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SOME INVARIANT POINT RESULTS FOR RATIONAL CONTRACTIONS INVOLVING CONTROL FUNCTIONS IN R-METRIC SPACE

Samriddhi Ghosh[1], Ramakant Bharadwaj[2], Amita Pal[3], Sonam[4], Purvee Bhardwaj[5] and Satyendra Narayan[6]

[1,2,4]Department of Mathematics, Amity University,
Kolkata, West Bengal, 700135, India
e-mail: [1]ritha98@gmail.com,[2]rkbhardwaj100@gmail.com,[4]sonam27msu@gmail.com

[3]Department of Engineering Sciences, MMIT, Pune,
Maharashtra, 411047, India
e-mail: amitapal20@gmail.com

[5]Department of Physics, JNCT Professional University,
Bhopal, Madhya Pradesh, 462038, India
e-mail: purvee.bhardwaj@jnctpu.edu.in

[6]School of Computer Science and Technology, Algoma University,
Brampton, ON L6V 1A3, Canada
e-mail: narayan.satyendra@gmail.com

Abstract. The main purpose of the research is to establish some invariant point results under the purview of R-Metric Spaces. For that purpose, the concept of control functions under the framework of R-metric space is used for rational type contraction. To the best of known literatures, this kind of work in the said space has not yet been found. Moreover, the concepts of R-preserving, R-continuous, R-contraction, R-convergence and R-Cauchy sequence have also been used for the said space. In addition to it, some corollaries have been represented as special cases of the established results. Also, some examples and applications have been developed to illustrate the established results and underscore its importance.

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[0]Corresponding author: R. Bhardwaj(rkbhardwaj100@gmail.com).

1. Introduction

The concept of metric spaces came to reality when a French mathematician named Maurice Fréchet [14, 15] initiated its study in the early twentieth century. Yet he did not coin the term "Metric spaces" though. This was done by another German mathematician named Felix Hausdorff [23] in 1914. Since then, the study of non-linear functional analysis took a kick-start as several more mathematicians took their signature steps into the topic.

For instance in 1922, a polish mathematician named Stefan Banach [5] established the concept of contraction, and proved a invariant point (fixed point) theorem which is commonly known as the Banach fixed-point theorem. Many other mathematicians such as Ravindran Kanan [25, 26] in 1968 and 1969, Ljubomir Ćirić [12] in 1974, Shoutir Kishore Chatterjea [10] in 1972 and Erdal Karapinar [27] in 2012, obtained motivation from Banach's discovery and established their own invariant point (fixed point) theorems.

Simultaneously many other generalizations of the concepts given by Fréchet and Hausdorff were established. Such as Kramosil [31] in 1975 discovered Fuzzy metric and Statistical metric space, Kirişci [30] in 2020 discovered Neutrosophic metric space, Park [33] in 2004 discovered Intuitionistic fuzzy metric space, Beaulaa [6] in 2015 introduced Soft fuzzy metric space, Das [13] in 2013 introduced Soft metric space, Jeyaraman [24] in 2023 introduced Neutrosophic soft metric space, Tian [43] in 2020 discovered Tripled fuzzy metric space, Backhtin [4] in 1989 gave the idea of b-metric space, George [16] in 2015 established Rectangular b-metric space, Sonam [40] in 2023 introduced Soft rectangular b-metric space along with an invariant point result in it, and Ghosh [20] in 2024 established the concept of Neutrosophic fuzzy metric space along with its topological characteristics. Later in 2025, Ghosh [18] also deduced some conventional fixed point theorems under the realm of his introduced space.

Combining the concepts of these new generalizations with the concepts of invariant point theory, many other mathematicians [1],[2],[11]-[21],[39],[44] have shown their interests in non-linear functional analysis. Berinde [7] in 2004 introduced the concept of weak contraction and utilized it to deduce a invariant point result, which marked a revolutionary step in the study of contraction mappings.

Now another generalization of Fréchet's concept of conventional metric spaces, has recently been introduced by Simak Khalehoghli [28] in 2020. In that research not only he extended Banach's and Brouwer's invariant point theorems for the newly introduced R-Metric space, but also he proved the existence and distinctiveness of solution of fractional integral equations of a particular type. After that, A. Malhotra [32] researched this topic in 2023, by deducing some fixed point theorems for multivalued mappings, under the purview of the said space. Then in 2025, S. Ghosh et. al. [22] introduced the ideology of integral type contractions under the purview of the said space. Recently in 2025, S. Raghuwanshi et. al. [35] also deduced two conventional fixed point theorems for contractions of Kannan and Chatterjea type under the framework of the mentioned space. To the best of known literatures, no other work has been done on the newly established space since then. So undoubtedly it can be said that, the concept of rational contraction and control functions in R-Metric space is novel and unique.

2. Preliminaries

Definition 2.1. ([22, 28]) **(R-Metric space)** Let (S, d) be a metric space, and R be a relation on S. Then the triplet (S, d, R) is said to be a R-metric space.

Example 2.2. Let $S = \mathbb{R}^+ \cup \{0\}$ be endowed with the standard distance metric d. Define $x\ R\ y$ if $x + y \geq xy$. Then, (S, d, R) becomes a R-metric space.

So, the only dissimilarity between the R-metric space and traditional metric space frameworks is that, the metric defined on a non empty set in the traditional metric space framework allows the usage of all the elements in the set. Whereas, the metric defined on a non empty set in the R-metric space framework, allows only the usage of elements which satisfy the mathematical ralation. And thus it can be said that, R-metric space framework is a more specified case of the traditional metric space framework.

Definition 2.3. ([22, 28]) **(R-Sequence)** Let $\{x_n\}$ be a sequence in (S, d, R). Then, it is said to be a R-sequence if and only if $x_n R x_{n+m}$ for all $m, n \in \mathbb{N}$.

Definition 2.4. ([22, 28]) **(R-Convergence)** A R-sequence $\{x_n\}$ in (S, d, R) R-converges to some $x \in S$ if for all $\epsilon > 0$ there exists a $K \in \mathbb{Z}$ such that $d(x_n, x) < \epsilon$, where $n \geq K$.

Definition 2.5. ([22, 28]) **(R-Cauchy sequence)** A R-sequence $\{x_n\}$ in a R-metric space $(S, \mathscr{D}, R)$ is said to be a R-Cauchy sequence if it is R-convergent to some $x \in S$. That is, for all $\epsilon > 0$ there exists a $K \in \mathbb{Z}$ such that $\mathscr{D}(x_m, x_n) < \epsilon$, where $m, n \geq K$. As a result, $x_n R x_m$ or $x_m R x_n$.

Definition 2.6. ([22, 28]) **(R-Continuous)** Let $T : S \to S$ be a mapping. T is proclaimed to be R-continuous at some point say $\mathfrak{v} \in S$, if for every R-sequence $\{\mathfrak{v}_n\}$ that R-converges to $\mathfrak{v} \in S$, it can be said that $\{T(\mathfrak{v}_n)\}$ converges to $T(\mathfrak{v})$. In this context it is said that, T is R-continuous on S if and only if T is R-continuous for all $\mathfrak{v} \in S$.

Remark 2.7. ([28]) Since every R-sequence is a sequence, it can be said that every continuous mapping $T : S \to S$ is R-continuous. The converse is not necessarily true.

