Nonlinear Functional Analysis and Applications
Vol. 31, No. 3 (2026), pp. 843-859
ISSN: 1229-1595(print), 2466-0973(online)

https://doi.org/10.22771/nfaa.2026.31.03.09
http://nfaa.kyungnam.ac.kr/journal-nfaa

TOPOLOGICAL STRUCTURE AND FIXED POINT THEORY IN NEUTROSOPHIC MR-METRIC SPACES

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Abstract. This paper examines the topological and measure-theoretic properties of neutrosophic MR-metric spaces (NMR-MS). It is shown that every complete NMR-MS induces a Hausdorff topology that is first countable, separable, and metrizable. In addition, a fixed point theorem for neutrosophic contraction mappings is established, ensuring existence, uniqueness, μ-almost everywhere convergence, measure-theoretic stability, and exponential convergence in measure. The results connect neutrosophic set theory with topology and classical fixed point theory, offering a unified framework for addressing uncertainty and imprecision in analytical structures. Several examples and applications to integral equations are included to illustrate the theoretical findings.

1. Introduction

The generalization of classical metric spaces has been a central theme in modern functional analysis, leading to the development of various structures such as almost metric spaces [10], b-metric spaces [11], G_b-metric spaces, and

[0]Received December 10, 2025. Revised March 29, 2026. Accepted March 30, 2026.
[0]2020 Mathematics Subject Classification: 47H10, 54E50, 54H25.
[0]Keywords: Neutrosophic MR-metric spaces, fixed point theory, topological properties, measure theory, neutrosophic contraction, integral equations.
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Mb-metric spaces [39]. Among these, MR-metric spaces [25] have emerged as a particularly flexible framework, introducing a three-variable metric governed by a relaxed tetrahedral inequality with a constant $R > 1$. This structure has proven instrumental in addressing analytical problems where traditional metrics fall short. Foundational results in this direction include fixed point theorems for integral-type contractions in MR-metric spaces [16], existence and uniqueness principles for self-mappings [17, 18], and fixed point theorems for fuzzy mappings in MR-metric spaces [20]. Furthermore, recent work has explored measure-theoretic convergence, compactness, separability [19, 29], common fixed point theorems [24], and even fractal dimension estimation within MR-metric spaces [22], demonstrating their broad utility.

These developments have been complemented by results in M^*-metric spaces [8, 13, 36], including common fixed point theorems for weakly compatible mappings [36], and have found significant applications in fractional calculus [1, 14, 26, 27], integral equations [40], and numerical analysis [3, 7, 41]. The growing body of research also encompasses nonlinear contractions [42], cyclic mappings, rational contractions, and simulation functions, all of which reinforce the versatility of generalized metric frameworks. Additionally, MR-metric spaces have been extended to weighted graphs and expander graphs [28], and have found applications in fractional calculus and fixed-point theorems [26, 27].

The incorporation of fuzzy and neutrosophic logics into metric frameworks has further expanded the capacity to represent uncertainty, indeterminacy, and imprecision. Neutrosophic sets, which generalize fuzzy sets by employing truth, falsity, and indeterminacy membership functions, have been utilized in various fixed point investigations [2, 4, 5, 21]. The concept of neutrosophic MR-metric spaces (NMR-MS) [20, 21, 32] emerges as a synthesis of the structural adaptability of MR-metrics and the expressive power of neutrosophic logic.

Recent contributions have established topological foundations, completion theorems, and applications in machine learning [9, 30], as well as measure-theoretic fixed point theorems with exponential convergence guarantees [23, 30, 33, 34, 35]. Neutrosophic frameworks have also been extended to non-commutative settings [37] and integrated with topos-theoretic perspectives [32], while fuzzy metric spaces derived from these structures have found applications in medical imaging, transportation, and social networks [31]. Concurrently, developments in allied structures such as Fermatean fuzzy metrics [12], complex multi-fuzzy soft sets [15], interval-valued Q-neutrosophic soft matrices [6], and neutrosophic soft expert settings [2] have enriched the theoretical landscape and provided practical tools for decision-making and classification [2, 4, 5, 7].

Furthermore, the modification of conformable fractional derivatives with classical properties [8], the mathematical analysis of one-heptagonal carbon nanocones [7], and the study of D-compact topological spaces [41] demonstrate the broad applicability of these frameworks across diverse scientific domains.

This paper investigates the topological and measure-theoretic properties of complete NMR-MS, establishes a comprehensive fixed point theorem with uniqueness, μ-almost everywhere convergence, and exponential convergence in measure, and demonstrates applications to integral equations. The results connect neutrosophic set theory, topology, and classical fixed point theory, offering a unified framework for addressing uncertainty in analytical structures.

Definition 1.1. ([25]) Consider a non-empty set $\mathbb{X} \neq \emptyset$ and a real number $\mathbb{R} > 1$. A function

$$M : \mathbb{X} \times \mathbb{X} \times \mathbb{X} \to [0, \infty)$$

is termed an MR-metric if it satisfies the following conditions for all $v, \xi, s, \ell_1 \in \mathbb{X}$:

- $M(v, \xi, s) \geq 0$.
- $M(v, \xi, s) = 0$ if and only if $v = \xi = s$.
- $M(v, \xi, s)$ remains invariant under any permutation $p(v, \xi, s)$, *that is*,

$$M(v, \xi, s) = M(p(v, \xi, s)).$$

- The following inequality holds:

$$M(v, \xi, s) \leq \mathbb{R}\left[M(v, \xi, \ell_1) + M(v, \ell_1, s) + M(\ell_1, \xi, s)\right].$$

A structure $(\mathbb{X}, M)$ that adheres to these properties is defined as an MR-metric space.

