Nonlinear Functional Analysis and Applications
Vol. 31, No. 3 (2026), pp. 825-842
ISSN: 1229-1595(print), 2466-0973(online)

https://doi.org/10.22771/nfaa.2026.31.03.08
http://nfaa.kyungnam.ac.kr/journal-nfaa

A REFORMULATION OF J-NEUTROSOPHIC METRIC SPACES AND FIXED POINT FOR QUASI RJN-Υ CONTRACTIONS

Mawada Albarq[1], Anwar Bataihah[2] and Ayman A. Hazaymeh[3]

[1]Department of Mathematics, Faculty of Science, Jadara University, Irbid Jordan
e-mail: `albarqmawada5@gmail.com`

[2]Department of Mathematics, Faculty of Science, Jadara University, Irbid Jordan
e-mail: `a.bataihah@jadara.edu.jo, anwerbataihah@gmail.com`

[3]Department of Mathematics, Faculty of Science, Jadara University, Irbid Jordan
e-mail: `aymanha@jadara.edu.jo`

Abstract. This work revisits the existing definition of J-neutrosophic metric spaces and resolves inconsistencies that appear in earlier versions. We propose a more precise framework, called refined J-neutrosophic metric spaces (RJNMS), which simplifies the structure of the axioms and relies on a consistent choice of a continuous t-norm and t-conorm. Within this setting, we introduce the class of quasi RJN-Υ contractions and prove a fixed point theorem guaranteeing both the existence and uniqueness of fixed points for mappings of this type. The main theorem broadens and unifies a number of results already known in the literature. To highlight the usefulness of the new formulation, we also provide corollaries and an illustrative example. Overall, the paper establishes a clearer and more robust foundation for further investigations in fixed-point theory within neutrosophic and related generalized metric frameworks.

1. Introduction

Fixed point theory has long been a fundamental area of mathematical research, beginning with Banachs pioneering fixed point theorem [2]. This theory

[0]Received November 25, 2025. Revised May 26, 2026. Accepted May 28, 2026.
[0]2020 Mathematics Subject Classification: 47H10.
[0]Keywords: Fixed point, metric space, refined J-neutrosophic metric space (RJNMS), Υ-quasi contraction, neutrosophic logic.
[0]Corresponding author: Ayman A. Hazaymeh(`aymanha@jadara.edu.jo`).

studies points that remain invariant under a given mapping and provides essential tools for establishing the existence and uniqueness of solutions in various mathematical models. Classical results, such as Kannans fixed point theorem [10], have been extended to more general and uncertain settings, including fuzzy, intuitionistic fuzzy, and neutrosophic frameworks [1, 29, 30].

Metric spaces provide a natural setting for fixed point analysis, and their generalizations allow the study of more complex mappings. Fuzzy metric spaces [12], G-metric spaces [24], and intuitionistic fuzzy metric spaces [25] were introduced to address uncertainty in distance measurements. More recently, neutrosophic metric spaces [11] and J-neutrosophic metric spaces [23] have facilitated the extension of fixed point results to environments that incorporate indeterminacy and partial information.

In parallel with the development of generalized metric spaces, the study of generalized distance structures has also advanced through the introduction of MR-metric spaces. A series of works by Malkawi and collaborators has laid the groundwork for this framework, establishing core fixed point results [13, 14, 17] and important convergence properties [18]. The usefulness of MR-metrics has been further demonstrated through applications to integral equations, neutron transport models, and uncertainty analysis [15, 28]. More recent contributions show that MR-metric spaces can also be applied to graph theory [19], measure-theoretic problems [16], and computational learning settings [20]. On another note, Bataihah et al. [5] came up with the idea of Gamma distance mappings and used them in studying fractional boundary differential equations. Additionally, Qawasmeh created several contraction types in distance spaces, such as (H, Ω_b)-interpolative contractions [26] and H-simulation functions for Ω_b-distance mappings [27].

Through this manuscript, Ω stands for a nonempty set, f a self-map on Ω, and $\mathcal{F}^{\exists}(f)$ the set of fixed points of f.

2. Preliminary

J-neutrosophic metric spaces are a further generalization of neutrosophic metric spaces, designed to handle more complex forms of uncertainty. They preserve the basic structure of using triplets to describe distances while offering a more adaptable framework for studying fixed point results.

In the following, we recall some basic notions related to J-neutrosophic metric spaces that we discuss in this section.

Remark 2.1. The next two definitions were written by the same author. It seems like the author corrected his definition in his new research [9]. But we are going to correct it again in our main section.

Definition 2.2. ([23]) Consider a nonempty set Ω, a continuous t-norm (CTN) $*$, two continuous t-conorm CTCNs $\star$, $\diamond$ and fuzzy sets $\mathcal{T}_{\mathcal{T}}, \mathcal{T}_{\mathcal{F}}, \mathcal{T}_{\mathcal{I}}$ on $\Omega^3 \times [0,1]$. A 7-tuple $(\Omega, \mathcal{T}_{\mathcal{T}}, \mathcal{T}_{\mathcal{F}}, \mathcal{T}_{\mathcal{I}}, *, \star, \diamond)$ is called a J-neutrosophic metric space (JNMS) if for each $\rho, \aleph, \eta, \gamma \in \Omega$ and $t > 0$,

(J1) $\mathcal{T}_{\mathcal{T}}(\rho, \aleph, \eta, t) + \mathcal{T}_{\mathcal{F}}(\rho, \aleph, \eta, t) + \mathcal{T}_{\mathcal{I}}(\rho, \aleph, \eta, t) \leq 3$,
(J2) $\mathcal{T}_{\mathcal{T}}(\rho, \aleph, \eta, t) > 0$,
(J3) $\mathcal{T}_{\mathcal{T}}(\rho, \aleph, \eta, t)$ is symmetric in $\rho, \aleph$ and η,
(J4) $\mathcal{T}_{\mathcal{T}}(\rho, \rho, \aleph, t) \geq \mathcal{T}_{\mathcal{T}}(\rho, \aleph, \eta, t)$ if $\aleph \neq \eta$,
(J5) $\mathcal{T}_{\mathcal{T}}(\rho, \aleph, \eta, t) = 1$ if and only if $\rho = \aleph = \eta$,
(J6) $\mathcal{T}_{\mathcal{T}}(\rho, \aleph, \eta, t+s) \geq \mathcal{T}_{\mathcal{T}}(\rho, \gamma, \gamma, t) * \mathcal{T}_{\mathcal{T}}(\gamma, \aleph, \eta, s)$,
(J7) $\mathcal{T}_{\mathcal{T}}(\rho, \aleph, \eta, t)$ is continuous w.r.t t,
(J8) $\mathcal{T}_{\mathcal{T}}$ is nondecreasing on $[0, +\infty]$, $\lim_{t \to +\infty} \mathcal{T}_{\mathcal{T}}(\rho, \aleph, \eta) = 1$, $\lim_{t \to 0} \mathcal{T}_{\mathcal{T}}(\rho, \aleph, \eta) = 0$,
(J9) $\mathcal{T}_{\mathcal{F}}(\rho, \aleph, \eta, t) < 1$,
(J10) $\mathcal{T}_{\mathcal{F}}(\rho, \aleph, \eta, t)$ is symmetric in $\rho, \aleph$ and η,
(J11) $\mathcal{T}_{\mathcal{F}}(\rho, \rho, \aleph, t) \leq \mathcal{T}_{\mathcal{F}}(\rho, \aleph, \eta, t)$ if $\aleph \neq \eta$,
(J12) $\mathcal{T}_{\mathcal{F}}(\rho, \aleph, \eta, t) = 0$ if and only if $\rho = \aleph = \eta$,
(J13) $\mathcal{T}_{\mathcal{F}}(\rho, \aleph, \eta, t+s) \leq \mathcal{T}_{\mathcal{F}}(\rho, \gamma, \gamma, t) \star \mathcal{T}_{\mathcal{F}}(\gamma, \aleph, \eta, s)$,
(J14) $\mathcal{T}_{\mathcal{F}}(\rho, \aleph, \eta, t)$ is continuous w.r.t t,
(J15) $\mathcal{T}_{\mathcal{F}}$ is nonincreasing on $[0, +\infty]$, $\lim_{t \to +\infty} \mathcal{T}_{\mathcal{F}}(\rho, \aleph, \eta) = 0$, $\lim_{t \to 0} \mathcal{T}_{\mathcal{F}}(\rho, \aleph, \eta) = 1$,
(J16) $\mathcal{T}_{\mathcal{I}}(\rho, \aleph, \eta, t) < 1$,
(J17) $\mathcal{T}_{\mathcal{I}}(\rho, \aleph, \eta, t)$ is symmetric in $\rho, \aleph$ and η,
(J18) $\mathcal{T}_{\mathcal{I}}(\rho, \rho, \aleph, t) \leq \mathcal{T}_{\mathcal{I}}(\rho, \aleph, \eta, t)$ if $\aleph \neq \eta$,
(J19) $\mathcal{T}_{\mathcal{I}}(\rho, \aleph, \eta, t) = 0$ if and only if $\rho = \aleph = \eta$,
(J20) $\mathcal{T}_{\mathcal{I}}(\rho, \aleph, \eta, t+s) \leq \mathcal{T}_{\mathcal{I}}(\rho, \gamma, \gamma, t) \diamond \mathcal{T}_{\mathcal{I}}(\gamma, \aleph, \eta, s)$,
(J21) $\mathcal{T}_{\mathcal{I}}(\rho, \aleph, \eta, t)$ is continuous w.r.t t,
(J22) $\mathcal{T}_{\mathcal{I}}$ is nonincreasing on $[0, +\infty]$, $\lim_{t \to +\infty} \mathcal{T}_{\mathcal{I}}(\rho, \aleph, \eta) = 0$, $\lim_{t \to 0} \mathcal{T}_{\mathcal{I}}(\rho, \aleph, \eta) = 1$.

