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BOUNDARY VALUES OF HANKEL AND TOEPLITZ DETERMINANTS INVOLVING (p, q)-DIFFERENTIAL OPERATOR

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Abstract. In this article, we introduce new subclasses of the generalized binomial series using (p, q)-differential operator. Further we find the values of the Hankel and Toeplitz determinants associated with these classes.

1. Introduction

Let $\mathcal{A}$ denote the set of all complex-valued functions $\psi(u)$ in the form:

$$\psi(u) = u + \sum_{j=2}^{\infty} v_j u^j. \tag{1.1}$$

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In the open unit disk $\mathbb{U} = \{u \in \mathbb{C} : |u| < 1\}$, these functions are analytic. In the domain $\mathbb{U}$, a function $\psi(u)$ is regarded as univalent. The subset of univalent functions of class $\mathcal{A}$ is represented by $\mathcal{S}$.

The quantum calculus has numerous applications in the fields of special functions and many further spaces [6]. We utilize essential notations and (p, q)-calculus is handed by

$$[j]_{p,q} = \frac{p^j - q^j}{p - q}, \quad p > 0, q > 0.$$

In particular, we attain:

$$\lim_{p \to 1} [j]_{p,q} = [j]_q.$$

Obviously, the notation is symmetric, that is,

$$[j]_{p,q} = [j]_{q,p}.$$

Definition 1.1. ([18]) The (p, q)-differential operator $D_{p,q} : \mathcal{A} \to \mathcal{A}$, of a function $\psi \in \mathcal{A}$, is defined by

$$D_{p,q}\psi(u) = \frac{\psi(pu) - \psi(qu)}{u(p - q)} \ (p \neq q, u \in \mathbb{U}). \tag{1.2}$$

The following is the (p, q)-derivative of $\psi(u)$ that appears in (1.1).

$$D_{p,q}\psi(u) = 1 + \sum_{j=2}^{\infty} [j]_{p,q} v_j u^{j-1}. \tag{1.3}$$

In addition we noting that

$$\lim_{q \to 1^-} \lim_{p \to 1^-} D_{p,q}\psi(u) = \psi'(u),$$
$$(D_{p,q}\psi)(0) = \psi'(0)$$

provided that ψ is differentiable at 0.

We introduce two new classes of (p, q)-convex functions, $C_{p,q}(u)$ and $C_{p,q}(\eta, \lambda; u)$, utilizing the (p, q)-differential operator $D_{p,q}\psi(u)$.

Definition 1.2. For $0 < q < p \leq 1$, we call $\psi(u)$ in $C_{p,q}(u)$ if and only if

$$\frac{D_{p,q}(uD_{p,q}\psi(u))}{D_{p,q}\psi(u)} \prec \mathcal{N}_1(u), \tag{1.4}$$

where $\mathcal{N}_1(u) = \dfrac{1 + u}{1 - pqu}$ and $u \in \mathbb{U}$.

Definition 1.3. For $0 < q < p \leq 1$, $0 < \eta \leq 1$ and $0 < \lambda \leq 1$, we call $\psi(u)$ in $C_{p,q}(\eta, \lambda; u)$ if and only if

$$\frac{D_{p,q}(uD_{p,q}\psi(u))}{D_{p,q}\psi(u)} \prec \mathcal{N}_2(u), \tag{1.5}$$

where $\mathcal{N}_2(u) = \dfrac{1+\lambda u}{1-\eta\lambda u}$.

The function ψ's $H_l(m)$ Hankel determinant was provided by Noonan and Thomas [13]. Here, $l \geq 1, m \geq 1$, and $c_1 = 1$

$$H_l(m) = \begin{vmatrix} c_m & c_{m+1} & \cdots & c_{m+l-1} \\ c_{m+1} & c_{m+2} & \cdots & c_{m+l} \\ \vdots & \vdots & \cdots & \vdots \\ c_{m+l-1} & c_{m+l} & \cdots & c_{m+2l-2} \end{vmatrix}.$$

The Hankel determinant $H_l(m)$ reduces to the famous Fekete-Szegö functional when $l = 2$ and $m = 1$:

$$H_2(1) = \begin{vmatrix} 1 & c_2 \\ c_2 & c_3 \end{vmatrix} = |c_3 - c_2^2|.$$

This functionality is further expanded in more generalized ways, such as:

$$|c_3 - \sigma c_2^2|,$$

a real or complex number can be represented by σ.

According to the theory of singularity [7], the Hankel determinant is a valuable tool for examining power series with integer coefficients (see [9, 11]). Within a range of analytic function subclasses, numerous scholars have established upper bounds for $H_l(m)$ for different combinations of m and n (See [2, 3, 5, 12, 14, 19]).

Additionally, our research explores particular cases of Toeplitz determinants, including the second order $\mathcal{T}_2(2)$ and third-order $\mathcal{T}_3(1)$ inequalities, giving a thorough examination of these mathematical concepts.

The symmetric Toeplitz determinant $\mathcal{T}_l(m)$ has the following definition.

$$\mathcal{T}_l(m) = \begin{vmatrix} c_m & c_{m+1} & \cdots & c_{m+l-1} \\ c_{m+1} & c_{m+2} & \cdots & c_{m+l-2} \\ \vdots & \vdots & \cdots & \vdots \\ c_{m+l-1} & c_{m+l-2} & \cdots & c_m \end{vmatrix}.$$

Thomas and Abdul Halim[17] deduced that $l \geq 1, m \geq 1$, and $c_m = 1$, respectively.

The limits of the Toeplitz determinant $\mathcal{T}_l(m)$ for different families analytic functions have recently been the subject of some investigation (see, e.g., [1, 16]). Toeplitz of analytic functions introduced by means of the Borel distribution is examined in [20], where as quantum calculus is included in the study of the Toeplitz determinant carried out in [15].

The following lemmas must be presented in order to illustrate Hankel determinants bounds for the classes $C_{p,q}(u)$ and $C_{p,q}(\eta, \lambda; u)$.

The familiar class of Caratheodory functions

$$\beta(u) = 1 + \sum_{j=1}^{\infty} \beta_j u^j$$

with $\mathcal{R}(\beta(u)) > 0$ is pointed by $\mathcal{P}$.

Lemma 1.4. ([4]) *If the function $\beta(u) \in \mathcal{P}$, then*

$$|\beta_j| \leq 2, \ (j \geq 2).$$

Lemma 1.5. ([8]) *If the function $\beta(u) \in \mathcal{P}$, then*

$$2\beta_2 = \beta_1^2 + \mu(4 - \beta_1^2)$$

and

$$4\beta_3 = \beta_1^3 + 2(4 - \beta_1^2)\beta_1\mu - (4 - \beta_1^2)\beta_1\mu^2 + 2(4 - \beta_1^2)(1 - |\mu|^2)u$$

in which, for some μ and u, $|\mu| \leq 1$ and $|u| \leq 1$.