Definition 2.8. ([22, 28]) **(R-Contraction)** A R-contraction is a self mapping $H : S \to S$ where for all $\mathfrak{v}, \mathfrak{w} \in S$ such that $\mathfrak{v}\ R\ \mathfrak{w}$, it can be said that,

$$\mathscr{D}(H(\mathfrak{v}), H(\mathfrak{w})) \leq \alpha \mathscr{D}(\mathfrak{v}, \mathfrak{w}),\ \forall\ \alpha \in (0, 1).$$

In this context, the constant α is also known as the Lipschitz contant.

Definition 2.9. ([22, 28]) **(R-Preserving)** A self mapping $T : S \to S$ is apparently R-preserving, if $x\ R\ y$ then $(T(x))\ R\ (T(y))$ for all $x, y \in S$.

Example 2.10. Let $S = \mathbb{R}^+$ with it's usual standard topology and $x\ R\ y$ is defined when $xy \in \{x, y\}$, where $x, y \in S$. Suppose the mapping $T : S \to S$ is defined as,

$$T(x) = \begin{cases} 1/x, & \text{if } x \leq 1, \\ 1, & \text{if } x > 1. \end{cases}$$

Now, let $x\ R\ y$. Thus $x = 1$ or $y = 1$. Therefore, either $T(x) = 1$ or $T(y) = 1$, implies that, $(T(x))\ R\ (T(y))$. From here it is clear that for $x\ R\ y$ $(T(x))\ R\ (T(y))$. Hence it can be said that, T is R-preserving.

Example 2.11. Let $S = \{1, 2, 3, 4, 5\}$ endowed with the standard euclidean metric d. And let $\mathfrak{v}\ R\ \mathfrak{w}$ is defined when $\mathfrak{v}\mathfrak{w} \in \{\mathfrak{v}, \mathfrak{w}\}$, where $\mathfrak{v}, \mathfrak{w} \in S$.
Suppose the mapping $H : S \to S$ is stipulated as,

$$H(\mathfrak{v}) = \begin{cases} 1, & \text{if } \mathfrak{v} = 1, \\ 6 - \mathfrak{v}, & \text{if } \mathfrak{v} \in \{2, 3, 4, 5\}. \end{cases}$$

Now let $\mathfrak{v}\ R\ \mathfrak{w}$. Thus $\mathfrak{v} = 1$ or $\mathfrak{w} = 1$. Therefore, either $H(\mathfrak{v}) = 1$ or $H(\mathfrak{w}) = 1$, implies that, $(H(\mathfrak{v}))R(H(\mathfrak{w}))$. In other words, for $\mathfrak{v}\ R\ \mathfrak{w}$, $(H(\mathfrak{v}))R(H(\mathfrak{w}))$. Hence it can be said that, H is R-preserving.

Example 2.12. Let $S = \mathbb{N}$ with it's usual standard topology and $x\ R\ y$ is defined when $xy \geq \min\{x, y\}$, where $x, y \in S$. Suppose the mapping $T : S \to S$ is defined as $T(x) = [e^x]$. Now choose $x_1, x_2 \in S$ such that $x_1 > x_2$. Then, $x_1x_2 \geq x_2$. From here it can be said that x_1Rx_2. Putting x_1, x_2 in $T(x) = [e^x]$ it is clear that,

$$(Tx_1 = [e^{x_1}]) > (Tx_2 = [e^{x_2}]).$$

Then clearly, $Tx_1Tx_2 \geq Tx_2$. Which implies, $(Tx_1)R(Tx_2)$. Therefore, for $x\ R\ y$, $(Tx)\ R\ (Ty)$ for all $x, y \in S$. Hence it can be said that, T is R-preserving.

Theorem 2.13. ([28]) *Suppose $(S, \mathscr{D}, R)$ is a R-complete R-metric space and $\alpha \in (0, 1)$. Let $H : S \to S$ be R-continuous, R-preserving, R-contraction with Lipschitz constant α. Assume that there exists a $x_0 \in S$ such that $x_0\ R\ y$ for all $y \in H(S)$. Then H has an unique invariant point. And also, H is a Picard operator.*

Now, some key points about the topology of R-metric spaces has been discussed below.

Definition 2.14. ([28]) **(R-Compactness)** A set $W \subseteq S$ is apparently R-compact if every R-sequence $\{x_n\} \subseteq W$ has a subsequence in W which is convergent.

Theorem 2.15. ([28]) *Let $S = \mathbb{R}^n$ be endowed with it's usual topology and $W \subseteq S$ be a strong R-compact set. Suppose $T : W \to W$ is R-continuous and R-preserving. Let $x_0 \in W$ be given such that x_0Ry for all $y \in T(W)$. Then T has a invariant point.*

Definition 2.16. ([28]) **(R-Limit point)** Let $E \subseteq S$. Then $x \in S$ is called an R-limit point of E if there exists an R-sequence $\{x_n\} \subseteq E : x_n \neq x$ for all $n \in \mathbb{N}$ such that $\{x_n\}$ R-converges to x.

Definition 2.17. ([28]) The collection of all R-limit points of E is denotes by E'^R.

Definition 2.18. ([28]) $E \subseteq S$ is said to be R-closed if $E'^R \subseteq E$.

Definition 2.19. ([28]) $E \subseteq S$ is said to be R-open if E^C is R-closed.

Theorem 2.20. ([28]) *$E \subseteq S$ is R-open if and only if for any $x \in E$ and any $\{x_n\}$ which is an R-sequence such that $\{x_n\}$ R-converges to x, then there exists $\mathfrak{N} \in \mathbb{N}$ such that $x_n \in E$ for all $n \geq \mathfrak{N}$.*

Theorem 2.21. ([28]) *The set $\{H \subseteq S : H^C$ is R-closed$\}$ is a topology on S. This is also known as R-topology, which is designated as τ_R.*

Theorem 2.22. ([28]) *Any metric topology $\tau_{\mathscr{D}}$ on S, is a subset of the R-topology defined as τ_R.*

Lemma 2.23. ([28]) *If $R := S \times S$, then $\tau_R = \tau_{\mathscr{D}}$.*

Remark 2.24. ([28]) Let $(S, \mathscr{D})$ be a metric space and $R := S \times S$. Then the R-metric space $(S, \mathscr{D}, R)$ is identical to the metric space $(S, \mathscr{D})$.

Definition 2.25. ([29]) **(Control function)** Any function applied to a contraction in invariant point theory, which modifies or generalizes the contractive condition under study, is known as a control function.

Definition 2.26. ([3, 22]) **(Picard Operator)** Any self-mapping on a metric space, possesing a unique invariant point where every repeated application of the mapping converges the sequence to its invariant point is known as Picard operator.

3. Main results

In this section two invariant point theorems have been established in R-metric space, one (Theorem 3.1) for a rational inequality without using any control function and the other (Theorem 3.3) for a rational inequality under the usage of control functions. In both the theorems the concept of R-preserving and R-continuous mappings have been taken. The established results are useful to find the solution of integral equations.

