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MULTIPLICITY RESULTS FOR A FOURTH-ORDER DIFFEERNTIAL EQUATION WITH INTEGRAL BOUNDARY CONDITIONS

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Abstract. This work examines a nonlinear two-point boundary value problem associated with a fourth-order differential equation. The problem is reformulated through inversion, enabling the explicit construction and subsequent analysis of the associated Greens function. Employing the Leggett-Williams fixed-point theorem alongside analytical estimates derived from Hölder inequality. A concrete example is constructed to illustrate the applicability of the theoretical results.

1. Introduction

This work investigates a fourth-order two-point boundary value problem. By applying the Leggett-Williams fixed-point theorem, supported by key estimates from Hölder's inequality, we demonstrate the existence of three positive

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solutions

$$\begin{cases} z^{(4)}(\xi) = \Lambda(\xi)\,\mathcal{F}(\xi, z(\xi)), \xi \in (0,1), \\ z(0) = z(1) = \displaystyle\int_0^1 \Phi(r) z(r)\,dr, \\ z''(0) = z''(1) = \displaystyle\int_0^1 \Psi(r) z''(r)\,dr, \end{cases} \tag{1.1}$$

where $\mathcal{F}$, Φ, Ψ and Λ satisfy

- (A_1) $\Lambda(\cdot) \in L^P[0,1]$ for some $p \in (0, +\infty)$ and there exists $\gamma' > 0$ such that $\Lambda(\xi) \geq \gamma'$ *a.e. on* $[0,1]$,
- (A_2) $\mathcal{F} \in C([0,1] \times [0,\infty), [0,\infty))$,
- (A_3) Φ, $\Psi \in L^1[0,1]$ are nonnegative and $\varkappa, \sigma \in [0,1)$, where

$$\varkappa = \int_0^1 \Phi(\xi)\,d\xi, \quad \sigma = \int_0^1 \Psi(\xi)\,d\xi.$$

The study of ordinary differential equations with multi-point boundary conditions finds substantial applications across the physical and engineering sciences. This framework is essential, for instance, in analyzing the elastic stability of structural systems. Similarly, higher-order boundary value problems are fundamental to modeling phenomena in fluid dynamics, hydromagnetics, and astrophysics, such as in the propagation of long waves or the design of induction motors. Given this broad utility, the investigation of higher-order boundary value problems has become a topic of considerable mathematical and practical interest, generating numerous significant theoretical results. In particular, third-order, fourth-order, higher-order and other kinds of nonlinear boundary value problems has been studied by many authors using different approaches, see [2, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 16] and the references therein.

Boundary value problems featuring integral boundary conditions represent a classic form of nonlocal problem. They emerge naturally in diverse applied contexts, including hydrodynamics, semiconductor theory, and the study of thermal conduction (see, e.g., [12]).

Recently, Lv et al. [18] studied the existence and nonexistence of concave and monotone positive solutions by using a fixed point theorem of cone expansion and compression of norm type for the following boundary value problems

$$\begin{cases} z^{(4)}(\xi) = \mathcal{F}\big(\xi, z(\xi), z'(\xi), z''(\xi)\big), \xi \in (0,1), \\ z''(0) = \displaystyle\int_0^1 \Phi(s)\, z''(s)\,ds, \\ z'(0) = z'(1) = 0,\ z^{(3)}(1) = 0, \end{cases} \tag{1.2}$$

where $\mathcal{F} \in C\Big([0,1] \times ([0,\infty))^2 \times (-\infty, 0], [0,\infty)\Big)$, $\Lambda(\cdot) \in L^1[0,1]$ is nonnegative and $\Phi \in C([0,1], [0,\infty))$.

Utilizing global bifurcation techniques, Shen and He [21] investigated the global structure of positive solutions for the singular boundary value problem

$$\begin{cases} z^{(4)}(\xi) - \gamma\, \mathcal{F}\big(\xi, z(\xi)\big) = 0, \quad \xi \in (0,1), \\ z(0) = \displaystyle\int_0^1 z(s)\, d\alpha(s), \\ z'(0) = z(1) = z'(1) = 0, \end{cases} \tag{1.3}$$

where $\gamma \in (0,\infty)$, $\mathcal{F}(\xi)$ has singularities at both $\xi = 0$ and $\xi = 1$, and $\int_0^1 z(s)\, d\alpha(s)$ is a Stieltjes integral, with $\alpha(\xi)$ being nonconstant on $[0,1]$.