Lemma 1.6. ([10]) *If the function $\beta(u) \in \mathcal{P}$, then*

$$|\beta_2 - \sigma\beta_1^2| \leq 2\max(1, |2\sigma - 1|), \ (\sigma \in \mathbb{C}).$$

Next, we should calculate the maximum value of the second-order Hankel determinant $\mathcal{H}_2(2)$ by first looking at the first-type Hankel determinant $\mathcal{H}_2(1)$.

2. Hankel determinants findings

We will determine the boundaries of initial coefficients in the subsequent theorems.

Theorem 2.1. *If $\psi(u) \in C_{p,q}(u)$, with $\psi(u)$ as in (1.1) then*

$$\begin{aligned} \mathcal{H}_2(1) &= |v_3 - v_2^2| \\ &\leq \frac{1 + pq}{A(A-1)}, \end{aligned}$$

where $A = (p^2 + pq + q^2)$.

Proof. Taking into account the subordination criterion given in equation (1.4), we obtain

$$\frac{D_{p,q}(uD_{p,q}\psi(u))}{D_{p,q}\psi(u)} = \mathcal{N}_1(w(u)). \tag{2.1}$$

The function $\beta(u)$ will now be illustrated using the subsequent technique:

$$\begin{aligned}\beta(u) &= \frac{1+w(u)}{1-w(u)}\\ &= 1+\beta_1 u+\beta_2 u^2+\beta_3 u^3+\beta_4 u^4+\cdots .\end{aligned}$$

Obviously, $\beta \in \mathcal{P}$, it achieves

$$w(u)=\frac{\beta(u)-1}{\beta(u)+1}$$

and

$$\mathcal{N}_1(w(u))=\frac{2\beta(u)}{1+(1-pq)\beta(u)+pq}.$$

A calculation produces

$$\begin{aligned}&\frac{2\beta(u)}{1+(1-pq)\beta(u)+pq}\\ &\quad=1+\frac{(1+pq)\beta_1}{2}u+\left\{\frac{(1+pq)\beta_2}{2}-\frac{(1-p^2q^2)\beta_1^2}{4}\right\}u^2\\ &\qquad+\left\{\frac{(1+pq)\beta_3}{2}-\frac{(1-p^2q^2)\beta_1\beta_2}{2}+\frac{(1+pq)(1-pq)^2\beta_1^3}{8}\right\}u^3+\cdots .\end{aligned} \tag{2.2}$$

From the second side of equation (2.1), we derive the following.

$$\begin{aligned}\frac{D_{p,q}(uD_{p,q}\psi(u))}{D_{p,q}\psi(u)} &= 1+(p+q)Bv_2u\\ &\quad+\{A[A-1]v_3-(p+q)^2[(p+q)-1]v_2^2\}u^2\\ &\quad+\{(p+q)^3[(p+q)-1]v_2^3\\ &\quad-A(p+q)[(p+q)+A-2]v_2v_3\\ &\quad+(p^2+q^2)(p+q)[(p^2+q^2)(p+q)-1]v_4\}u^3+\cdots ,\end{aligned} \tag{2.3}$$

where $B=(p+q-1)$.

Compared to that, we deduce

$$v_2=\frac{1+pq}{2(p+q)B}\beta_1, \tag{2.4}$$

$$v_3=\frac{1+pq}{2A(A-1)}\left(\beta_2+\frac{(1+pq)-(1-pq)B}{2B}\beta_1^2\right) \tag{2.5}$$

and

$$
\begin{aligned}
v_4 &= \left(\frac{1+pq}{2(p^2+q^2)(p+q)((p^2+q^2)(p+q)-1)}\right) \\
&\quad \times \left(\beta_3 - \frac{\Upsilon}{2B(A-1)}\beta_1\beta(1-pq)^2B^2(A-1) - (1+pq)^2(A-1)\right. \\
&\quad \left. + \frac{[(p+q)+A-2][(1+pq)^2-(1-p^2q^2)B]}{4B^2(A-1)}\beta_1^3\right),
\end{aligned} \tag{2.6}
$$

where

$$\Upsilon = 2(1-pq)B(A-1) - (1+pq)[(p+q)+(A-2)].$$

Here, the equations of v_2 and v_3 can be used to find the functional $|v_3 - v_2^2|$.

$$
|v_3 - v_2^2| = \frac{(1+pq)}{2A(A-1)}\left(\beta_2 - \left(\frac{\begin{array}{l}(1+pq)A(A-1) \\ -[(1+pq)(p+q)^2B \\ -(p+q)^2(1-pq)B^2]\end{array}}{2(p+q)^2B^2}\right)\beta_1^2\right). \tag{2.7}
$$

Considering Lemma 1.5 together with $\beta_1 \leq 2$, we obtain

$$
|v_3 - v_2^2| = \frac{(1+pq)}{2A(A-1)}\left(\frac{\mu(4-\beta_1^2)}{2} - \left(\frac{\begin{array}{l}(1+pq)A(A-1) \\ -[(1+pq)(p+q)^2B \\ -(p+q)^2(1-pq)B^2] \\ +(p+q)^2B^2\end{array}}{2(p+q)^2B^2}\right)\beta_1^2\right). \tag{2.8}
$$

Setting $\beta_1 = \beta$ ($\beta \in [0,2]$) and $|\mu| = \theta$, it achieves that

$$|v_3 - v_2^2| = \frac{(1+pq)}{2A(A-1)} \left(\frac{\theta(4-\beta^2)}{2} + \left(\frac{\begin{array}{l}(1+pq)A(A-1) \\ -[(1+pq)(p+q)^2 B \\ -(p+q)^2(1-pq)B^2] \\ +(p+q)^2B^2\end{array}}{2(p+q)^2B^2} \right) \beta^2 \right) \tag{2.9}$$
$$= X(\beta, \theta).$$

Through partial differentiation of the function $X(\beta,\theta)$ with respect to θ, we arrive to the following result

$$\frac{\delta X(\beta,\theta)}{\delta\theta} > 0.$$

Consequently, when θ falls within the interval $[0,1]$, the function $X(\beta,\theta)$ becomes an increasing function of θ. Therefore, the maximum value of $X(\beta,\theta)$ at $\theta = 1$ satisfies the equation.

$$\max\{X(\beta,\theta)\} = X(\beta,1) = K_1(\beta)$$

with

$$K_1(\beta) = \frac{(1+pq)}{2A(A-1)} \left(2 + \left(\frac{\begin{array}{l}(1+pq)A(A-1) \\ -[(1+pq)(p+q)^2 B \\ -(p+q)^2(1-pq)B^2]\end{array}}{2(p+q)^2B^2} \right) \beta^2 \right).$$

Given that $K_1(\beta)$ obviously permits for a maximal record at $\beta = 0$, it follows that

$$|v_3 - v_2^2| \le K_1(\beta) = \frac{(1+pq)}{A(A-1)}.$$

□

Theorem 2.2. *If $\psi(u) \in C_{p,q}(\eta, \lambda; u)$, with $\psi(u)$ as in* (1.1), *then*

$$\begin{aligned}\mathcal{H}_2(1) &= |v_3 - v_2^2| \\ &\leq \frac{\lambda(1+\eta)}{A(A-1)},\end{aligned}$$

where $A = (p^2 + pq + q^2)$.