Theorem 3.1. *Suppose $(S, \mathscr{D}, R)$ is a R-metric space which is R-complete and $H : S \to S$ is a R-preserving, R-continuous self-map that satisfies the R-contractive condition,*

$$\mathscr{D}(H\mathfrak{v}, H\mathfrak{w}) \leq \alpha \left[\frac{\mathscr{D}(\mathfrak{w}, H^2\mathfrak{v}) \{\mathscr{D}(\mathfrak{v}, H\mathfrak{w}) + \mathscr{D}(\mathfrak{v}, H\mathfrak{v}) + \mathscr{D}(\mathfrak{w}, H\mathfrak{w})\}}{\mathscr{D}(\mathfrak{w}, H\mathfrak{v}) + \mathscr{D}(\mathfrak{v}, \mathfrak{w}) + \mathscr{D}(\mathfrak{v}, H\mathfrak{w})} \right]$$
$$-\beta \left[\frac{\mathscr{D}(\mathfrak{v}, H\mathfrak{v}) \{\mathscr{D}(\mathfrak{w}, H\mathfrak{v}) + \mathscr{D}(\mathfrak{v}, H\mathfrak{v})\}}{\mathscr{D}(\mathfrak{v}, \mathfrak{w})} \right],$$

where α and β are constants such that $2\alpha - \beta < 1/2$ and $v \neq \mathfrak{w}$. Suppose there exist $\mathfrak{v}_0 \in S$ such that $\mathfrak{v}_0 \; R \; \mathfrak{w}$ for all $\mathfrak{w} \in H(S)$. Then H has an unique invariant point in $(S, \mathscr{D}, R)$.

Proof. Let $\mathfrak{v}_0 \in S$ be arbitrary. Define by induction $H^n : S \to S$ by $H^0\mathfrak{v}_0 = \mathfrak{v}_0$, $H^1\mathfrak{v}_0 = \mathfrak{v}_1$, $H^2\mathfrak{v}_0 = H(H\mathfrak{v}_0) = H\mathfrak{v}_1 = \mathfrak{v}_2$, ... ,$H^n\mathfrak{v}_0 = \mathfrak{v}_n$ where $n \geq 0$. Thus an iterative sequence $\{\mathfrak{v}_n\}$ associated with H is obtained.

Let $m, n \in \mathbb{N}$ and $n < m$, substitute $m - n = k$. We have $\mathfrak{v}_0 \ R \ H^k(\mathfrak{v}_0)$. Since H is R-preserving,

$$[H^n(\mathfrak{v}_0)] \ R \ \left[H^{n+k}(\mathfrak{v}_0)\right],$$

which gives an implication that, $\mathfrak{v}_n \ R \ \mathfrak{v}_m$. Hence $\{\mathfrak{v}_n\}$ is a R-sequence.

Now from the inequality given above it can be said that,

$$\begin{aligned}
&\mathscr{D}(\mathfrak{v}_n, \mathfrak{v}_{n+1}) = \mathscr{D}(H\mathfrak{v}_{n-1}, H\mathfrak{v}_n) \\
&\quad \leq \alpha \left[\frac{\mathscr{D}(\mathfrak{v}_n, H(H\mathfrak{v}_{n-1}))\left\{\mathscr{D}(\mathfrak{v}_{n-1}, H\mathfrak{v}_n) + \mathscr{D}(\mathfrak{v}_n, H\mathfrak{v}_n) + \mathscr{D}(\mathfrak{v}_{n-1}, H\mathfrak{v}_{n-1})\right\}}{\mathscr{D}(\mathfrak{v}_n, H\mathfrak{v}_{n-1}) + \mathscr{D}(\mathfrak{v}_{n-1}, \mathfrak{v}_n) + \mathscr{D}(\mathfrak{v}_{n-1}, H\mathfrak{v}_n)}\right] \\
&\qquad - \beta \left[\frac{\mathscr{D}(\mathfrak{v}_{n-1}, H\mathfrak{v}_{n-1})\left\{\mathscr{D}(\mathfrak{v}_n, H\mathfrak{v}_{n-1}) + \mathscr{D}(\mathfrak{v}_{n-1}, H\mathfrak{v}_{n-1})\right\}}{\mathscr{D}(\mathfrak{v}_{n-1}, \mathfrak{v}_n)}\right] \\
&\quad \leq \alpha \left[\mathscr{D}(\mathfrak{v}_{n-1}, \mathfrak{v}_{n+1}) + \mathscr{D}(\mathfrak{v}_n, \mathfrak{v}_{n+1}) + \mathscr{D}(\mathfrak{v}_{n-1}, \mathfrak{v}_n)\right] - \beta \left[\mathscr{D}(\mathfrak{v}_{n-1}, \mathfrak{v}_n)\right] \\
&\quad \leq 2\alpha \left[\mathscr{D}(\mathfrak{v}_n, \mathfrak{v}_{n+1}) + \mathscr{D}(\mathfrak{v}_{n-1}, \mathfrak{v}_n)\right] - \beta \left[\mathscr{D}(\mathfrak{v}_{n-1}, \mathfrak{v}_n) + \mathscr{D}(\mathfrak{v}_n, \mathfrak{v}_{n+1})\right] \\
&\quad \leq (2\alpha - \beta) \left[\mathscr{D}(\mathfrak{v}_n, \mathfrak{v}_{n+1}) + \mathscr{D}(\mathfrak{v}_{n-1}, \mathfrak{v}_n)\right],
\end{aligned}$$

it implies that,

$$\frac{\mathscr{D}(\mathfrak{v}_n, \mathfrak{v}_{n+1})}{2\alpha - \beta} \leq \mathscr{D}(\mathfrak{v}_n, \mathfrak{v}_{n+1}) + \mathscr{D}(\mathfrak{v}_{n-1}, \mathfrak{v}_n)$$

or

$$\left[\frac{1 - (2\alpha - \beta)}{2\alpha - \beta}\right] \mathscr{D}(\mathfrak{v}_n, \mathfrak{v}_{n+1}) \leq \mathscr{D}(\mathfrak{v}_{n-1}, \mathfrak{v}_n),$$

that is,

$$\mathscr{D}(\mathfrak{v}_n, \mathfrak{v}_{n+1}) \leq \left(\frac{2\alpha - \beta}{1 - 2\alpha + \beta}\right) \mathscr{D}(\mathfrak{v}_{n-1}, \mathfrak{v}_n).$$

Continuing in this way,

$$\mathscr{D}(\mathfrak{v}_n, \mathfrak{v}_{n+1}) \leq \left[\frac{2\alpha - \beta}{1 - 2\alpha + \beta}\right]^n \mathscr{D}(\mathfrak{v}_0, \mathfrak{v}_1).$$

Then,

$$\begin{aligned}
\mathscr{D}(\mathfrak{v}_n, \mathfrak{v}_m) &\leq \mathscr{D}(\mathfrak{v}_n, \mathfrak{v}_{n+1}) + \mathscr{D}(\mathfrak{v}_{n+1}, \mathfrak{v}_{n+2}) + \cdots + \mathscr{D}(\mathfrak{v}_{m-1}, \mathfrak{v}_m) \\
&\leq c^n \mathscr{D}(\mathfrak{v}_0, \mathfrak{v}_1) + c^{n+1} \mathscr{D}(\mathfrak{v}_0, \mathfrak{v}_1) + \cdots + c^{m-1} \mathscr{D}(\mathfrak{v}_0, \mathfrak{v}_1) \\
&\leq \left[c^n + c^{n+1} + \cdots + c^{m-1}\right] \mathscr{D}(\mathfrak{v}_0, \mathfrak{v}_1) \\
&\leq \frac{c^n}{1 - c} \mathscr{D}(\mathfrak{v}_0, \mathfrak{v}_1),
\end{aligned}$$

where

$$c = \frac{2\alpha - \beta}{1 - 2\alpha + \beta} < 1.$$

Therefore,

$$\mathscr{D}(\mathfrak{v}_n, \mathfrak{v}_m) \to 0 \text{ as } m, n \to 0.$$

Hence, $\{\mathfrak{v}_n\}$ is R-Cauchy.

