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A MODIFIED INERTIAL PARALLEL-TYPE SUBGRADIENT EXTRAGRADIENT METHOD FOR FINDING THE COMMON SOLUTION OF VARIATIONAL INEQUALITY PROBLEMS

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Abstract. This paper introduces an inertial parallel-type subgradient extragradient algorithm, featuring inertial extrapolation terms and a self-adaptive step size rule, for finding a common solution of variational inequality problems in real Hilbert spaces. We establish both weak and strong convergence theorems for the proposed algorithm under suitable conditions. Furthermore, this method is extended to find a common element of the common solution set of variational inequality problems and the fixed point set of a quasi-nonexpansive mapping. Finally, we present numerical experiments to illustrate the convergence behavior of our algorithm and demonstrate its advantages over several existing methods in the literature.

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1. Introduction

Throughout this paper, let H be a real Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$, and let C be a nonempty closed and convex subset of H.

Let $T : C \to C$ be a mapping. A point $x \in C$ is a *fixed point* of T if $Tx = x$. The set of all fixed points of T is denoted by $\mathrm{Fix}(T)$, that is, $\mathrm{Fix}(T) = \{x \in C : Tx = x\}$. A mapping $T : C \to C$ with a nonempty fixed-point set ($\mathrm{Fix}(T) \neq \emptyset$) is called *quasi-nonexpansive* if $\|Tx - p\| \leq \|x - p\|$ for all $x \in C$ and $p \in \mathrm{Fix}(T)$. A mapping T of C into itself is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. Note that the mapping $I - T$ is demiclosed at zero if $x \in \mathrm{Fix}(T)$ whenever $x_n \rightharpoonup x$ and $x_n - Tx_n \to 0$. It is widely known that if $T : H \to H$ is nonexpansive, then $I - T$ is demiclosed at zero.

Fixed point theory for nonlinear mappings has received significant attention in recent years due to its broad applicability in areas such as compressed sensing, image recovery, and data classification (see, for instance, [6, 8, 19, 21] and the references therein). In particular, fixed-point algorithms have been effectively applied to medical data classification for instance, in predicting cancer [5], Parkinsons disease [16], and chronic kidney disease [9]—demonstrating their efficacy in constructing reliable predictive models.

Another fundamental problem is the *variational inequality problem* (VIP). For a given operator $A : C \to H$, the VIP is to find a point $x \in C$ such that

$$\langle Ax, y - x \rangle \geq 0, \quad \forall y \in C.$$

The solution set of the VIP is denoted by $\mathrm{VI}(C, A)$. One of the most prominent methods for solving the VIP is the extragradient method, introduced by Korpelevich [12] in 1976 for operators in Euclidean space. For an operator A that is *monotone* and *L-Lipschitz continuous* with a constant $L > 0$, the extragradient method generates a sequence $\{x_n\}$ via the iterative scheme:

$$\begin{cases} y_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = P_C(x_n - \lambda A y_n), \quad \forall n \geq 0, \end{cases} \tag{1.1}$$

where P_C is the metric projection onto C and the step size λ is a constant in $(0, 1/L)$. If the solution set $\mathrm{VI}(C, A)$ is nonempty, the sequence $\{x_n\}$ generated by (1.1) converges weakly to a solution of the VIP.

In 2011, Censor et al. [4] introduced a *subgradient extragradient method* where the second projection onto the set C is replaced by a projection onto a strategically constructed half-space. The algorithm is defined as follows:

$$\begin{cases} y_n = P_C(x_n - \lambda A x_n), \\ T_n = \{x \in H : \langle x_n - \lambda A x_n - y_n, x - y_n \rangle \leq 0\}, \\ x_{n+1} = P_{T_n}(x_n - \lambda A y_n), \quad \forall n \geq 0, \end{cases} \tag{1.2}$$

for all $\lambda \in (0, 1/L)$. They proved that if the solution set $\mathrm{VI}(C, A)$ is nonempty, the sequence $\{x_n\}$ generated by (1.2) converges weakly to a solution. This method has since been modified and generalized in various directions by numerous researchers (see, e.g., [10, 11, 14, 27, 30] and the references therein).

In 2020, Suantai et al. [20] introduced a parallel algorithm for finding a common solution to a family of variational inequalities, which incorporates a viscosity method and a line search. Given constants $\rho, \mu \in (0,1)$, a sequence of real numbers $\{\alpha_n\} \subset (0,1)$, $r_\gamma(x) := x - P_C(x - \gamma A x), \gamma > 0$ and an initial point $x_1 \in H$, the method is described as follows:

$$\begin{cases} y_n^i = P_C(x_n - \lambda_n^i A_i x_n), \lambda_n^i = \rho^{l_n^i}, \\ (l_n^i \text{ is the smallest non-negative integer } l^i \text{ such that} \\ \lambda_n^i \|A_i x_n - A_i y_n^i\| \leq \mu \|r_{\rho^{l_n^i}}(x_n)\|), \\ z_n^i = P_{T_n}(x_n - \lambda_n^i A_i y_n^i), \\ T_n := \{z \in H : \langle x_n - \lambda_n^i A_i x_n - y_n^i, z - y_n^i \rangle \leq 0\}, \\ x_{n+1} = (1 - \alpha_n) f(x_n) + \alpha_n z_n^i, \quad \forall n \geq 1, \end{cases} \tag{1.3}$$

where for each $i \in \{1, \ldots, N\}$, $A_i : C \to H$ is a monotone and L_i-Lipschitz continuous operator, and $f : C \to C$ is a strict contraction mapping with constant $k \in (0,1]$. Under suitable conditions, they proved that the sequence $\{x_n\}$ generated by (1.3) converges strongly to $x^* \in \bigcap_{i=1}^N \mathrm{VI}(C, A_i)$, where $x^* = P_{\bigcap_{i=1}^N \mathrm{VI}(C,A_i)} f(x^*)$ is the unique solution of the variational inequality

$$\langle (I - f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \bigcap_{i=1}^N \mathrm{VI}(C, A_i).$$

To improve the convergence rate of such algorithms, researchers have incorporated various acceleration techniques. Among the most effective is the *inertial method*, first introduced by Polyak [17]. The combination of inertial effects with the subgradient extragradient method has been a fruitful area of research, developed by many authors (see, e.g., [1, 3, 23, 24, 25, 26, 29] and the references therein). A key component in the design of efficient inertial algorithms is the choice of the *inertial parameter*, θ_n. In many foundational studies, this parameter, which controls the momentum of the iterates, was restricted to the interval $[0, 1)$, effectively limiting the influence of the inertial step (see, e.g., [26, 29]).

Challenging this convention, Shehu et al. [18] in 2021 introduced a subgradient extragradient algorithm with a more flexible inertial term, allowing the parameter θ_n to be chosen from the interval $[0,1]$. Their method also features a self-adaptive step size and is formulated as follows:

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = P_C(w_n - \lambda_n A w_n), \\ T_n := \{z \in H : \langle w_n - \lambda_n A w_n - y_n, z - y_n\rangle \le 0\}, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n P_{T_n}(w_n - \lambda_n A y_n), \qquad \forall n \ge 1, \end{cases} \tag{1.4}$$

where the parameters satisfy $0 \le \theta_n \le \theta_{n+1} \le 1$ and $0 < \alpha \le \alpha_n \le \alpha_{n+1} < \dfrac{1}{2+\delta}$ for some $\delta > 0$. The step size λ_n is updated via the following rule:

$$\lambda_{n+1} = \begin{cases} \min\left\{\dfrac{\mu\|w_n - y_n\|}{\|Aw_n - Ay_n\|}, \lambda_n\right\}, & \text{if } Aw_n \ne Ay_n, \\ \lambda_n, & \text{otherwise.} \end{cases} \tag{1.5}$$

They proved that under appropriate conditions, the sequence $\{x_n\}$ generated by (1.4) converges weakly to a solution in $\mathrm{VI}(C,A)$ for a monotone and L-Lipschitz continuous operator A. Furthermore, they established a strong convergence result for the special case where the operator A is strongly monotone and Lipschitz continuous.

Motivated by the recent research of Suantai et al. [20] and Shehu et al. [18], we introduce a new class of inertial parallel-type subgradient extragradient methods. We first propose an algorithm for finding a common solution set of variational inequality problems and subsequently extend it to find a common element of the common solution set of variational inequality problems and the fixed point set of a quasi-nonexpansive mapping.

The main contributions and features of our work are summarized as follows. First, our method adopts the inertial condition $0 \le \theta_n \le 1$, allowing for a potentially larger and more effective momentum step. Second, we employ a self-adaptive step size rule that does not require prior knowledge of the operator's Lipschitz constant, making the algorithm more practical. Finally, the proposed algorithms are designed to be computationally efficient by avoiding any line search procedures.

We establish both weak and strong convergence theorems for the proposed methods under suitable conditions. Finally, we provide numerical experiments to illustrate the convergence behavior and demonstrate the effectiveness of our algorithms in comparison to existing methods. Our results extend and unify several recent results in the literature.

The remainder of this paper is organized as follows. Section 2 reviews fundamental definitions and lemmas. In Section 3, we introduce our proposed algorithms and provide a thorough convergence analysis. Section 4 presents numerical experiments and provides remarks on the results to validate our theoretical findings. Finally, Section 5 provides a summary of our work and concluding remarks.

2. Preliminaries

This section recalls some fundamental definitions and lemmas required for our subsequent analysis. Throughout, we denote strong and weak convergence by "$\to$" and "$\rightharpoonup$", respectively.

Definition 2.1. A mapping $A : H \to H$ is called:

(i) *monotone*, if for all $x, y \in H$,

$$\langle Ax - Ay, x - y\rangle \geq 0;$$

(ii) *η-strongly monotone*, for some $\eta > 0$, if for all $x, y \in H$,

$$\langle Ax - Ay, x - y\rangle \geq \eta\|x - y\|^2;$$

(iii) *pseudo-monotone*, if for all $x, y \in H$,

$$\langle Ax, y - x\rangle \geq 0 \implies \langle Ay, y - x\rangle \geq 0;$$

(iv) *δ-strongly pseudo-monotone*, for some $\delta > 0$, if for all $x, y \in H$,

$$\langle Ay, x - y\rangle \geq 0 \Rightarrow \langle Ax, x - y\rangle \geq \delta\|x - y\|^2;$$

(v) *L-Lipschitz continuous*, for some $L > 0$, if for all $x, y \in H$,

$$\|Ax - Ay\| \leq L\|x - y\|.$$

It is well known that every monotone mapping is also pseudo-monotone, but the converse is not true, as the following example illustrates.

Example 2.2. ([28]) The mapping $A : (0, \infty) \to (0, \infty)$, defined by $Ax = \dfrac{a}{a + x}$ with $a > 0$, is pseudo-monotone but not monotone.

The metric projection is a fundamental tool in our analysis. For all $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$ such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C.$$

P_C is called the metric projection of H onto C.

The projection operator is characterized by the following well-known properties.

Lemma 2.3. ([22]) *For any $x \in H$ and $y \in C$, we have $P_C x = y$ if and only if the following variational inequality holds:*

$$\langle x - y, y - z\rangle \geq 0, \quad \forall z \in C.$$

Lemma 2.4. ([2]) *The metric projection satisfies the following inequality for all $x \in H$ and $y \in C$:*

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2. \tag{2.1}$$

The following lemma, often referred to as Minty's Lemma, provides an alternative formulation for solutions to the VIP under specific conditions.