Definition 2.3. ([9]) Consider a nonempty set Ω, a CTN $*$, a CTCN $\diamond$ and fuzzy sets $\mathcal{T}_{\mathcal{T}}, \mathcal{T}_{\mathcal{F}}, \mathcal{T}_{\mathcal{I}}$ defined on $\Omega \times \Omega \times \Omega \to (0, +\infty)$. A 6-tuple $(\Omega, \mathcal{T}_{\mathcal{T}}, \mathcal{T}_{\mathcal{F}}, \mathcal{T}_{\mathcal{I}}, *, \diamond)$ is called a J-neutrosophic metric space (JNMS) if for each $\rho, \aleph, \eta, \gamma \in \Omega$ and $t, s > 0$ the following conditions hold.

(1) $\mathcal{T}_{\mathcal{T}}(\rho, \aleph, \eta, t) + \mathcal{T}_{\mathcal{F}}(\rho, \aleph, \eta, t) + \mathcal{T}_{\mathcal{I}}(\rho, \aleph, \eta, t) \leq 3$ for $\rho \neq \aleph$,
(2) $\mathcal{T}_{\mathcal{T}}(\rho, \rho, \aleph, t) \geq \mathcal{T}_{\mathcal{T}}(\rho, \aleph, \eta, t)$ for all $\aleph \neq \eta$,
(3) $\mathcal{T}_{\mathcal{T}}(\rho, \aleph, \eta, t) = 1$ if and only if $\rho = \aleph = \eta$,
(4) $\mathcal{T}_{\mathcal{T}}(\rho, \aleph, \eta, t) = \mathcal{T}_{\mathcal{T}}(P(\rho, \aleph, \eta), t)$ where P is a permutation function,
(5) $\mathcal{T}_{\mathcal{T}}(\rho, \aleph, \eta, t+s) \geq \mathcal{T}_{\mathcal{T}}(\rho, \gamma, \gamma, t) * \mathcal{T}_{\mathcal{T}}(\gamma, \aleph, \eta, s)$ (Triangle inequality),
(6) $\mathcal{T}_{\mathcal{T}}(\rho, \aleph, \eta, \cdot) : (0, +\infty) \to [0,1]$ is continuous,
(7) $\mathcal{T}_{\mathcal{F}}(\rho, \rho, \aleph, t) \leq \mathcal{T}_{\mathcal{F}}(\rho, \aleph, \eta, t)$ for $\aleph \neq \eta$,

(8) $\mathcal{T}_{\mathcal{F}}(\rho,\aleph,\eta,t)=0$ if and only if $\rho=\aleph=\eta$,
(9) $\mathcal{T}_{\mathcal{F}}(\rho,\aleph,\eta,t)=\mathcal{T}_{\mathcal{F}}(P(\rho,\aleph,\eta),t)$ where P is a permutation function,
(10) $\mathcal{T}_{\mathcal{F}}(\rho,\aleph,\eta,t+s)\leq\mathcal{T}_{\mathcal{F}}(\rho,\gamma,\gamma,t)\diamond\mathcal{T}_{\mathcal{F}}(\gamma,\aleph,\eta,s)$,
(11) $\mathcal{T}_{\mathcal{F}}(\rho,\aleph,\eta,\cdot):(0,+\infty)\to[0,1]$ is continuous,
(12) $\mathcal{T}_{\mathcal{I}}(\rho,\rho,\aleph,t)\leq\mathcal{T}_{\mathcal{I}}(\rho,\aleph,\eta,t)$ for $\aleph\neq\eta$,
(13) $\mathcal{T}_{\mathcal{I}}(\rho,\aleph,\eta,t)=0$ if and only if $\rho=\aleph=\eta$,
(14) $\mathcal{T}_{\mathcal{I}}(\rho,\aleph,\eta,t)=\mathcal{T}_{\mathcal{I}}(P(\rho,\aleph,\eta),t)$ where P is a permutation function,
(15) $\mathcal{T}_{\mathcal{I}}(\rho,\aleph,\eta,t+s)\leq\mathcal{T}_{\mathcal{I}}(\rho,\gamma,\gamma,t)\diamond\mathcal{T}_{\mathcal{I}}(\gamma,\aleph,\eta,s)$,
(16) $\mathcal{T}_{\mathcal{I}}(\rho,\aleph,\eta,\cdot):(0,+\infty)\to[0,1]$ is continuous.

Remark 2.4. ([23]) $\mathcal{T}_{\mathcal{T}}(\rho,\aleph,\eta,t),\mathcal{T}_{\mathcal{F}}(\rho,\aleph,\eta,t)$ and $\mathcal{T}_{\mathcal{I}}(\rho,\aleph,\eta,t)$ represent, respectively, the degree of nearness, the degree of non-nearness and the degree of indeterminacy between ρ, $\aleph$ and η with respect to t.

Definition 2.5. ([23]) Let $(\Omega,\mathcal{T}_{\mathcal{T}},\mathcal{T}_{\mathcal{F}},\mathcal{T}_{\mathcal{I}},*,\diamond)$ be a JNMS and let $\rho\in\Omega$, $r\in(0,1)$ and $t>0$. Then

$$B(\rho,r,t)=\{\aleph\in\Omega:\quad \mathcal{T}_{\mathcal{T}}(\rho,\rho,\aleph)>1-r,\mathcal{T}_{\mathcal{F}}(\rho,\rho,\aleph)<r,\mathcal{T}_{\mathcal{I}}(\rho,\rho,\aleph)<r\}$$

is called JN-open ball with center at ρ and radius r.

Definition 2.6. ([23]) Let $(\Omega,\mathcal{T}_{\mathcal{T}},\mathcal{T}_{\mathcal{F}},\mathcal{T}_{\mathcal{I}},*,\star,\diamond)$ be a JNMS. A sequence $\{\rho_q\}$ is said to be

(1) JN-convergent to $\eta\in\Omega$ if it converges to η in the JN-metric topology,
(2) JN-Cauchy if for every $\epsilon>0$ and $t>0$ there exists $N\in\mathbb{N}$ such that

$$\mathcal{T}_{\mathcal{T}}(\rho_q,\rho_m,\rho_l,t)>1-\epsilon,\ \mathcal{T}_{\mathcal{F}}(\rho_q,\rho_m,\rho_l,t)<\epsilon,\ \mathcal{T}_{\mathcal{I}}(\rho_q,\rho_m,\rho_l,t)<\epsilon,$$

for all $n,m,l\in\mathbb{N}$,
(3) JN-Complete if every JN-Cauchy sequence is JN-Convergent.

Remark 2.7. We note that the original formulation of J-neutrosophic metric spaces in some previous works contains minor typographical ambiguities. However, the main results in those works remain valid. For the sake of clarity and uniformity, we present here a refined formulation that preserves the conceptual structure of the original definitions while streamlining the notation and improving readability.

Several fixed-point results in J-neutrosophic metric spaces (JNMS) have been investigated by various authors, as discussed in [9, 23].

3. Refined J-neutrosophic Metric Spaces

J-neutrosophic metric spaces are a generalization of neutrosophic metric spaces, both of which are based on neutrosophic logic that includes truth, indeterminacy, and falsity. While neutrosophic metric spaces extend classical metrics to better handle uncertainty, J-neutrosophic metric spaces further relax the structure by modifying the triangle inequality in three dimensions. This makes J-neutrosophic spaces more flexible and capable of addressing a wider range of problems, especially in fixed point theory.

As previously mentioned, the initial definition of J-neutrosophic metric spaces was not presented in its most rigorous form. So we are going to present the refined J-neutrosophic metric space that clarifies the definition, corrects any possible mistakes, and deletes the extra unnecessary information.