Now by R-completeness of S it can be said that, $\{\mathfrak{v}_n\}$ R-converges to some $u \in S$. Now since H is R-continuous, $\{H\mathfrak{v}_n\}$ R-converges to Hu. Therefore,

$$\begin{aligned} Hu &= H\left(\lim_{n\to\infty} \mathfrak{v}_n\right) \\ &= \lim_{n\to\infty} H\mathfrak{v}_n \\ &= \lim_{n\to\infty} \mathfrak{v}_{n+1} = u. \end{aligned}$$

Hence, u is a invariant point of H.

For the uniqueness, let $a \in S$ be an invariant point of H. As a result, $\mathfrak{v}_0 R a$. Hence $(H^n\mathfrak{v}_0 = \mathfrak{v}_n)\ R\ (H^n a = a)$ for all $n \in \mathbb{W}$. So by the virtue of triangular inequality it can be written that,

$$\begin{aligned} \mathscr{D}(u, a) &= \mathscr{D}(H^n u, H^n a) \\ &\leq \mathscr{D}(H^n u, H^n x_0) + \mathscr{D}(H^n x_0, H^n a) \\ &\leq c^n \mathscr{D}(u, x_0) + c^n \mathscr{D}(x_0, a), \end{aligned}$$

which tends to 0 as $n \to \infty$ because $c < 1$. And thus, $u = a$. Hence, the invariant point is unique. □

Lemma 3.2. *The mapping defined in Theorem* 3.1 *is a Picard operator.*

Proof. Let $\mathfrak{v} \in S$ be arbitrary. Since $\mathfrak{v}_0\ R\ (H\mathfrak{v})$ and H is R-preservative, it can be written that $(H^n\mathfrak{v}_0)\ R\ (H^{n+1}\mathfrak{v})$ for all $n \in \mathbb{W}$.
Therefore,

$$\begin{aligned} \mathscr{D}(u, H^n\mathfrak{v}) &= \mathscr{D}(H^n u, H^n \mathfrak{v}) \\ &\leq \mathscr{D}(H^{n-1}u, H^{n-1}\mathfrak{v}_0) + \mathscr{D}(H^{n-1}\mathfrak{v}_0, H^{n-1}(H\mathfrak{v})) \\ &\leq c^{n-1}\mathscr{D}(u, \mathfrak{v}_0) + c^{n-1}\mathscr{D}(\mathfrak{v}_0, H\mathfrak{v}), \end{aligned}$$

which tends to 0 as $n \to \infty$, because $c < 1$. Hence,

$$\lim_{n\to\infty} H^n\mathfrak{v} = u.$$

□

Theorem 3.3. *Let $(S, \mathscr{D}, R)$ be a R-metric space which is R-complete, and $H : S \to S$ be a R-preserving, R-continuous self-map that satisfies the R-contractive condition,*

$$\mathscr{D}(H\mathfrak{v}, H\mathfrak{w}) \leq \alpha\phi\left[\frac{\mathscr{D}(\mathfrak{w}, H\mathfrak{w})\left\{\mathscr{D}(H\mathfrak{v}, \mathfrak{w}) + \mathscr{D}(\mathfrak{w}, H\mathfrak{w}) + \mathscr{D}(\mathfrak{v}, H\mathfrak{v})\right\}}{1 + \mathscr{D}(H\mathfrak{v}, H^2\mathfrak{v})}\right]$$
$$- \psi\left\{\mathscr{D}(\mathfrak{v}, H\mathfrak{w}) + \mathscr{D}(\mathfrak{w}, H\mathfrak{v})\right\},$$

where the control functions $\phi, \psi : (0, \infty) \to (0, \infty)$ are continuous such that $\phi(t) \leq t$ and $\phi(0) = \psi(0) = 0$. Also, α is a constant such that $\alpha \in (0, 1/2)$ and $\mathfrak{v} \neq \mathfrak{w}$. Suppose there exists $\mathfrak{v}_0 \in S$ such that $\mathfrak{v}_0\ R\ \mathfrak{w}$ for all $\mathfrak{w} \in H(S)$. Then H has a unique invariant point in $(S, \mathscr{D}, R)$.

Proof. Let $\mathfrak{v}_0 \in S$ be arbitrary. Define by induction $H^n : S \to S$ by $H^0\mathfrak{v}_0 = \mathfrak{v}_0$, $H^1\mathfrak{v}_0 = \mathfrak{v}_1$, $H^2\mathfrak{v}_0 = H(H\mathfrak{v}_0) = H\mathfrak{v}_1 = \mathfrak{v}_2, \cdots, H^n\mathfrak{v}_0 = \mathfrak{v}_n$, where $n \geq 0$. Thus an iterative sequence $\{\mathfrak{v}_n\}$ associated with H is obtained.

Let $m, n \in \mathbb{N}$ and $n < m$, substitute $m - n = k$. Then we have $\mathfrak{v}_0\ R\ H^k(\mathfrak{v}_0)$. Since H is R-preserving, $[H^n(\mathfrak{v}_0)]\ R\ \left[H^{n+k}(\mathfrak{v}_0)\right]$, which implies, $\mathfrak{v}_n\ R\ \mathfrak{v}_m$. Hence $\{\mathfrak{v}_n\}$ is a R-sequence.

Now from the R-contractive condition of H it can be said that,

$$\begin{aligned}
\mathscr{D}(\mathfrak{v}_n,\mathfrak{v}_{n+1}) &= \mathscr{D}(H\mathfrak{v}_{n-1}, H\mathfrak{v}_n)\\
&\leq \alpha\phi\left[\frac{\mathscr{D}(\mathfrak{v}_n, H\mathfrak{v}_n)\left\{\mathscr{D}(H\mathfrak{v}_{n-1}, \mathfrak{v}_n) + \mathscr{D}(\mathfrak{v}_{n-1}, H\mathfrak{v}_{n-1}) + \mathscr{D}(\mathfrak{v}_n, H\mathfrak{v}_n)\right\}}{1 + \mathscr{D}(H\mathfrak{v}_{n-1}, H(H\mathfrak{v}_{n-1}))}\right]\\
&\quad - \psi\left\{\mathscr{D}(\mathfrak{v}_{n-1}, H\mathfrak{v}_n) + \mathscr{D}(\mathfrak{v}_n, H\mathfrak{v}_{n-1})\right\}\\
&\leq \alpha\left[\phi\left\{\mathscr{D}(H\mathfrak{v}_{n-1}, \mathfrak{v}_n) + \mathscr{D}(\mathfrak{v}_{n-1}, H\mathfrak{v}_{n-1}) + \mathscr{D}(\mathfrak{v}_n, H\mathfrak{v}_n)\right\}\right]\\
&\leq \alpha\left[\phi\left\{\mathscr{D}(\mathfrak{v}_{n-1}, H\mathfrak{v}_{n-1}) + \mathscr{D}(\mathfrak{v}_n, H\mathfrak{v}_n)\right\}\right]\\
&\leq \alpha\left[\mathscr{D}(\mathfrak{v}_n, \mathfrak{v}_{n+1}) + \mathscr{D}(\mathfrak{v}_{n-1}, \mathfrak{v}_n)\right],
\end{aligned}$$

it implies that,

$$\frac{\mathscr{D}(\mathfrak{v}_n, \mathfrak{v}_{n+1})}{\alpha} \leq \mathscr{D}(\mathfrak{v}_{n-1}, \mathfrak{v}_n) + \mathscr{D}(\mathfrak{v}_n, \mathfrak{v}_{n+1})$$

and so,

$$\left[\frac{1-\alpha}{\alpha}\right]\mathscr{D}(\mathfrak{v}_n, \mathfrak{v}_{n+1}) \leq \mathscr{D}(\mathfrak{v}_{n-1}, \mathfrak{v}_n)$$

that is,

$$\mathscr{D}(\mathfrak{v}_n, \mathfrak{v}_{n+1}) \leq \left(\frac{\alpha}{1-\alpha}\right)\mathscr{D}(\mathfrak{v}_{n-1}, \mathfrak{v}_n).$$