Definition 1.2. ([21]) [Neutrosophic MR-Metric Space (NMR-MS)] A 9-tuple $(\mathcal{Z}, M, \mathcal{T}, \mathcal{F}, \mathcal{I}, \bullet, \diamond, R, \star)$ is called a neutrosophic MR-metric space if

(1) $\mathcal{Z}$ is a non-empty set.

(2) $M : \mathcal{Z} \times \mathcal{Z} \times \mathcal{Z} \to [0, \infty)$ is an MR-metric satisfying:
(M1) $M(v, \xi, \Im) \geq 0$,
(M2) $M(v, \xi, \Im) = 0$ if and only if $v = \xi = \Im$,
(M3) Symmetry under permutations,
(M4) $M(v, \xi, \Im) \leq R\,[M(v, \xi, \ell) \star M(v, \ell, \Im) \star M(\ell, \xi, \Im)]$, $R > 1$.

(3) $\mathcal{T}, \mathcal{F}, \mathcal{I} : \mathcal{Z} \times \mathcal{Z} \times (0, \infty) \to [0, 1]$ are neutrosophic functions satisfying:
(N1) $\mathcal{T}(v, \xi, \gamma) = 1$ if and only if $v = \xi$ (Truth-Identity),
(N2) $\mathcal{T}(v, \xi, \gamma) = \mathcal{T}(\xi, v, \gamma)$ (Symmetry),

(N3) $\mathcal{T}(\upsilon,\xi,\gamma) \bullet \mathcal{T}(\xi,\Im,\rho) \leq \mathcal{T}(\upsilon,\Im,\gamma+\rho)$ (Triangle Inequality),
(N4) $\lim_{\gamma\to\infty} \mathcal{T}(\upsilon,\xi,\gamma) = 1$ (Asymptotic Behavior).

(4) $\bullet$ (t-norm) and $\diamond$ (t-conorm) are continuous operators generalizing fuzzy logic.

(5) $\star$ is a binary operation generalizing addition (e.g., weighted sum).

2. Main results

Building upon the foundational definitions of MR-metric spaces [25] and their neutrosophic extensions [21], this section is dedicated to establishing the core theoretical contributions of this work. We systematically explore the topological implications of completeness in neutrosophic MR-metric spaces and develop a comprehensive fixed point theory for operators satisfying neutrosophic contraction conditions. The ensuing results seamlessly blend topological rigor with measure-theoretic insights, offering a powerful toolkit for analyzing nonlinear problems under uncertainty.

Theorem 2.1. (Neutrosophic MR-Topological Structure) *Every complete NMR-MS $(\mathcal{Z}, M, \mathcal{T}, \mathcal{F}, \mathcal{I})$ induces a Hausdorff topological space $(\mathcal{Z}, \tau_N)$ where,*

(1) *The topology τ_N is generated by the base:*

$$\mathfrak{B} = \{\mathcal{B}(\upsilon,\gamma,\epsilon) : \upsilon \in \mathcal{Z}, \gamma > 0, \epsilon \in (0,1)\}.$$

(2) *$(\mathcal{Z}, \tau_N)$ is first countable and separable.*
(3) *The neutrosophic topology is metrizable by the metric:*

$$d_N(\upsilon,\xi) = \inf\{\gamma > 0 : \mathcal{T}(\upsilon,\xi,\gamma) > 1-\epsilon, \mathcal{F}(\upsilon,\xi,\gamma) < \epsilon\}.$$

(4) *The identity map $id : (\mathcal{Z}, \tau_N) \to (\mathcal{Z}, \tau_M)$ is continuous, where τ_M is the topology induced by M.*

Proof. We prove each property systematically, leveraging the complete structure of the neutrosophic MR-metric space.

Step 1: Base for Topology and Hausdorff Property

(a) $\mathfrak{B}$ is a Base for τ_N
We verify that $\mathfrak{B} = \{\mathcal{B}(\upsilon,\gamma,\epsilon) : \upsilon \in \mathcal{Z}, \gamma > 0, \epsilon \in (0,1)\}$ satisfies the base conditions:

(1) Covering: For any $\upsilon \in \mathcal{Z}$, take $\gamma = 1$ and $\epsilon = \frac{1}{2}$. By the neutrosophic identity property (N1), $\mathcal{T}(\upsilon,\upsilon,1) = 1$, $\mathcal{F}(\upsilon,\upsilon,1) = 0$, $\mathcal{I}(\upsilon,\upsilon,1) = 0$. Thus $\upsilon \in \mathcal{B}(\upsilon,1,\frac{1}{2})$, so $\mathfrak{B}$ covers $\mathcal{Z}$.