Proof. Considering the subordination requirement stated in equation (1.5) we achieve.

$$\frac{D_{p,q}(uD_{p,q}\psi(u))}{D_{p,q}\psi(u)} = \mathcal{N}_2(w(u)). \tag{2.10}$$

When $\mathcal{N}_2(u)$ is simplified, we obtain

$$\mathcal{N}_2(u) = 1 + \lambda(1+\eta)u + \eta\lambda^2(1+\eta)u^2 + \eta^2\lambda^3(1+\eta)u^3 + \cdots . \tag{2.11}$$

We now use the following method to illustrate the behavior of the function:

$$\begin{aligned}\beta(u) &= \frac{1+w(u)}{1-w(u)} \\ &= 1 + \beta_1 u + \beta_2 u^2 + \beta_3 u^3 + \beta_4 u^4 + \cdots .\end{aligned}$$

Distinctly, $\beta \in \mathcal{P}$, then

$$\begin{aligned}w(u) &= \frac{\beta(u)-1}{\beta(u)+1} \\ &= \frac{1}{2}\beta_1 u + \frac{1}{2}\left(\beta_2 - \frac{1}{2}\beta_1^2\right)u^2 + \frac{1}{2}\left(\beta_3 - \beta_1\beta_2 + \frac{1}{4}\beta_1^3\right)u^3 + \cdots .\end{aligned}$$

Considering of $\mathcal{N}_2(u)$ and $\beta(u)$, we get

$$\begin{aligned}\mathcal{N}_2(w(u)) = 1 &+ \frac{\lambda(1+\eta)}{2}\beta_1 u + \left\{\frac{\lambda(1+\eta)}{2}\beta_2 + \frac{\lambda(1+\eta)(\lambda\eta - 1)}{4}\beta_1^2\right\}u^2 \\ &+ \frac{\lambda(1+\eta)}{2}\left\{\beta_3 + (\eta\lambda - 1)\beta_1\beta_2 + \frac{1 - 2\eta\lambda + \eta^2\lambda^2}{4}\beta_1^3\right\}u^3 + \cdots .\end{aligned} \tag{2.12}$$

Similarly, the equation (2.3) gives

$$\begin{aligned}\frac{D_{p,q}(uD_{p,q}\psi(u))}{D_{p,q}\psi(u)} &= 1 + (p+q)Bv_2 u \\ &+ \{A[A-1]v_3 - (p+q)^2[(p+q)-1]v_2^2\}u^2 \\ &+ \{(p+q)^3[(p+q)-1]v_2^3 - A(p+q)[(p+q)+A-2]v_2 v_3 \\ &+ (p^2+q^2)(p+q)[(p^2+q^2)(p+q)-1]v_4\}u^3 + \cdots ,\end{aligned}$$

where $B = (p + q - 1)$.

By substituting into the equation (2.10) the outcomes are

$$v_2 = \frac{\lambda(1+\eta)}{2(p+q)B}\beta_1, \tag{2.13}$$

$$v_3 = \frac{\lambda(1+\eta)}{2A(A-1)}\Big\{\beta_2 + \Big(\frac{\eta\lambda - 1}{2} + \frac{\lambda(1+\eta)}{2B}\Big)\beta_1^2\Big\} \tag{2.14}$$

and

$$\begin{aligned} v_4 = {} & \frac{\lambda(1+\eta)}{2(p^2+q^2)(p+q)[(p^2+q^2)B]} \\ & \times \left\{\beta_3 + \left(\eta\lambda - 1 + \frac{\lambda(1+\eta)}{2}\frac{[(p+q)+A-2]}{B(A-1)}\right)\beta_1\beta_2 + \Omega_1\beta_1^3\right\}, \end{aligned} \tag{2.15}$$

where

$$\begin{aligned} \Omega_1 = \Bigg\{ & \frac{1}{4} - \frac{\eta\lambda}{2} + \frac{\eta^2\lambda^2}{4} \\ & + \frac{\lambda(1+\eta)}{2}\frac{[(p+q)+A-2]}{B(A-1)}\left(\frac{\eta\lambda-1}{2} + \frac{\lambda(1+\eta)}{2B}\right) \\ & - \frac{\lambda^2(1+\eta)^2}{4B^2}\Bigg\}. \end{aligned} \tag{2.16}$$

The procedure of Theorem 2.1 is followed in order to verify the requested value of inequality $\mathcal{H}_2(1)$. $\square$

For the classes $C_{p,q}(u)$ and $C_{p,q}(\eta, \lambda; u)$ respectively, the following results establish the upper bound of Hankel determinant $\mathcal{H}_2(2)$.

Theorem 2.3. *If $\psi(u) \in C_{p,q}(u)$ with $\psi(u)$ as in* (1.1), *then*

$$\begin{aligned} \mathcal{H}_2(2) &= |v_2 v_4 - v_3^2| \\ &\leq \frac{(1+pq)^2}{A^2(A-1)^2}, \end{aligned}$$

where $A = (p^2 + pq + q^2)$.

Proof. Taking Theorem 2.1's coefficients $v_j (j = 2, 3, 4)$ into consideration, it follows that

$$
\begin{aligned}
v_2v_4 - v_3^2 = &\left(\frac{(1+pq)^2}{4(p^2+q^2)(p+q)[(p^2+q^2)(p+q)-1]A(A-1)}\right)\beta_1\beta_3 \\
&- \left(\frac{(1+pq)^2\Big[2(1-pq)B(A-1) - (1+pq)[(p+q) + (A-2)]\Big]}{8B^2(A-1)(p^2+q^2)(p+q) \times [(p^2+q^2)(p+q)-1]} + \frac{(1+pq)^2[(1+pq) - (1-pq)B]}{4BA^2(A-1)^2}\right)\beta_1^2\beta_2 \\
&- \left(\frac{(1+pq)^2}{4A^2(A-1)^2}\right)\beta_2^2 \\
&+ \left(\frac{(1+pq)^2\Big[(1-pq)^2B^2(A-1) - (A-1)(1+pq)^2 + [(p+q) + A-2][(1+pq)^2 - (1-p^2q^2)B]\Big]}{16B^3(A-1)(p^2+q^2)(p+q)^2 \times [(p^2+q^2)(p+q)-1]} - \frac{(1+pq)^2\Big[(1+pq) + (1-pq)^2B^2 - 2(1-p^2q^2)B\Big]}{16B^2A^2(A-1)^2}\right)\beta_1^4,
\end{aligned}
$$

where $B = (p + q - 1)$.