Continuing in this way, we have

$$\mathscr{D}(\mathfrak{v}_n, \mathfrak{v}_{n+1}) \leq \left[\frac{\alpha}{1-\alpha}\right]^n\mathscr{D}(\mathfrak{v}_0, \mathfrak{v}_1).$$

Then,

$$\begin{aligned}\mathscr{D}(\mathfrak{v}_n,\mathfrak{v}_m) &\leq \mathscr{D}(\mathfrak{v}_n,\mathfrak{v}_{n+1})+\mathscr{D}(\mathfrak{v}_{n+1},\mathfrak{v}_{n+2})+\cdots+\mathscr{D}(\mathfrak{v}_{m-1},\mathfrak{v}_m)\\ &\leq p^n\mathscr{D}(\mathfrak{v}_0,\mathfrak{v}_1)+p^{n+1}\mathscr{D}(\mathfrak{v}_0,\mathfrak{v}_1)+\cdots+p^{m-1}\mathscr{D}(\mathfrak{v}_0,\mathfrak{v}_1)\\ &\leq \left[p^n+p^{n+1}+\cdots+p^{m-1}\right]\mathscr{D}(\mathfrak{v}_0,\mathfrak{v}_1)\\ &\leq \frac{p^n}{1-p}\,\mathscr{D}(\mathfrak{v}_0,\mathfrak{v}_1),\end{aligned}$$

where

$$p=\frac{\alpha}{1-\alpha}<1.$$

Therefore,

$$\mathscr{D}(\mathfrak{v}_n,\mathfrak{v}_m)\to 0 \text{ as } m,n\to 0.$$

Hence $\{\mathfrak{v}_n\}$ is R-Cauchy. Now by R-completeness of S it can be said that, $\{\mathfrak{v}_n\}$ R-converges to some $u\in S$. Now since H is R-continuous, $\{H\mathfrak{v}_n\}$ R-converges to Hu. Therefore,

$$\begin{aligned}Hu &= H\left(\lim_{n\to\infty}\mathfrak{v}_n\right)\\ &=\lim_{n\to\infty}H\mathfrak{v}_n\\ &=\lim_{n\to\infty}\mathfrak{v}_{n+1}=u.\end{aligned}$$

Hence, u is a invariant point of H.

For the uniqueness, let $b\in S$ be an invariant point of H. As a result, $\mathfrak{v}_0\ R\ b$. Hence $(H^n\mathfrak{v}_0=\mathfrak{v}_n)\ R\ (H^nb=b)\ \forall\ n\in\mathbb{W}$. So by the virtue of triangular inequality it can be written that,

$$\begin{aligned}\mathscr{D}(u,b) &= \mathscr{D}(H^nu,H^nb)\\ &\leq \mathscr{D}(H^nu,H^n\mathfrak{v}_0)+\mathscr{D}(H^n\mathfrak{v}_0,H^nb)\\ &\leq p^n\mathscr{D}(u,\mathfrak{v}_0)+p^n\mathscr{D}(\mathfrak{v}_0,b).\end{aligned}$$

Which tends to 0 as $n\to\infty$. [Since, $p<1$]
As a result, $u=b$. Hence, the invariant point is unique. □

Lemma 3.4. *The mapping defined in Theorem* (3.3) *is a Picard operator.*

Proof. Let $\mathfrak{v}\in S$ be arbitrary. Since $\mathfrak{v}_0\ R\ (H\mathfrak{v})$ and H is R-preservative, it can be written that $(H^n\mathfrak{v}_0)\ R\ (H^{n+1}\mathfrak{v})$ for all $n\in\mathbb{W}$. Therefore, we have

$$\begin{aligned}\mathscr{D}(u,H^n\mathfrak{v}) &= \mathscr{D}(H^nu,H^n\mathfrak{v})\\ &\leq \mathscr{D}(H^{n-1}u,H^{n-1}\mathfrak{v}_0)+\mathscr{D}(H^{n-1}\mathfrak{v}_0,H^{n-1}(H\mathfrak{v}))\\ &\leq p^{n-1}\mathscr{D}(u,\mathfrak{v}_0)+p^{n-1}\mathscr{D}(\mathfrak{v}_0,H\mathfrak{v}),\end{aligned}$$

which tends to 0 as $n \to \infty$, because $p < 1$. Hence,

$$\lim_{n\to\infty} H^n \mathfrak{v} = u.$$

$\square$

4. Illustration

Example 4.1. Let $S = \mathbb{N}$, $\mathscr{D} = |\mathfrak{w} - \mathfrak{v}|$ for all $\mathfrak{v}, \mathfrak{w} \in S$ and R be a relation defined on S when $vx \in \{\mathfrak{v}, \mathfrak{w}\}$. Then, $(S, \mathscr{D}, R)$ is a R-metric space. Suppose $H : S \to S$ is stipulated by

$$H(\mathfrak{v}) = \begin{cases} 1, & \text{if } \mathfrak{v} = 1, \\ [1 + \frac{1}{v}], & \text{if } v > 1. \end{cases}$$

Now let $\{\mathfrak{v}_n\}$ be a R-sequence in S. Then it is clear that there exists $K \in \mathbb{N}$ such that,

$$\mathfrak{v}_n = 1, \ \forall \ n \geq K, \ n \in \mathbb{N}$$

Hence $\{\mathfrak{v}_n\}$ R-converges to 1. Therefore $(S, \mathscr{D}, R)$ is R-complete and T is R-continuous. Suppose $v \ R \ w$ then $\mathfrak{v} = 1$ or $\mathfrak{w} = 1$. Therefore $H(\mathfrak{v}) = 1$ or $H(\mathfrak{w}) = 1$. Which means that, $(H(\mathfrak{v})) \ R \ (H(\mathfrak{w}))$. That is, for $\mathfrak{v} \ R \ \mathfrak{w}$, $(H(\mathfrak{v})) \ R \ (H(\mathfrak{w}))$ for all $\mathfrak{v}, \mathfrak{w} \in S$. Hence T is R-preserving.