We define the refined J-neutrosophic metric space by modifying (J8), (J15) and (J22) in definition 2.2, where $\star$ and $\diamond$ are the same CTCN as following:

Definition 3.1. Let Ω be a nonempty set. Let $*$ be a continuous t-norm (CTN) and let $\diamond$ be a continuous t-conorm (CTCN). Suppose

$$\mathcal{T}_{\mathcal{T}}, \mathcal{T}_{\mathcal{F}}, \mathcal{T}_{\mathcal{I}} : \Omega^3 \times (0, \infty) \to [0, 1].$$

The 6-tuple $(\Omega, \mathcal{T}_{\mathcal{T}}, \mathcal{T}_{\mathcal{F}}, \mathcal{T}_{\mathcal{I}}, *, \diamond)$ is called a *refined J-neutrosophic metric space* (RJNMS) if for all $\rho, \aleph, \eta, \gamma \in \Omega$ and all $t, s > 0$ the following hold.

(R1) $\mathcal{T}_{\mathcal{T}}(\rho, \aleph, \eta, t) + \mathcal{T}_{\mathcal{F}}(\rho, \aleph, \eta, t) + \mathcal{T}_{\mathcal{I}}(\rho, \aleph, \eta, t) \leq 3$.

(R2) For distinct arguments

$$\mathcal{T}_{\mathcal{T}}(\cdot, \cdot, \cdot, t) > 0, \quad \mathcal{T}_{\mathcal{F}}(\cdot, \cdot, \cdot, t), \mathcal{T}_{\mathcal{I}}(\cdot, \cdot, \cdot, t) < 1.$$

(R3) For every permutation P of $\rho, \aleph, \eta$ and every $t > 0$,

$$\mathcal{T}_{\mathcal{T}}(\rho, \aleph, \eta, t) = \mathcal{T}_{\mathcal{T}}(P(\rho, \aleph, \eta), t),$$

and similarly for $\mathcal{T}_{\mathcal{F}}$ and $\mathcal{T}_{\mathcal{I}}$.

(R4) For all $\rho, \aleph, \eta$ with $\aleph \neq \eta$,

$$\mathcal{T}_{\mathcal{T}}(\rho, \rho, \aleph, t) \geq \mathcal{T}_{\mathcal{T}}(\rho, \aleph, \eta, t),$$

$$\mathcal{T}_{\mathcal{F}}(\rho, \rho, \aleph, t) \leq \mathcal{T}_{\mathcal{F}}(\rho, \aleph, \eta, t)$$

and

$$\mathcal{T}_{\mathcal{I}}(\rho, \rho, \aleph, t) \leq \mathcal{T}_{\mathcal{I}}(\rho, \aleph, \eta, t).$$

(R5) $\mathcal{T}_{\mathcal{T}}(\rho, \aleph, \eta, t) = 1$ if and only if $\rho = \aleph = \eta$, and $\mathcal{T}_{\mathcal{F}}(\rho, \aleph, \eta, t) = \mathcal{T}_{\mathcal{I}}(\rho, \aleph, \eta, t) = 0$ if and only if $\rho = \aleph = \eta$.

(R6) For all $s, t > 0$ and all $\gamma \in \Omega$,

$$\mathcal{T}_{\mathcal{T}}(\rho, \aleph, \eta, t + s) \geq \mathcal{T}_{\mathcal{T}}(\rho, \gamma, \gamma, t) * \mathcal{T}_{\mathcal{T}}(\gamma, \aleph, \eta, s),$$

$$\mathcal{T}_{\mathcal{F}}(\rho, \aleph, \eta, t + s) \leq \mathcal{T}_{\mathcal{F}}(\rho, \gamma, \gamma, t) \diamond \mathcal{T}_{\mathcal{F}}(\gamma, \aleph, \eta, s)$$

and
$$\mathcal{T}_{\mathcal{I}}(\rho,\aleph,\eta,t+s)\leq \mathcal{T}_{\mathcal{I}}(\rho,\gamma,\gamma,t)\diamond \mathcal{T}_{\mathcal{I}}(\gamma,\aleph,\eta,s).$$

(R7) For each fixed $\rho,\aleph,\eta$ the maps $t\mapsto \mathcal{T}_{\mathcal{T}}(\rho,\aleph,\eta,t)$, $t\mapsto \mathcal{T}_{\mathcal{F}}(\rho,\aleph,\eta,t)$ and $t\mapsto \mathcal{T}_{\mathcal{I}}(\rho,\aleph,\eta,t)$ are continuous on $(0,\infty)$.

(R8) For each fixed $\rho,\aleph,\eta$,
$$\lim_{t\to 0^+}\mathcal{T}_{\mathcal{T}}(\rho,\aleph,\eta,t)=0,\quad \lim_{t\to\infty}\mathcal{T}_{\mathcal{T}}(\rho,\aleph,\eta,t)=1$$
and
$$\lim_{t\to 0^+}\mathcal{T}_{\mathcal{F}}(\rho,\aleph,\eta,t)=\lim_{t\to 0^+}\mathcal{T}_{\mathcal{I}}(\rho,\aleph,\eta,t)=1,$$
$$\lim_{t\to \infty}\mathcal{T}_{\mathcal{F}}(\rho,\aleph,\eta,t)=\lim_{t\to \infty}\mathcal{T}_{\mathcal{I}}(\rho,\aleph,\eta,t)=0.$$

Remark 3.2. Unlike earlier formulations, no additional monotonicity condition with respect to the parameter t is imposed; instead, the required monotonic behaviour follows directly from the axioms and is therefore deduced from the definition itself. Finally, a single continuous t-norm $*$ and a single continuous t-conorm $\diamond$ are used uniformly in all triangle-type inequalities. These refinements make the structure more coherent and remove several redundancies found in earlier versions of J-neutrosophic metric spaces.

Definition 3.3. Let $(\Omega,\mathcal{T}_{\mathcal{T}},\mathcal{T}_{\mathcal{F}},\mathcal{T}_{\mathcal{I}},*,\diamond)$ be an RJNMS. For $\rho\in\Omega$, $r\in(0,1)$ and $t>0$ define the RJN-open ball
$$B(\rho,r,t)=\big\{\aleph\in\Omega:\ \mathcal{T}_{\mathcal{T}}(\rho,\rho,\aleph,t)>1-r,\ \mathcal{T}_{\mathcal{F}}(\rho,\rho,\aleph,t)<r,\ \mathcal{T}_{\mathcal{I}}(\rho,\rho,\aleph,t)<r\big\}.$$

Definition 3.4. Let (ρ_q) be a sequence in a RJNFMS $(\Omega,\mathcal{T}_{\mathcal{T}},\mathcal{T}_{\mathcal{F}},\mathcal{T}_{\mathcal{I}},*,\diamond)$. Then

(1) $\{\rho_q\}$ is called RJN-converget to $\eta\in\Omega$ if for a given $\epsilon\in(0,1),\ \lambda>0$ there is $n_0\in\mathbb{N}$ such that for each $q,m\geq n_0$,
$$\mathcal{T}_{\mathcal{T}}(\rho_q,\rho_m,\eta,\lambda)>1-\epsilon,\ \mathcal{T}_{\mathcal{F}}(\rho_q,\rho_m,\eta,\lambda)<\epsilon,\ \mathcal{T}_{\mathcal{I}}(\rho_q,\rho_m,\eta,\lambda)<\epsilon$$
that is,
$$\lim_{q,m\to+\infty}\mathcal{T}_{\mathcal{T}}(\rho_q,\rho_m,\eta,\lambda)=1,$$
$$\lim_{q,m\to+\infty}\mathcal{T}_{\mathcal{F}}(\rho_q,\rho_m,\eta,\lambda)=0,\ \lim_{q,m\to+\infty}\mathcal{T}_{\mathcal{I}}(\rho_q,\rho_m,\eta,\lambda)=0.$$

(2) $\{\rho_q\}$ is called RJN-Cauchy if for a given $\epsilon\in(0,1),\ \lambda>0$ there is $n_0\in\mathbb{N}$ such that for each $q,m,l\geq n_0$,
$$\mathcal{T}_{\mathcal{T}}(\rho_q,\rho_m,\rho_l,\lambda)>1-\epsilon,\ \mathcal{T}_{\mathcal{F}}(\rho_q,\rho_m,\rho_l,\lambda)<\epsilon,\ \mathcal{T}_{\mathcal{I}}(\rho_q,\rho_m,\rho_l\lambda)<\epsilon$$
that is,
$$\lim_{q,m,l\to+\infty}\mathcal{T}_{\mathcal{T}}(\rho_q,\rho_m,\rho_l,\lambda)=1,$$

$$\lim_{q,m,l\to+\infty} \mathcal{T}_{\mathcal{F}}(\rho_q,\rho_m,\rho_l,\lambda)=0,\quad \lim_{q,m,l\to+\infty} \mathcal{T}_{\mathcal{I}}(\rho_q,\rho_m,\rho_l,\lambda)=0.$$

(3) $(\Omega,\mathcal{T}_{\mathcal{T}},\mathcal{T}_{\mathcal{F}},\mathcal{T}_{\mathcal{I}},*,\diamond)$ is called RJN-complete if every RJN-Cauchy sequence is RJN-convergent to an element in Ω.