Lemma 2.5. ([7]) *Let $A : C \to H$ be a pseudo-monotone and continuous operator. A point x^* is a solution of $\mathrm{VI}(C, A)$ if and only if it satisfies*

$$\langle Ax, x - x^*\rangle \geq 0, \quad \forall x \in C.$$

We will also use the following fundamental properties of a Hilbert space H.

Lemma 2.6. *The following statements hold for all $x, y \in H$:*

(1) $\|x + y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2$;
(2) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$;
(3) *for any $a, b \in \mathbb{R}$, $\|ax + by\|^2 \leq a(a+b)\|x\|^2 + b(a+b)\|y\|^2 - ab\|x - y\|^2$.*

To establish the convergence of our proposed algorithms, we require several key analytical tools. For weak convergence, we use the well-known Opial's Lemma.

Lemma 2.7. ([15]) *Let C be a nonempty subset of H and let $\{x_n\}$ be a sequence in H. If $\{x_n\}$ satisfies the following two conditions:*

(1) *for any $x \in C$, the limit $\lim_{n\to\infty} \|x_n - x\|$ exists;*
(2) *every sequential weak cluster point of $\{x_n\}$ is in C;*

then $\{x_n\}$ converges weakly to a point in C.

The next two lemmas are essential for analyzing the convergence of real-valued sequences that arise from iterative algorithms.

Lemma 2.8. ([13]) *Let $\{\varphi_n\}$, $\{\delta_n\}$ and $\{\theta_n\}$ be sequences of non-negative real numbers such that*

$$\varphi_{n+1} \leq \varphi_n + \theta_n(\varphi_n - \varphi_{n-1}) + \delta_n, \quad \forall n \geq 1$$

with $\sum_{n=1}^{\infty} \delta_n < +\infty$, and there exists a real number θ such that $0 \leq \theta_n \leq \theta < 1$ for all $n \in \mathbb{N}$. Then, the following assertions hold:

(1) $\sum_{n=1}^{+\infty}[\varphi_n - \varphi_{n-1}]_+ < +\infty$, *where $[t]_+ := \max\{t, 0\}$;*
(2) *The sequence $\{\varphi_n\}$ converges to some $\varphi^* \in [0, +\infty)$.*

3. Main result

This section is dedicated to the development and analysis of our proposed algorithms. We begin by introducing an inertial parallel-type subgradient extragradient method (IPSE) designed to find a common solution of variational inequality problems. We then establish both weak and strong convergence theorems for this method. Subsequently, we will extend this framework to solve a more general problem that also includes finding a fixed point of a quasi-nonexpansive mapping.

To establish our results, we operate under the following assumptions on the problem and the algorithm's parameters.

Assumption 3.1. The operators, solution set and parameters of the algorithm satisfy the following:

(a) The feasible set C is a nonempty closed and convex subset of H.
(b) For each $i \in \{1, \dots, N\}$, the operator $A_i : H \to H$ is monotone and L_i-Lipschitz continuous with $L = \max\{L_i\}$.
(c) The set of common solutions $\bigcap_{i=1}^{N} \mathrm{VI}(C, A_i)$ is nonempty.
(d) The inertial sequence $\{\theta_n\}$ is non-decreasing with $0 \leq \theta_n \leq 1$.
(e) The sequence $\{\alpha_n\}$ is non-decreasing and satisfies $0 < \alpha \leq \alpha_n < \frac{1}{2+\delta}$ for some $\delta > 0$.
(f) The weights $\{a_i\}$ satisfy $\sum_{i=1}^{N} a_i = 1$ and $0 < p \leq a_i \leq q < 1$ for some constants p, q.
(g) The positive sequence $\{\beta_n\}$ satisfies $0 < c \leq \beta_n \leq d < 1$ for some constants c, d and for all $n \geq 1$.

Our main algorithm, the Inertial Parallel-Type Subgradient Extragradient method (IPSE), is formally presented in Algorithm 1.

Algorithm 1 (Inertial Parallel-Type Subgradient Extragradient Method)**.**

Initialization: Choose the parameters $\mu \in (0, 1)$ and $\lambda_1^i > 0$ for all $i = 1, 2, ..., N$. Let $x_0, x_1 \in H$ be given starting points.

Iterative Steps: Calculate x_{n+1} as follows:

Step 1. Compute y_n^i for all $i = 1, 2, ..., N$ by

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n^i = P_C(w_n - \lambda_n^i A_i w_n), \end{cases} \tag{3.1}$$

where

$$\lambda_{n+1}^i = \begin{cases} \min\left\{ \dfrac{\mu\|w_n - y_n^i\|}{\|A_i w_n - A_i y_n^i\|}, \lambda_n^i \right\}, & \text{if } A_i w_n \neq A_i y_n^i, \\ \lambda_n^i, & \text{otherwise.} \end{cases} \tag{3.2}$$

If $w_n = y_n^i = x_n$, STOP. Otherwise

Step 2. Compute

$$u_n^i = \beta_n w_n + (1-\beta_n) P_{T_n^i}(w_n - \lambda_n^i A_i y_n^i), \tag{3.3}$$

where

$$T_n^i := \{z \in H : \langle w_n - \lambda_n^i A_i w_n - y_n^i, z - y_n^i \rangle \leq 0\}.$$

Step 3. Compute

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n \sum_{i=1}^{N} a_i u_n^i, \quad n \geq 1. \tag{3.4}$$

Step 4. Set $n \leftarrow n+1$, and **go to Step 1**.

Remark 3.2. The algorithm terminates if the condition $x_n = w_n = y_n^i$ holds for any $i = 1, 2 \ldots, N$. From the equations (3.1), it follows that $x_n = P_C(x_n - \lambda_n^i A_i x_n)$. Thus $x_n \in VI(C, A_i)$ for all $i = 1, 2, ..., N$ and so $x_n \in F := \bigcap_{i=1}^{N} VI(C, A_i)$.

Remark 3.3. The step size sequence $\{\lambda_n^i\}$ for each $i \in \{1, \ldots, N\}$ is well-defined, positive, and convergent. By construction in (3.2), the sequence is non-increasing, so $\lambda_{n+1}^i \leq \lambda_n^i$. If $A_i w_n \neq A_i y_n^i$, the Lipschitz continuity of A_i implies

$$\frac{\mu \|w_n - y_n^i\|}{\|A_i w_n - A_i y_n^i\|} \geq \frac{\mu \|w_n - y_n^i\|}{L_i \|w_n - y_n^i\|} = \frac{\mu}{L_i} \geq \frac{\mu}{L}.$$

This means that for all $n \geq 1$, we have $\lambda_{n+1}^i \geq \min\{\frac{\mu}{L}, \lambda_n^i\}$ and so $\lambda_{n+1}^i \leq \lambda_n^i$. This implies that $\lambda_n^i \leq \lambda_1^i, \forall n \geq 1$. Since $\min\left\{\frac{\mu}{L}, \lambda_n^i\right\} \leq \lambda_{n+1}^i$, we have $\min\left\{\frac{\mu}{L}, \lambda_1^i\right\} \leq \lambda_2^i$ and $\min\left\{\frac{\mu}{L}, \lambda_2^i\right\} \leq \lambda_3^i$. It implies that

$$\min\left\{\frac{\mu}{L}, \min\left\{\frac{\mu}{L}, \lambda_1^i\right\}\right\} \leq \min\left\{\frac{\mu}{L}, \lambda_2^i\right\} \leq \lambda_3^i.$$

Then, we obtain $\min\left\{\dfrac{\mu}{L}, \lambda_1^i\right\} \leq \lambda_3^i$. Continuing, for every $i = 1, 2, ..., N$, we get

$$0 < \min\left\{\frac{\mu}{L}, \lambda_1^i\right\} \leq \lambda_n^i, \quad \forall n \geq 1.$$

Since $\{\lambda_n^i\}$ is non-increasing and bounded below by a positive constant, the limit $\lambda^i := \lim_{n\to\infty} \lambda_n^i$ exists and is positive.

Lemma 3.4. *Let $\{x_n\}$ be the sequence generated by Algorithm* 1. *If Assumption* 3.1 *holds, then $\{x_n\}$ is bounded. Furthermore, for any $x^* \in \bigcap_{i=1}^{N} \mathrm{VI}(C, A_i)$, the limit* $\lim_{n\to\infty} \|x_n - x^*\|$ *exists.*

Proof. Let $x^* \in \bigcap_{i=1}^N \mathrm{VI}(C, A_i)$. We define

$$z_n^i := P_{T_n^i}(w_n - \lambda_n^i A_i y_n^i) \tag{3.5}$$

for each $i = 1, 2, ..., N$. Since $x^* \in \bigcap_{i=1}^N \mathrm{VI}(C, A_i)$ and the operator A_i is monotone, it follows that $\langle A_i y_n^i, y_n^i - x^* \rangle \geq 0$ for all $i = 1, 2, ..., N$. From this, we can obtain

$$\langle A_i y_n^i, x^* - z_n^i \rangle \leq \langle A_i y_n^i, y_n^i - z_n^i \rangle. \tag{3.6}$$

From the definition of the half-space T_n^i, we have

$$\langle w_n - \lambda_n^i A_i w_n - y_n^i, z_n^i - y_n^i \rangle \leq 0.$$

Therefore,

$$\begin{aligned}
\langle w_n - \lambda_n^i A_i y_n^i - y_n^i, z_n^i - y_n^i \rangle &= \langle w_n - \lambda_n^i A_i w_n - y_n^i, z_n^i - y_n^i \rangle \\
&\quad + \lambda_n^i \langle A_i w_n - A_i y_n^i, z_n^i - y_n^i \rangle \\
&\leq \lambda_n^i \langle A_i w_n - A_i y_n^i, z_n^i - y_n^i \rangle.
\end{aligned} \tag{3.7}$$

From (2.1) and (3.6), we have

$$\begin{aligned}
\|z_n^i - x^*\|^2 &\leq \|w_n - \lambda_n^i A_i y_n^i - x^*\|^2 - \|w_n - \lambda_n^i A_i y_n^i - z_n^i\|^2 \\
&= \|w_n - x^*\|^2 - \|w_n - z_n^i\|^2 + 2\lambda_n^i \langle A_i y_n^i, z_n^i - x^* \rangle \\
&\leq \|w_n - x^*\|^2 - \|w_n - z_n^i\|^2 + 2\lambda_n^i \langle A_i y_n^i, y_n^i - z_n^i \rangle \\
&= \|w_n - x^*\|^2 - \|w_n - y_n^i\|^2 - \|y_n^i - z_n^i\|^2 \\
&\quad + 2\langle w_n - \lambda_n^i A_i y_n^i - y_n^i, z_n^i - y_n^i \rangle.
\end{aligned} \tag{3.8}$$