Utilizing Lemma 1.5 we attain

$$v_2v_4 - v_3^2 = \left(\frac{\begin{array}{l}(1+pq)^2\Big[(1-pq)^2B^2(A-1)\\ -(A-1)(1+pq)^2\\ +[(p+q)+A-2][(1+pq)^2\\ -(1-p^2q^2)B]\Big]\end{array}}{\begin{array}{c}16B^3(A-1)(p^2+q^2)\\ \times(p+q)^2[(p^2+q^2)\\ \times(p+q)-1]\end{array}} - \frac{\begin{array}{l}(1+pq)^2\Big[(1+pq)\\ +(1-pq)^2B^2\\ -2(1-p^2q^2)B\Big]\end{array}}{16B^2A^2(A-1)^2} \right)\beta_1^4$$

$$+ \left(\frac{(1+pq)^2}{\begin{array}{l}16(p^2+q^2)(p+q)[(p^2+q^2)(p+q)-1]\\ \times A(A-1)\end{array}} \right)$$

$$\times \beta_1\{\beta_1^3 + 2\beta_1(4-\beta_1^2)\mu - \beta_1(4-\beta_1^2)\mu^2 + 2(4-\beta_1^2)(1-|\mu|^2)u\}$$

$$- \left(\frac{\begin{array}{l}(1+pq)^2\Big[2(1-pq)\\ \times B(A-1)-(1+pq)\\ \times[(p+q)+(A-2)]\Big]\end{array}}{\begin{array}{l}16B^2(A-1)\\ \times(p^2+q^2)(p+q)\\ \times[(p^2+q^2)(p+q)-1]\end{array}} + \frac{\begin{array}{l}(1+pq)^2[(1+pq)\\ -(1-pq)B]\end{array}}{8BA^2(A-1)^2} \right)$$

$$\times \beta_1^2\{\beta_1^2 + \mu(4-\beta_1^2)\}$$

$$- \left(\frac{(1+pq)^2}{4A^2(A-1)^2} \right)\left\{\beta_1^4 + 2\mu(4-\beta_1^2)\beta_1^2 + (4-\beta_1^2)^2\mu^2\right\}.$$

Setting $\beta_1 = \beta$ and $|\mu| = \theta$, then

$$\begin{aligned}
|v_2 v_4 - v_3^2| \leq \frac{1}{\Phi^1_{p,q}} \Bigg[& \Phi^2_{p,q}\beta^4 + 2(p+q)B^3 A(A-1)\beta(4-\beta^2) + \Phi^3_{p,q}(4-\beta^2)\beta^2\theta \\
& + \Big[(p+q)B^3 A(A-1)\beta^2 + (p+q)^2 B^3 (p^2+q^2)((p^2+q^2) \\
& \times (p+q)-1)(4-\beta^2) - 2(p+q)B^3 A(A-1)\beta\Big](4-\beta^2)\theta^2 \Bigg] \\
= Y(\beta,\theta), &
\end{aligned}$$

where

$$\begin{aligned}
\Phi^1_{p,q} &= 16B^3 A^2 (A-1)^2 (p+q)^2 (p^2+q^2)((p^2+q^2)(p+q)-1), \\
\Phi^2_{p,q} &= \Big| A^2(A-1)\Big[(A-1)((1-pq)^2 B^2 - (1+pq)^2) \\
&\quad + ((p+q)+A-2)(1+pq)^2 - (1-p^2q^2)B)\Big] \\
&\quad + B(A-1)\Big[(p+q)B^2 - 2(1-pq)B(A-1) \\
&\quad + (1+pq)((p+q)+A-2)\Big] \\
&\quad - (p+q)^2 B(p^2+q^2)((p^2+q^2)(p+q)-1) \\
&\quad \times \Big[(1+pq) - (1-pq)^2 B^2 + 2(1-p^2q^2)B \\
&\quad + 2B(1+pq) + 2(1-pq)B^2 - 1\Big]\Big|
\end{aligned}$$

and

$$\begin{aligned}
\Phi^3_{p,q} &= \Big| B(A-1)\Big[2(p+q)B^2 A - 2(1-pq)B(A-1) \\
&\quad + (1+pq)((p+q)+A-2)\Big] \\
&\quad - 2(p+q)^2 B^2 (p^2+q^2)((p^2+q^2)(p+q)-1) \\
&\quad \times \Big[B + ((1+pq) - (1-pq)B)\Big]\Big|.
\end{aligned}$$

The partial differentiation of $Y(\beta,\theta)$ with respect to θ yields the following results:

$$\begin{aligned}
\frac{\delta Y(\beta,\theta)}{\delta\theta} = \frac{1}{\Phi^1_{p,q}} \Big[& \Phi^3_{p,q}(4-\beta^2)\beta^2 + 2\Big[(p+q)B^3 A(A-1)\beta^2 \\
& + (p+q)^2 B^3 (p^2+q^2)((p^2+q^2)(p+q)-1)(4-\beta^2) \\
& - 2(p+q)B^3 A(A-1)\beta\Big](4-\beta^2)\theta\Big] > 0.
\end{aligned}$$

Consequently, $Y(\beta, \theta)$ is an increasing function of $\theta(\theta \in [0,1])$, and we obtain

$$\max\{Y(\beta,\theta)\} = Y(\beta,1) = L(\beta),$$

where

$$\begin{aligned} L(\beta) = \frac{1}{\Phi^1_{p,q}} \Big[\Big(& \Phi^2_{p,q} - \Phi^3_{p,q} - (p+q)B^3A(A-1) \\ & + (p+q)^2B^3(p^2+q^2)((p^2+q^2)(p+q)-1)\Big)\beta^4 \\ & + \Big(4\Phi^3_{p,q} + 4(p+q)B^3A(A-1) \\ & - 8(p+q)^2B^3(p^2+q^2)((p^2+q^2)(p+q)-1)\Big)\beta^2 \\ & + 16(p+q)^2B^3(p^2+q^2)((p^2+q^2)(p+q)-1)\Big] \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} L'(\beta) = \frac{1}{\Phi^1_{p,q}} \Big[4\Big(& \Phi^2_{p,q} - \Phi^3_{p,q} - (p+q)B^3A(A-1) \\ & + (p+q)^2B^3(p^2+q^2)((p^2+q^2)(p+q)-1)\Big)\beta^3 \\ & + 2\Big(4\Phi^3_{p,q} + 4(p+q)B^3A(A-1) \\ & - 8(p+q)^2B^3(p^2+q^2)((p^2+q^2)(p+q)-1)\Big)\beta\Big]. \end{aligned} \tag{2.18}$$

