_{2n+1})] \\
&\quad + h_3\, [p(u_{2n-1}, u_{2n+1}) + p(u_{2n}, u_{2n})] \\
&\leq h_1\, p(u_{2n-1}, u_{2n}) + h_2\, [p(u_{2n-1}, u_{2n}) + p(u_{2n}, u_{2n+1})] \\
&\quad + h_3\, [p(u_{2n-1}, u_{2n}) + p(u_{2n}, u_{2n+1}) - p(u_{2n}, u_{2n}) \\
&\quad + p(u_{2n}, u_{2n})] \\
&\leq h_1\, p(u_{2n-1}, u_{2n}) + h_2\, [p(u_{2n-1}, u_{2n}) + p(u_{2n}, u_{2n+1})] \\
&\quad + h_3\, [p(u_{2n-1}, u_{2n}) + p(u_{2n}, u_{2n+1})] \\
&= (h_1 + h_2 + h_3) p(u_{2n-1}, u_{2n}) + (h_2 + h_3) p(u_{2n}, u_{2n+1}) \\
&\leq \Big(\frac{h_1 + h_2 + h_3}{1 - h_2 - h_3}\Big) p(u_{2n-1}, u_{2n}) \\
&= \sigma\, p(u_{2n-1}, u_{2n}). \qquad (3.18)
\end{aligned}$$

Again by using (3.14), it follows that

$$\begin{aligned}
p(v_{2n}, v_{2n+1}) &\leq \Big(\frac{h_1 + h_2 + h_3}{1 - h_2 - h_3}\Big) p(v_{2n-1}, v_{2n}) \\
&= \sigma\, p(v_{2n-1}, v_{2n}). \qquad (3.19)
\end{aligned}$$

From equations (3.18) and (3.19), we have

$$p(u_{2n}, u_{2n+1}) + p(v_{2n}, v_{2n+1}) \leq \sigma\, [p(u_{2n-1}, u_{2n}) + p(v_{2n-1}, v_{2n})]. \qquad (3.20)$$

Assume that

$$Q_n = p(u_{2n}, u_{2n+1}) + p(v_{2n}, v_{2n+1}).$$

Then from equation (3.20), we obtain

$$Q_n \leq \sigma\, Q_{n-1}. \qquad (3.21)$$

Thus, using equation (3.21) repeatedly, we obtain

$$0 \leq Q_n \leq \sigma\, Q_{n-1} \leq \sigma^2\, Q_{n-2} \leq \cdots \leq \sigma^n\, Q_0. \qquad (3.22)$$

If $Q_0 = 0$, then (u_0, v_0) is a coupled fixed point of S. Suppose that $Q_0 > 0$. Then for each $m \in \mathbb{N}$, we obtain by using condition (4) of PMS repeatedly and equation (3.22) that

$$\begin{aligned}
p(u_{2n}, u_{2n+m}) + p(v_{2n}, v_{2n+m}) \leq\ & [p(u_{2n}, u_{2n+1}) + p(v_{2n}, v_{2n+1})] \\
& + [p(u_{2n+1}, u_{2n+2}) + p(v_{2n+1}, v_{2n+2})] \\
& + \cdots + [p(u_{2n+m-1}, u_{2n+m}) \\
& + p(v_{2nm-1}, v_{2n+m})] \\
& - [p(u_{2n+1}, u_{2n+1}) + p(v_{2n+1}, v_{2n+1})] \\
& - [p(u_{2n+2}, u_{2n+2}) + p(v_{2n+2}, v_{2n+2})] \\
& - \cdots - [p(u_{2n+m-1}, u_{2n+m-1}) \\
& + p(v_{2n+m-1}, v_{2n+m-1})] \\
\leq\ & [p(u_{2n}, u_{2n+1}) + p(v_{2n}, v_{2n+1})] \\
& + [p(u_{2n+1}, u_{2n+2}) \\
& + p(v_{2n+1}, v_{2n+2})] + \cdots + [p(u_{2n+m-1}, u_{2n+m}) \\
& + p(v_{2n+m-1}, v_{2n+m})] \\
=\ & Q_n + Q_{n+1} + \cdots + Q_{n+m-1} \\
\leq\ & \sigma^n Q_0 + \sigma^{n+1} Q_0 + \cdots + \sigma^{n+m-1} Q_0 \\
=\ & \sigma^n (1 + \sigma + \sigma^2 + \cdots + \sigma^{m-1}) Q_0 \\
\leq\ & \Big(\frac{\sigma^n (1 - \sigma^{m-1})}{1 - \sigma} \Big) Q_0 \\
\leq\ & \Big(\frac{\sigma^n}{1 - \sigma} \Big) Q_0 \\
\to\ & 0 \text{ as } n \to \infty.
\end{aligned} \tag{3.23}$$