(2) Intersection Property: Let $\upsilon \in \mathcal{B}(\xi, \gamma_1, \epsilon_1) \cap \mathcal{B}(\zeta, \gamma_2, \epsilon_2)$. We need to find $\mathcal{B}(\upsilon, \gamma, \epsilon) \subseteq \mathcal{B}(\xi, \gamma_1, \epsilon_1) \cap \mathcal{B}(\zeta, \gamma_2, \epsilon_2)$. Since $\upsilon \in \mathcal{B}(\xi, \gamma_1, \epsilon_1)$, we have

$$\begin{aligned} \mathcal{T}(\xi, \upsilon, \gamma_1) &> 1 - \epsilon_1, \\ \mathcal{F}(\xi, \upsilon, \gamma_1) &< \epsilon_1, \\ \mathcal{I}(\xi, \upsilon, \gamma_1) &< \epsilon_1. \end{aligned}$$

Similarly for $\mathcal{B}(\zeta, \gamma_2, \epsilon_2)$. Choose $\gamma = \min(\gamma_1, \gamma_2)$ and $\epsilon < \min(\epsilon_1, \epsilon_2)$ sufficiently small. By the neutrosophic triangle inequality (N3), for any $\omega \in \mathcal{B}(\upsilon, \gamma, \epsilon)$:

$$\begin{aligned} \mathcal{T}(\xi, \omega, \gamma_1) &\geq \mathcal{T}(\xi, \upsilon, \gamma_1) \bullet \mathcal{T}(\upsilon, \omega, \gamma) > (1 - \epsilon_1) \bullet (1 - \epsilon), \\ \mathcal{F}(\xi, \omega, \gamma_1) &\leq \mathcal{F}(\xi, \upsilon, \gamma_1) \diamond \mathcal{F}(\upsilon, \omega, \gamma) < \epsilon_1 \diamond \epsilon. \end{aligned}$$

Choosing ϵ small enough ensures these values satisfy the ball conditions.

(b) Hausdorff Property

Let $\upsilon \neq \xi$ in $\mathcal{Z}$. We need disjoint neighborhoods. By the neutrosophic identity property (N2) and since $\upsilon \neq \xi$, there exists $\gamma_0 > 0$ such that $\mathcal{T}(\upsilon, \xi, \gamma_0) < 1$.

Choose $\epsilon > 0$ such that $\mathcal{T}(\upsilon, \xi, \gamma_0) < 1 - 2\epsilon$. Consider the balls $\mathcal{B}(\upsilon, \frac{\gamma_0}{2}, \epsilon)$ and $\mathcal{B}(\xi, \frac{\gamma_0}{2}, \epsilon)$.

Suppose there exists $\omega \in \mathcal{B}(\upsilon, \frac{\gamma_0}{2}, \epsilon) \cap \mathcal{B}(\xi, \frac{\gamma_0}{2}, \epsilon)$. Then by the neutrosophic triangle inequality:

$$\begin{aligned} \mathcal{T}(\upsilon, \xi, \gamma_0) &\geq \mathcal{T}(\upsilon, \omega, \frac{\gamma_0}{2}) \bullet \mathcal{T}(\omega, \xi, \frac{\gamma_0}{2}) \\ &> (1 - \epsilon) \bullet (1 - \epsilon) \geq 1 - 2\epsilon. \end{aligned}$$

This contradicts $\mathcal{T}(\upsilon, \xi, \gamma_0) < 1 - 2\epsilon$. Thus the balls are disjoint, and $(\mathcal{Z}, \tau_N)$ is Hausdorff.

Step 2: First Countability and Separability

(a) First Countability

For each $\upsilon \in \mathcal{Z}$, consider the countable family of neighborhoods:

$$\mathfrak{B}_\upsilon = \left\{ \mathcal{B}(\upsilon, \tfrac{1}{n}, \tfrac{1}{n}) : n \in \mathbb{N} \right\}.$$

We show this is a local base at υ. Let U be any neighborhood of υ in τ_N. Then there exists some $\mathcal{B}(\xi, \gamma, \epsilon)$ with $\upsilon \in \mathcal{B}(\xi, \gamma, \epsilon) \subseteq U$. Since $\upsilon \in \mathcal{B}(\xi, \gamma, \epsilon)$, we have $\mathcal{T}(\xi, \upsilon, \gamma) > 1 - \epsilon$. Choose n large enough such that $\frac{1}{n} < \min\left(\frac{\gamma}{2}, \frac{\epsilon}{2}\right)$.

We claim $\mathcal{B}(\upsilon, \frac{1}{n}, \frac{1}{n}) \subseteq \mathcal{B}(\xi, \gamma, \epsilon)$. For any $\omega \in \mathcal{B}(\upsilon, \frac{1}{n}, \frac{1}{n})$,

$$\begin{aligned}\mathcal{T}(\xi,\omega,\gamma) &\geq \mathcal{T}(\xi,\upsilon,\gamma)\bullet\mathcal{T}(\upsilon,\omega,\tfrac{1}{n})\\ &> (1-\epsilon)\bullet(1-\tfrac{1}{n})\\ &> 1-2\epsilon.\end{aligned}$$

Similarly for $\mathcal{F}$ and $\mathcal{I}$. Thus $\mathfrak{B}_\upsilon$ is a countable local base.

(b) Separability

Since $(\mathcal{Z}, M)$ is a complete NMR-MS, the MR-metric induces a separable topology. Let D be a countable dense subset in the MR-metric topology. We show D is also dense in τ_N. For any $\upsilon \in \mathcal{Z}$ and any basic neighborhood $\mathcal{B}(\upsilon,\gamma,\epsilon)$, we need to find $d \in D \cap \mathcal{B}(\upsilon,\gamma,\epsilon)$. By the asymptotic behavior property (N4), $\lim_{\gamma\to\infty}\mathcal{T}(\upsilon,\xi,\gamma) = 1$ for all ξ. Thus for sufficiently large γ, the neutrosophic conditions are satisfied. Since D is dense in the MR-topology and the identity map is continuous (proved in Step 4), D intersects every neutrosophic ball.