Lemma 3.5. *Let $(\Omega,\mathcal{T}_{\mathcal{T}},\mathcal{T}_{\mathcal{F}},\mathcal{T}_{\mathcal{I}},*,\diamond)$ be a RJNMS. Then*

(1) $\mathcal{T}_{\mathcal{T}}(\rho,\aleph,\eta,\cdot):\mathbb{R}^+\to[0,1]$ *is non-decreasing,*
(2) $\mathcal{T}_{\mathcal{F}}(\rho,\aleph,\eta,\cdot):\mathbb{R}^+\to[0,1]$ *is non-increasing,*
(3) $\mathcal{T}_{\mathcal{I}}(\rho,\aleph,\eta,\cdot):\mathbb{R}^+\to[0,1]$ *is non-increasing.*

Proof. (1) Let $t,s>0$ with $t>s$. Then, there is $\epsilon>0$ such that $t=s+\epsilon$ that gives

$$\begin{aligned}\mathcal{T}_{\mathcal{T}}(\rho,\aleph,\eta,t)&=\mathcal{T}_{\mathcal{T}}(\rho,\aleph,\eta,s+\epsilon)\\&\geq \mathcal{T}_{\mathcal{T}}(\rho,\rho,\rho,s)*\mathcal{T}_{\mathcal{T}}(\rho,\aleph,\eta,\epsilon)\\&=\mathcal{T}_{\mathcal{T}}(\rho,\aleph,\eta,s).\end{aligned}$$

The proofs of (2) and (3) are similar to that of (1). □

4. Fixed Point Results

Definition 4.1. ([21]) A function $\Upsilon:\mathbb{R}^+\to\mathbb{R}^+$ is called a comparison function if it satisfies the following:

(1) $\Upsilon:\mathbb{R}^+\to\mathbb{R}^+$ is monotone increasing,
(2) $\lim_{n\to\infty}\Upsilon^n(\tau)=0$ for all $\tau>0$.

Remark 4.2. ([21]) If Υ is comparison function, then

(1) $\Upsilon(0)=0$,
(2) $\Upsilon(\tau)<\tau$ fo rall $\tau>0$.

Next, the concept of quasi RJN-Υ contractions is presented.

Definition 4.3. Let $(\Omega,\mathcal{T}_{\mathcal{T}},\mathcal{T}_{\mathcal{F}},\mathcal{T}_{\mathcal{I}},*,\diamond)$ be a RJNMS. A mapping $f:\Omega\to\Omega$ is called quasi RJN-Υ contraction if there exists a comparison function $\Upsilon:\mathbb{R}^+\to\mathbb{R}^+$ and for each $\rho,\aleph,\eta\in\Omega$ and $\gamma>0$, we have

$$\begin{aligned}\frac{1}{\mathcal{T}_{\mathcal{T}}(f\rho,f\aleph,f\eta,\gamma)}-1&\leq\Upsilon(\mathscr{M}_{\mathcal{T}_{\mathcal{T}},f}(\rho,\aleph,\eta,\gamma)),\\ \mathcal{T}_{\mathcal{F}}(f\rho,f\aleph,f\eta,\gamma)&\leq\Upsilon(\mathscr{N}_{\mathcal{T}_{\mathcal{F}},f}(\rho,\aleph,\eta,\gamma)),\\ \mathcal{T}_{\mathcal{I}}(f\rho,f\aleph,f\eta,\gamma)&\leq\Upsilon(\mathscr{N}_{\mathcal{T}_{\mathcal{I}},f}(\rho,\aleph,\eta,\gamma)).\end{aligned}\tag{4.1}$$

If $\mathcal{G} : \Omega^3 \times \mathbb{R} \to \mathbb{R}$, Then

$$\begin{aligned}\mathscr{M}_{\mathcal{G},f}(\rho,\aleph,\eta,\gamma) = \max\Bigg\{&\frac{1}{\mathcal{G}(\rho,\aleph,\eta,\gamma)}-1, \frac{1}{\mathcal{G}(\rho,f\rho,f\rho,\gamma)}-1,\\ &\frac{1}{\mathcal{G}(\aleph,f\aleph,f\aleph,\gamma)}-1, \frac{1}{\mathcal{G}(\rho,f\aleph,f\aleph,\gamma)}-1,\\ &\frac{1}{\mathcal{G}(\aleph,f\rho,f\rho,\gamma)}-1, \frac{1}{\mathcal{G}(\eta,f\eta,f\eta,\gamma)}-1,\\ &\frac{1}{\mathcal{G}(\aleph,f\eta,f\eta,\gamma)}-1, \frac{1}{\mathcal{G}(\rho,f\eta,f\eta,\gamma)}-1,\\ &\frac{1}{\mathcal{G}(\eta,f\rho,f\rho,\gamma)}-1, \frac{1}{\mathcal{G}(\eta,f\aleph,f\aleph,\gamma)}-1\Bigg\},\end{aligned}$$

$$\begin{aligned}\mathscr{N}_{\mathcal{G},f}(\rho,\aleph,\eta,\gamma) = \max\{&\mathcal{G}(\rho,\aleph,\eta,\gamma), \mathcal{G}(\rho,f\rho,f\rho,\gamma), \mathcal{G}(\aleph,f\aleph,f\aleph,\gamma),\\ &\mathcal{G}(\rho,f\aleph,f\aleph,\gamma), \mathcal{G}(\aleph,f\rho,f\rho,\gamma), \mathcal{G}(\eta,f\eta,f\eta,\gamma),\\ &\mathcal{G}(\aleph,f\eta,f\eta,\gamma), \mathcal{G}(\rho,f\eta,f\eta,\gamma), \mathcal{G}(\eta,f\rho,f\rho,\gamma),\\ &\mathcal{G}(\eta,f\aleph,f\aleph,\gamma)\}.\end{aligned}$$

And

(1) $\delta_{\mathcal{G}}(\rho,f,\gamma) = \max\limits_{i,j,k\in\mathbb{N}} \left\{\frac{1}{\mathcal{G}(f^i\rho, f^j\rho, f^k\rho,\gamma)} - 1\right\}$,

(2) $\sigma_{\mathcal{G}}(\rho,f,\gamma) = \max\limits_{i,j,k\in\mathbb{N}} \left\{\mathcal{G}(f^i\rho, f^j\rho, f^k\rho,\gamma)\right\}$.

Theorem 4.4. *Let $(\Omega, \mathcal{T}_{\mathcal{T}}, \mathcal{T}_{\mathcal{F}}, \mathcal{T}_{\mathcal{I}}, *, \diamond)$ be a complete RJNMS, Suppose that $f : \Omega \to \Omega$ is quasi RJN-Υ contraction. Furthermore, suppose that at least one of the following conditions is satisfied:*

(a) *The function f is sequentially continuous that is, if $\rho_n \to \rho$, then $f\rho_n \to f\rho$,*

(b) *$\mathcal{T}_{\mathcal{T}}, \mathcal{T}_{\mathcal{F}}$ and $\mathcal{T}_{\mathcal{I}}$ are continuous in the first three coordinates.*

Then the fixed point set of f, $\mathcal{F}^{\exists}(f)$ is singleton.

Proof. Let $\rho_0 \in \Omega$ represents an arbitrary point. We examine the Picard sequence $\{\rho_n\}$ characterized by $\rho_{n+1} = f^n\rho_0$ where $n \geq 0$. For all $i,j,k \in \mathbb{N}$, we have from Condition (4.1) in Definition 4.3,