Continue differentiating the function $L'_2(\beta)$ concerning to β, then

$$\begin{aligned} L''(\beta) = \frac{1}{\Phi^1_{p,q}} \Big[12\Big(& \Phi^2_{p,q} - \Phi^3_{p,q} - (p+q)B^3A(A-1) \\ & + (p+q)^2B^3(p^2+q^2)((p^2+q^2)(p+q)-1)\Big)\beta^2 \\ & + 2\Big(4\Phi^3_{p,q} + 4(p+q)B^3A(A-1) \\ & - 8(p+q)^2B^3(p^2+q^2)((p^2+q^2)(p+q)-1)\Big)\Big]. \end{aligned} \tag{2.19}$$

This shows that the maximum value of $L(\beta)$ occurs when $\beta = 0$. Thus, we get

$$|v_2v_4 - v_3^2| \leq \frac{(1+pq)^2}{A^2(A-1)^2}.$$

□

Theorem 2.4. *If $\psi(u) \in C_{p,q}(\eta, \lambda; u)$ with $\psi(u)$ as in (1.1), then*

$$\mathcal{H}_2(2) = |v_2v_4 - v_3^2| \leq \frac{\lambda^2(1+\eta)^2}{A^2(A-1)^2},$$

where $A = (p^2 + pq + q^2)$.

Proof. With the assistance of (2.13), (2.14) and (2.15), it will be obtained that

$$v_2v_4 - v_3^2 = \frac{\lambda^2(1+\eta)^2}{4}\Bigg\{\mathcal{X}_1\beta_1\beta_3 + \Bigg(\mathcal{X}_1\Bigg(\eta\lambda - 1 + \frac{\lambda(1+\eta)}{2}\frac{[(p+q)+A-2]}{B(A-1)}\Bigg) - 2\mathcal{X}_2\Omega_3\Bigg)\beta_1^2\beta_2 - \mathcal{X}_2\beta_2^2 + (\mathcal{X}_1\Omega_1 - \mathcal{X}_2\Omega_3^2)\beta_1^4\Bigg\}, \tag{2.20}$$

where $B = (p+q-1)$ and Ω_1 was concluded in (2.16),

$$\mathcal{X}_1 = \frac{1}{(p+q)^2B(p^2+q^2)((p^2+q^2)(p+q)-1)},$$

$$\mathcal{X}_2 = \frac{1}{A^2(A-1)^2}$$

and

$$\Omega_3 = \frac{\eta\lambda - 1}{2} + \frac{\lambda(1+\eta)}{2B}.$$

By employing a comparable method as demonstrated in Theorem 2.3, We achieve the desired result. □

Theorem 2.5. *If* $\psi(u) \in C_{p,q}(u)$*, with* $\psi(u)$ *as in* (1.1)*, then*

$$|v_2v_3 - v_4| \le \left(\frac{\Psi_{p,q}^1(1+pq)}{B^2A(A-1)(p^2+q^2)(p+q)[(p^2+q^2)(p+q)-1]}\right),$$

where

$$\begin{aligned}\Psi_{p,q}^1 = \Big|&(1+pq)(p^2+q^2)[(p^2+q^2)(p+q)-1][1+p^2q+pq^2] \\ &- A(A-1)[(1-pq)^2 - (1+pq)^2 + B^2] \\ &- A[(p+q)+A-2][((1+pq)^2 - (1-p^2q^2)B) + B] \\ &+ 2(1-pq)B(A-1)\Big|,\end{aligned} \tag{2.21}$$

where $A = (p^2 + pq + q^2)$ *and* $B = (p + q - 1)$.

Proof. Simplifying from (2.4) to (2.6) of Theorem 2.1, we procure

$$v_2v_3 - v_4 = \left(\frac{(1+pq)^2[(1+pq) - (1-pq)B]}{8(p+q)B^2A(A-1)} - \frac{(1+pq)\Big[(1-pq)^2B^2(A-1) - (1+pq)^2(A-1) + [(p+q) + A-2][(1+pq)^2 - (1-p^2q^2)B]\Big]}{8B^2(A-1)(p^2+q^2)(p+q) \times [(p^2+q^2)(p+q)-1]} \right)\beta_1^3$$
$$- \frac{1+pq}{2(p^2+q^2)(p+q)[(p^2+q^2)(p+q)-1]}\beta_3$$
$$+ \left(\frac{(1+pq)^2}{4(p+q)BA(A-1)} - \frac{(1+pq)[(p+q)+A-2] + 2(1-p^2q^2)B(A-1)}{4B(A-1)(p^2+q^2)(p+q) \times [(p^2+q^2)(p+q)-1]} \right)\beta_1\beta_2. \tag{2.22}$$

Employing Lemma 1.5, we get

$$v_2v_3 - v_4 = \left(\frac{(1+pq)^2[(1+pq) - (1-pq)B]}{8(p+q)B^2A(A-1)} - \frac{(1+pq)\Big[(1-pq)^2B^2(A-1) - (1+pq)^2(A-1) + [(p+q) + A - 2][(1+pq)^2 - (1-p^2q^2)B]\Big]}{8B^2(A-1)(p^2+q^2) \times (p+q)[(p^2+q^2)(p+q)-1]} \right)\beta_1^3$$
$$- \frac{1+pq}{8(p^2+q^2)(p+q)[(p^2+q^2)(p+q)-1]}$$
$$\times [\beta_1^3 + 2\beta_1(4-\beta_1^2)\mu - \beta_1(4-\beta_1^2)\mu^2 + 2(4-\beta_1^2)(1-|\mu|^2)u]$$
$$+ \left(\frac{(1+pq)^2}{8(p+q)BA(A-1)} - \frac{(1+pq)[(p+q)+A-2] + 2(1-p^2q^2)B(A-1)}{8B(A-1) \times (p^2+q^2)(p+q) \times [(p^2+q^2)(p+q)-1]} \right)$$
$$\times \beta_1\{\beta_1^2 + \mu(4-\beta_1^2)\}.$$