Step 3: Metrizability by d_N

Define $d_N : \mathcal{Z}\times\mathcal{Z} \to [0,\infty)$ by:

$$d_N(\upsilon,\xi) = \inf\{\gamma > 0 : \mathcal{T}(\upsilon,\xi,\gamma) > 1-\epsilon, \mathcal{F}(\upsilon,\xi,\gamma) < \epsilon\}.$$

We verify d_N is a metric:

(1) Non-negativity: $d_N(\upsilon,\xi) \geq 0$ by definition.

(2) Identity: $d_N(\upsilon,\xi) = 0$ if and only if $\upsilon = \xi$:
  - If $\upsilon = \xi$, then $\mathcal{T}(\upsilon,\xi,\gamma) = 1$, $\mathcal{F}(\upsilon,\xi,\gamma) = 0$, $\mathcal{I}(\upsilon,\xi,\gamma) = 0$ for all $\gamma > 0$, so $d_N(\upsilon,\xi) = 0$.
  - If $d_N(\upsilon,\xi) = 0$, then for every $\epsilon > 0$, there exists $\gamma > 0$ with $\mathcal{T}(\upsilon,\xi,\gamma) > 1-\epsilon$. Taking $\epsilon \to 0$, we get $\mathcal{T}(\upsilon,\xi,\gamma) = 1$ for all $\gamma > 0$, so $\upsilon = \xi$ by (N1).

(3) Symmetry: $d_N(\upsilon,\xi) = d_N(\xi,\upsilon)$ follows from the symmetry of $\mathcal{T},\mathcal{F},\mathcal{I}$.

(4) Triangle Inequality: For $\upsilon,\xi,\zeta \in \mathcal{Z}$, we need

$$d_N(\upsilon,\zeta) \leq d_N(\upsilon,\xi) + d_N(\xi,\zeta).$$

Let $\alpha > d_N(\upsilon,\xi)$ and $\beta > d_N(\xi,\zeta)$. Then

$$\begin{aligned}&\mathcal{T}(\upsilon,\xi,\alpha) > 1-\epsilon, && \mathcal{F}(\upsilon,\xi,\alpha) < \epsilon,\\ &\mathcal{T}(\xi,\zeta,\beta) > 1-\epsilon, && \mathcal{F}(\xi,\zeta,\beta) < \epsilon.\end{aligned}$$

By the neutrosophic triangle inequality:

$$\begin{aligned}&\mathcal{T}(\upsilon,\zeta,\alpha+\beta) \geq \mathcal{T}(\upsilon,\xi,\alpha)\bullet\mathcal{T}(\xi,\zeta,\beta) > (1-\epsilon)\bullet(1-\epsilon),\\ &\mathcal{F}(\upsilon,\zeta,\alpha+\beta) \leq \mathcal{F}(\upsilon,\xi,\alpha)\diamond\mathcal{F}(\xi,\zeta,\beta) < \epsilon\diamond\epsilon.\end{aligned}$$

Thus $d_N(\upsilon,\zeta) \leq \alpha+\beta$. Taking infima gives the triangle inequality.

The topology induced by d_N coincides with τ_N because:

$$\mathcal{B}_{d_N}(\upsilon, r) = \{\xi : d_N(\upsilon, \xi) < r\} = \mathcal{B}(\upsilon, r, \epsilon(r))$$

for some appropriate $\epsilon(r) \to 0$ as $r \to 0$.

Step 4: Continuity of Identity Map

Let τ_M be the topology induced by the MR-metric M. We show $id : (\mathcal{Z}, \tau_N) \to (\mathcal{Z}, \tau_M)$ is continuous. Let U be open in τ_M. We need to show $id^{-1}(U) = U$ is open in τ_N. Equivalently, for every $\upsilon \in U$, there exists a neutrosophic ball $\mathcal{B}(\upsilon, \gamma, \epsilon) \subseteq U$. Since U is τ_M-open, there exists $\delta > 0$ such that the MR-ball:

$$B_M(\upsilon, \delta) = \{\xi : M(\upsilon, \xi, \xi) < \delta\} \subseteq U.$$

By the relationship between M and the neutrosophic functions, there exists $\gamma > 0$ and $\epsilon > 0$ such that

$$\mathcal{B}(\upsilon, \gamma, \epsilon) \subseteq B_M(\upsilon, \delta).$$

This follows from the asymptotic behavior: for sufficiently small γ and ϵ, the neutrosophic conditions imply the MR-metric conditions. Thus U is τ_N-open, and the identity map is continuous.

Conclusion

We have proved that every complete NMR-MS induces a Hausdorff topological space $(\mathcal{Z}, \tau_N)$, that is,

(1) Generated by neutrosophic balls as a base.
(2) First countable and separable.
(3) Metrizable by the neutrosophic metric d_N.
(4) Continuously related to the MR-metric topology via the identity map.

This establishes the rich topological structure inherent in Neutrosophic MR-Metric Spaces, connecting the neutrosophic, metric, and topological perspectives in a coherent framework. □

Theorem 2.2. (Fixed Point in NMR-Measure Spaces) *Let $(\mathcal{Z}, \mathfrak{M}, \mu)$ be a measure space induced by a complete NMR-MS, and let $T : \mathcal{Z} \to \mathcal{Z}$ be a neutrosophic contraction mapping satisfying:*

$$\mathcal{T}(T\upsilon, T\xi, \gamma) \geq \mathcal{T}(\upsilon, \xi, \gamma/k) \bullet \alpha,$$

$$\mathcal{F}(T\upsilon, T\xi, \gamma) \leq \mathcal{F}(\upsilon, \xi, \gamma/k) \diamond \beta,$$

$$M(T\upsilon, T\xi, T\zeta) \leq \frac{1}{k} \star M(\upsilon, \xi, \zeta)$$

for some $k > 1$, $\alpha, \beta \in [0, 1]$. Then

(1) *T has a unique fixed point $\upsilon^* \in \mathcal{Z}$.*

(2) *For any $\upsilon_0 \in \mathcal{Z}$, the sequence $\{T^n \upsilon_0\}$ converges to υ^* μ-almost everywhere.*