$$
\begin{aligned}
&\frac{1}{\mathcal{T}_{\mathcal{T}}(f^{n+i}\rho_0, f^{n+j}\rho_0, f^{n+k}\rho_0, \gamma)} - 1\\
&\quad \leq \Upsilon \ \max\left\{\frac{1}{\mathcal{T}_{\mathcal{T}}(f^{n+i-1}\rho_0, f^{n+j-1}\rho_0, f^{n+k-1}\rho_0, \gamma)} - 1,\right.\\
&\qquad\qquad \frac{1}{\mathcal{T}_{\mathcal{T}}(f^{n+i-1}\rho_0, f^{n+i}\rho_0, f^{n+i}\rho_0, \gamma)} - 1,\\
&\qquad\qquad \frac{1}{\mathcal{T}_{\mathcal{T}}(f^{n+j-1}\rho_0, f^{n+j}\rho_0, f^{n+j}\rho_0, \gamma)} - 1,\\
&\qquad\qquad \frac{1}{\mathcal{T}_{\mathcal{T}}(f^{n+i-1}\rho_0, f^{n+j}\rho_0, f^{n+j}\rho_0, \gamma)} - 1,\\
&\qquad\qquad \frac{1}{\mathcal{T}_{\mathcal{T}}(f^{n+j-1}\rho_0, f^{n+i}\rho_0, f^{n+i}\rho_0, \gamma)} - 1,\\
&\qquad\qquad \frac{1}{\mathcal{T}_{\mathcal{T}}(f^{n+i-1}\rho_0, f^{n+k}\rho_0, f^{n+k}\rho_0, \gamma)} - 1,\\
&\qquad\qquad \frac{1}{\mathcal{T}_{\mathcal{T}}(f^{n+j-1}\rho_0, f^{n+k}\rho_0, f^{n+k}\rho_0, \gamma)} - 1,\\
&\qquad\qquad \frac{1}{\mathcal{T}_{\mathcal{T}}(f^{n+k-1}\rho_0, f^{n+k}\rho_0, f^{n+k}\rho_0, \gamma)} - 1,\\
&\qquad\qquad \frac{1}{\mathcal{T}_{\mathcal{T}}(f^{n+k-1}\rho_0, f^{n+i}\rho_0, f^{n+i}\rho_0, \gamma)} - 1,\\
&\qquad\qquad \left.\frac{1}{\mathcal{T}_{\mathcal{T}}(f^{n+k-1}\rho_0, f^{n+j}\rho_0, f^{n+j}\rho_0, \gamma)} - 1\right\},
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{T}_{\mathcal{F}}(f^{n+i}\rho_0, f^{n+j}\rho_0, f^{n+k}\rho_0, \gamma)\\
&\quad \leq \Upsilon \ \max\Big\{\mathcal{T}_{\mathcal{F}}(f^{n+i-1}\rho_0, f^{n+j-1}\rho_0, f^{n+k-1}\rho_0, \gamma),\\
&\qquad\qquad \mathcal{T}_{\mathcal{F}}(f^{n+i-1}\rho_0, f^{n+i}\rho_0, f^{n+i}\rho_0, \gamma),\\
&\qquad\qquad \mathcal{T}_{\mathcal{F}}(f^{n+j-1}\rho_0, f^{n+j}\rho_0, f^{n+j}\rho_0, \gamma),\\
&\qquad\qquad \mathcal{T}_{\mathcal{F}}(f^{n+i-1}\rho_0, f^{n+j}\rho_0, f^{n+j}\rho_0, \gamma),\\
&\qquad\qquad \mathcal{T}_{\mathcal{F}}(f^{n+j-1}\rho_0, f^{n+i}\rho_0, f^{n+i}\rho_0, \gamma),\\
&\qquad\qquad \mathcal{T}_{\mathcal{F}}(f^{n+i-1}\rho_0, f^{n+k}\rho_0, f^{n+k}\rho_0, \gamma),\\
&\qquad\qquad \mathcal{T}_{\mathcal{F}}(f^{n+j-1}\rho_0, f^{n+k}\rho_0, f^{n+k}\rho_0, \gamma),\\
&\qquad\qquad \mathcal{T}_{\mathcal{F}}(f^{n+k-1}\rho_0, f^{n+k}\rho_0, f^{n+k}\rho_0, \gamma),
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{T}_{\mathcal{F}}(f^{n+k-1}\rho_0, f^{n+i}\rho_0, f^{n+i}\rho_0, \gamma),\\
&\mathcal{T}_{\mathcal{F}}(f^{n+k-1}\rho_0, f^{n+j}\rho_0, f^{n+j}\rho_0, \gamma)\Big\},
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{T}_{\mathcal{I}}(f^{n+i}\rho_0, f^{n+j}\rho_0, f^{n+k}\rho_0, \gamma)\\
&\quad \le \Upsilon \ \max\Big\{\mathcal{T}_{\mathcal{I}}(f^{n+i-1}\rho_0, f^{n+j-1}\rho_0, f^{n+k-1}\rho_0, \gamma),\\
&\qquad \mathcal{T}_{\mathcal{I}}(f^{n+i-1}\rho_0, f^{n+i}\rho_0, f^{n+i}\rho_0, \gamma),\\
&\qquad \mathcal{T}_{\mathcal{I}}(f^{n+j-1}\rho_0, f^{n+j}\rho_0, f^{n+j}\rho_0, \gamma),\\
&\qquad \mathcal{T}_{\mathcal{I}}(f^{n+i-1}\rho_0, f^{n+j}\rho_0, f^{n+j}\rho_0, \gamma),\\
&\qquad \mathcal{T}_{\mathcal{I}}(f^{n+j-1}\rho_0, f^{n+i}\rho_0, f^{n+i}\rho_0, \gamma),\\
&\qquad \mathcal{T}_{\mathcal{I}}(f^{n+i-1}\rho_0, f^{n+k}\rho_0, f^{n+k}\rho_0, \gamma),\\
&\qquad \mathcal{T}_{\mathcal{I}}(f^{n+j-1}\rho_0, f^{n+k}\rho_0, f^{n+k}\rho_0, \gamma),\\
&\qquad \mathcal{T}_{\mathcal{I}}(f^{n+k-1}\rho_0, f^{n+k}\rho_0, f^{n+k}\rho_0, \gamma),\\
&\qquad \mathcal{T}_{\mathcal{I}}(f^{n+k-1}\rho_0, f^{n+i}\rho_0, f^{n+i}\rho_0, \gamma),\\
&\qquad \mathcal{T}_{\mathcal{I}}(f^{n+k-1}\rho_0, f^{n+j}\rho_0, f^{n+j}\rho_0, \gamma)\Big\}.
\end{aligned}
$$

Thus, we conclude

$$
\begin{aligned}
\delta_{\mathcal{T}_{\mathcal{T}}}(f^n\rho_0, f, \gamma) &\le \Upsilon \ \delta_{\mathcal{T}_{\mathcal{T}}}(f^{n-1}\rho_0, f, \gamma),\\
\sigma_{\mathcal{T}_{\mathcal{F}}}(f^n\rho_0, f, \gamma) &\le \Upsilon \ \sigma_{\mathcal{T}_{\mathcal{F}}}(f^{n-1}\rho_0, f, \gamma)
\end{aligned}
$$

and

$$
\sigma_{\mathcal{T}_{\mathcal{I}}}(f^n\rho_0, f, \gamma) \le \Upsilon \ \sigma_{\mathcal{T}_{\mathcal{I}}}(f^{n-1}\rho_0, f, \gamma).
$$

Hence, by induction we have for $n \ge 1$,

$$
\begin{aligned}
\delta_{\mathcal{T}_{\mathcal{T}}}(f^n\rho_0, f, \gamma) &\le \Upsilon^n \ \delta_{\mathcal{T}_{\mathcal{T}}}(\rho_0, f, \gamma),\\
\sigma_{\mathcal{T}_{\mathcal{F}}}(f^n\rho_0, f, \gamma) &\le \Upsilon^n \ \sigma_{\mathcal{T}_{\mathcal{F}}}(\rho_0, f, \gamma)
\end{aligned}
$$

and

$$
\sigma_{\mathcal{T}_{\mathcal{I}}}(f^n\rho_0, f, \gamma) \le \Upsilon^n \ \sigma_{\mathcal{T}_{\mathcal{I}}}(\rho_0, f, \gamma).
$$

From the above inequalities, we have

$$
\begin{aligned}
&\frac{1}{\mathcal{T}_{\mathcal{T}}(f^n\rho_0, f^{n+m}\rho_0, f^{n+l}\rho_0, \gamma)} - 1 \le \delta_{\mathcal{T}_{\mathcal{T}}}(f^n\rho_0, f, \gamma) \le \Upsilon^n \ \delta_{\mathcal{T}_{\mathcal{T}}}(\rho_0, f, \gamma),\\
&\mathcal{T}_{\mathcal{F}}(f^n\rho_0, f^{n+m}\rho_0, f^{n+l}\rho_0, \gamma) \le \sigma_{\mathcal{T}_{\mathcal{F}}}(f^n\rho_0, f, \gamma) \le \Upsilon^n \ \sigma_{\mathcal{T}_{\mathcal{F}}}(\rho_0, f, \gamma)
\end{aligned}
$$

and

$$\mathcal{T}_{\mathcal{I}}(f^n\rho_0, f^{n+m}\rho_0, f^{n+l}\rho_0, \gamma) \leq \sigma_{\mathcal{T}_{\mathcal{I}}}(f^n\rho_0, f, \gamma) \leq \Upsilon^n \ \sigma_{\mathcal{T}_{\mathcal{I}}}(\rho_0, f, \gamma).$$

By taking the limit when $n, m, l \to \infty$, we get

$$\lim_{n,m,l\to\infty} \mathcal{T}_{\mathcal{T}}(f^n\rho_0, f^{n+m}\rho_0, f^{n+l}\rho_0, \gamma) = 1,$$

$$\lim_{n,m,l\to\infty} \mathcal{T}_{\mathcal{F}}(f^n\rho_0, f^{n+m}\rho_0, f^{n+l}\rho_0, \gamma) = 0$$

and

$$\lim_{n,m,l\to\infty} \mathcal{T}_{\mathcal{I}}(f^n\rho_0, f^{n+m}\rho_0, f^{n+l}\rho_0, \gamma) = 0.$$

Hence, $\{f^n\rho_0\}$ is a RJN-Cauchy sequence. Thus, it is convergent, and so, an element $u \in \Omega$ exists such that $f^n\rho_0 \to u$.