Afterwards, assuming $\beta_1 = \beta$ and using $|\mu| = \theta$, we attain

$$|v_2 v_3 - v_4| \leq M(\beta, \theta),$$

where

$$M(\beta,\theta) = \left(\frac{1+pq}{8B^2A(A-1)(p^2+q^2)(p+q) \times [(p^2+q^2)(p+q)-1]}\right)\Big(\Psi^1_{p,q}\beta^3 + \Psi^2_{p,q}\beta(4-\beta^2)\theta + 2B^2A(A-1)(4-\beta^2) + B^2A(A-1)(\beta-2)(4-\beta^2)\theta^2\Big)$$

for

$$\Psi^2_{p,q} = \Big|2B^2A(A-1) - \Big[(1+pq)B(p^2+q^2)[(p^2+q^2)(p+q)-1] - BA\Big([(p+q)A-2] + 2(1-pq)B(A-1)\Big)\Big]\Big|$$

and $\Psi^1_{p,q}$ known in (2.21).

When the function $M(\beta,\theta)$ is differentiated with respect to θ, we obtain

$$M'(\beta,\theta) = \left(\frac{1+pq}{8B^2A(A-1)(p^2+q^2) \times (p+q)[(p^2+q^2)(p+q)-1]}\right) \times \Big(\Psi^2_{p,q}\beta(4-\beta^2) + 2B^2A(A-1)(\beta-2)(4-\beta^2)\theta\Big) > 0.$$

This leads to the function $M(\beta,\theta)$ being an increasing function of θ ($\theta \in [0,1]$), and acquire

$$M(\beta,\theta) \leq M(\beta,1).$$

Subsequently,

$$\max\{M(\beta,\theta) = M(\beta,1) \leq N(\beta),$$

where

$$N(\beta) = \left(\frac{1 + pq}{8B^2A(A-1)(p^2+q^2) \times (p+q)[(p^2+q^2)(p+q)-1]} \right) \times \left[\left(\Psi^1_{p,q} + \Psi^2_{p,q} - B^2A(A-1) \right)\beta^3 + \left(4\Psi^2_{p,q} + 4B^2A(A-1) \right)\beta \right].$$

Given that $0 \leq \beta \leq 2$, where $\beta = 2$ is the maximal point, therefore

$$N(\beta) \leq \left(\frac{\Psi^1_{p,q}(1+pq)}{B^2A(A-1)(p^2+q^2)(p+q)[(p^2+q^2)(p+q)-1]} \right).$$

This pertains to the desired limit, where $\beta = 2$ and $\theta = 1$. □

Theorem 2.6. *If $\psi(u) \in C_{p,q}(\eta, \lambda; u)$ with $\psi(u)$ as in (1.1), then*

$$|v_2v_3 - v_4| \leq \left(\frac{(1+\eta)\lambda\Theta^1_{p,q}}{B^2A(A-1)(p^2+q^2)(p+q)[(p^2+q^2)(p+q)-1]} \right), \quad (2.23)$$

where

$$\begin{aligned}\Theta^1_{p,q} = \Big| &\lambda(1+\eta)(p^2+q^2)[(p^2+q^2)(p+q)-1][(\lambda\eta-1)B + \lambda^2(1+\eta)^2] \\ &- \mathcal{Y}^1_q - 4B^2A(A-1) + 2\lambda(1+\eta)B(p^2+q^2)[(p^2+q^2)(p+q)-1] \\ &- 2BA\Big[(\lambda\eta-1)B(A-1) + \lambda(1+\eta)[(p+q)+A-2]\Big]\Big|\end{aligned}$$

for $A = (p^2+pq+q^2)$ and $B = (p+q-1)$.

Proof. Simplifying from (2.13) to (2.15) of Theorem 2.2, we get

$$v_2v_3 - v_4 = \left(\frac{\lambda^2(1+\eta)^2[(\lambda\eta-1)B + \lambda^2(1+\eta)^2]}{8(p+q)B^2A(A-1)} - \frac{\lambda(1+\eta)\mathcal{Y}^1_{p,q}}{8B^2(A-1)(p^2+q^2) \times (p+q)[(p^2+q^2)(p+q)-1]} \right)\beta_1^3$$
$$- \frac{\lambda(1+\eta)}{2(p^2+q^2)(p+q)[(p^2+q^2)(p+q)-1]}\beta_3$$
$$+ \left(\frac{\lambda^2(1+\eta)^2}{4(p+q)BA(A-1)} - \frac{2\lambda(1+\eta)(\eta\lambda-1)B(A-1) + \lambda^2(1+\eta)^2[(p+q)+A-2]}{4B(A-1)(p^2+q^2) \times (p+q)[(p^2+q^2)(p+q)-1]} \right)\beta_1\beta_2, \tag{2.24}$$

where

$$\mathcal{Y}^1_{p,q} = (1+2\eta\lambda+\eta^2\lambda^2)B^2(A-1) - \lambda^2(1+\eta)^2(A-1) + \lambda(1+\eta)[(\lambda\eta-1)B + \lambda(1+\eta)][(p+q)+A-2].$$

The inequality (2.23) is derived using a similar methodology as shown in Theorem 2.5. □

3. Toeplitz determinants findings

This section finds the bound values for the Toeplitz matrix's second $\mathcal{T}_2(2)$ and third $\mathcal{T}_3(1)$ orders inequalities.

Theorem 3.1. *If $\psi(u) \in C_{p,q}(u)$ with $\psi(u)$ as in* (1.1), *then*

$$\mathcal{T}_2(2) = |v_3^2 - v_2^2| \le \frac{(1+pq)^2}{A^2(A-1)^2}\left(\frac{B^2 + [(p+q)(pq-1)+2](1+pq(p+q))}{B^2} + \frac{A^2(A-1)^2}{(p+q)^2B^2} \right), \tag{3.1}$$

where $A = (p^2+pq+q^2)$ and $B = (p+q-1)$.