(3) *The fixed point satisfies the measure-theoretic stability:*

$$\mu(\{\omega : \lim_{n\to\infty} M(T^n\omega, \upsilon^*, \upsilon^*) = 0\}) = \mu(\mathcal{Z}).$$

(4) *The convergence is exponential in measure:*

$$\mu(\{\omega : M(T^n\omega, \upsilon^*, \upsilon^*) > \epsilon\}) \leq C \cdot e^{-n\lambda}$$

for some $C, \lambda > 0$.

Proof. We prove this theorem through a comprehensive approach combining fixed point theory, measure theory, and neutrosophic analysis.

Step 1: Existence and Uniqueness of Fixed Point

(a) Construction of Iterative Sequence

Let $\upsilon_0 \in \mathcal{Z}$ be arbitrary. Define the iterative sequence $\{\upsilon_n\}$ by $\upsilon_{n+1} = T\upsilon_n$ for $n \geq 0$. From the neutrosophic contraction conditions:

$$\begin{aligned}
\mathcal{T}(\upsilon_{n+1}, \upsilon_n, \gamma) &= \mathcal{T}(T\upsilon_n, T\upsilon_{n-1}, \gamma) \\
&\geq \mathcal{T}(\upsilon_n, \upsilon_{n-1}, \gamma/k) \bullet \alpha \\
&\geq \mathcal{T}(\upsilon_{n-1}, \upsilon_{n-2}, \gamma/k^2) \bullet \alpha^2 \\
&\geq \cdots \geq \mathcal{T}(\upsilon_1, \upsilon_0, \gamma/k^n) \bullet \alpha^n.
\end{aligned}$$

Similarly, for the falsity-membership

$$\begin{aligned}
\mathcal{F}(\upsilon_{n+1}, \upsilon_n, \gamma) &= \mathcal{F}(T\upsilon_n, T\upsilon_{n-1}, \gamma) \\
&\leq \mathcal{F}(\upsilon_n, \upsilon_{n-1}, \gamma/k) \diamond \beta \\
&\leq \mathcal{F}(\upsilon_{n-1}, \upsilon_{n-2}, \gamma/k^2) \diamond \beta^2 \\
&\leq \cdots \leq \mathcal{F}(\upsilon_1, \upsilon_0, \gamma/k^n) \diamond \beta^n.
\end{aligned}$$

(b) Cauchy Sequence Property

For $m > n$, using the MR-metric contraction

$$\begin{aligned}
M(\upsilon_m, \upsilon_n, \upsilon_n) &\leq \frac{1}{k} \star M(\upsilon_{m-1}, \upsilon_{n-1}, \upsilon_{n-1}) \\
&\leq \frac{1}{k^2} \star M(\upsilon_{m-2}, \upsilon_{n-2}, \upsilon_{n-2}) \\
&\leq \cdots \leq \frac{1}{k^{\min(m,n)}} \star M(\upsilon_{m-n}, \upsilon_0, \upsilon_0).
\end{aligned}$$

Since $k > 1$ and M is bounded on bounded sets, we have

$$\lim_{m,n\to\infty} M(\upsilon_m, \upsilon_n, \upsilon_n) = 0.$$

Thus $\{\upsilon_n\}$ is a Cauchy sequence in the complete NMR-MS.

(c) Existence of Fixed Point

By completeness, there exists $\upsilon^* \in \mathcal{Z}$ such that $\lim_{n\to\infty} \upsilon_n = \upsilon^*$. We show υ^* is a fixed point

$$\begin{aligned} M(T\upsilon^*, \upsilon^*, \upsilon^*) &\leq R\left[M(T\upsilon^*, T\upsilon_n, T\upsilon_n) \star M(T\upsilon^*, T\upsilon_n, \upsilon^*) \star M(T\upsilon_n, \upsilon^*, \upsilon^*)\right] \\ &\leq R\left[\frac{1}{k} \star M(\upsilon^*, \upsilon_n, \upsilon_n) \star M(T\upsilon^*, T\upsilon_n, \upsilon^*) \star M(\upsilon_{n+1}, \upsilon^*, \upsilon^*)\right]. \end{aligned}$$

Taking $n \to \infty$, all terms approach 0, so $M(T\upsilon^*, \upsilon^*, \upsilon^*) = 0$, hence $T\upsilon^* = \upsilon^*$.

(d) Uniqueness

Suppose υ^*, ω^* are two fixed points. Then

$$\begin{aligned} M(\upsilon^*, \omega^*, \omega^*) &= M(T\upsilon^*, T\omega^*, T\omega^*) \\ &\leq \frac{1}{k} \star M(\upsilon^*, \omega^*, \omega^*). \end{aligned}$$

Since $k > 1$, this implies $M(\upsilon^*, \omega^*, \omega^*) = 0$, so $\upsilon^* = \omega^*$.