If (a) holds, then $f^{n+1}\rho_0 = ff^n\rho_0 \to fu$ and so, $u = fu$.

If (b) holds, then Definition 4.3 implies that

$$\begin{aligned}
&\mathcal{T}_{\mathcal{I}}(fu, f^{n+1}\rho_0, f^{n+1}\rho_0, \gamma) \\
&\quad \leq \Upsilon \ \max\{\mathcal{T}_{\mathcal{I}}(u, f^n\rho_0, f^n\rho_0, \gamma), \mathcal{T}_{\mathcal{I}}(u, fu, fu, \gamma), \\
&\qquad\qquad \mathcal{T}_{\mathcal{I}}(f^n\rho_0, f^{n+1}\rho_0, f^{n+1}\rho_0, \gamma), \mathcal{T}_{\mathcal{I}}(u, f^{n+1}\rho_0, f^{n+1}\rho_0, \gamma), \\
&\qquad\qquad \mathcal{T}_{\mathcal{I}}(f^n\rho_0, fu, fu, \gamma), \mathcal{T}_{\mathcal{I}}(f^n\rho_0, f^{n+1}\rho_0, f^{n+1}\rho_0, \gamma), \\
&\qquad\qquad \mathcal{T}_{\mathcal{I}}(f^n\rho_0, f^{n+1}\rho_0, f^{n+1}\rho_0, \gamma), \mathcal{T}_{\mathcal{I}}(u, f^{n+1}\rho_0, f^{n+1}\rho_0, \gamma), \\
&\qquad\qquad \mathcal{T}_{\mathcal{I}}(f^n\rho_0, fu, fu, \gamma), \mathcal{T}_{\mathcal{I}}(f^n\rho_0, f^{n+1}\rho_0, f^{n+1}\rho_0, \gamma)\}.
\end{aligned}$$

By taking the limit, we get

$$\mathcal{T}_{\mathcal{I}}(fu, u, u, \gamma) \leq \Upsilon(\mathcal{T}_{\mathcal{I}}(fu, u, u, \gamma)).$$

If $\mathcal{T}_{\mathcal{I}}(fu, u, u, \gamma)$ is not equal 0, then

$$\mathcal{T}_{\mathcal{I}}(fu, u, u, \gamma) < \mathcal{T}_{\mathcal{I}}(fu, u, u, \gamma),$$

which is a contradiction. Hence $u = fu$. Note that the proofs of $\mathcal{T}_{\mathcal{T}}$ and $\mathcal{T}_{\mathcal{F}}$ are as same as $\mathcal{T}_{\mathcal{I}}$

Now, let $w \in \Omega$ be so that $w = fw$. If we assume $u \neq w$, then depending on Definition 4.3, we get

$$\begin{aligned}\mathcal{T}_{\mathcal{I}}(u,w,w,\gamma) = \mathcal{T}_{\mathcal{I}}(fu,fw,fw,\gamma) \leq &\Upsilon \ \max\{\mathcal{T}_{\mathcal{I}}(u,w,w,\gamma), \mathcal{T}_{\mathcal{I}}(u,u,u,\gamma), \\ &\quad \mathcal{T}_{\mathcal{I}}(w,w,w,\gamma), \mathcal{T}_{\mathcal{I}}(u,w,w,\gamma), \\ &\quad \mathcal{T}_{\mathcal{I}}(w,u,u,\gamma), \mathcal{T}_{\mathcal{I}}(w,w,w,\gamma), \\ &\quad \mathcal{T}_{\mathcal{I}}(w,w,w,\gamma), \mathcal{T}_{\mathcal{I}}(u,w,w,\gamma), \\ &\quad \mathcal{T}_{\mathcal{I}}(w,u,u,\gamma), \mathcal{T}_{\mathcal{I}}(w,w,w,\gamma)\} \\ =&\Upsilon \ \max\{\mathcal{T}_{\mathcal{I}}(u,w,w,\gamma), \mathcal{T}_{\mathcal{I}}(w,u,u,\gamma)\}.\end{aligned}$$

But,

$$\begin{aligned}\mathcal{T}_{\mathcal{I}}(u,u,w,\gamma) = \mathcal{T}_{\mathcal{I}}(fu,fu,fw,\gamma) \leq &\Upsilon \ \max\{\mathcal{T}_{\mathcal{I}}(u,u,w,\gamma), \mathcal{T}_{\mathcal{I}}(u,u,u,\gamma), \\ &\quad \mathcal{T}_{\mathcal{I}}(u,u,u,\gamma), \mathcal{T}_{\mathcal{I}}(u,u,u,\gamma), \\ &\quad \mathcal{T}_{\mathcal{I}}(u,u,u,\gamma), \mathcal{T}_{\mathcal{I}}(w,w,w,\gamma), \\ &\quad \mathcal{T}_{\mathcal{I}}(u,w,w,\gamma), \mathcal{T}_{\mathcal{I}}(u,w,w,\gamma), \\ &\quad \mathcal{T}_{\mathcal{I}}(w,u,u,\gamma), \mathcal{T}_{\mathcal{I}}(w,u,u,\gamma)\} \\ =&\Upsilon \ \max\{\mathcal{T}_{\mathcal{I}}(u,w,w,\gamma), \mathcal{T}_{\mathcal{I}}(w,u,u,\gamma)\}.\end{aligned}$$

Then,

$$\max\{\mathcal{T}_{\mathcal{I}}(u,w,w,\gamma), \mathcal{T}_{\mathcal{I}}(w,u,u,\gamma)\} \leq \Upsilon \ \max\{\mathcal{T}_{\mathcal{I}}(u,w,w,\gamma), \mathcal{T}_{\mathcal{I}}(w,u,u,\gamma)\}.$$

If

$$\max\{\mathcal{T}_{\mathcal{I}}(u,w,w,\gamma), \mathcal{T}_{\mathcal{I}}(w,u,u,\gamma)\} > 0,$$

it implies

$$\begin{aligned}\max\{\mathcal{T}_{\mathcal{I}}(u,w,w,\gamma), \mathcal{T}_{\mathcal{I}}(w,u,u,\gamma)\} &\leq \Upsilon \ \max\{\mathcal{T}_{\mathcal{I}}(u,w,w,\gamma), \mathcal{T}_{\mathcal{I}}(w,u,u,\gamma)\} \\ &< \max\{\mathcal{T}_{\mathcal{I}}(u,w,w,\gamma), \mathcal{T}_{\mathcal{I}}(w,u,u,\gamma)\},\end{aligned}$$

which is a contradiction. Hence,

$$\max\{\mathcal{T}_{\mathcal{I}}(u,w,w,\gamma), \mathcal{T}_{\mathcal{I}}(w,u,u,\gamma)\} = 0.$$

So,

$$\mathcal{T}_{\mathcal{I}}(u,w,w,\gamma) = \mathcal{T}_{\mathcal{I}}(w,u,u,\gamma) = 0.$$

Therefore, $u = w$. $\square$

5. Example and Corollaries

Example 5.1. Let $\Omega = [0,1]$, and define the refined J-neutrosophic structure by

$$\mathcal{T}_{\mathcal{T}}(\rho, \aleph, \eta, \gamma) = \frac{1}{1+\gamma(|\rho-\aleph|+|\aleph-\eta|+|\eta-\rho|)},$$

$$\mathcal{T}_{\mathcal{F}}(\rho, \aleph, \eta, \gamma) = 1-\mathcal{T}_{\mathcal{T}}(\rho, \aleph, \eta, \gamma), \quad \mathcal{T}_{\mathcal{I}}(\rho, \aleph, \eta, \gamma) = 0.$$

Let the operations $*$ and $\diamond$ be the standard continuous t-norm and t-conorm:

$$a * b = \min\{a, b\}, \qquad a \diamond b = \max\{a, b\}.$$

Clearly, $\mathcal{T}_{\mathcal{T}}(\rho, \aleph, \eta, \gamma) \in (0,1]$ and the triple $(\mathcal{T}_{\mathcal{T}}, \mathcal{T}_{\mathcal{F}}, \mathcal{T}_{\mathcal{I}})$ satisfies the axioms of refined J-neutrosophic metric structure. Since $[0,1]$ is complete with respect to the usual metric, the space $(\Omega, \mathcal{T}_{\mathcal{T}}, \mathcal{T}_{\mathcal{F}}, \mathcal{T}_{\mathcal{I}}, *, \diamond)$ is a complete RJNMS.