From (3.7) and (3.8), we get

$$\begin{aligned}
\|z_n^i - x^*\|^2 &\leq \|w_n - x^*\|^2 - \|w_n - y_n^i\|^2 - \|y_n^i - z_n^i\|^2 \\
&\quad + 2\langle w_n - \lambda_n^i A_i y_n^i - y_n^i, z_n^i - y_n^i \rangle \\
&\leq \|w_n - x^*\|^2 - \|w_n - y_n^i\|^2 - \|y_n^i - z_n^i\|^2 \\
&\quad + 2\lambda_n^i \langle A_i w_n - A_i y_n^i, z_n^i - y_n^i \rangle \\
&\leq \|w_n - x^*\|^2 - \|w_n - y_n^i\|^2 - \|y_n^i - z_n^i\|^2 \\
&\quad + 2\lambda_n^i \|A_i w_n - A_i y_n^i\| \|z_n^i - y_n^i\| \\
&\leq \|w_n - x^*\|^2 - \|w_n - y_n^i\|^2 - \|y_n^i - z_n^i\|^2 \\
&\quad + \frac{2\lambda_n^i \mu}{\lambda_{n+1}^i} \|w_n - y_n^i\| \|z_n^i - y_n^i\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \|w_n - x^*\|^2 - \|w_n - y_n^i\|^2 - \|y_n^i - z_n^i\|^2 \\
&\quad + \frac{\lambda_n^i \mu}{\lambda_{n+1}^i} \left(\|w_n - y_n^i\|^2 + \|z_n^i - y_n^i\|^2\right) \\
&= \|w_n - x^*\|^2 - \left(1 - \frac{\lambda_n^i \mu}{\lambda_{n+1}^i}\right) \left(\|w_n - y_n^i\|^2 + \|z_n^i - y_n^i\|^2\right). \qquad (3.9)
\end{aligned}
$$

We note that since $\lim_{n\to\infty} \lambda_n^i$ exists and is positive,

$$
\lim_{n\to\infty} \left(1 - \frac{\lambda_n^i \mu}{\lambda_{n+1}^i}\right) = 1 - \mu > 0.
$$

Therefore, there exists a natural number $n_0 \geq 1$ such that this term is positive for all $n \geq n_0$. It follows from (3.9) that

$$
\|z_n^i - x^*\| \leq \|w_n - x^*\|, \quad \forall n \geq n_0,\ \forall i = 1, 2, ..., N. \qquad (3.10)
$$

It follows from (3.10) that

$$
\begin{aligned}
\|\sum_{i=1}^N a_i u_n^i - x^*\| &\leq \sum_{i=1}^N a_i \|u_n^i - x^*\| \\
&= \sum_{i=1}^N a_i \|\beta_n w_n + (1-\beta_n) z_n^i - x^*\| \\
&\leq \sum_{i=1}^N a_i \left(\beta_n \|w_n - x^*\| + (1-\beta_n)\|z_n^i - x^*\|\right) \\
&\leq \sum_{i=1}^N a_i \left(\beta_n \|w_n - x^*\| + (1-\beta_n)\|w_n - x^*\|\right) \\
&= \sum_{i=1}^N a_i \|w_n - x^*\| = \|w_n - x^*\|. \qquad (3.11)
\end{aligned}
$$

Now, using the result from (3.11) and the definition of x_n, we get

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|(1-\alpha_n)x_n + \alpha_n \sum_{i=1}^N a_i u_n^i - x^*\|^2 \\
&= (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n \|\sum_{i=1}^N a_i u_n^i - x^*\|^2 \\
&\quad - (1-\alpha_n)\alpha_n \|x_n - \sum_{i=1}^N a_i u_n^i\|^2
\end{aligned}
$$

$$\leq (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n\|w_n - x^*\|^2 \\ - (1-\alpha_n)\alpha_n\|x_n - \sum_{i=1}^{N} a_i u_n^i\|^2, \quad \forall n \geq n_0. \tag{3.12}$$

Note that from the definition of x_n, we can write:

$$\sum_{i=1}^{N} a_i u_n^i - x_n = \frac{1}{\alpha_n}(x_{n+1} - x_n), \quad \forall n \geq 1. \tag{3.13}$$

Substituting (3.13) into (3.12), we have

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n\|w_n - x^*\|^2 \\ &\quad - (1-\alpha_n)\alpha_n \left\| \frac{1}{\alpha_n}(x_{n+1} - x_n) \right\|^2 \\ &= (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n\|w_n - x^*\|^2 \\ &\quad - \frac{1-\alpha_n}{\alpha_n}\|x_{n+1} - x_n\|^2. \end{aligned} \tag{3.14}$$

Next, applying Lemma 2.6 (iii) to the inertial term yields

$$\begin{aligned} \|w_n - x^*\|^2 &= \|x_n + \theta_n(x_n - x_{n-1}) - x^*\|^2 \\ &= \|(1+\theta_n)(x_n - x^*) - \theta_n(x_{n-1} - x^*)\|^2 \\ &= (1+\theta_n)\|x_n - x^*\|^2 - \theta_n\|x_{n-1} - x^*\|^2 \\ &\quad + \theta_n(1+\theta_n)\|x_n - x_{n-1}\|^2, \quad \forall n \geq 1. \end{aligned} \tag{3.15}$$

Combining (3.14) and (3.15), we obtain the following inequality

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq (1-\alpha_n)\|x_n - x^*\|^2 \\ &\quad + \alpha_n\big((1+\theta_n)\|x_n - x^*\|^2 - \theta_n\|x_{n-1} - x^*\|^2 \\ &\quad + \theta_n(1+\theta_n)\|x_n - x_{n-1}\|^2\big) - \frac{1-\alpha_n}{\alpha_n}\|x_{n+1} - x_n\|^2 \\ &= (1+\alpha_n\theta_n)\,\|x_n - x^*\|^2 - \alpha_n\theta_n\|x_{n-1} - x^*\|^2 \\ &\quad + \alpha_n\theta_n(1+\theta_n)\|x_n - x_{n-1}\|^2 \\ &\quad - \frac{1-\alpha_n}{\alpha_n}\|x_{n+1} - x_n\|^2, \quad \forall n \geq n_0. \end{aligned} \tag{3.16}$$

Now, we define the sequence

$$\Gamma_n := \|x_n - x^*\|^2 - \alpha_n\theta_n\|x_{n-1} - x^*\|^2 \\ + \alpha_n\theta_n(1+\theta_n)\|x_n - x_{n-1}\|^2, \quad n \geq 1.$$

Since $\alpha_n \leq \alpha_{n+1}$ and $\theta_n \leq \theta_{n+1}$, it follows that $\alpha_n\theta_n \leq \alpha_{n+1}\theta_{n+1}$.

We then analyze the difference $\Gamma_{n+1} - \Gamma_n$:

$$\begin{aligned}\Gamma_{n+1} - \Gamma_n &= \|x_{n+1} - x^*\|^2 - \alpha_{n+1}\theta_{n+1}\|x_n - x^*\|^2 \\ &\quad + \alpha_{n+1}\theta_{n+1}(1+\theta_{n+1})\|x_{n+1} - x_n\|^2 - \|x_n - x^*\|^2 \\ &\quad + \alpha_n\theta_n\|x_{n-1} - x^*\|^2 - \alpha_n\theta_n(1+\theta_n)\|x_n - x_{n-1}\|^2 \\ &\le \|x_{n+1} - x^*\|^2 - (1+\alpha_n\theta_n)\|x_n - x^*\|^2 + \alpha_n\theta_n\|x_{n-1} - x^*\|^2 \\ &\quad + \alpha_{n+1}\theta_{n+1}(1+\theta_{n+1})\|x_{n+1} - x_n\|^2 \\ &\quad - \alpha_n\theta_n(1+\theta_n)\|x_n - x_{n-1}\|^2. \end{aligned} \tag{3.17}$$

Substituting (3.16) into (3.17) and simplifying gives

$$\Gamma_{n+1} - \Gamma_n \le -\left(\frac{1-\alpha_n}{\alpha_n} - \alpha_{n+1}\theta_{n+1}(1+\theta_{n+1})\right)\|x_{n+1} - x_n\|^2. \tag{3.18}$$

From the conditions in Assumption 3.1, we get

$$\begin{aligned}\frac{1-\alpha_n}{\alpha_n} - \alpha_{n+1}\theta_{n+1}(1+\theta_{n+1}) &= \frac{1}{\alpha_n} - 1 - \alpha_{n+1}\theta_{n+1}(1+\theta_{n+1}) \\ &\ge (2+\delta) - 1 - \frac{1}{2+\delta}(1)(2) \\ &= \delta + \frac{\delta}{2+\delta} \\ &\ge \delta. \end{aligned} \tag{3.19}$$

Substituting (3.19) into (3.18), we have

$$\Gamma_{n+1} - \Gamma_n \le -\delta\|x_{n+1} - x_n\|^2 \tag{3.20}$$

for some $\delta > 0$. This shows that the sequence $\{\Gamma_n\}$ is non-increasing for $n \ge n_0$. From the definition of Γ_n, we also have

$$\begin{aligned}\Gamma_n &= \|x_n - x^*\|^2 - \alpha_n\theta_n\|x_{n-1} - x^*\|^2 + \alpha_n\theta_n(1+\theta_n)\|x_n - x_{n-1}\|^2 \\ &\ge \|x_n - x^*\|^2 - \alpha_n\theta_n\|x_{n-1} - x^*\|^2. \end{aligned} \tag{3.21}$$

Observe that

$$\alpha_n\theta_n < \frac{1}{2+\delta} =: \epsilon < 1.$$

Using (3.21), we obtain

$$\begin{aligned}
\|x_n - x^*\|^2 &\le \alpha_n\theta_n\|x_{n-1} - x^*\|^2 + \Gamma_n \\
&\le \epsilon\|x_{n-1} - x^*\|^2 + \Gamma_1 \\
&\le \epsilon\left(\epsilon\|x_{n-2} - x^*\|^2 + \Gamma_1\right) + \Gamma_1 \\
&= \epsilon^2\|x_{n-2} - x^*\|^2 + \epsilon\Gamma_1 + \Gamma_1 \\
&\vdots \\
&\le \epsilon^n\|x_0 - x^*\|^2 + (1 + \cdots + \epsilon^{n-1})\Gamma_1 \\
&\le \epsilon^n\|x_0 - x^*\|^2 + \frac{\Gamma_1}{1-\epsilon}. \qquad (3.22)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\Gamma_{n+1} &= \|x_{n+1} - x^*\|^2 - \alpha_{n+1}\theta_{n+1}\|x_n - x^*\|^2 \\
&\quad + \alpha_{n+1}\theta_{n+1}(1+\theta_{n+1})\|x_{n+1} - x_n\|^2 \\
&\ge -\alpha_{n+1}\theta_{n+1}\|x_n - x^*\|^2. \qquad (3.23)
\end{aligned}$$

It follows from (3.23) that

$$\begin{aligned}
-\Gamma_{n+1} &\le \alpha_{n+1}\theta_{n+1}\|x_{n-1} - x^*\|^2 \\
&\le \epsilon\|x_{n-1} - x^*\|^2 \\
&\le \epsilon^{n+1}\|x_0 - x^*\|^2 + \frac{\epsilon\Gamma_1}{1-\epsilon}. \qquad (3.24)
\end{aligned}$$

From (3.20) and (3.24), we have

$$\begin{aligned}
\delta\sum_{n=1}^{k}\|x_{n+1} - x_n\|^2 &\le \Gamma_1 - \Gamma_{k+1} \\
&\le \epsilon^{k+1}\|x_0 - x^*\|^2 + \frac{\Gamma_1}{1-\epsilon}. \qquad (3.25)
\end{aligned}$$

This implies that

$$\sum_{n=1}^{\infty}\|x_{n+1} - x_n\|^2 \le \frac{\Gamma_1}{\delta(1-\epsilon)} < +\infty. \qquad (3.26)$$