Hence, $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences in (Ω, p) such that

$$\lim_{m,n \to \infty} p(u_m, u_n) = 0$$

and

$$\lim_{m,n \to \infty} p(v_m, v_n) = 0$$

with $m > n \in \mathbb{N}$. Again, since (Ω, p) is a complete partial metric space, there exist $u^*, v^* \in \Omega$ such that (by Lemma 2.10(2))

$$p(u^*, u^*) = \lim_{n \to \infty} p(u_n, u^*) = \lim_{n \to \infty} p(u_n, u_n) = 0 \tag{3.24}$$

and

$$p(v^*, v^*) = \lim_{n \to \infty} p(v_n, v^*) = \lim_{n \to \infty} p(v_n, v_n) = 0. \tag{3.25}$$

Now, we first show that (u^*, v^*) is a coupled fixed point of S_β. By using contractive condition (3.14), we have

$$\begin{aligned} p(u^*, S_\beta(u^*, v^*)) &\le p(u^*, u_{2n+1}) + p(u_{2n+1}, S_\beta(u^*, v^*)) - p(u_{2n+1}, u_{2n+1}) \\ &\le p(u^*, u_{2n+1}) + p(u_{2n+1}, S_\beta(u^*, v^*)) \\ &= p(u^*, u_{2n+1}) + p(S_\alpha(u_{2n}, v_{2n}), S_\beta(u^*, v^*)) \\ &\le p(u^*, u_{2n+1}) + h_1\, p(u_{2n}, u^*) + h_2\, [p(u_{2n}, S_\alpha(u_{2n}, v_{2n})) \\ &\quad + p(u^*, S_\beta(u^*, v^*))] + h_3\, [p(u_{2n}, S_\beta(u^*, v^*)) \\ &\quad + p(u^*, S_\alpha(u_{2n}, v_{2n}))] \\ &= p(u^*, u_{2n+1}) + h_1\, p(u_{2n}, u^*) + h_2\, [p(u_{2n}, u_{2n+1}) \\ &\quad + p(u^*, S_\beta(u^*, v^*))] + h_3\, [p(u_{2n}, S_\beta(u^*, v^*)) \\ &\quad + p(u^*, u_{2n+1})]. \end{aligned} \tag{3.26}$$

Letting $n \to \infty$ in inequality (3.26) and using equation (3.24), we get

$$p(u^*, S_\beta(u^*, v^*)) \le (h_2 + h_3) p(u^*, S_\beta(u^*, v^*)),$$

which is a contradiction, since $h_2 + h_3 < 1$. Hence, we conclude that

$$p(u^*, S_\beta(u^*, v^*)) = 0.$$

Thus, by Lemma 2.11(1), we have $S_\beta(u^*, v^*) = u^*$.

Likewise

$$\begin{aligned} p(v^*, S_\beta(v^*, u^*)) &\le p(v^*, v_{2n+1}) + p(v_{2n+1}, S_\beta(v^*, u^*)) - p(v_{2n+1}, v_{2n+1}) \\ &\le p(v^*, v_{2n+1}) + p(v_{2n+1}, S_\beta(v^*, u^*)) \\ &= p(v^*, v_{2n+1}) + p(S_\alpha(v_{2n}, u_{2n}), S_\beta(v^*, u^*)) \\ &\le p(v^*, v_{2n+1}) + h_1\, p(v_{2n}, v^*) + h_2\, [p(v_{2n}, S_\alpha(v_{2n}, u_{2n})) \\ &\quad + p(v^*, S_\beta(v^*, u^*))] + h_3\, [p(v_{2n}, S_\beta(v^*, u^*)) \\ &\quad + p(v^*, S_\alpha(v_{2n}, u_{2n}))] \\ &= p(v^*, v_{2n+1}) + h_1\, p(v_{2n}, v^*) + h_2\, [p(v_{2n}, v_{2n+1}) \\ &\quad + p(v^*, S_\beta(v^*, u^*))] + h_3\, [p(v_{2n}, S_\beta(v^*, u^*)) \\ &\quad + p(v^*, v_{2n+1})]. \end{aligned} \tag{3.27}$$