From the definition of $H(\mathfrak{v})$ it also follows that,

$$\mathscr{D}(H\mathfrak{v}, H\mathfrak{w}) = |H(\mathfrak{v}) - H(\mathfrak{w})| = 0, \ \forall \ \mathfrak{v}, \mathfrak{w} \in S.$$

So by taking $\alpha = 0.5$ and $\beta = 0.51$ it can be claimed that,

$$\mathscr{D}(H\mathfrak{v}, H\mathfrak{w}) \leq \alpha \left[\frac{\mathscr{D}(\mathfrak{w}, H^2\mathfrak{v}) \{ \mathscr{D}(\mathfrak{v}, H\mathfrak{w}) + \mathscr{D}(\mathfrak{v}, H\mathfrak{v}) + \mathscr{D}(\mathfrak{w}, H\mathfrak{w}) \}}{\mathscr{D}(\mathfrak{w}, H\mathfrak{v}) + \mathscr{D}(\mathfrak{v}, \mathfrak{w}) + \mathscr{D}(\mathfrak{v}, H\mathfrak{w})} \right]$$
$$-\beta \left[\frac{\mathscr{D}(\mathfrak{v}, H\mathfrak{v}) \{ \mathscr{D}(\mathfrak{w}, H\mathfrak{v}) + \mathscr{D}(\mathfrak{v}, H\mathfrak{v}) \}}{\mathscr{D}(\mathfrak{v}, \mathfrak{w})} \right],$$

where $\mathfrak{w} \neq \mathfrak{v}$ and $\mathfrak{v}\mathfrak{w} \in \{\mathfrak{v}, \mathfrak{w}\}$ such that $\mathfrak{v}, \mathfrak{w} \in \mathbb{N}$. Thus H satisfies the R-contractive condition, and with that note, all the conditions of Theorem 3.1 hold true. Therefore, H has an unique invariant point. For this example it is evident that, H has an unique invariant point at $\mathfrak{v} = 1$.

Example 4.2. Let $S = [1, \infty)$, $d = |\mathfrak{w} - \mathfrak{v}|$ for all $\mathfrak{v}, \mathfrak{w} \in S$ and R be a relation defined on S when $\mathfrak{w} + \mathfrak{v} \in \{\mathfrak{v} + 1, \mathfrak{w} + 1\}$. Then, $(S, \mathscr{D}, R)$ is a R-metric space.

Suppose $H : S \to S$ is stipulated by

$$H(\mathfrak{v}) = \begin{cases} 1, & \text{if } \mathfrak{v} = 1, \\ \mathfrak{v}^{[1/\mathfrak{v}]}, & \text{if } \mathfrak{v} > 1. \end{cases}$$

Now let $\{\mathfrak{v}_n\}$ be a R-sequence in S. Then it is comprehensible that there exists $K \in \mathbb{N}$ such that,

$$\mathfrak{v}_n = 1, \ \forall\ n \geq K, \ n \in \mathbb{N}.$$

Hence $\{\mathfrak{v}_n\}$ R-converges to 1. Therefore $(S, \mathscr{D}, R)$ is R-complete and H is R-continuous.

Suppose $\mathfrak{v}\ R\ \mathfrak{w}$ then $\mathfrak{v} = 1$ or $\mathfrak{w} = 1$. Therefore, $H(\mathfrak{v}) = 1$ or $H(\mathfrak{w}) = 1$, which means that $(H(\mathfrak{v}))\ R\ (H(\mathfrak{w}))$. That is, for $\mathfrak{v}\ R\ \mathfrak{w}$, $(H(\mathfrak{v}))\ R\ (H(\mathfrak{w}))$ for all $\mathfrak{v}, \mathfrak{w} \in S$. Hence, H is R-preserving. From the definition of $H(\mathfrak{v})$ it also follows that,

$$\mathscr{D}(H\mathfrak{v}, H\mathfrak{w}) = |H(\mathfrak{v}) - H(\mathfrak{w})| = 0, \ \forall\ \mathfrak{v}, \mathfrak{w} \in S.$$

So by taking $\alpha = \frac{499}{1000}$ and defining the control functions $\phi, \psi : (0, \infty) \to (0, \infty)$ as $\phi(t) = \frac{99t}{100}$ and $\psi(t) = \frac{t}{10000}$, it can be claimed that

$$\begin{aligned}\mathscr{D}(H\mathfrak{v}, H\mathfrak{w}) \leq \alpha\phi &\left[\frac{\mathscr{D}(\mathfrak{w}, H\mathfrak{w})\left\{\mathscr{D}(H\mathfrak{v}, \mathfrak{w}) + \mathscr{D}(\mathfrak{w}, H\mathfrak{w}) + \mathscr{D}(\mathfrak{v}, H\mathfrak{v})\right\}}{1 + \mathscr{D}(H\mathfrak{v}, H^2\mathfrak{v})}\right] \\ &- \psi\left\{\mathscr{D}(\mathfrak{v}, H\mathfrak{w}) + \mathscr{D}(\mathfrak{w}, H\mathfrak{v})\right\},\end{aligned}$$

where $\mathfrak{v} \neq \mathfrak{w}$ and $\mathfrak{w} + \mathfrak{v} \in \{\mathfrak{v} + 1, \mathfrak{w} + 1\}$ such that $\mathfrak{v}, \mathfrak{w} \in [1, \infty)$. Thus H satisfies the R-contractive condition, and with that note, all the conditions of Theorem 3.3 hold true. Therefore H has an unique invariant point. For this example it is evident that, H has a unique invariant point at $\mathfrak{v} = 1$.

5. Consequences

Corollary 5.1. *Let $(S, \mathscr{D}, R)$ be a R-metric space which is R-complete and $H : S \to S$ be a R-continuous, R-preserving self-map that satisfies the R-contractive condition,*

$$\mathscr{D}(H\mathfrak{v}, H\mathfrak{w}) \leq \alpha\left[\frac{\mathscr{D}(\mathfrak{w}, H^2\mathfrak{v})\left\{\mathscr{D}(\mathfrak{v}, H\mathfrak{w}) + \mathscr{D}(\mathfrak{w}, H\mathfrak{w}) + \mathscr{D}(\mathfrak{v}, H\mathfrak{v})\right\}}{\mathscr{D}(\mathfrak{w}, H\mathfrak{v}) + \mathscr{D}(\mathfrak{v}, \mathfrak{w}) + \mathscr{D}(\mathfrak{v}, H\mathfrak{w})}\right],$$

where α is a constant such that $\alpha \in (0, 1/4)$ and $\mathfrak{v} \neq \mathfrak{w}$. Suppose there exists $\mathfrak{v}_0 \in S$ such that $\mathfrak{v}_0\ R\ \mathfrak{w}$ for all $\mathfrak{w} \in H(S)$. Then H has a unique invariant point in $(S, \mathscr{D}, R)$.

Corollary 5.2. *Let $(S, \mathscr{D}, R)$ be a R-metric space which is R-complete and $H : S \to S$ be a R-continuous, R-preserving self-mapping satisfying the R-contractive condition,*

$$\mathscr{D}(H\mathfrak{v}, H\mathfrak{w}) \leq \beta\left[\frac{\mathscr{D}(\mathfrak{v}, H\mathfrak{v})\left\{\mathscr{D}(\mathfrak{w}, H\mathfrak{v}) + \mathscr{D}(\mathfrak{v}, H\mathfrak{v})\right\}}{\mathscr{D}(\mathfrak{v}, \mathfrak{w})}\right],$$

where β is a constant such that $\beta \in (0, 1/2)$ and $\mathfrak{v} \neq \mathfrak{w}$. Suppose there exists $\mathfrak{v}_0 \in S$ such that $\mathfrak{v}_0 \ R \ \mathfrak{w}$ for all $\mathfrak{w} \in H(S)$. Then H has a unique invariant point in $(S, \mathscr{D}, R)$.