Therefore,

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \qquad (3.27)$$

From the definition of w_n, we obtain

$$\|w_n - x_n\| = \theta_n\|x_n - x_{n-1}\| \le \|x_n - x_{n-1}\| \to 0 \qquad (3.28)$$

as $n \to \infty$. From (3.16), it follows that

$$\|x_{n+1} - x^*\|^2 \leq (1 + \alpha_n \theta_n) \|x_n - x^*\|^2 - \alpha_n \theta_n \|x_{n-1} - x^*\|^2 \\ + 2\|x_n - x_{n-1}\|^2. \tag{3.29}$$

Since all the conditions of Lemma 2.8 are satisfied, we conclude that

$$\lim_{n\to\infty} \|x_n - x^*\|^2 = l < \infty. \tag{3.30}$$

This completes the proof. □

Theorem 3.5. *Let $\{x_n\}$ be the sequence generated by Algorithm 1. If Assumption 3.1 holds. Then the sequence $\{x_n\}$ converges weakly to a point in $\bigcap_{i=1}^N \mathrm{VI}(C, A_i)$.*

Proof. From (3.13) and the fact that $\lim_{n\to\infty} \|x_n - x_{n-1}\| = 0$, it follows that

$$\begin{aligned} \|\sum_{i=1}^N a_i u_n^i - x_n\| &= \frac{1}{\alpha_n} \|x_{n+1} - x_n\| \\ &\leq \frac{1}{\alpha} \|x_{n+1} - x_n\| \to 0 \end{aligned} \tag{3.31}$$

as $n \to \infty$. It follows from (3.28) and (3.31) that

$$\lim_{n\to\infty} \|w_n - \sum_{i=1}^N a_i u_n^i\| = 0. \tag{3.32}$$

From (3.3), we have

$$\begin{aligned} \|\sum_{i=1}^N a_i u_n^i - x^*\|^2 &\leq \sum_{i=1}^N a_i \|u_n^i - x^*\|^2 \\ &= \sum_{i=1}^N a_i \|\beta_n w_n + (1 - \beta_n) z_n^i - x^*\|^2 \\ &= \sum_{i=1}^N a_i \big(\beta_n \|w_n - x^*\|^2 + (1 - \beta_n)\|z_n^i - x^*\|^2 \\ &\quad - \beta_n (1 - \beta_n) \|w_n - z_n^i\|^2\big) \\ &\leq \beta_n \|w_n - x^*\|^2 + (1 - \beta_n)\|z_n^i - x^*\|^2 \\ &\quad - \beta_n (1 - \beta_n) \sum_{i=1}^N a_i \|w_n - z_n^i\|^2 \end{aligned}$$

$$\leq \|w_n - x^*\|^2 + (1-\beta_n)\|z_n^i - x^*\|^2 - \beta_n(1-\beta_n)\sum_{i=1}^{N} a_i\|w_n - z_n^i\|^2.$$

It implies that

$$\begin{aligned}\beta_n(1-\beta_n)\sum_{i=1}^{N} a_i\|w_n - z_n^i\|^2 &\leq \|w_n - x^*\|^2 - \|\sum_{i=1}^{N} a_i u_n^i - x^*\|^2 \\ &\leq \left(\|w_n - x^*\| + \|\sum_{i=1}^{N} a_i u_n^i - x^*\|\right) \\ &\quad \times \|w_n - \sum_{i=1}^{N} a_i u_n^i\|. \end{aligned} \tag{3.33}$$

From Assumption 3.1 (g), (3.32) and (3.33), we have

$$\lim_{n\to\infty} \|w_n - z_n^i\| = 0, \ \forall i = 1, 2, ..., N. \tag{3.34}$$

Now, by (3.9), we get for some $M > 0$

$$\begin{aligned}\left(1 - \frac{\lambda_n^i \mu}{\lambda_{n+1}^i}\right)\|w_n - y_n^i\|^2 &\leq \|w_n - x^*\|^2 - \|z_n^i - x^*\|^2 \\ &= \left(\|w_n - x^*\| + \|z_n^i - x^*\|\right)\left(\|w_n - x^*\| - \|z_n^i - x^*\|\right) \\ &\leq M\|w_n - z_n^i\|.\end{aligned}$$

Using (3.34) (and noting that $\lim_{n\to\infty}\lambda_n^i = \lambda^i$ for all $i = 1, 2, ..., N$), we see that

$$\lim_{n\to\infty} \|w_n - y_n^i\| = 0. \tag{3.35}$$

Hence, we have

$$\|x_n - y_n^i\| \leq \|w_n - y_n^i\| + \|w_n - x_n\|$$

and as a consequence, it yields

$$\lim_{n\to\infty} \|x_n - y_n^i\| = 0. \tag{3.36}$$

Since $\{x_n\}$ is bounded, it has a subsequence $\{x_{n_k}\}$ such that $\{x_{n_k}\}$ converges weakly to some $q \in H$. We now show that $q \in \bigcap_{i=1}^{N} \mathrm{VI}(C, A_i)$. Since $\|x_n - y_n^i\| \to 0$ as $i = 1, 2, ..., N$ implies that $y_{n_k}^i \rightharpoonup q$ and since $y_{n_k}^i \in C$, we have $q \in C$. For all $x \in C$ and using the property of the projection P_C, we have

(A_i is monotone)

$$\begin{aligned}
0 &\leq \langle y^i_{n_k} - x_{n_k} + \lambda^i_{n_k} A_i x_{n_k}, x - y^i_{n_k} \rangle \\
&= \langle y^i_{n_k} - x_{n_k}, x - y^i_{n_k} \rangle + \lambda^i_{n_k} \langle A_i x_{n_k}, x_{n_k} - y^i_{n_k} \rangle + \lambda^i_{n_k} \langle A_i x_{n_k}, x - x_{n_k} \rangle \\
&\leq \langle y^i_{n_k} - x_{n_k}, x - y^i_{n_k} \rangle + \lambda^i_{n_k} \langle A_i x_{n_k}, x_{n_k} - y^i_{n_k} \rangle + \lambda^i_{n_k} \langle A_i x, x - x_{n_k} \rangle.
\end{aligned}$$

Letting $k \to \infty$ and using the facts $\lim_{k\to\infty} \|y^i_{n_k} - x_{n_k}\| = 0$ and $\lim_{k\to\infty} \lambda^i_{n_k} = \lambda^i > 0$ for all $i = 1, 2, ..., N$, we get

$$\langle A_i x, x - q \rangle \geq 0, \ \forall x \in C, \ i = 1, 2, ..., N.$$

By Lemma 2.5, we have $q \in \mathrm{VI}(C, A_i)$ for all $i = 1, 2, ..., N$. This implies that $q \in \bigcap_{i=1}^N \mathrm{VI}(C, A_i)$. Applying Lemma 2.7, we can conclude that the sequence $\{x_n\}$ converges weakly to a point in $\bigcap_{i=1}^N \mathrm{VI}(C, A_i)$. □

We now establish a strong convergence theorem for the proposed algorithm. This result requires the stronger condition that each operator A_i is not just monotone, but also strongly monotone and Lipschitz continuous.

Theorem 3.6. *Suppose that Assumption* 3.1 *(a), (d), (e), (f) and (g) holds and that for every $i = 1, 2, ..., N$, A_i is strongly monotone and Lipschitz continuous on H. Then the sequence $\{x_n\}$ generated by Algorithm* 1 *converges strongly to the unique solution $x^* \in \bigcap_{i=1}^N \mathrm{VI}(C, A_i)$.*

Proof. Since A_i is strongly monotone, it follows that $\mathrm{VI}(C, A_i)$ has a unique solution for all $i = 1, 2, ..., N$. We denote this unique solution by x^*. Consequently,

$$\langle A_i x^*, y^i_n - x^* \rangle \geq 0, \ \forall i = 1, 2, ..., N.$$

Using the strong monotonicity of A_i, we have

$$\langle A_i y^i_n, y^i_n - x^* \rangle \geq \eta_i \|y^i_n - x^*\|^2, \ \forall i = 1, 2, ..., N,$$

where η_i is some positive constant. Thus

$$\langle A_i y^i_n, y^i_n - z^i_n + z^i_n - x^* \rangle \geq \eta_i \|y^i_n - x^*\|^2, \ \forall i = 1, 2, ..., N$$

and

$$\langle A_i y^i_n, x^* - z^i_n \rangle \leq \langle A_i y^i_n, y^i_n - z^i_n \rangle - \eta_i \|y^i_n - x^*\|^2, \ \forall i = 1, 2, ..., N. \tag{3.37}$$

From (2.1) and (3.37), we get

$$\begin{aligned}
\|z_n^i - x^*\|^2 &\leq \|w_n - \lambda_n^i A_i y_n^i - x^*\|^2 - \|w_n - \lambda_n^i A_i y_n^i - z_n^i\|^2 \\
&= \|w_n - x^*\|^2 - \|w_n - z_n^i\|^2 + 2\lambda_n^i \langle A_i y_n^i, x^* - z_n^i \rangle \\
&\leq \|w_n - x^*\|^2 - \|w_n - z_n^i\|^2 + 2\lambda_n^i \langle A_i y_n^i, y_n^i - z_n^i \rangle \\
&\quad - 2\eta_i \lambda_n^i \|y_n^i - x^*\|^2 \\
&= \|w_n - x^*\|^2 - \|w_n + y_n^i - y_n^i - z_n^i\|^2 \\
&\quad + 2\lambda_n^i \langle A_i y_n^i, y_n^i - z_n^i \rangle - 2\eta_i \lambda_n^i \|y_n^i - x^*\|^2 \\
&= \|w_n - x^*\|^2 - \|w_n - y_n^i\|^2 - \|y_n^i - z_n^i\|^2 \\
&\quad - 2\eta_i \lambda_n^i \|y_n^i - x^*\|^2 + 2\langle w_n - \lambda_n^i A_i y_n^i - y_n^i, z_n^i - y_n^i \rangle. \qquad (3.38)
\end{aligned}$$

From (3.7), we have

$$\langle w_n - \lambda_n^i A_i y_n^i - y_n^i, z_n^i - y_n^i \rangle \leq \lambda_n^i \langle A_i w_n - A_i y_n^i, z_n^i - y_n^i \rangle, \ \forall i = 1, 2, ..., N. \quad (3.39)$$

From (3.38), we get

$$\begin{aligned}
\|z_n^i - x^*\|^2 &\leq \|w_n - x^*\|^2 - \|w_n - y_n^i\|^2 - \|y_n^i - z_n^i\|^2 \\
&\quad - 2\eta_i \lambda_n^i \|y_n^i - x^*\|^2 + 2\langle w_n - \lambda_n^i A_i y_n^i - y_n^i, z_n^i - y_n^i \rangle \\
&\leq \|w_n - x^*\|^2 - \|w_n - y_n^i\|^2 - \|y_n^i - z_n^i\|^2 \\
&\quad - 2\eta_i \lambda_n^i \|y_n^i - x^*\|^2 + 2\lambda_n^i \langle A_i w_n - A_i y_n^i, z_n^i - y_n^i \rangle \\
&\leq \|w_n - x^*\|^2 - \|w_n - y_n^i\|^2 - \|y_n^i - z_n^i\|^2 \\
&\quad - 2\eta_i \lambda_n^i \|y_n^i - x^*\|^2 + 2\lambda_n^i \|A_i w_n - A_i y_n^i\| \|z_n^i - y_n^i\| \\
&\leq \|w_n - x^*\|^2 - \|w_n - y_n^i\|^2 - \|y_n^i - z_n^i\|^2 \\
&\quad - 2\eta_i \lambda_n^i \|y_n^i - x^*\|^2 + \frac{2\lambda_n^i \mu}{\lambda_{n+1}^i} \|w_n - y_n^i\| \|z_n^i - y_n^i\| \\
&\leq \|w_n - x^*\|^2 - \|w_n - y_n^i\|^2 - \|y_n^i - z_n^i\|^2 \\
&\quad - 2\eta_i \lambda_n^i \|y_n^i - x^*\|^2 + \frac{\lambda_n^i \mu}{\lambda_{n+1}^i} \|w_n - y_n^i\|^2 + \frac{\lambda_n^i \mu}{\lambda_{n+1}^i} \|z_n^i - y_n^i\|^2.
\end{aligned} \quad (3.40)$$