Now define the self-map $f : \Omega \to \Omega$ by

$$f(\rho) = \frac{\rho}{2}.$$

Let the comparison function be:

$$\Upsilon(a) = \frac{a}{2},$$

which is clearly monotone increasing and satisfies $\Upsilon^n(a) \to 0$ for all $a > 0$.

$$\frac{1}{\mathcal{T}_{\mathcal{T}}(f\rho, f\aleph, f\eta, \gamma)} - 1 \leq \Upsilon\left(\mathscr{M}_{\mathcal{T}_{\mathcal{T}}, f}(\rho, \aleph, \eta, \gamma)\right),$$

where

$$\begin{aligned}\mathscr{M}_{\mathcal{T}_{\mathcal{T}}, f}(\rho, \aleph, \eta, \gamma) = \max\Bigg\{ & \frac{1}{\mathcal{T}_{\mathcal{T}}(\rho, \aleph, \eta, \gamma)} - 1, \frac{1}{\mathcal{T}_{\mathcal{T}}(\rho, f\rho, f\rho, \gamma)} - 1, \\ & \frac{1}{\mathcal{T}_{\mathcal{T}}(\aleph, f\aleph, f\aleph, \gamma)} - 1, \frac{1}{\mathcal{T}_{\mathcal{T}}(\rho, f\aleph, f\aleph, \gamma)} - 1, \\ & \frac{1}{\mathcal{T}_{\mathcal{T}}(\aleph, f\rho, f\rho, \gamma)} - 1, \frac{1}{\mathcal{T}_{\mathcal{T}}(\eta, f\eta, f\eta, \gamma)} - 1, \\ & \frac{1}{\mathcal{T}_{\mathcal{T}}(\aleph, f\eta, f\eta, \gamma)} - 1, \frac{1}{\mathcal{T}_{\mathcal{T}}(\rho, f\eta, f\eta, \gamma)} - 1, \\ & \frac{1}{\mathcal{T}_{\mathcal{T}}(\eta, f\rho, f\rho, \gamma)} - 1, \frac{1}{\mathcal{T}_{\mathcal{T}}(\eta, f\aleph, f\aleph, \gamma)} - 1 \Bigg\}.\end{aligned}$$

For example, using $f(\rho) = \frac{\rho}{2}$ and choosing $\rho = 1$, $\aleph = 0.5$, $\eta = 0$, and $\gamma = 1$, we get

$$f\rho = 0.5, \quad f\aleph = 0.25, \quad f\eta = 0.$$

Then,

$$\begin{aligned}\mathcal{T}_{\mathcal{T}}(f\rho, f\aleph, f\eta, 1) &= \frac{1}{1+|0.5-0.25|+|0.25-0|+|0-0.5|}\\ &= \frac{1}{1+0.25+0.25+0.5}\\ &= \frac{1}{2}.\end{aligned}$$

So,

$$\frac{1}{\mathcal{T}_{\mathcal{T}}(f\rho, f\aleph, f\eta, 1)} - 1 = 1.$$

Also,

$$\begin{aligned}\mathscr{M}_{\mathcal{T}_{\mathcal{T}},f}(1, 0.5, 0, 1) &\geq \frac{1}{\mathcal{T}_{\mathcal{T}}(1, 0.5, 0, 1)} - 1\\ &= \frac{1}{1+1+0.5+1} - 1\\ &= \frac{1}{3.5} - 1 \ (< 1).\end{aligned}$$

Thus,

$$1 = \frac{1}{\mathcal{T}_{\mathcal{T}}(f\rho, f\aleph, f\eta, 1)} - 1 \leq \Upsilon(\mathscr{M}_{\mathcal{T}_{\mathcal{T}},f}) = \frac{\mathscr{M}_{\mathcal{T}_{\mathcal{T}},f}}{2},$$

which holds for general values when γ is sufficiently small. Also, we have

$$\begin{aligned}\mathcal{T}_{\mathcal{F}}(f\rho, f\aleph, f\eta, \gamma) &= 1 - \mathcal{T}_{\mathcal{T}}(f\rho, f\aleph, f\eta, \gamma)\\ &\leq \Upsilon(\mathscr{N}_{\mathcal{T}_{\mathcal{F}},f}(\rho, \aleph, \eta, \gamma))\end{aligned}$$

and

$$\mathcal{T}_{\mathcal{I}} = 0 \leq \Upsilon(\cdot).$$

Hence, all conditions of Definition 4.3 are satisfied, and since f is continuous and Ω is complete, the fixed point theorem applies. Therefore, f has a unique fixed point. In fact, we have $f(\rho) = \rho$ then $\frac{\rho}{2} = \rho$ so, we have $\rho = 0$.

Corollary 5.2. *Let $(\Omega, \mathcal{T}_{\mathcal{T}}, \mathcal{T}_{\mathcal{F}}, \mathcal{T}_{\mathcal{I}}, *, \diamond)$ be a complete RJNMS, and consider $f : \Omega \to \Omega$. Suppose that there exists a comparison function $\Upsilon : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $\rho, \aleph, \eta \in \Omega$ and $\gamma > 0$, the following conditions are satisfied:*

$$\frac{1}{\mathcal{T}_{\mathcal{T}}(f\rho, f\aleph, f\eta, \gamma)} - 1 \leq \Upsilon\left(\mathscr{W}_{\mathcal{T}_{\mathcal{T}},f}(\rho, \aleph, \eta, \gamma)\right),$$

$$\mathcal{T}_{\mathcal{F}}(f\rho, f\aleph, f\eta, \gamma) \leq \Upsilon\left(\mathscr{H}_{\mathcal{T}_{\mathcal{F}},f}(\rho, \aleph, \eta, \gamma)\right)$$

and

$$\mathcal{T}_{\mathcal{I}}(f\rho, f\aleph, f\eta, \gamma) \leq \Upsilon\left(\mathscr{H}_{\mathcal{T}_{\mathcal{I}},f}(\rho, \aleph, \eta, \gamma)\right),$$

where

$$\mathscr{W}_{\mathcal{G},f}(\rho,\aleph,\eta,\gamma)=\max\Big\{\frac{1}{\mathcal{G}(\rho,\aleph,\eta,\gamma)}-1,\frac{1}{\mathcal{G}(\rho,f\rho,f\rho,\gamma)}-1,\frac{1}{\mathcal{G}(\aleph,f\aleph,f\aleph,\gamma)}-1,\frac{1}{\mathcal{G}(\eta,f\eta,f\eta,\gamma)}-1\Big\},$$

$$\mathscr{H}_{\mathcal{G},f}(\rho,\aleph,\eta,\gamma)=\max\{\mathcal{G}(\rho,\aleph,\eta,\gamma),\mathcal{G}(\rho,f\rho,f\rho,\gamma),\mathcal{G}(\aleph,f\aleph,f\aleph,\gamma),\mathcal{G}(\eta,f\eta,f\eta,\gamma)\}.$$

Additionally, assume that either:

(a) *The function f is sequentially continuous, or*
(b) $\mathcal{T}_{\mathcal{T}},\mathcal{T}_{\mathcal{F}},\mathcal{T}_{\mathcal{I}}$ *are continuous in the first three coordinates.*

Then $\mathcal{F}^{\exists}(f)$ is singleton.

Corollary 5.3. *Let $(\Omega,\mathcal{T}_{\mathcal{T}},\mathcal{T}_{\mathcal{F}},\mathcal{T}_{\mathcal{I}},*,\diamond)$ be a complete RJNMS, Suppose that a real number $k\in[0,1)$ exists so that for all $\rho,\aleph,\eta\in\Omega$ and each $\gamma>0$ the self-map $f:\Omega\to\Omega$ satisfies the following:*

$$\frac{1}{\mathcal{T}_{\mathcal{T}}(f\rho,f\aleph,f\eta,\gamma)}-1\le k\ \mathscr{W}_{\mathcal{T}_{\mathcal{T}},f}(\rho,\aleph,\eta,\gamma),$$

$$\mathcal{T}_{\mathcal{F}}(f\rho,f\aleph,f\eta,\gamma)\le k\ \mathscr{H}_{\mathcal{T}_{\mathcal{F}},f}(\rho,\aleph,\eta,\gamma)$$

and

$$\mathcal{T}_{\mathcal{I}}(f\rho,f\aleph,f\eta,\gamma)\le k\ \mathscr{H}_{\mathcal{T}_{\mathcal{I}},f}(\rho,\aleph,\eta,\gamma).$$

Furthermore, suppose that at least one of the following conditions is satisfied:

(1) *The function f is sequential continuous,*
(2) $\mathcal{T}_{\mathcal{T}},\mathcal{T}_{\mathcal{F}}$ *and* $\mathcal{T}_{\mathcal{I}}$ *are continuous in the first three coordinates.*

Therefore, $\mathcal{F}^{\exists}(f)$ is singleton.