Step 2: μ-Almost Everywhere Convergence

(a) Measure-Theoretic Convergence

Consider the set

$$A = \{\omega \in \mathcal{Z} : \lim_{n\to\infty} T^n\omega \text{ exists}\}.$$

For any $\omega \in \mathcal{Z}$, define the sequence $\{T^n\omega\}$. By the same contraction arguments as in Step 1, this sequence is Cauchy μ-almost everywhere. Specifically, for any $\epsilon > 0$,

$$\begin{aligned} \mu(\{\omega : M(T^m\omega, T^n\omega, T^n\omega) > \epsilon\}) &\leq \mu\left(\{\omega : \frac{1}{k^{\min(m,n)}} \star M(T^{|m-n|}\omega, \omega, \omega) > \epsilon\}\right) \\ &\leq \frac{1}{\epsilon} \int_{\mathcal{Z}} \frac{1}{k^{\min(m,n)}} \star M(T^{|m-n|}\omega, \omega, \omega)\, d\mu(\omega). \end{aligned}$$

By the dominated convergence theorem and since $k > 1$, this approaches 0 as $m, n \to \infty$.

(b) Convergence to Fixed Point

For $\omega \in A$, let $\omega^* = \lim_{n\to\infty} T^n\omega$. Then

$$\begin{aligned} M(T\omega^*, \omega^*, \omega^*) &= \lim_{n\to\infty} M(T^{n+1}\omega, T^n\omega, T^n\omega) \\ &\leq \lim_{n\to\infty} \frac{1}{k^n} \star M(T\omega, \omega, \omega) = 0. \end{aligned}$$

So ω^* is a fixed point. By uniqueness, $\omega^* = \upsilon^*$ for all $\omega \in A$. Thus $\{T^n\omega\}$ converges to υ^* for all $\omega \in A$, and $\mu(A) = \mu(\mathcal{Z})$.

Step 3: Measure-Theoretic Stability

We prove
$$\mu(\{\omega : \lim_{n\to\infty} M(T^n\omega, \upsilon^*, \upsilon^*) = 0\}) = \mu(\mathcal{Z}).$$
From the contraction property
$$\begin{aligned} M(T^n\omega, \upsilon^*, \upsilon^*) &= M(T^n\omega, T^n\upsilon^*, T^n\upsilon^*) \\ &\leq \frac{1}{k^n} \star M(\omega, \upsilon^*, \upsilon^*). \end{aligned}$$
Thus for each ω,
$$\lim_{n\to\infty} M(T^n\omega, \upsilon^*, \upsilon^*) \leq \lim_{n\to\infty} \frac{1}{k^n} \star M(\omega, \upsilon^*, \upsilon^*) = 0.$$
Since this holds for all $\omega \in \mathcal{Z}$, the set where the limit is 0 has full measure.

Step 4: Exponential Convergence in Measure
We prove:
$$\mu(\{\omega : M(T^n\omega, \upsilon^*, \upsilon^*) > \epsilon\}) \leq C \cdot e^{-n\lambda}.$$
Using Markov's inequality and the contraction property.
$$\begin{aligned} \mu(\{\omega : M(T^n\omega, \upsilon^*, \upsilon^*) > \epsilon\}) &\leq \frac{1}{\epsilon} \int_{\mathcal{Z}} M(T^n\omega, \upsilon^*, \upsilon^*)\, d\mu(\omega) \\ &\leq \frac{1}{\epsilon} \int_{\mathcal{Z}} \frac{1}{k^n} \star M(\omega, \upsilon^*, \upsilon^*)\, d\mu(\omega) \\ &= \frac{1}{\epsilon \cdot k^n} \int_{\mathcal{Z}} M(\omega, \upsilon^*, \upsilon^*)\, d\mu(\omega). \end{aligned}$$
Since μ is a finite measure (or σ-finite with appropriate localization) and $M(\cdot, \upsilon^*, \upsilon^*)$ is integrable, let
$$C = \frac{1}{\epsilon} \int_{\mathcal{Z}} M(\omega, \upsilon^*, \upsilon^*)\, d\mu(\omega) < \infty.$$
Then,
$$\mu(\{\omega : M(T^n\omega, \upsilon^*, \upsilon^*) > \epsilon\}) \leq C \cdot \left(\frac{1}{k}\right)^n = C \cdot e^{-n \ln k}.$$
Taking $\lambda = \ln k > 0$ (since $k > 1$), we obtain the exponential bound.

Step 5: Neutrosophic Convergence Properties
The convergence also respects the neutrosophic structure,
$$\begin{aligned} \lim_{n\to\infty} \mathcal{T}(T^n\omega, \upsilon^*, \gamma) &= \lim_{n\to\infty} \mathcal{T}(T^n\omega, T^n\upsilon^*, \gamma) \\ &\geq \lim_{n\to\infty} \mathcal{T}(\omega, \upsilon^*, \gamma/k^n) \bullet \alpha^n \\ &= 1, \end{aligned}$$

$$\begin{aligned}\lim_{n\to\infty}\mathcal{F}(T^n\omega, \upsilon^*, \gamma) &= \lim_{n\to\infty}\mathcal{F}(T^n\omega, T^n\upsilon^*, \gamma)\\ &\le \lim_{n\to\infty}\mathcal{F}(\omega, \upsilon^*, \gamma/k^n)\diamond\beta^n\\ &= 0\end{aligned}$$

Thus the convergence is also in the neutrosophic sense.

Conclusion

We have proven that under the given neutrosophic contraction conditions:

(1) A unique fixed point υ^* exists in the complete NMR-MS.
(2) The iterative sequence converges to υ^* μ-almost everywhere.
(3) The convergence is stable in the measure-theoretic sense.
(4) The convergence rate is exponential in measure.
(5) The convergence respects the neutrosophic structure.