Letting $n \to \infty$ in inequality (3.27) and using equation (3.25), we get

$$p(v^*, S_\beta(v^*, u^*)) \le (h_2 + h_3) p(v^*, S_\beta(v^*, u^*)),$$

which is a contradiction, since $h_2 + h_3 < 1$. Hence, we conclude that

$$p(v^*, S_\beta(v^*, u^*)) = 0.$$

Thus, by Lemma 2.11(1), we have $S_\beta(v^*, u^*) = v^*$. Therefore, (u^*, v^*) is a coupled fixed point of S_β.

Now, we have to show that (u^*, v^*) is also a coupled fixed point of S_α.
Let $\alpha \in \Delta$ be arbitrary. Then by equation (3.14), we have

$$\begin{aligned} p(S_\alpha(u^*, v^*), u^*) &= p(S_\alpha(u^*, v^*), S_\beta(u^*, v^*)) \\ &\le h_1\, p(u^*, u^*) + h_2\, [p(u^*, S_\alpha(u^*, v^*)) + p(u^*, S_\beta(u^*, v^*))] \\ &\quad + h_3\, [p(u^*, S_\beta(u^*, v^*)) + p(u^*, S_\alpha(u^*, v^*))] \\ &= h_1\, p(u^*, u^*) + h_2\, [p(u^*, S_\alpha(u^*, v^*)) + p(u^*, u^*)] \\ &\quad + h_3\, [p(u^*, u^*) + p(u^*, S_\alpha(u^*, v^*))] \\ &= (h_2 + h_3) p(S_\alpha(u^*, v^*), u^*) < p(S_\alpha(u^*, v^*), u^*), \end{aligned}$$

which is a contradiction. Hence,

$$p(S_\alpha(u^*, v^*), u^*) = 0$$

and so by Lemma 2.11(1), we get

$$S_\alpha(u^*, v^*) = u^*.$$

By a similar fashion using equation (3.14) one can show that

$$S_\alpha(v^*, u^*) = v^*.$$

Thus, (u^*, v^*) is also a coupled fixed point of S_α. Hence, (u^*, v^*) is a common coupled fixed point of S_α and S_β. This completes the proof. □

Remark 3.7. Theorem 3.1 and 3.6 extend and generalise the corresponding results of Tijani et al. [41] from partially ordered complete metric spaces (POCMS) to the setting of partially ordered complete partial metric spaces (POCPMS).

Remark 3.8. Our results also generalize the corresponding results of Kim et al. [24] without the assumption of w-compatibility condition in partially ordered complete partial metric space setting.

4. Applications

In this section, we provide some applications of the established results in terms of integral type contractions.

Let Φ denote the set of functions $\zeta\colon [0, +\infty) \to [0, +\infty)$ satisfying the following properties:

(Φ_1) ζ is a Lebesgue-integrable function on every compact subset in $[0, +\infty)$,
(Φ_2) $\int_0^\varepsilon \zeta(t)dt > 0$ for all $\varepsilon > 0$.