Since $\lim_{n\to\infty} \lambda_n^i = \lambda^i$ for all $i = 1, 2, ..., N$ and the sequence $\{\lambda_n^i\}$ is monotonically decreasing, we have $\lambda_n^i \geq \lambda^i$ for all $n \geq 1, i = 1, 2, ..., N$. Let ρ be a fixed number in the interval $(\mu, 1)$. Since $\lim_{n\to\infty} \dfrac{\lambda_n^i \mu}{\lambda_{n+1}^i} = \mu < \rho$ for all $i = 1, 2, ..., N$, there exists a natural number M such that $\dfrac{\lambda_n^i \mu}{\lambda_{n+1}^i} < \rho$ for all

$n \geq M$, we have

$$\lambda_n^i > \lambda^i, \ \frac{\lambda_n^i \mu}{\lambda_{n+1}^i} < \rho, \ \forall i = 1, 2, ..., N. \tag{3.41}$$

From (3.40) and (3.41), we have

$$\begin{aligned}
\|z_n^i - x^*\|^2 &\leq \|w_n - x^*\|^2 - \|w_n - y_n^i\|^2 - \|y_n^i - z_n^i\|^2 - 2\eta_i \lambda_n^i \|y_n^i - x^*\|^2 \\
&\quad + \frac{\lambda_n^i \mu}{\lambda_{n+1}^i} \|w_n - y_n^i\|^2 + \frac{\lambda_n^i \mu}{\lambda_{n+1}^i} \|z_n^i - y_n^i\|^2 \\
&\leq \|w_n - x^*\|^2 - \|w_n - y_n^i\|^2 - \|y_n^i - z_n^i\|^2 - 2\eta_i \lambda^i \|y_n^i - x^*\|^2 \\
&\quad + \rho \|w_n - y_n^i\|^2 + \rho \|z_n^i - y_n^i\|^2 \\
&= \|w_n - x^*\|^2 - (1-\rho)\|w_n - y_n^i\|^2 - (1-\rho)\|y_n^i - z_n^i\|^2 \\
&\quad - 2\eta_i \lambda^i \|y_n^i - x^*\|^2, \quad \forall n \geq M.
\end{aligned} \tag{3.42}$$

Observe that

$$\begin{aligned}
\Big\|\sum_{i=1}^N a_i u_n^i - x^*\Big\|^2 &\leq \sum_{i=1}^N a_i \|u_n^i - x^*\|^2 \\
&\leq \sum_{i=1}^N a_i \|\beta_n w_n + (1-\beta_n) z_n^i - x^*\|^2 \\
&= \sum_{i=1}^N a_i \|\beta_n (w_n - x^*) + (1-\beta_n)(z_n^i - x^*)\|^2 \\
&\leq \sum_{i=1}^N a_i \left(\beta_n \|w_n - x^*\|^2 + (1-\beta_n)\|z_n^i - x^*\|^2\right) \\
&\leq \beta_n \|w_n - x^*\|^2 + (1-\beta_n) \sum_{i=1}^N a_i \|z_n^i - x^*\|^2.
\end{aligned} \tag{3.43}$$

From the inequalities (3.43) and (3.42), we have

$$\begin{aligned}
\Big\|\sum_{i=1}^N a_i u_n^i - x^*\Big\|^2 &\leq \beta_n \|w_n - x^*\|^2 + (1-\beta_n) \sum_{i=1}^N a_i \|z_n^i - x^*\|^2 \\
&\leq \beta_n \|w_n - x^*\|^2 + (1-\beta_n) \sum_{i=1}^N a_i \Big(\|w_n - x^*\|^2
\end{aligned}$$

$$
\begin{aligned}
&- (1-\rho)\|w_n - y_n^i\|^2 - (1-\rho)\|y_n^i - z_n^i\|^2 \\
&- 2\eta_i\lambda^i\|y_n^i - x^*\|^2\Big) \\
=&\beta_n\|w_n - x^*\|^2 + (1-\beta_n)\|w_n - x^*\|^2 \\
&- \sum_{i=1}^N a_i(1-\rho)\|w_n - y_n^i\|^2 - \sum_{i=1}^N a_i(1-\rho)\|y_n^i - z_n^i\|^2 \\
&- 2\sum_{i=1}^N a_i\eta_i\lambda^i\|y_n^i - x^*\|^2 \\
=&\|w_n - x^*\|^2 - \sum_{i=1}^N a_i(1-\rho)\|w_n - y_n^i\|^2 \\
&- \sum_{i=1}^N a_i(1-\rho)\|y_n^i - z_n^i\|^2 - 2\sum_{i=1}^N a_i\eta_i\lambda^i\|y_n^i - x^*\|^2 \\
=&\|(1+\theta_n)(x_n - x^*) - \theta_n(x_{n-1} - x^*)\|^2 \\
&- \sum_{i=1}^N a_i(1-\rho)\|w_n - y_n^i\|^2 - \sum_{i=1}^N a_i(1-\rho)\|y_n^i - z_n^i\|^2 \\
&- 2\sum_{i=1}^N a_i\eta_i\lambda^i\|y_n^i - x^*\|^2 \\
=&(1+\theta_n)\|x_n - x^*\|^2 - \theta_n\|x_{n-1} - x^*\|^2 \\
&+ (1+\theta_n)\theta_n\|x_n - x_{n-1}\|^2 - \sum_{i=1}^N a_i(1-\rho)\|w_n - y_n^i\|^2 \\
&- \sum_{i=1}^N a_i(1-\rho)\|y_n^i - z_n^i\|^2 - 2\sum_{i=1}^N a_i\eta_i\lambda^i\|y_n^i - x^*\|^2. \qquad (3.44)
\end{aligned}
$$

It follows from (3.44) that

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|(1-\alpha_n)x_n + \alpha_n\sum_{i=1}^N a_i u_n^i - x^*\|^2 \\
&\le (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n\|\sum_{i=1}^N a_i u_n^i - x^*\|^2 \\
&\le (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n\Big((1+\theta_n)\|x_n - x^*\|^2
\end{aligned}
$$

$$
\begin{aligned}
&- \theta_n\|x_{n-1} - x^*\|^2 + (1+\theta_n)\theta_n\|x_n - x_{n-1}\|^2 \\
&- \sum_{i=1}^{N} a_i(1-\rho)\|w_n - y_n^i\|^2 - \sum_{i=1}^{N} a_i(1-\rho)\|y_n^i - z_n^i\|^2 \\
&- 2\sum_{i=1}^{N} a_i\eta_i\lambda^i\|y_n^i - x^*\|^2\Big) \\
=&(1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n(1+\theta_n)\|x_n - x^*\|^2 \\
&- \alpha_n\theta_n\|x_{n-1} - x^*\|^2 + \alpha_n(1+\theta_n)\theta_n\|x_n - x_{n-1}\|^2 \\
&- (1-\rho)\sum_{i=1}^{N} a_i\alpha_n\|w_n - y_n^i\|^2 - (1-\rho)\sum_{i=1}^{N} a_i\alpha_n\|y_n^i - z_n^i\|^2 \\
&- 2\sum_{i=1}^{N} a_i\eta_i\lambda^i\alpha_n\|y_n^i - x^*\|^2 \\
\le&(1+\alpha_n\theta_n)\|x_n - x^*\|^2 - \alpha_n\theta_n\|x_{n-1} - x^*\|^2 \\
&+ \alpha_n(1+\theta_n)\theta_n\|x_n - x_{n-1}\|^2 - 2\sum_{i=1}^{N} a_i\eta_i\lambda^i\alpha\|y_n^i - x^*\|^2. \qquad (3.45)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
2\alpha\sum_{i=1}^{N} a_i\eta_i\lambda^i\|y_n^i - x^*\|^2 \le&\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
&+ \alpha_n\theta_n\left(\|x_n - x^*\|^2 - \|x_{n-1} - x^*\|^2\right) \\
&+ 2\|x_n - x_{n-1}\|^2. \qquad (3.46)
\end{aligned}
$$

Summing the above inequality from $k = M$ to n, we obtain

$$
\begin{aligned}
2\alpha\sum_{k=M}^{n}\sum_{i=1}^{N} a_i\eta_i\lambda^i\|y_k^i - x^*\|^2 \le \|x_M - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \alpha_n\theta_n\|x_n - x^*\|^2 \\
- \alpha_M\theta_M\|x_{M-1} - x^*\|^2 + 2\sum_{k=M}^{n}\|x_k - x_{k-1}\|^2.
\end{aligned}
$$

Since the sequence $\{x_n\}$ is bounded and $\sum_{k=M}^{\infty}\|x_k - x_{k-1}\|^2 < \infty$ by (3.28), letting $n \to \infty$, yields

$$
2\alpha\sum_{k=M}^{\infty}\sum_{i=1}^{N} a_i\eta_i\lambda^i\|y_k^i - x^*\|^2 < \infty.
$$

This implies that

$$\sum_{k=M}^{\infty} \|y_k^i - x^*\|^2 < \infty, \ \forall i = 1, 2, ..., N.$$

Hence

$$\lim_{n\to\infty} \|y_n^i - x^*\| = 0$$

for all $i = 1, 2, ..., N$. Observe that by the triangle inequality,

$$\|x_n - x^*\| \le \|x_n - y_n^i\| + \|y_n^i - x^*\|, \ \forall i = 1, 2, ..., N.$$

From (3.36) and the result above, we have

$$\lim_{n\to\infty} \|x_n - x^*\| = 0.$$

This concludes the proof. □

Remark 3.7. Here are two key observations about Algorithm 1 and its connection to existing literature:

(i) The condition on our inertial parameter, $0 \le \theta_n \le \theta_{n+1} \le 1$, is adopted from the recent work of Shehu et al. [18]. This allows for a more flexible and potentially faster-accelerating momentum term.

(ii) Our algorithm represents a direct generalization. Specifically, if we consider the non-parallel case (i.e., setting $A_i = A$ for all $i = 1, \dots, N$) and remove the relaxation step by setting $\beta_n = 0$, then Algorithm 1 reduces exactly to the method proposed by Shehu et al. [18], as described in (1.4).

We now extend our framework to solve a more complex problem: finding a common element of the common solution set of variational inequality problems and the fixed point set of a quasi-nonexpansive mapping $T : H \to H$. The modified method for this combined task is presented in Algorithm 2.

Algorithm 2 (Inertial Parallel-Type Subgradient Extragradient Method for Finding Fixed Points)**.**

Initialization: Choose the parameters $\mu \in (0, 1)$ and $\lambda_1^i > 0, \ \forall i = 1, 2, ..., N$. Let $x_0, x_1 \in H$ be given starting points.