Proof. Given the values of v_2 and v_3 in Theorem 2.1, we have

$$|v_3^2 - v_2^2| = \left| \left(\frac{1+pq}{2A(A-1)}\left[\beta_2 + \frac{(1+pq)-(1-pq)B}{2B}\beta_1^2\right] \right)^2 - \frac{(1+pq)^2}{4(p+q)^2B^2}\beta_1^2 \right|$$

and

$$|v_3^2 - v_2^2| = \frac{(1+pq)^2}{4A^2(A-1)^2}\left|\beta_2^2 + \frac{[(1+pq)-(1-pq)B]^2}{4B^2}\beta_1^4 \times \left(\frac{[(1+pq)-(1-pq)B](p+q)^2 B\beta_2 - A^2(A-1)^2}{(p+q)^2B^2}\right)\beta_1^2\right|.$$

Applying Lemma 1.5 with $\beta_1 \leq 2$, we attain

$$|v_3^2 - v_2^2| = \frac{(1+pq)^2}{4A^2(A-1)^2}\left|\frac{1}{4}\mu^2\mathcal{X}^2 + \frac{pq(p+q)+1}{2B}\beta_1^2\mu\mathcal{X} + \frac{B^2+[(p+q)(pq-1)+2](pq(p+q)+1)}{4B^2}\beta_1^4 - \frac{A^2(A-1)^2}{(p+q)^2B^2}\beta_1^2\right|,$$

where $\mathcal{X} = (4-\beta_1^2)$.

Afterwards assuming $\beta_1 = \beta$ with $0 \leq \beta \leq 2$, and using $|\mu| = \theta$, we obtain

$$\begin{aligned}|v_3^2 - v_2^2| &\leq E_{p,q}(\theta)\\ &= \frac{(1+pq)^2}{4A^2(A-1)^2}\left(\left|\frac{B^2+[(p+q)(pq-1)+2](pq(p+q)+1)}{4B^2}\beta^4 - \frac{A^2(A-1)^2}{(p+q)^2B^2}\beta^2\right| + \frac{1}{4}\theta^2\mathcal{X}^2 + \frac{pq(p+q)+1}{2B}\beta^2\theta\mathcal{X}\right)\end{aligned} \tag{3.2}$$

with $\mathcal{X} = 4-\beta^2$.

Differentiating the function $E_{p,q}(\theta)$ with respect to $\theta(0 \leq \theta \leq 1)$, we consider that

$$\frac{\delta E_{p,q}}{\delta\theta} = \frac{(1+pq)^2}{4A^2(A-1)^2}\left(\frac{1}{2}\theta\mathcal{X}^2 + \frac{pq(p+q)+1}{2B}\beta^2\mathcal{X}\right) > 0.$$

The function $E(\theta)$ is therefore an increasing function of θ when $\theta = 1$.

$$\max\{E_{p,q}\} = E_{p,q}(1),$$

where

$$|v_3^2 - v_2^2| \leq \frac{(1+pq)^2}{4A^2(A-1)^2}\left(\left|\frac{B^2+[(p+q)(pq-1)+2](pq(p+q)+1)}{4B^2}\beta^4 - \frac{A^2(A-1)^2}{(p+q)^2B^2}\beta^2\right| + \frac{1}{4}\mathcal{X}^2 + \frac{pq(p+q)+1}{2B}\beta^2\mathcal{X}\right).$$

Since $0 \leq \beta \leq 2$, $\beta = 2$ is the maximal point, so it follows that

$$|v_3^2 - v_2^2| \leq \frac{(1+pq)^2}{A^2(A-1)^2}\left(\frac{B^2 + [(p+q)(pq-1)+2](1+pq(p+q))}{B^2} + \frac{A^2(A-1)^2}{(p+q)^2B^2}\right).$$

We derive now the function $|v_3^2 - v_2^2|$ for the class $C_{p,q}(\eta, \lambda; u)$. □

Theorem 3.2. *If $\psi(u) \in C_{p,q}(\eta, \lambda; u)$ with $\psi(u)$ as in (1.1), then*

$$\mathcal{T}_2(2) = |v_3^2 - v_2^2| \leq \frac{(1+\eta)^2\lambda^2}{A^2(A-1)^2}\left(\frac{\Lambda_\eta^\lambda}{B^2} + \frac{A^2(A-1)^2}{(p+q)^2B^2}\right), \tag{3.3}$$

where $\Lambda_\eta^\lambda = |2B(1+\eta)\eta\lambda^2 + B^2\eta^2\lambda^2 + (1+\eta)^2\lambda^2|$, $A = (p^2+pq+q^2)$ *and* $B = (p+q-1)$.

Proof. In Theorem 2.2 given the values of v_2 and v_3, we have

$$|v_3^2 - v_2^2| = \left|\left(\frac{\lambda(1+\eta)}{2A(A-1)}\left\{\beta_2 + \left(\frac{\eta\lambda - 1}{2} + \frac{\lambda(1+\eta)}{2B}\right)\beta_1^2\right\}\right)^2 - \frac{\lambda^2(1+\eta)^2}{4(p+q)^2B^2}\beta_1^2\right|$$

and

$$|v_3^2 - v_2^2| = \frac{\lambda^2(1+\eta)^2}{4A^2(A-1)^2}\left|\beta_2^2 + \frac{\Lambda_{1,\eta}^\lambda}{4B^2}\beta_1^4 + \left(\frac{\Lambda_{2,\eta}^\lambda(p+q)^2B\beta_2 - A^2(A-1)^2}{(p+q)^2B^2}\right)\beta_1^2\right|,$$

where

$$\Lambda_{1,\eta}^\lambda = 2B(\eta\lambda^2 - \lambda)(1+\eta) + (1+\eta)^2\lambda^2 + B^2(\eta^2\lambda^2 + 1 - 2\eta\lambda)$$

and

$$\Lambda_{2,\eta}^\lambda = B(\eta\lambda - 1) + \lambda + \eta\lambda.$$

Applying Lemma 1.5 with $\beta_1 \leq 2$, we get

$$|v_3^2 - v_2^2| = \frac{\lambda^2(1+\eta)^2}{4A^2(A-1)^2}\left|\frac{1}{4}\mu^2\mathcal{X}^2 + \frac{B + \Lambda_{2,\eta}^\lambda}{2B}\beta_1^2\mu\mathcal{X} + \frac{\Lambda_\eta^\lambda}{4B^2}\beta_1^4 - \frac{A^2(A-1)^2}{(p+q)^2B^2}\beta_1^2\right|,$$

where $\mathcal{X} = (4 - \beta_1^2)$ and $\Lambda_\eta^\lambda = 2B(1+\eta)\eta\lambda^2 + B^2\eta^2\lambda^2 + (1+\eta)^2\lambda^2$.

The identical technique of Theorem 3.1 is used to derive the inequality (3.3). □

Theorem 3.3. *If $\psi(u) \in C_{p,q}(u)$ with $\psi(u)$ as in* (1.1), *then*

$$\mathcal{T}_3(1) = \begin{vmatrix} 1 & v_2 & v_3 \\ v_2 & 1 & v_2 \\ v_3 & v_2 & 1 \end{vmatrix} \leq \left(1 + \Delta_{p,q} + \frac{2(1+pq)^2}{(p+q)^2 B}\right),$$

where

$$\begin{aligned}\Delta_{p,q} =& \frac{(1+pq)^2}{(p+q)^2 B^3 A^2 (A-1)^2} \Big| \Big(2(1+pq)A(A-1)[pq(p+q)+1] \\ & - (p+q)^2 B[(p+q)(p^2q^2(p+q)+6pq-4)+9] \Big) \Big|\end{aligned} \tag{3.4}$$

for $A = (p^2+pq+q^2)$ and $B = (p+q-1)$.