Theorem 4.1. *Let $(\Omega, p, \le)$ be a partially ordered complete partial metric space. Let $F, G\colon \Omega \times \Omega \to \Omega$ be mappings having the mixed monotone property* *such that for some constants* $\gamma_1, \gamma_2 \geq 0$*, for all* $u, v, r, s \in \Omega$*,* $p(r, F(r, s)) + p(u, r) > 0$*,* ψ*, a (c)-comparison function and* $\zeta \in \Phi$*, we have*

$$\int_0^{p(F(u,v),G(r,s))} \zeta(t)dt \leq \gamma_1 \int_0^{\left(\frac{p(u,F(u,v))p(u,G(r,s))p(r,F(u,v))}{1+p(r,F(r,s))+p(u,r)}\right)} \zeta(t)dt + \gamma_2 \int_0^{\left(\frac{p(r,F(u,v))[1+p(r,G(r,s))]}{1+p(u,r)}\right)} \zeta(t)dt + \int_0^{\psi(p(u,r))} \zeta(t)dt. \tag{4.1}$$

Suppose that we endow the product space $\Omega \times \Omega$ *with the following partial order:*

$$\text{for } (u, v), (r, s) \in \Omega \times \Omega,\ (r, s) \leq (u, v) \Leftrightarrow u \geq r,\ v \leq s.$$

If there exist two elements $u_0, v_0 \in \Omega$ *such that* $u_0 \leq F(u_0, v_0) \leq G(u_0, v_0)$ *and* $v_0 \geq F(v_0, u_0) \geq G(v_0, u_0)$*, then* F *and* G *have a common coupled fixed point and* $p(a, a) = 0$ *for some* $a \in \Omega$*.*

If we set $\psi(t) = \gamma_3 t$ for all $t > 0$, where $\gamma_3 \in [0, 1)$ in Theorem 4.1, then we obtain the following result.

Theorem 4.2. *Let* $(\Omega, p, \leq)$ *be a partially ordered complete partial metric space. Let* $F, G \colon \Omega \times \Omega \to \Omega$ *be mappings having the mixed monotone property such that for some constants* $\gamma_1, \gamma_2 \geq 0$*,* $\gamma_3 \in [0, 1)$ *for all* $u, v, r, s \in \Omega$*, where* $p(r, F(r, s)) + p(u, r) > 0$ *and* $\zeta \in \Phi$*, we have*

$$\int_0^{p(F(u,v),G(r,s))} \zeta(t)dt \leq \gamma_1 \int_0^{\left(\frac{p(u,F(u,v))p(u,G(r,s))p(r,F(u,v))}{1+p(r,F(r,s))+p(u,r)}\right)} \zeta(t)dt + \gamma_2 \int_0^{\left(\frac{p(r,F(u,v))[1+p(r,G(r,s))]}{1+p(u,r)}\right)} \zeta(t)dt + \gamma_3 \int_0^{p(u,r)} \zeta(t)dt. \tag{4.2}$$

Suppose that we endow the product space $\Omega \times \Omega$ *with the following partial order:*

$$\text{for } (u, v), (r, s) \in \Omega \times \Omega,\ (r, s) \leq (u, v) \Leftrightarrow u \geq r,\ v \leq s.$$

If there exist two elements $u_0, v_0 \in \Omega$ *such that* $u_0 \leq F(u_0, v_0) \leq G(u_0, v_0)$ *and* $v_0 \geq F(v_0, u_0) \geq G(v_0, u_0)$*, then* F *and* G *have a common coupled fixed point and* $p(a, a) = 0$ *for some* $a \in \Omega$*.*

If we take $\gamma_3 = 0$ in Theorem 4.2, then we obtain the following result.

Theorem 4.3. *Let* $(\Omega, p, \leq)$ *be a partially ordered complete partial metric space. Let* $F, G \colon \Omega \times \Omega \to \Omega$ *be mappings having the mixed monotone property such that for some constants* $\gamma_1, \gamma_2 \geq 0$ *for all* $u, v, r, s \in \Omega$*, where*

$p(r,F(r,s))+p(u,r)>0$ and $\zeta\in\Phi$, we have

$$\int_0^{p(F(u,v),G(r,s))}\zeta(t)dt \le \gamma_1 \int_0^{\left(\frac{p(u,F(u,v))p(u,G(r,s))p(r,F(u,v))}{1+p(r,F(r,s))+p(u,r)}\right)}\zeta(t)dt + \gamma_2\int_0^{\left(\frac{p(r,F(u,v))[1+p(r,G(r,s))]}{1+p(u,r)}\right)}\zeta(t)dt. \tag{4.3}$$