Iterative Steps: Calculate x_{n+1} as follows:

Step 1. Compute y_n^i for all $i = 1, 2, ..., N$ by

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n^i = P_C(w_n - \lambda_n^i A_i w_n), \end{cases} \tag{3.47}$$

where

$$\lambda_{n+1}^i = \begin{cases} \min\left\{ \dfrac{\mu\|w_n - y_n^i\|}{\|A_i w_n - A_i y_n^i\|}, \lambda_n^i \right\}, & \text{if } A_i w_n \neq A_i y_n^i, \\ \lambda_n^i, & \text{otherwise.} \end{cases} \tag{3.48}$$

If $w_n = y_n^i = x_n$, STOP. Otherwise.

Step 2. Compute

$$u_n^i = \beta_n w_n + (1-\beta_n) T P_{T_n^i}(w_n - \lambda_n^i A_i y_n^i), \tag{3.49}$$

where

$$T_n^i := \{z \in H : \langle w_n - \lambda_n^i A_i w_n - y_n^i, z - y_n^i \rangle \leq 0\}.$$

Step 3. Compute

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n \sum_{i=1}^N a_i u_n^i, \quad n \geq 1. \tag{3.50}$$

Step 4. Set $n \leftarrow n+1$, and **go to Step 1**.

Remark 3.8. This remark clarifies the termination condition of Algorithm 2. First, observe that if $x_n = w_n = y_n^i$ for all $i = 1, 2, ..., N$, then from (3.1) it implies that $x_n = P_C(x_n - \lambda_n^i A_i x_n)$ for all $i = 1, 2, ..., N$. In this case, $x_n \in \mathrm{VI}(C, A_i)$ for all $i = 1, 2, ..., N$, and hence $x_n \in \bigcap_{i=1}^N \mathrm{VI}(C, A_i)$.

Furthermore, the condition $w_n = y_n^i$ also simplifies the projection onto the half-space T_n^i. Since $w_n = y_n^i$, the defining inequality for T_n^i is satisfied by y_n^i, which means $y_n^i = P_{T_n^i}(w_n - \lambda_n^i A_i y_n^i)$ for all $i = 1, 2, ..., N$. With these facts established, if we assume $x_n = w_n = y_n^i$, then the update step in (3.49) becomes $u_n^i = \beta_n x_n + (1-\beta_n)Tx_n$ for all $i = 1, 2, ..., N$.

Consequently, if $x_{n+1} = x_n$, then the main update rule in (3.50) becomes:

$$\begin{aligned} x_n &= (1-\alpha_n)x_n + \alpha_n \sum_{i=1}^N a_i u_n^i \\ &= (1-\alpha_n)x_n + \alpha_n(\beta_n x_n + (1-\beta_n)Tx_n) \\ &= (1-\alpha_n)x_n + \alpha_n\beta_n x_n + \alpha_n(1-\beta_n)Tx_n \\ &= x_n - \alpha_n x_n + \alpha_n\beta_n x_n + \alpha_n(1-\beta_n)Tx_n. \end{aligned}$$

It follows from that

$$\begin{aligned} x_n &= \frac{1}{\alpha_n}(\alpha_n\beta_n x_n + \alpha_n(1-\beta_n)Tx_n) \\ &= \beta_n x_n + (1-\beta_n)Tx_n. \end{aligned}$$

This implies $x_n = Tx_n$. Hence, we have shown that $x_n \in \mathrm{Fix}(T)$. Combining our findings, x_n is a common solution to both problems, that is,

$$x_n \in \bigcap_{i=1}^{N} \mathrm{VI}(C, A_i) \cap \mathrm{Fix}(T).$$

Theorem 3.9. *Assume that $\{x_n\}$ is generated by Algorithm 2. Then under Assumption 3.1 holds with $\Omega := \bigcap_{i=1}^{N} VI(C, A_i) \cap \mathrm{Fix}(T) \neq \emptyset$, the sequence $\{x_n\}$ converges weakly to a point in Ω if $I - T$ is demiclosed.*

Proof. Let $x^* \in \Omega$. Following the lines of the arguments in Lemma 3.4 and Theorem 3.5, we can show that $\{x_n\}$ is bounded, $\lim_{n\to\infty} \|x_n - x^*\|$ exists,

$$\begin{aligned}\|Tz_n^i - x^*\|^2 &\le \|z_n^i - x^*\|^2 \\ &\le \|w_n - x^*\|^2 - \left(1 - \frac{\lambda_n^i \mu}{\lambda_{n+1}^i}\right) \|w_n - y_n^i\|^2 \\ &\quad - \left(1 - \frac{\lambda_n^i \mu}{\lambda_{n+1}^i}\right) \|u_n^i - y_n^i\|^2, \end{aligned} \tag{3.51}$$

$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$ and $\lim_{n\to\infty} \|w_n - x_n\| = 0$. Hence

$$\begin{aligned}\left\|\sum_{i=1}^{N} a_i u_n^i - x_n\right\| &\le \frac{1}{\alpha_n} \|x_{n+1} - x_n\| \\ &\le \frac{1}{\alpha} \|x_{n+1} - x_n\|.\end{aligned}$$

It follows that

$$\lim_{n\to\infty} \left\|\sum_{i=1}^{N} a_i u_n^i - x_n\right\| = 0, \ \forall i = 1, 2, ..., N.$$

Since $\lim_{n\to\infty} \|w_n - x_n\| = 0$ and $\lim_{n\to\infty} \|\sum_{i=1}^{N} a_i u_n^i - x_n\| = 0$ for all $i = 1, 2, ..., N$, we have

$$\lim_{n\to\infty} \left\|\sum_{i=1}^{N} a_i u_n^i - w_n\right\| = 0, \ \forall i = 1, 2, ..., N.$$

Observe that

$$
\begin{aligned}
\|\sum_{i=1}^{N} a_i u_n^i - x^*\|^2 &\leq \sum_{i=1}^{N} a_i \|u_n^i - x^*\|^2 \\
&= \sum_{i=1}^{N} a_i \|\beta_n w_n + (1-\beta_n) T z_n^i - x^*\|^2 \\
&= \sum_{i=1}^{N} a_i \big(\beta_n \|w_n - x^*\|^2 + (1-\beta_n)\|T z_n^i - x^*\|^2 \\
&\quad - \beta_n(1-\beta_n)\|w_n - T z_n^i\|^2\big) \\
&\leq \beta_n \|w_n - x^*\|^2 + (1-\beta_n)\|z_n^i - x^*\|^2 \\
&\quad - \beta_n(1-\beta_n) \sum_{i=1}^{N} a_i \|w_n - T z_n^i\|^2 \\
&\leq \|w_n - x^*\|^2 + (1-\beta_n)\|z_n^i - x^*\|^2 \\
&\quad - \beta_n(1-\beta_n) \sum_{i=1}^{N} a_i \|w_n - T z_n^i\|^2.
\end{aligned}
$$

It implies that

$$
\begin{aligned}
\beta_n(1-\beta_n) \sum_{i=1}^{N} a_i \|w_n - T z_n^i\|^2 &\leq \|w_n - x^*\|^2 - \|\sum_{i=1}^{N} a_i u_n^i - x^*\|^2 \\
&\leq \left(\|w_n - x^*\| + \|\sum_{i=1}^{N} a_i u_n^i - x^*\| \right) \\
&\quad \times \|w_n - \sum_{i=1}^{N} a_i u_n^i\|.
\end{aligned}
$$

From Assumption 3.1 (g) and $\lim_{n\to\infty} \|\sum_{i=1}^{N} a_i u_n^i - w_n\| = 0$ for all $i = 1, 2, ..., N$, we have

$$
\lim_{n\to\infty} \|w_n - T z_n^i\| = 0, \ \forall i = 1, 2, ..., N. \tag{3.52}
$$

We can also show that $\lim_{n\to\infty} \|w_n - y_n^i\| = 0$, $\lim_{n\to\infty} \|w_n - z_n^i\| = 0$ and $\lim_{n\to\infty} \|x_n - y_n^i\| = 0$. Furthermore,

$$
\|z_n^i - x_n\| \leq \|z_n^i - y_n^i\| + \|y_n^i - x_n\|, \ \forall i = 1, 2, ..., N.
$$

It follows that

$$\lim_{n\to\infty} \|z_n^i - x_n\| = 0,\ \forall i = 1, 2, ..., N$$

and

$$\|Tz_n^i - z_n^i\| \leq \|Tz_n^i - x_n\| + \|u_n^i - x_n\|,\ \forall i = 1, 2, ..., N.$$

It implies that

$$\lim_{n\to\infty} \|Tz_n^i - z_n^i\| = 0,\ \forall i = 1, 2, ..., N.$$

The boundedness of the sequence $\{x_n\}$ implies that there exist subsequences $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \xi \in H$. Let $\{x_n\}$ be such a subsequence. Since $\lim_{n\to\infty} \|z_n^i - x_n\| = 0$ for all $i = 1, 2, ..., N$, it also follows that $z_{n_k}^i \rightharpoonup \xi$. Now the demiclosedness of $I - T$ implies that $\xi \in \mathrm{Fix}(T)$.

On the other hand, for every $i = 1, 2, ..., N$, since $\|x_n - y_n^i\| \to 0$ as $n \to \infty$ implies that $y_{n_k}^i \rightharpoonup \xi$ and since $y_{n_k}^i \in C$, we have $\xi \in C$. For all $x \in C$ and using the property of the projection P_C, we have (A_i is monotone)

$$\begin{aligned}
0 &\leq \langle y_{n_k}^i - x_{n_k} + \lambda_{n_k}^i A_i x_{n_k}, x - y_{n_k}^i \rangle \\
&= \langle y_{n_k}^i - x_{n_k}, x - y_{n_k}^i \rangle + \lambda_{n_k}^i \langle A_i x_{n_k}, x_{n_k} - y_{n_k}^i \rangle + \lambda_{n_k}^i \langle A_i x_{n_k}, x - x_{n_k} \rangle \\
&\leq \langle y_{n_k}^i - x_{n_k}, x - y_{n_k}^i \rangle + \lambda_{n_k}^i \langle A_i x_{n_k}, x_{n_k} - y_{n_k}^i \rangle + \lambda_{n_k}^i \langle A_i x, x - x_{n_k} \rangle.
\end{aligned}$$

Letting $k \to \infty$ and using the facts $\lim_{k\to\infty} \|y_{n_k}^i - x_{n_k}\| = 0$ and $\lim_{k\to\infty} \lambda_{n_k}^i = \lambda^i > 0$ for all $i = 1, 2, ..., N$, we get

$$\langle A_i x, x - \xi \rangle \geq 0,\ \forall x \in C,\ i = 1, 2, ..., N.$$

By Lemma 2.5, we have

$$\xi \in \mathrm{VI}(C, A_i),\ \forall i = 1, 2, ..., N.$$

This implies that

$$\xi \in \bigcap_{i=1}^{N} \mathrm{VI}(C, A_i).$$

Hence

$$\xi \in \bigcap_{i=1}^{N} \mathrm{VI}(C, A_i) \cap \mathrm{Fix}(T) = \Omega.$$

It follows from Lemma 2.7 that $\{x_n\}$ converges weakly to a point in Ω. This completes the proof. $\square$

4. Numerical experiments

This section is dedicated to validating our theoretical results and demonstrating the computational efficiency of the proposed algorithm. We present a numerical experiment designed to showcase the convergence behavior of our method, Algorithm 1 (IPSE), in comparison with several other recent and relevant algorithms from the literature.