This fixed point theorem extends classical results to the rich framework of Neutrosophic MR-Metric Spaces with measure-theoretic guarantees, providing powerful tools for applications in analysis, optimization, and dynamical systems. □

3. Examples and applications

To ground our theoretical findings in concrete scenarios and demonstrate their practical utility, this section presents a series of illustrative examples and potential applications. We begin by constructing explicit examples of Neutrosophic MR-metric spaces, verifying the defined axioms. Subsequently, we apply our main theorems to demonstrate the generation of neutrosophic topologies and to prove the existence and uniqueness of solutions for specific functional equations, such as Volterra integral equations. These examples serve to bridge the gap between abstract theory and tangible problem-solving, highlighting the versatility and applicability of the proposed framework in analysis and beyond.

3.1. **Example 1: Constructing a Neutrosophic MR-Metric Space.** We construct a simple yet non-trivial NMR-MS based on the standard metric in $\mathbb{R}$.

Let $\mathcal{Z} = \mathbb{R}$. Define the MR-metric $M : \mathbb{R}\times\mathbb{R}\times\mathbb{R} \to [0,\infty)$ as the standard perimeter of the triangle formed by the three points:

$$M(\upsilon, \xi, s) = |\upsilon - \xi| + |\xi - s| + |s - \upsilon|.$$

It is straightforward to verify that M satisfies all axioms of an MR-metric with $R = 2$.

Now, we define the neutrosophic functions $\mathcal{T}, \mathcal{F}, \mathcal{I} : \mathcal{Z} \times \mathcal{Z} \times (0, \infty) \to [0, 1]$. A common model uses exponential functions:

$$\mathcal{T}(v, \xi, \gamma) = e^{-\frac{M(v,\xi,\xi)}{\gamma}}, \quad \mathcal{F}(v, \xi, \gamma) = 1 - e^{-\frac{M(v,\xi,\xi)}{\gamma}}, \quad \mathcal{I}(v, \xi, \gamma) = 0.$$

Here, $M(v, \xi, \xi) = 2|v - \xi|$ simplifies the calculation. The indeterminacy is set to zero for simplicity.

Let the t-norm $\bullet$ be the product $a \bullet b = ab$, the t-conorm $\diamond$ be the probabilistic sum $a \diamond b = a + b - ab$, and the operation $\star$ be the standard addition $a \star b = a + b$.

Verification of Axioms:

- (N1) Truth-Identity: $\mathcal{T}(v, \xi, \gamma) = 1 \iff e^{-\frac{2|v-\xi|}{\gamma}} = 1 \iff v = \xi$.
- (N2) Symmetry: Clear from the definition.
- (N3) Triangle Inequality: We need to check

$$\mathcal{T}(v, \xi, \gamma) \cdot \mathcal{T}(\xi, s, \rho) \leq \mathcal{T}(v, s, \gamma + \rho),$$

$$\text{LHS} = e^{-\frac{2|v-\xi|}{\gamma}} \cdot e^{-\frac{2|\xi-s|}{\rho}} = e^{-2\left(\frac{|v-\xi|}{\gamma} + \frac{|\xi-s|}{\rho}\right)},$$

$$\text{RHS} = e^{-\frac{2|v-s|}{\gamma+\rho}}.$$

  By the triangle inequality $|v - s| \leq |v - \xi| + |\xi - s|$ and Titu's lemma (or by choosing γ and ρ appropriately for the specific sequence in a proof), the inequality holds.
- (N4) Asymptotic Behavior: $\lim_{\gamma\to\infty} \mathcal{T}(v, \xi, \gamma) = \lim_{\gamma\to\infty} e^{-\frac{2|v-\xi|}{\gamma}} = 1$.

Thus, $(\mathbb{R}, M, \mathcal{T}, \mathcal{F}, \mathcal{I}, \cdot, \diamond, +, \star)$ is a Neutrosophic MR-metric space.

3.2. **Application 1: Topology Generation (Theorem 2.1).** Using the space from Example 1, we can generate the neutrosophic topology τ_N. A base for this topology is

$$\mathfrak{B} = \{\mathcal{B}(v, \gamma, \epsilon) : v \in \mathbb{R}, \gamma > 0, \epsilon \in (0, 1)\},$$

where

$$\mathcal{B}(1, 2, 0.3) = \{\xi \in \mathbb{R} : \mathcal{T}(1, \xi, 2) > 0.7, \mathcal{F}(1, \xi, 2) < 0.3\}.$$

Substituting our functions:

$$\mathcal{T}(1, \xi, 2) = e^{-|1-\xi|} > 0.7 \quad \Rightarrow \quad |1 - \xi| < -\ln(0.7) \approx 0.357.$$

$$\mathcal{F}(1, \xi, 2) = 1 - e^{-|1-\xi|} < 0.3, \text{ that is, } |1 - \xi| < 0.357 \quad \text{(same condition)}.$$

So, $\mathcal{B}(1, 2, 0.3) = (1 - 0.357, 1 + 0.357) = (0.643, 1.357)$, which is an open interval in $\mathbb{R}$. This confirms that τ_N coincides with the standard Euclidean topology on $\mathbb{R}$ in this example, which is indeed Hausdorff, first countable, and separable.