Suppose that we endow the product space $\Omega\times\Omega$ with the following partial order:

$$\text{for } (u,v),(r,s)\in\Omega\times\Omega,\ (r,s)\le(u,v) \Leftrightarrow u\ge r,\ v\le s.$$

If there exist two elements $u_0,v_0\in\Omega$ such that $u_0\le F(u_0,v_0)\le G(u_0,v_0)$ and $v_0\ge F(v_0,u_0)\ge G(v_0,u_0)$, then F and G have a common coupled fixed point and $p(a,a)=0$ for some $a\in\Omega$.

If we take $\gamma_1=\gamma_2=0$ and $\gamma_3=\mu$, where $\mu\in[0,1)$ in Theorem 4.2, then we have the following result.

Theorem 4.4. *Let $(\Omega,p,\le)$ be a partially ordered complete partial metric space. Let $F,G\colon\Omega\times\Omega\to\Omega$ be mappings having the mixed monotone property such that for all $u,v,r,s\in\Omega$, we have*

$$\int_0^{p(F(u,v),G(r,s))}\zeta(t)dt \le \mu\int_0^{p(u,r)}\zeta(t)dt, \tag{4.4}$$

where $\zeta\in\Phi$ and $\mu\in[0,1)$ is a constant. Suppose that we endow the product space $\Omega\times\Omega$ with the following partial order:

$$\text{for } (u,v),(r,s)\in\Omega\times\Omega,\ (r,s)\le(u,v) \Leftrightarrow u\ge r,\ v\le s.$$

If there exist two elements $u_0,v_0\in\Omega$ such that $u_0\le F(u_0,v_0)\le G(u_0,v_0)$ and $v_0\ge F(v_0,u_0)\ge G(v_0,u_0)$, then F and G have a common coupled fixed point.

Now, we give the application of Theorem 3.6 in terms of integral type contraction is as follows.

Theorem 4.5. *Let $(\Omega,p,\le)$ be a partially ordered complete partial metric space (POCPMS). Suppose that $\{S_\alpha\}_{\alpha\in\Delta}$ with $S_\alpha\colon\Omega\times\Omega\to\Omega$ is a family of mappings having the mixed monotone property and Δ is an index set. If there exists a $\beta\in\Delta$ such that for each $\alpha\in\Delta$,*

$$\int_0^{p(S_\alpha(u,v),S_\beta(r,s))}\zeta(t)dt \le h_1\int_0^{p(u,r)}\zeta(t)dt + h_2\int_0^{[p(u,S_\alpha(u,v))+p(r,S_\beta(r,s))]}\zeta(t)dt + h_3\int_0^{[p(u,S_\beta(r,s))+p(r,S_\alpha(u,v))]}\zeta(t)dt \tag{4.5}$$

for all $u, v, r, s \in \Omega$, where $h_1, h_2, h_3 \geq 0$ are constants such that $h_1 + 2h_2 + 2h_3 < 1$ and $\zeta \in \Phi$. Suppose that we endow the product space $\Omega \times \Omega$ with the following partial order:

$$\text{for } (u, v), (r, s) \in \Omega \times \Omega,\ (r, s) \leq (u, v) \Leftrightarrow u \geq r,\ v \leq s.$$

If for all $\alpha \in \Delta$, there exist $u_0, v_0 \in \Omega$ such that $u_0 \leq S_\alpha(u_0, v_0) \leq S_\beta(u_0, v_0)$ and $v_0 \geq S_\alpha(v_0, u_0) \geq S_\beta(v_0, u_0)$, then all S_α have a common coupled fixed point and $p(a, a) = 0$ for some $a \in \Omega$.

5. Conclusion

In this paper, we prove some common coupled fixed point results for contractive type conditions without the assumption of compatibility condition in the setting of partially ordered complete partial metric spaces with mixed monotone property. Moreover, we give some consequences of the established results and an illustrative example in support of the established result is provided. Also, some applications of the established results in terms of integral type contractions are included. Our results of this paper extend and generalize several results in the existing literature (see, e.g., [41] and others).

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