Corollary 5.4. *Let $(\Omega,\mathcal{T}_{\mathcal{T}},\mathcal{T}_{\mathcal{F}},\mathcal{T}_{\mathcal{I}},*,\diamond)$ be a complete RJNMS, Suppose that there is $k\in[0,1)$ such that for each $\rho,\aleph,\eta\in\Omega$ and each $\gamma>0$ the map $f:\Omega\to\Omega$ satisfies the following:*

$$\frac{1}{\mathcal{T}_{\mathcal{T}}(f\rho,f\aleph,f\eta,\gamma)}-1\le\frac{k}{3}\ \Big(\frac{1}{\mathcal{T}_{\mathcal{T}}(\rho,f\rho,f\rho,\gamma)}-1+\frac{1}{\mathcal{T}_{\mathcal{T}}(\aleph,f\aleph,f\aleph,\gamma)}-1+\frac{1}{\mathcal{T}_{\mathcal{T}}(\eta,f\eta,f\eta,\gamma)}-1\Big),$$

$$\mathcal{T}_{\mathcal{F}}(f\rho,f\aleph,f\eta,\gamma)\le\frac{k}{3}\ (\mathcal{T}_{\mathcal{F}}(\rho,f\rho,f\rho,\gamma)+\mathcal{T}_{\mathcal{F}}(\aleph,f\aleph,f\aleph,\gamma)+\mathcal{T}_{\mathcal{F}}(\eta,f\eta,f\eta,\gamma))$$

and

$$\mathcal{T}_{\mathcal{I}}(f\rho, f\aleph, f\eta, \gamma) \leq \frac{k}{3}\ (\mathcal{T}_{\mathcal{I}}(\rho, f\rho, f\rho, \gamma) + \mathcal{T}_{\mathcal{I}}(\aleph, f\aleph, f\aleph, \gamma) + \mathcal{T}_{\mathcal{I}}(\eta, f\eta, f\eta, \gamma)).$$

Furthermore, suppose that at least one of the following conditions is satisfied:

(1) *The function f is sequentially continuous,*

(2) $\mathcal{T}_{\mathcal{T}}, \mathcal{T}_{\mathcal{F}}$ *and* $\mathcal{T}_{\mathcal{I}}$ *are continuous in the first three coordinates.*

Then, $\mathcal{F}^{\exists}(f)$ *is singleton.*

6. Conclusion

This work has introduced a refined and rigorous framework for J-neutrosophic metric spaces. The proposed Refined J-Neutrosophic Metric Space (RJNMS) corrects and unifies previous definitions, offering a more robust and streamlined structure for handling uncertainty through truth, falsity, and indeterminacy membership functions. Within this new setting, we have investigated a significant class of mappings, namely quasi RJN-Υ contractions. Our central result is a fixed point theorem that guarantees the existence and uniqueness of fixed points for these contractions, generalizing several classical and recent results in fixed point theory. The validity and utility of our findings were demonstrated through concrete examples and specialized corollaries.

This study not only clarifies the foundational aspects of J-neutrosophic metrics but also opens up promising avenues for future research. Potential directions include extending these results to other contraction types, studying common fixed points, and exploring applications in integral equations, dynamic systems, and other domains where uncertainty and indeterminacy are inherent.

Acknowledgments: The authors are grateful to the Deanship of Scientific Research at Jadara University for providing financial support for this publication.

References

[1] K. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets and Syst., **20**(1) (1986), 87–96.

[2] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math., **3** (1922), 133–181.

[3] A. Bataihah and A. Hazaymeh, *Quasi Contractions and Fixed Point Theorems in the Context of Neutrosophic Fuzzy Metric Spaces*, Euro. J. Pure Appl. Math., **18**(1) (2025), 57–85.

[4] A. Bataihah, A. A. Hazaymeh, Y. Al-Qudah and F. Al-Sharqi, *Some fixed point theorems in complete neutrosophic metric spaces for neutrosophic Υ-quasi contractions*, Neutrosophic Sets and Syst., **82**(1) (2025), pp. 1.

[5] A. Bataihah, T. Qawasmeh, I. Batiha, I. M. Batiha and T. Abdeljawad, *Gamma Distance Mappings with Application to Fractional Boundary Differential Equation*, J. Math. Anal., **15**(5) (2024), 99–106.
[6] A. Hazaymeh and A. Bataihah, *Results on fixed points in neutrosophic metric spaces through the use of simulation functions*, J. Math. Anal., **15**(6) (2024).
[7] M.V. Jeyaraman, R. Muthuraj, M. Sornavalli and M. Jeyabharathi, *Common fixed point theorems in G-fuzzy metric spaces*, J. New Theory, **10** (2016), 12–18.
[8] M. Jleli and B. Samet, *A generalized metric space and related fixed point theorems*, Fixed Point Theory Appl., (2015), pp. 1–14.
[9] J. Johnsy and M. Jeyaraman, *Fixed point theorems for $(\Upsilon$-$\varphi)$-contractions in generalized neutrosophic metric spaces*, Bull. Math. Anal. Appl., **16**(1) (2024).
[10] R. Kannan, *Some results on fixed points*, Bull. Calcu. Math. Soc., **60** (1968), 71–76.
[11] M. Kirişci and N. Şimşek, *Neutrosophic metric spaces*, Math. Sci., **14**(3) (2020), 241–248.
[12] I. Kramosil and J. Michálek, *Fuzzy metrics and statistical metric spaces*, Kybernetika, **11**(5) (1975), 336–344.
[13] A.A.R.M. Malkawi, *Convergence and Fixed Points of Self-Mappings in MR-Metric Spaces: Theory and Applications*, Euro. J. Pure Appl. Math., **18**(2) (2025), Article Number 5952.
[14] A.A.R.M. Malkawi, *Existence and Uniqueness of Fixed Points in MR-Metric Spaces and Their Applications*, Euro. J. Pure Appl. Math., **18**(2) (2025), Article Number 6077.
[15] A. Malkawi, *Enhanced uncertainty modeling through neutrosophic MR-metrics: a unified framework with fuzzy embedding and contraction principles*, Euro. J. Pure Appl. Math., **18**(3) (2025), Article number 6475.
[16] A. Malkawi, *Applications of MR-metric spaces in measure theory and convergence analysis*, Euro. J. Pure Appl. Math., **18**(3) (2025), Artical Number 6528.
[17] A.A.R.M. Malkawi, *Fixed Point Theorem in MR-metric Spaces Via Integral Type Contraction*, WSEAS Trans. Math., **24** (2025), 295–299.
[18] A.A.R.M. Malkawi, D. Mahmoud, A.M. Rabaiah, R. Al-Deiakeh and W. Shatanawi, *On fixed point theorems in MR-metric spaces*, Nonlinear Funct. Anal. Appl., **29**(4) (2024), 1125–1136.
[19] A. Malkawi and A. Rabaiah, *MR-metric spaces: theory and applications in weighted graphs, expander graphs, and fixed-point theorems*, Euro. J. Pure Appl. Math., **18**(3) (2025), Artical Number 6525.
[20] A. Malkawi and A. Rabaiah, *Compactness and separability in MR-metric spaces with applications to deep learning*, Euro. J. Pure Appl. Math., **18**(3) (2025), Artical Number 6592.
[21] J. Matkowski, *Integrable solutions of functional equations*, Diss. Math., **127** (1975), 1–68.
[22] K. Menger, Statistical Metrics, Proc. National Acad. Sci., **28**(12) (1942), 535–537.
[23] S. Mookiah and M. Jeyaraman, *A generalized neutrosophic metric space and coupled coincidence point results*, Neutrosophic Sets Syst., **42** (2021), 253–269.
[24] Z. Mustafa and B. Sims, *A new approach to generalized metric spaces*, J. Nonlinear Convex Anal., **7**(2) (2006), 289–298.
[25] J.H. Park, *Intuitionistic fuzzy metric spaces*, Chaos, Solitons Fractals, **22**(5) (2004), 1039–1046.
[26] T. Qawasmeh, (H, Ω_b)*-Interpolative Contractions in* Ω_b*-Distance Mappings with Applications*, Euro. J. Pure Appl. Math., **16**(3) (2023), 1717–1730.

[27] T. Qawasmeh, *H-Simulation functions and Ω_b-distance mappings in the setting of G_b-metric spaces and application*, Nonlinear Funct. Anal. Appl., **28**(2) (2023), 557–570.

[28] T. Qawasmeh and A. Malkawi, *Fixed point theory in MR-metric spaces: fundamental theorems and applications to integral equations and neutron transport*, Euro. J. Pure Appl. Math., **18**(3) (2025), Artical Number 6440.

[29] F. Smarandache, *Neutrosophic set-a generalization of the intuitionistic fuzzy set*, Int. J. Pure Appl. Math., **24**(3) (2005), 287–297.

[30] L.A. Zadeh, Fuzzy sets, Infor. Control, **8**(3) (1965), 338–353.