Corollary 5.3. *Let $(S, \mathscr{D}, R)$ be a R-metric space which is R-complete and $H : S \to S$ be a R-continuous, R-preserving self-mapping satisfying the R-contractive condition,*

$$\mathscr{D}(H\mathfrak{v}, H\mathfrak{w}) \leq \alpha\phi\left[\frac{\mathscr{D}(H\mathfrak{w}, \mathfrak{w})\left\{\mathscr{D}(H\mathfrak{v}, \mathfrak{w}) + \mathscr{D}(\mathfrak{v}, H\mathfrak{v}) + \mathscr{D}(\mathfrak{w}, H\mathfrak{w})\right\}}{1 + \mathscr{D}(H\mathfrak{v}, H^2\mathfrak{v})}\right],$$

where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous such that $\phi(t) \leq t$, $\phi(0) = 0$. Also, α is a constant such that $\alpha \in (0, 1/2)$ and $\mathfrak{v} \neq \mathfrak{w}$. Suppose there exists $\mathfrak{v}_0 \in S$ such that $\mathfrak{v}_0 \ R \ \mathfrak{w}$ for all $\mathfrak{w} \in H(S)$. Then H has a unique invariant point in $(S, \mathscr{D}, R)$.

Corollary 5.4. *Suppose $(S, \mathscr{D}, R)$ be a R-metric space which is R-complete and $H : S \to S$ be a R-continuous, R-preserving self- mapping satisfying the R-contractive condition,*

$$\mathscr{D}(H\mathfrak{v}, H\mathfrak{w}) \leq \alpha\phi\left[\frac{\mathscr{D}(H\mathfrak{w}, \mathfrak{w})\left\{\mathscr{D}(\mathfrak{w}, H\mathfrak{w}) + \mathscr{D}(\mathfrak{v}, H\mathfrak{v})\right\}}{\mathscr{D}(H\mathfrak{v}, H^2\mathfrak{v}) + \mathscr{D}(\mathfrak{w}, H^2\mathfrak{w})}\right],$$

where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous such that $\phi(t) \leq t$, $\phi(0) = 0$. Also, α is a constant such that $\alpha \in (0, 1/2)$ and $\mathfrak{v} \neq \mathfrak{w}$. Suppose there exists $\mathfrak{v}_0 \in S$ such that $\mathfrak{v}_0 \ R \ \mathfrak{w}$ for all $\mathfrak{w} \in H(S)$. Then H has a unique invariant point in $(S, \mathscr{D}, R)$.

Corollary 5.5. *Let $(S, \mathscr{D}, R)$ be a R-metric space which is R-complete and $H : S \to S$ be a R-continuous, R-preserving self-mapping satisfying the R-contractive condition,*

$$\mathscr{D}(H\mathfrak{v}, H\mathfrak{w}) \leq \alpha\phi\left[\frac{\mathscr{D}(\mathfrak{v}, H\mathfrak{v})\mathscr{D}(\mathfrak{w}, H\mathfrak{w})}{\mathscr{D}(H\mathfrak{v}, H^2\mathfrak{v}) + \mathscr{D}(\mathfrak{w}, H^2\mathfrak{w})}\right],$$

where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous such that $\phi(t) \leq t$, $\phi(0) = 0$. Also, α is a constant such that $\alpha \in (0, 1)$ and $\mathfrak{v} \neq \mathfrak{w}$. Suppose there exists $\mathfrak{v}_0 \in S$ such that $\mathfrak{v}_0 \ R \ \mathfrak{w}$ for all $\mathfrak{w} \in H(S)$. Then H has a unique invariant point in $(S, \mathscr{D}, R)$.

Corollary 5.6. *Let $(S, \mathscr{D}, R)$ be a R-metric space which is R-complete and $H : S \to S$ be a R-continuous, R-preserving self-mapping satisfying the R-contractive condition,*

$$\mathscr{D}(H\mathfrak{v}, H\mathfrak{w}) \leq \alpha\mathscr{D}(\mathfrak{v}, \mathfrak{w}),$$

where $\alpha \in (0, 1)$ and $\mathfrak{v} \neq \mathfrak{w}$. Suppose there exists $\mathfrak{v}_0 \in S$ such that $\mathfrak{v}_0 \ R \ \mathfrak{w}$ for all $\mathfrak{w} \in H(S)$. Then H has a unique invariant point in $(S, \mathscr{D}, R)$.

Remark 5.7. Corollary 5.6 is an improved version of Khalehoghli [28].

Corollary 5.8. *Let* $(S, \mathscr{D}, R)$ *be a R-metric space which is R-complete and* $H : S \to S$ *be a R-continuous, R-preserving self mapping satisfying the R-contractive condition,*

$$\mathscr{D}(H\mathfrak{v}, H\mathfrak{w}) \leq \alpha \left[\mathscr{D}(\mathfrak{v}, H\mathfrak{v}) + \mathscr{D}(\mathfrak{w}, H\mathfrak{w})\right],$$

where $\alpha \in (0, 1/2)$ *and* $\mathfrak{v} \neq \mathfrak{w}$. *Suppose there exists* $\mathfrak{v}_0 \in S$ *such that* $\mathfrak{v}_0 \ R \ \mathfrak{w}$ *for all* $\mathfrak{w} \in H(S)$. *Then* H *has a unique invariant point in* $(S, \mathscr{D}, R)$.

Corollary 5.9. *Let* $(S, \mathscr{D}, R)$ *be a R-metric space which is R-complete and* $H : S \to S$ *bes a R-continuous, R-preserving self mapping satisfying the R-contractive condition,*

$$\mathscr{D}(H\mathfrak{v}, H\mathfrak{w}) \leq \alpha \left[\mathscr{D}(\mathfrak{v}, H\mathfrak{w}) + \mathscr{D}(\mathfrak{w}, H\mathfrak{v})\right],$$

where $\alpha \in (0, 1/2)$ *and* $\mathfrak{v} \neq \mathfrak{w}$. *Suppose there exists* $\mathfrak{v}_0 \in S$ *such that* $\mathfrak{v}_0 \ R \ \mathfrak{w}$ *for all* $\mathfrak{w} \in H(S)$. *Then* H *has a unique invariant point in* $(S, \mathscr{D}, R)$.

Corollary 5.10. *Let* $(S, \mathscr{D}, R)$ *be a R-metric space which is R-complete and* $H : S \to S$ *be a R-continuous, R-preserving self-mapping satisfying the R-contractive condition,*

$$\mathscr{D}(H\mathfrak{v}, H\mathfrak{w}) \leq \alpha \left[\mathscr{D}(\mathfrak{v}, H\mathfrak{w}) + \mathscr{D}(\mathfrak{w}, H\mathfrak{v}) + \mathscr{D}(\mathfrak{v}, H\mathfrak{v}) + \mathscr{D}(\mathfrak{w}, H\mathfrak{w})\right],$$

where $\alpha \in (0, 1/4)$ *and* $\mathfrak{v} \neq \mathfrak{w}$. *Suppose there exists* $\mathfrak{v}_0 \in S$ *such that* $\mathfrak{v}_0 \ R \ \mathfrak{w}$ *for all* $\mathfrak{w} \in H(S)$. *Then* H *has a unique invariant point in* $(S, \mathscr{D}, R)$.

6. Applications

In this section, the outcomes of Therorem 3.3 are used to demonstrate the existence and distinctiveness of solution of the integral equation given below.

$$n(r) = z(r) + \int_{\kappa}^{\rho} \mathfrak{M}(r, \mu, n(\mu))d\mu, \tag{6.1}$$

where $\rho, \kappa \in \mathbb{R} : \kappa \leq \rho$, $n \in C[\kappa, \rho]$ (the set of all continuous real functions on the interval $[\kappa, \rho]$). Also $z : [\kappa, \rho] \to [\kappa, \rho]$ and $M : [\kappa, \rho]^3 \to [\kappa, \rho]$ are functions which are continuous.