The competing methods chosen for this benchmark are the classic subgradient extragradient method, CENSOR (see Algorithm 3 in [4]), the parallel method SUANTAI (see Algorithm 1 in [20]), and the recent inertial method SHEHU (see Algorithm 1 in [18]).

Example 4.1. Consider the Hilbert space $H = \mathbb{R}$ with the feasible set $C = [-2, 2]$. We define the operator $A : \mathbb{R} \to \mathbb{R}$ by:

$$A(x) = \frac{3x}{5} - \frac{3}{5}, \quad \forall x \in \mathbb{R}.$$

It can be readily verified that the operator A is monotone and $\frac{3}{5}$-Lipschitz continuous. For this experiment, the algorithm parameters were chosen as detailed in Table 1.

Table 1. Parameter settings for iterative algorithms used in Example 4.1.

Algorithms	**Parameters**
CENSOR	$\tau = 0.5$
SUANTAI	$\mu = 0.99, \quad \lambda_1^1 = 0.5, \quad \alpha_n^0 = \frac{1}{n^3},$ $\alpha_n^1 = 1 - \frac{1}{n^3}.$
SHEHU	$\mu = 0.99, \quad \lambda_1 = 0.5, \quad \theta_n = 0.9 - \frac{0.9}{n+20},$ $\alpha_n = 0.49 - \frac{0.49}{n+50}.$
IPSE	$\mu = 0.99, \quad \lambda_1^1 = 0.5, \quad \theta_n = 0.9 - \frac{0.9}{n+20},$ $\alpha_n = 0.49 - \frac{0.49}{n+50}, \quad \beta_n = 0.35 + \frac{0.001n}{n+10}.$

The experiment was run using the initial values $x_0 = -0.25$ and $x_1 = 0.5$, for which the exact solution is $x^* = 1$. As guaranteed by the strong convergence result in Theorem 3.6, the sequence $\{x_n\}$ generated by IPSE is confirmed to converge to this solution.

The numerical results of the comparison are presented in Table 2 and visualized in Figure 1. Both the table and the figure clearly demonstrate that IPSE converges faster than the other three methods.

TABLE 2. Computational results of x_n for Example 4.1 with $x_0 = -0.25$, $x_1 = 0.5$ and $n = 25$.

n	CENSOR	SUANTAI	SHEHU	IPSE
0	-0.25000	-0.25000	-0.25000	-0.25000
1	0.50000	0.50000	0.50000	0.50000
2	0.60500	0.50000	0.79441	0.79946
3	0.68795	0.59187	0.91118	0.91937
⋮	⋮	⋮	⋮	⋮
24	0.99779	0.99705	0.99968	1.000034
25	0.99825	0.99767	0.99973	1.000031

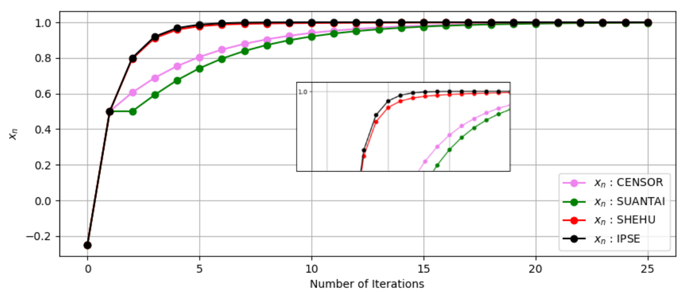


FIGURE 1. Convergence behaviour of x_n for Example 4.1.

Example 4.2. Consider a problem in the two-dimensional Hilbert space $H = \mathbb{R}^2$, equipped with the standard inner product $\langle \mathbf{x}, \mathbf{y} \rangle = x_1y_1 + x_2y_2$ and the induced norm $\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2}$, where $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$. The feasible set is the closed, convex half-space defined by:

$$C = \{(x_1, x_2) \in H \mid -2x_1 + x_2 \le 1\}.$$

We analyze the performance of the algorithms in two separate cases.

Case 1. We define two operators $A_i : C \to \mathbb{R}^2$ by:

$$A_i(x_1, x_2) = \left(\frac{x_1}{2i} - \frac{1}{2i}, \frac{x_2}{2i} - \frac{1}{2i} \right), \quad i = 1, 2.$$

For this problem, the common solution set $\mathrm{VI}(C, A_1) \cap \mathrm{VI}(C, A_2)$ contains the point $(1, 1)$. To evaluate the effectiveness of IPSE, a comparative numerical experiment was performed against SUANTAI. The parameters for each algorithm are specified in Table 3.

TABLE 3. Parameter settings for iterative algorithms used in Case 1 of Example 4.2.

Algorithms	Parameters
SUANTAI	$\mu = 0.99, \quad \lambda_1^1 = 0.5, \quad \alpha_n^0 = \frac{1}{(n+1)^{0.3}},$ $\alpha_n^1 = \frac{1}{2n}, \quad \alpha_n^2 = 1 - \frac{1}{(n+1)^{0.3}} - \frac{1}{2n}.$
IPSE	$\mu = 0.99, \quad \lambda_1^1 = 0.5, \quad \theta_n = 0.9 - \frac{0.9}{n+20},$ $\alpha_n = 0.49 - \frac{0.49}{n+50}, \quad \beta_n = 0.35 + \frac{0.001n}{n+10}.$

Case 2. We now consider three operators $A_i : C \to \mathbb{R}^2$ defined by:

$$A_i(x_1, x_2) = \left(\frac{x_1}{3i} - \frac{1}{3i}, \frac{x_2}{3i} - \frac{1}{3i} \right), \quad i = 1, 2, 3.$$

The common solution set $\mathrm{VI}(C, A_1) \cap \mathrm{VI}(C, A_2) \cap \mathrm{VI}(C, A_3)$ again includes the point $(1, 1)$. The parameters used in this case are listed in Table 4.

TABLE 4. Parameter settings for iterative algorithms used in Case 2 of Example 4.2.

Algorithms	Parameters
SUANTAI	$\mu = 0.99, \quad \lambda_1^1 = 0.5, \quad \alpha_n^0 = \frac{1}{(n+1)^{0.3}},$ $\alpha_n^1 = \frac{1}{2n}, \quad \alpha_n^2 = \frac{n}{n+1}, \quad \alpha_n^3 = 1 - \frac{1}{(n+1)^{0.3}} - \frac{1}{2n} - \frac{n}{n+1}.$
IPSE	$\mu = 0.99, \quad \lambda_1^1 = 0.5, \quad \theta_n = 0.9 - \frac{0.9}{n+20},$ $\alpha_n = 0.49 - \frac{0.49}{n+50}, \quad \beta_n = 0.35 + \frac{0.001n}{n+10}.$

For both cases, the algorithms are initialized with $\mathbf{x}^0 = (-0.25, 2)$ and $\mathbf{x}^1 = (0.5, 1.25)$. The operators A_i defined in this example are Lipschitz continuous, satisfying the conditions of our strong convergence theorem. Therefore, as established in Theorem 3.6, the sequence $\{\mathbf{x}^n\}$ generated by IPSE is guaranteed to converge strongly to the unique common solution, $\mathbf{x}^* = (1, 1)$.

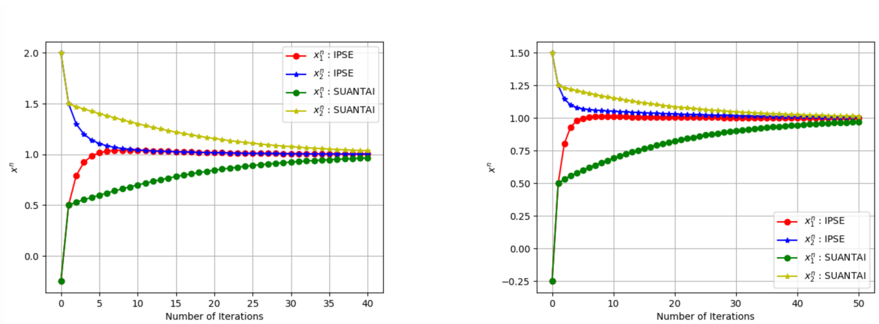
The detailed computational results are presented in Table 5 for Case 1 and Table 6 for Case 2. A graphical comparison of the convergence trajectories for both cases is shown in Figure 2.

Table 5. Computational results of $\mathbf{x}^n = (x_1^n, x_2^n)$ with error norm $\|(1,1) - (x_1^n, x_2^n)\|$ for Example 4.2 in **Case 1** with $\mathbf{x}^0 = (-0.25, 2)$, $\mathbf{x}^1 = (0.5, 1.25)$ and $n = 40$.

n	**SUANTAI**			**IPSE**		
	x_1^n	x_2^n	**Error**	x_1^n	x_2^n	**Error**
0	−0.25000	2.00000	1.60078	−0.25000	2.00000	1.60078
1	0.50000	1.25000	0.70711	0.50000	1.25000	0.70711
2	0.52980	1.47020	0.66497	0.79210	1.29847	0.36374
3	0.55342	1.44658	0.63156	0.92184	1.19860	0.21343
⋮	⋮	⋮	⋮	⋮	⋮	⋮
39	0.96162	1.03838	0.05428	1.00314	1.00312	0.00443
40	0.96446	1.03554	0.05025	1.00287	1.00287	0.00406

Table 6. Computational results of $\mathbf{x}^n = (x_1^n, x_2^n)$ with error norm $\|(1,1) - (x_1^n, x_2^n)\|$ for Example 4.2 in **Case 2** with $\mathbf{x}^0 = (-0.25, 2)$, $\mathbf{x}^1 = (0.5, 1.25)$ and $n = 40$.

n	**SUANTAI**			**IPSE**		
	x_1^n	x_2^n	**Error**	x_1^n	x_2^n	**Error**
0	−0.25000	2.00000	1.34629	−0.25000	2.00000	1.34629
1	0.50000	1.25000	0.55902	0.50000	1.25000	0.55902
2	0.53251	1.23374	0.52267	0.80484	1.14606	0.24376
3	0.55695	1.22152	0.49534	0.92884	1.10157	0.12402
⋮	⋮	⋮	⋮	⋮	⋮	⋮
39	0.94386	1.02807	0.06276	1.00263	1.01240	0.01268
40	0.94721	1.02640	0.05903	1.00250	1.01180	0.01206



(A) Case 1 : A_1 and A_2. (B) Case 2 : A_1, A_2 and A_3

Figure 2. Convergence behavior of $\mathbf{x}^n$ for Example 4.2.

To demonstrate the broader applicability and power of our proposed method beyond finite-dimensional Euclidean spaces, we now consider a numerical experiment in an infinite-dimensional Hilbert space. This test case is set in the function space $L_2([-1,1])$ and is designed to assess the robustness of IPSE's convergence when initiated from several different starting functions.

Example 4.3. Consider the Hilbert space $H = L_2([-1,1])$ with the inner product $\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)\,dt$ and the induced norm $\|f\| := \sqrt{\int_{-1}^{1} f^2(t)\,dt}$. We define the feasible set as the closed ball $C = \{x \in H : \|x\| \leq 2\}$. The projection onto this set, P_C is given by:

$$P_C(f(t)) = \begin{cases} f(t), & \text{if } \|f(t)\| \leq 2; \\ \dfrac{2f(t)}{\|f(t)\|}, & \text{if } \|f(t)\| > 2. \end{cases}$$

We define three operators $A_i : H \to H$ as $A_1(h(t)) = h(t) - 2t$, $A_2(h(t)) = \frac{3h(t)}{2} - 3t$, and $A_3(h(t)) = \frac{h(t)}{3} - \frac{2t}{3}$ for all $t \in [-1,1]$.