Proof. In Theorem 2.1 given the values v_2 and v_3, we have

$$\mathcal{T}_3(1) = |1 + 2v_2^2(v_3 - 1) - v_3^2|,$$

$$\mathcal{T}_3(1) = \left| 1 + \Gamma^1_{p,q}\beta_1^2\beta_2 + \Gamma^2_{p,q}\beta_1^4 - \frac{(1+pq)^2}{2(p+q)^2B^2}\beta_1^2 - \frac{(1+pq)^2}{4A^2(A-1)^2}\beta_2^2 \right|,$$

where

$$\begin{aligned}\Gamma^1_{p,q} = & \frac{(1+pq)^2}{4(p+q)^2B^2A^2(A-1)^2}\Big((1+pq)A(A-1) \\ & - (p+q)^2 B[(p+q)(pq-1)+2]\Big)\end{aligned}$$

and

$$\begin{aligned}\Gamma^2_{p,q} = & \frac{(1+pq)^2}{16(p+q)^2B^3A^2(A-1)^2}\Big(2(1+pq)A(A-1)[(p+q)(pq-1)+2] \\ & - (p+q)^2 B[(p+q)(pq-1)+2]^2\Big).\end{aligned}$$

Applying Lemma 1.5, we obtain

$$\mathcal{T}_3(1) = \left| 1 + \frac{\Delta_{p,q}}{16}\beta_1^4 - \frac{(1+pq)^2}{2(p+q)^2B}\beta_1^2 - \frac{(1+pq)^2}{16A^2(A-1)^2}\mu^2\mathcal{X}^2 + \Gamma^3_{p,q}\beta_1^2\mu\mathcal{X} \right|,$$

where $\mathcal{X} = (4 - \beta_1^2)$ with

$$\begin{aligned}\Gamma^3_{p,q} = & \frac{(1+pq)^2}{8(p+q)^2B^2A^2(A-1)^2}\Big(2(1+pq)A(A-1) \\ & + (p+q)^2 B[(p+q)(2pq-1)+3]\Big)\end{aligned}$$

and $\Delta_{p,q}$ well known in (3.4). Afterwards, assuming $\beta_1 = \beta$ with $0 \le \beta \le 2$, and using $|\mu| = \theta$, we get

$$\begin{aligned}\mathcal{T}_3(1) &\le F_{p,q}(\theta)\\ &= \left|1 + \frac{\Delta_{p,q}}{16}\beta^4 - \frac{(1+pq)^2}{2(p+q)^2 B}\beta^2\right|\\ &\quad + \frac{(1+pq)^2}{16A^2(A-1)^2}\theta^2\mathcal{X}^2 + \Gamma^4_{p,q}\beta_1^2\theta\mathcal{X}.\end{aligned}$$

Differentiating the function $F_{p,q}(\theta)$ with respect to $\theta (0 \le \theta \le 1)$, consider that

$$\frac{\delta F_{p,q}(\theta)}{\delta\theta} > 0.$$

This shows that the function $F_{p,q}(\theta)$ is increasing and reaches its maximum value at $\theta = 1$.

$$\max\{F_{p,q}(\theta)\} = F_{p,q}(1),$$

where

$$\mathcal{T}_3(1) \le \left|1 + \frac{\Delta_{p,q}}{16}\beta^4 - \frac{(1+pq)^2}{2(p+q)^2 B}\beta^2\right| + \frac{(1+pq)^2}{16A^2(A-1)^2}\mathcal{X}^2 + \Gamma^4_{p,q}\beta^2\mathcal{X}.$$

Since $0 \le \beta \le 2$, the maximal point is $\beta = 2$. The intended result is achieved. □

Theorem 3.4. *If $\psi(u) \in C_{p,q}(\eta, \lambda; u)$ with $\psi(u)$ as in (1.1), then*

$$\begin{aligned}\mathcal{T}_3(1) &= |1 + 2v_2^2(v_3 - 1) - v_3^2|\\ &\le \left(1 + \Delta^{\lambda}_{(p,q);\eta} + \frac{2\lambda^2(1+\eta)^2}{(p+q)^2 B^2}\right),\end{aligned} \tag{3.5}$$

where

$$\begin{aligned}\Delta^{\lambda}_{(p,q);\eta} = \frac{\lambda^2(1+\eta)^2}{(p+q)^2 B^3 A^2 (A-1)^2}\Big|\Big(&2\lambda(1+\eta)A(A-1)(-\eta\lambda(p+q))\\ &- (p+q)^2 B[((p+q)(\eta\lambda+1)+1)((p+q)(\eta\lambda+3)+1)+B]\Big)\Big|\end{aligned}$$

for $A = (p^2 + pq + q^2)$ and $B = (p + q - 1)$.

Proof. In Theorem 2.2 with v_2 and v_3, we have

$$\mathcal{T}_3(1) = \left|1 + \Gamma^4_{p,q}\beta_1^2\beta_2 + \Gamma^5_{p,q}\beta_1^4 - \frac{\lambda^2(1+\eta)^2}{2(p+q)^2 B^2}\beta_1^2 - \frac{\lambda^2(1+\eta)^2}{4A^2(A-1)^2}\beta_2^2\right|,$$

where

$$\Gamma^4_{p,q} = \frac{\lambda^2(1+\eta)^2}{4(p+q)^2B^2A^2(A-1)^2}\Big(\lambda(1+\eta)A(A-1) - (p+q)^2B[(p+q)(\eta\lambda+1)+1]\Big)$$

and

$$\Gamma^5_{p,q} = \frac{\lambda^2(1+\eta)^2}{16(p+q)^2B^3A^2(A-1)^2}\Big(2\lambda(1+\eta)A(A-1)[(p+q)(\eta\lambda+1)+1] - (p+q)^2B[(p+q)(\eta\lambda+1)+1]^2\Big).$$

The inequality (3.5) is generated using the same method as Theorem 3.3. □

4. Conclusion

In this work we have defined new subclasses $C_{p,q}(u)$ and $C_{p,q}(\eta,\lambda;u)$ of generalized binomial series using (p,q)-differential operator. Further more using these classes we find the boundary values of the Hankel and Toeplitz determinants. Further study may include differentiable subordination problems and many more problems in geometric function theory.

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