Let $S = C[\kappa, \rho]$ be endowed with the metric defined as,

$$\mathscr{D}(n, q) = \sup_{r \in [\kappa, \rho]} |n(r) - q(r)|, \text{ for any } n, q \in S.$$

And suppose, $H : S \to S$ is defined as

$$Hn(r) = z(r) + \int_{\kappa}^{\rho} \mathfrak{M}(r, \mu, n(\mu))d\mu, \text{ where } r \in [\kappa, \rho]. \tag{6.2}$$

Then, evidently $(S, \mathscr{D})$ is a metric space which is complete. Now let R be a relation defined in S such that the R-metric space $(S, \mathscr{D}, R)$ is R-complete, R-continuous and R-preserving. It is to be noted that, the solution of Equation (6.1) is similar to the fixed point of H in Equation (6.2). Accordingly, the following results have been deduced.

Theorem 6.1. *The integral equation* (6.1) *has a unique solution if* $\exists\ \rho > 0$ *such that,*

$$|\mathfrak{M}(r, \mu, n(\mu)) - \mathfrak{M}(r, \mu, q(\mu))| \leq \frac{\alpha}{(\rho - \kappa)}|n - q|,$$

where α *is a constant in the interval* $(0, 1)$.

Proof. It is known that,

$$\begin{aligned}
|Hn - Hq| &= \left| \int_{\kappa}^{\rho} \mathfrak{M}(r, \mu, n(\mu))d\mu - \mathfrak{M}(r, \mu, q(\mu))d\mu \right| \\
&\leq \int_{\kappa}^{\rho} \big|\mathfrak{M}(r, \mu, n(\mu)) - \mathfrak{M}(r, \mu, q(\mu))\big| d\mu \\
&\leq \frac{\alpha}{(\rho - \kappa)} \int_{\kappa}^{\rho} |n - q| d\mu \\
&= \alpha |n - q|.
\end{aligned}$$

Therefore,

$$|Hn - Hq| \leq \alpha |n - q|. \tag{6.3}$$

Now, taking supremum on both sides in the above equation,

$$\begin{aligned}
\mathscr{D}(Hn, Hq) &= \sup_{r \in [\kappa, \rho]} |Hn - Hq| \\
&\leq \sup_{r \in [\kappa, \rho]} \alpha |n - q| \\
&= \alpha \sup_{r \in [\kappa, \rho]} |n - q| \\
&= \alpha \mathscr{D}(n, q).
\end{aligned}$$

It implies that,

$$\mathscr{D}(Hn, Hq) \leq \alpha \mathscr{D}(n, q),$$

which means that, the mapping H fulfills all the requirements of Corollary 5.6. Therefore the existence of fixed point is guaranteed. Hence there exists $n(r) \in C[\kappa, \rho] : n(r) = Hn(r)$, which is a unique solution of Equation (6.1). □

Theorem 6.2. *The integral equation* (6.1) *has a unique solution if there exists* $\rho > 0$ *such that,*

$$\begin{aligned}|\mathfrak{M}(r,\mu,n(\mu)) - \mathfrak{M}(r,\mu,q(\mu))| \leq \frac{\alpha}{(\rho-\kappa)}&[|n-Hq| + |Hn-q| \\ &+ |n-Hn| + |q-Hq|],\end{aligned}$$

where α *is a constant in the interval* $(0, 1/4)$.

Proof. It is known that,

$$\begin{aligned}|Hn - Hq| &= \left| \int_\kappa^\rho \mathfrak{M}(r,\mu,n(\mu))d\mu - \mathfrak{M}(r,\mu,q(\mu))d\mu \right| \\ &\leq \int_\kappa^\rho \big|\mathfrak{M}(r,\mu,n(\mu)) - \mathfrak{M}(r,\mu,q(\mu))\big| d\mu \\ &\leq \frac{\alpha}{(\rho-\kappa)} \int_\kappa^\rho \left[|n-Hq| + |Hn-q| + |n-Hn| + |q-Hq|\right] d\mu \\ &= \alpha \left[|n-Hq| + |Hn-q| + |n-Hn| + |q-Hq|\right].\end{aligned}$$

Therefore,

$$|Hn - Hq| \leq \alpha \left[|n-Hq| + |Hn-q| + |n-Hn| + |q-Hq|\right]. \tag{6.4}$$

Now taking supremum on both sides in the above equation,

$$\begin{aligned}\mathscr{D}(Hn,Hq) &= \sup_{r\in[\kappa,\rho]} |Hn-Hq| \\ &\leq \sup_{r\in[\kappa,\rho]} \alpha \left[|n-Hq| + |Hn-q| + |n-Hn| + |q-Hq|\right] \\ &= \alpha \sup_{r\in[\kappa,\rho]} \left[|n-Hq| + |Hn-q| + |n-Hn| + |q-Hq|\right] \\ &= \alpha \left[\mathscr{D}(n,Hq) + \mathscr{D}(Hn,q) + \mathscr{D}(n,Hn) + \mathscr{D}(q,Hq)\right].\end{aligned}$$

It implies that,

$$\mathscr{D}(Hn,Hq) \leq \alpha\mathscr{D}(n,q),$$

which means that, the mapping H fulfills all the requirements of Corollary 5.10. Therefore the existence of fixed point is guaranteed. Hence there exists $n(r) \in C[\kappa,\rho] : n(r) = Hn(r)$, which is a unique solution of Equation (6.1). □

7. Conclusion

R-metric space is a generalization of metric space and unlike metric space it uses the concepts of relations combined with the set and a distance metric $\mathscr{D}$.

The current research fulfills the objective by establishing two invariant point results in R-metric space, one for a rational R-contractive condition without the usage of any control function, and the other for a rational R-contractive condition under the usage of control functions. For this purpose, some concepts such as R-preserving and R-continuous mappings, R-convergence, R-sequence, R-Cauchy sequence, R-contraction are brought to use. Some corollaries have also been represented as special cases of the established results.

Corollary 5.6 obtained from Theorem 3.3 is the generalization of Banach contraction principle, whereas Corollaries 5.8 and 5.9 are the generalizations of Kannan and Chatterjea fixed point theorems respectively, for the case of R-metric space where the mapping considered is R-continuous and R-preserving. In this manuscript, Corollaries 5.6 and 5.10 have been used to show the existence and uniqueness of solution of Fredholm integral equations of second kind.

Previously, this kind of result has been utilized by Khalehoghli in [28] to prove the existence and distinctiveness of solution for fractional integral equations of a specific type. Similarly, Corollaries 5.6 and 5.9 can be useful for solving fractional integral equations, initial value problems, boundary value problems and real life scenarios related to optimization and uncertainity. Also the obtained results can be useful for the establishment of Ćirić, Suzuki and Eldestein fixed point theorems, and notably, if the mapping considered here is not a self mapping, the results established for fixed point will be converted to best proximity point.

Now since, the applications of the established results has been deduced for integral equations, other fields and topics taking integral equations into consideration such as, Laplace and Fourier transforms, scattering techniques in quantum mechanics, rendering equations in computer graphics, fluid dynamics, image and signal processing can be benefited by the utilization of the results hence obtained.

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