To evaluate the effectiveness of IPSE, we perform a comparative experiment against SUANTAI. The parameters for both algorithms are specified in Table 7.

TABLE 7. Parameter settings for iterative algorithms used in Example 4.3.

Algorithms	Parameters
SUANTAI	$\mu = 0.99, \quad \lambda_1^1 = 0.5, \quad \alpha_n^0 = \frac{1}{(n+1)^{0.3}},$ $\alpha_n^1 = \frac{1}{2n}, \quad \alpha_n^2 = \frac{n}{n+1}, \quad \alpha_n^3 = 1 - \frac{1}{(n+1)^{0.3}} - \frac{1}{2n} - \frac{n}{n+1}.$
IPSE	$\mu = 0.99, \quad \lambda_1^1 = 0.5, \quad \theta_n = 0.9 - \frac{0.9}{n+20},$ $\alpha_n = 0.49 - \frac{0.49}{n+50}, \quad \beta_n = 0.35 + \frac{0.001n}{n+10}.$

We test the performance of the algorithms using three different sets of starting functions:

Case 1. $x_0(t) = 0.5t$ and $x_1(t) = -0.8t^2$;

Case 2. $x_0(t) = t^2 - 0.9t$ and $x_1(t) = \dfrac{e^{-2t}}{48}$;

Case 3. $x_0(t) = t^2$ and $x_1(t) = -\cos(t) + 4t$.

The chosen parameter sequences $\{\theta_n\}$, $\{\alpha_n\}$, and $\{\beta_n\}$ satisfy the conditions outlined in Theorem 3.6. For this problem, it can be verified that the function $x^*(t) = 2t$ is a common solution, that is,

$$2t \in \mathrm{VI}(C, A_1) \cap \mathrm{VI}(C, A_2) \cap \mathrm{VI}(C, A_3).$$

Therefore, as established by Theorem 3.6, the sequence $\{x_n(t)\}$ generated by IPSE is guaranteed to converge strongly to $2t$.

The computational results, which track the error term $\|x_{n+1} - x_n\|$ to illustrate the rate of convergence, are presented for all three cases in Table 8. The corresponding convergence behavior is visualized in Figures 3, 4, and 5.

TABLE 8. Computational results of the error norm $\|x_{n+1}(t) - x_n(t)\|$ for Example 4.3, where $n = 40$.

n	**Case 1.**		**Case 2.**		**Case 3.**	
	SUANTAI	**IPSE**	**SUANTAI**	**IPSE**	**SUANTAI**	**IPSE**
1	0.65013	0.65013	0.89606	0.89606	3.67122	3.67122
2	0.73854	0.59547	0.67028	0.62347	3.49230	2.80517
3	0.59088	0.413951	0.79224	0.60163	2.91275	1.03059
4	0.66773	0.40113	0.59831	0.39377	2.76324	0.82295
⋮	⋮	⋮	⋮	⋮	⋮	⋮
39	0.04679	0.01357	0.06835	0.01924	0.77661	0.00632
40	0.05562	0.01363	0.04596	0.01796	0.76939	0.00562

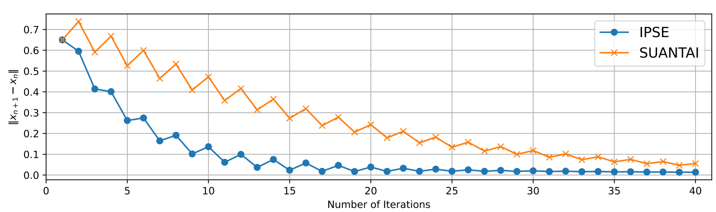


FIGURE 3. Convergence behavior of $\|x_{n+1} - x_n\|$ for Case 1 in Example 4.3.

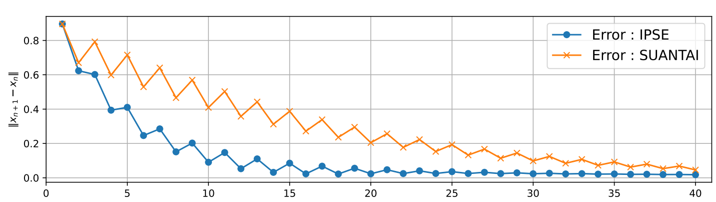


FIGURE 4. Convergence behavior of $\|x_{n+1} - x_n\|$ for Case 2 in Example 4.3.

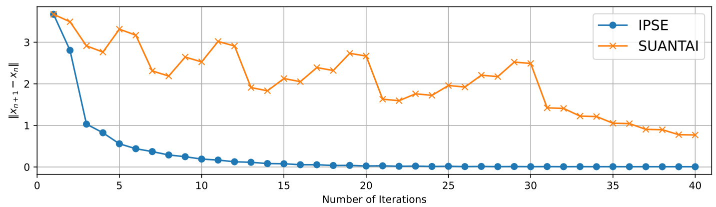


FIGURE 5. Convergence behavior of $\|x_{n+1} - x_n\|$ for Case 3 in Example 4.3.

A significant and practical application of algorithms for solving variational inequalities is in the field of convex minimization. The connection is straightforward: the first-order optimality condition for minimizing a differentiable convex function over a convex set is equivalent to a variational inequality where the operator is the function's gradient. This allows us to directly apply IPSE, to solve such optimization problems, as we demonstrate in the following example.

Example 4.4. Consider the common convex minimization problem, where the goal is to find a point that minimizes multiple convex functions. Let $f_i : C \to \mathbb{R}$ be a convex function for $i = 1, 2$. We consider the problem:

$$\min_{x \in C} f_i(x), \quad \text{for } i = 1, 2. \tag{4.1}$$

It is well known that a point $x^* \in C$ solves (4.1) for both functions if and only if x^* satisfies the following variational inequality for each function's gradient:

$$\langle \nabla f_i(x^*), x - x^* \rangle \geq 0, \quad \forall x \in C, \quad i = 1, 2, \tag{4.2}$$

which is equivalent to finding a point $x^* \in \mathrm{VI}(C, \nabla f_1) \cap \mathrm{VI}(C, \nabla f_2)$. We can therefore apply our algorithm to this problem by setting $A_i = \nabla f_i$.

For this specific example, we consider the Hilbert space $H = \mathbb{R}$ and take the feasible set as $C = [0.1, 10]$. We define two functions, $f_1, f_2 : [0.1, 10] \to \mathbb{R}$, as follows:

$$f_1(x) = \frac{(x-1)^2}{3} + 1 \quad \text{and} \quad f_2(x) = \frac{x^2}{2} - \ln x - \frac{1}{2}.$$

Clearly, ∇f_1 and ∇f_2 are Lipschitz continuous with constants $\frac{2}{3}$ and 1, and both are monotone. We test the performance of IPSE against SUANTAI using the initial points $x_0 = 4$ and $x_1 = 2.25$. The parameters for each algorithm are chosen as shown in Table 9.

The chosen parameter sequences $\{\theta_n\}, \{\alpha_n\}$, and $\{\beta_n\}$ satisfy the conditions required by Theorem 3.6. For this problem, the unique common solution is

TABLE 9. Parameter settings for iterative algorithms used in Example 4.4.

Algorithms	**Parameters**
SUANTAI	$\mu = 0.99, \quad \lambda_1^1 = 0.5, \quad \alpha_n^0 = \frac{1}{(n+1)^{0.3}},$ $\alpha_n^1 = \frac{1}{2n}, \quad \alpha_n^2 = 1 - \frac{1}{(n+1)^{0.3}} - \frac{1}{2n}.$
IPSE	$\mu = 0.99, \quad \lambda_1^1 = 0.5, \quad \theta_n = 0.9 - \frac{0.9}{n+20},$ $\alpha_n = 0.49 - \frac{0.49}{n+50}, \quad \beta_n = 0.35 + \frac{0.001n}{n+10}.$

$x^* = 1$, which is an element of $\mathrm{VI}(C, \nabla f_1) \cap \mathrm{VI}(C, \nabla f_2)$. As guaranteed by Theorem 3.6, the sequence $\{x_n\}$ generated by IPSE therefore converges strongly to 1. The computational results in Table 10 and the convergence plot in Figure 6 confirm this theoretical outcome and illustrate the performance of the tested algorithms.

TABLE 10. Computational results of x_n with error norm $\|1 - x_n\|$ for Example 4.4 with $x_0 = 4$, $x_1 = 2.25$ and $n = 40$.

n	**SUANTAI**		**IPSE**	
	x_n	**Error**	x_n	**Error**
0	4.00000	3.00000	4.00000	3.00000
1	2.25000	1.25000	2.25000	1.25000
2	1.23180	0.23180	1.53171	0.53271
3	0.90323	0.09677	1.23909	0.23909
4	0.77180	0.22820	1.11851	0.11851
⋮	⋮	⋮	⋮	⋮
39	0.93857	0.05984	1.00396	0.00396
40	0.94017	0.05832	1.00371	0.00371

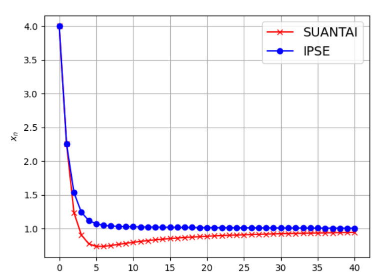


FIGURE 6. Convergence behaviour of x_n for Example 4.4.

The numerical experiments lead to several key conclusions regarding the performance of our proposed algorithm, IPSE. First and foremost, the results demonstrate a consistently superior performance by IPSE. Across all test casesin one dimension, multiple dimensions, and an infinite-dimensional spaceour algorithm achieved a faster convergence rate and higher accuracy than the competing methods within the same number of iterations. This clear advantage is shown in the initial test against CENSOR, SUANTAI, and SHEHU (see Table 2 and Figure 1) and is reinforced in all subsequent comparisons (Tables 5-10, Figures 2-6).

Another key takeaway is the robustness of the IPSE algorithm across different problem structures. The experiments highlight that its efficiency is remarkably stable. For instance, in the multi-dimensional problem of Example 4.2 (Tables 5, 6, Figure 2), IPSE's performance advantage was sustained even when the number of operators increased from two to three. Furthermore, the results from the infinite-dimensional space $L_2([-1,1])$ in Example 4.3 (Table 8, Figures 3, 4, 5) confirm that the algorithm's effectiveness is not confined to simple Euclidean settings.

5. Conclusion

In this work, we introduced and analyzed a novel inertial parallel-type subgradient extragradient method (IPSE) for finding a common solution set of variational inequality problems in real Hilbert spaces. A key feature of our method is a self-adaptive step size rule that does not require prior knowledge of the operators' Lipschitz constants, combined with a flexible inertial parameter.

We rigorously established both weak and strong convergence theorems for our main algorithm. Furthermore, we extended this framework to solve a more general problem involving a common element of the common solution set of variational inequality problems and the fixed point set of a quasi-nonexpansive mapping.

The numerical experiments confirmed our theoretical findings, illustrating that the proposed algorithm is highly effective and outperforms several recent methods in the literature. This work improves upon and generalizes many existing results, offering a robust and efficient tool for solving a wide class of optimization and feasibility problems.

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