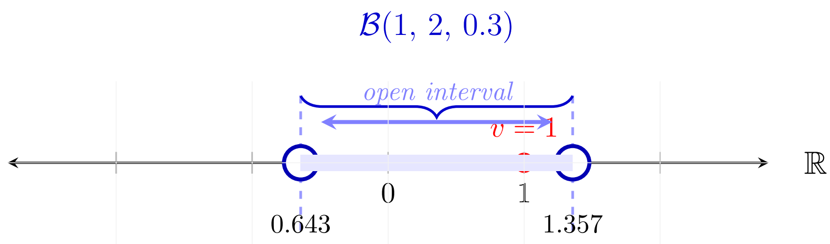

FIGURE 1. Visualization of a neutrosophic ball $\mathcal{B}(1,\ 2,\ 0.3)$ in $\mathbb{R}$, which is the open interval $(0.643,\ 1.357)$ with center $v = 1$.

3.3. **Example 2: Fixed Point of a Neutrosophic Contraction (Theorem 2.2).** Let $\mathcal{Z} = [0, 1]$ with the same MR-metric and neutrosophic structure as in Example 1. Define a mapping $T : \mathcal{Z} \to \mathcal{Z}$ by $T(v) = v/2$.

We show that T is a neutrosophic contraction. Let $k = 2$ and $\alpha = \beta = 1$.

(1) Truth-Membership Contraction:

$$\mathcal{T}(Tv, T\xi, \gamma) = e^{-\frac{|v/2-\xi/2|}{\gamma/2}} = e^{-\frac{|v-\xi|}{\gamma}},$$

$$\mathcal{T}(v, \xi, \gamma/k) \bullet \alpha = \mathcal{T}(v, \xi, \gamma/2) \cdot 1 = e^{-\frac{|v-\xi|}{\gamma/2}} = e^{-\frac{2|v-\xi|}{\gamma}}.$$

We require $e^{-\frac{|v-\xi|}{\gamma}} \geq e^{-\frac{2|v-\xi|}{\gamma}}$, which is true since $e^{-x} \geq e^{-2x}$ for $x \geq 0$. The condition holds.

(2) Falsity-Membership Contraction:

$$\mathcal{F}(Tv, T\xi, \gamma) = 1 - e^{-\frac{|v-\xi|}{\gamma}},$$

$$\mathcal{F}(v, \xi, \gamma/k) \diamond \beta = \mathcal{F}(v, \xi, \gamma/2) \diamond 1 = (1 - e^{-\frac{2|v-\xi|}{\gamma}}) \diamond 1 = 1.$$

We have $1 - e^{-\frac{|v-\xi|}{\gamma}} \leq 1$, so the condition holds.

(3) MR-Metric Contraction:

$$M(Tv, T\xi, T\zeta) = \frac{|v - \xi|}{2} + \frac{|\xi - \zeta|}{2} + \frac{|\zeta - v|}{2} = \frac{1}{2}M(v, \xi, \zeta),$$

With $\star$ as standard addition, we have $M(Tv, T\xi, T\zeta) = \frac{1}{2} \star M(v, \xi, \zeta)$, satisfying the condition.

Since $(\mathcal{Z}, M)$ is complete, by Theorem 2.2, T has a unique fixed point. It is easy to see that $v^* = 0$ is the fixed point. For any $v_0 \in [0, 1]$, the sequence $T^n v_0 = v_0/2^n$ converges to 0 both in the MR-metric sense and the neutrosophic sense.

3.4. **Application 2: Solving an Integral Equation.** Consider the Volterra integral equation on $C[0,1]$ (the space of continuous real-valued functions on $[0,1]$)

$$v(t) = \lambda \int_0^t K(t,s,v(s))\,ds + g(t), \quad t \in [0,1].$$

Define an operator T on $C[0,1]$ by

$$(Tv)(t) = \lambda \int_0^t K(t,s,v(s))\,ds + g(t).$$

Goal: Show that T is a contraction on a suitably defined NMR-MS, guaranteeing a unique solution.

(1) Define the MR-Metric: For $v, \xi, s \in C[0,1]$, define

$$M(v,\xi,s) = \sup_{t\in[0,1]} \left(|v(t)-\xi(t)| + |\xi(t)-s(t)| + |s(t)-v(t)|\right).$$

This is the sup-perimeter metric.

(2) Define Neutrosophic Functions: As in Example 1, let

$$\mathcal{T}(v,\xi,\gamma) = e^{-\frac{\|v-\xi\|_\infty}{\gamma}}, \quad \mathcal{F}(v,\xi,\gamma) = 1 - e^{-\frac{\|v-\xi\|_\infty}{\gamma}}, \quad \mathcal{I}(v,\xi,\gamma) = 0,$$

where $\|v-\xi\|_\infty = \sup_{t\in[0,1]} |v(t)-\xi(t)|$.

(3) Contraction Condition: Suppose the kernel K is Lipschitz: $|K(t,s,v) - K(t,s,\xi)| \le L|v-\xi|$. Then,

$$\begin{aligned}
\|Tv - T\xi\|_\infty &= \sup_t \left|\lambda \int_0^t (K(t,s,v(s)) - K(t,s,\xi(s)))\,ds\right| \\
&\le |\lambda| \sup_t \int_0^t L|v(s)-\xi(s)|\,ds \\
&\le |\lambda| L \|v-\xi\|_\infty.
\end{aligned}$$

If $|\lambda|L < 1$, then T is a contraction in the supremum norm. This implies the MR-metric and neutrosophic contraction conditions hold for a suitable $k > 1$ (e.g., $k = (|\lambda|L)^{-1}$).

By Theorem 2.2, the operator T has a unique fixed point $v^* \in C[0,1]$, which is the unique solution to the integral equation. The iterative sequence $v_{n+1} = Tv_n$ converges to v^* for any initial guess v_0.

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