Wang et al. [22] investigated the existence of positive solutions to the following fourth-order boundary value problem with integral boundary conditions by applying the Krasnoselskii fixed point theorem

$$\begin{cases} z^{(4)}(\xi) = \Lambda(\xi)\, \mathcal{F}\big(\xi, z(\xi), z'(\xi)\big), \xi \in (0,1), \\ z(0) = \displaystyle\int_0^1 \Phi_1(s)\, ds,\ z(1) = \int_0^1 \Psi_1(s)\, z(s)\, ds, \\ z''(0) = \displaystyle\int_0^1 \Phi_2(s)\, ds,\ z''(1) = \int_0^1 \Psi_2(s)\, z(s)\, ds, \end{cases} \tag{1.4}$$

where $\Lambda(\cdot) \in L^1[0,1]$ is nonnegative, Φ_i, $\Psi_i \in L^1[0,1]$, $\Phi_i(s) \geq 0$, $h_i(s) \geq 0$, $i = 1,2$ and $\mathcal{F} \in C([0,1] \times [0,\infty) \times (-\infty,0], [0,\infty))$.

Zhang and Ge [24] employed the Krasnoselskii fixed point theorem to investigate the existence of positive solutions to the following fourth-order boundary value problem with integral boundary conditions

$$\begin{cases} z^{(4)}(\xi) = \Lambda(\xi)\, \mathcal{F}\big(\xi, z(\xi), z'(\xi)\big), \xi \in (0,1), \\ z(0) = \displaystyle\int_0^1 \Phi(s)\, z(s)\, ds,\ z(1) = 0, \\ z''(0) = \displaystyle\int_0^1 \Psi(s)\, z''(s)\, ds,\ z''(1) = 0 \end{cases} \tag{1.5}$$

or

$$\begin{cases} z^{(4)}(\xi) = \Lambda(\xi)\, \mathcal{F}\big(\xi, z(\xi), z'(\xi)\big), \xi \in (0,1), \\ z(1) = \displaystyle\int_0^1 \Phi(s)\, z(s)\, ds,\ z(0) = 0, \\ z''(1) = \displaystyle\int_0^1 \Psi(s)\, z''(s)\, ds,\ z''(0) = 0, \end{cases} \tag{1.6}$$

where $\Lambda(\cdot) \in L^1[0,1]$ is nonnegative, $\Phi,\ \Psi \in L^1[0,1]$, $\Phi(s) \geq 0$, $\Psi(s) \geq 0$ and $\mathcal{F} \in C([0,1] \times [0,\infty) \times (-\infty,0], [0,\infty))$.

Zhang et al. [23] investigated the existence of positive solutions for the following fourth-order boundary value problem with integral boundary conditions:

$$\begin{cases} z^{(4)}(\xi) - \gamma\,\mathcal{F}\big(\xi, z(\xi)\big) = 0, \xi \in (0,1), \\ z(0) = z(1) = \displaystyle\int_0^1 \Phi(s)\,z(s)\,ds, \\ z''(0) = z''(1) = \displaystyle\int_0^1 \Psi(s)\,z(s)\,ds, \end{cases}$$

where $\gamma > 0$, $0 < \beta < \pi^2$, and $\Phi, \Psi \in L^1[0,1]$ are nonnegative functions satisfying $\Phi(s) \geq 0$ and $\Psi(s) \geq 0$.

Building on the aforementioned research, this paper investigates the existence of triple positive solutions for the boundary value problem (BVP) given in (1.1). The analysis employs a fixed point theorem established by Leggett and Williams, which concerns fixed points of cone-preserving operators acting on ordered Banach spaces [21].

The paper is structured as follows: Section 2 presents essential lemmas required for proving our main results concerning BVP (1.1). In Section 3, the principal theorems for BVP (1.1) are stated and proved, accompanied by an illustrative example that validates our findings.

2. Preliminaries

Let $C[0,1]$ denote the Banach space endowed with the supremum norm

$$\|z\| = \max_{0 \leq \xi \leq 1} |z(\xi)|.$$

The set $C^+[0,1]$, consisting of all nonnegative functions in $C[0,1]$, forms a cone within this space.

Definition 2.1. ([19]) Let $\mathfrak{E}$ be a real Banach space. A nonempty closed set $\mathfrak{P} \subset \mathfrak{E}$ is called a cone if it satisfies the following conditions:

(i) $a_1 w + a_2 v \in \mathfrak{P}$ for all $a_1 \geq 0,\ a_2 \geq 0$, and
(ii) $w, -w \in \mathfrak{P}$ implies $w = 0$.

Every cone $\mathfrak{P}$ defines a partial ordering on $\mathfrak{E}$ via the relation $w \preceq v$ if and only if $v - w \in \mathfrak{P}$.

Definition 2.2. ([3]) A functional β is called a nonnegative continuous concave functional on a cone $\mathfrak{P}$ of a real Banach space $\mathfrak{E}$ if $\beta : \mathfrak{P} \to [0, \infty)$ is continuous and satisfies

$$\beta\big(\xi w + (1-\xi)v\big) \geq \xi\beta(w) + (1-\xi)\beta(v) \quad \text{for all } w, v \in \mathfrak{P} \text{ and } 0 \leq \xi \leq 1.$$

The boundary value problem (1.1) will be reformulated as an equivalent integral equation. This is achieved through the auxiliary substitution

$$z''(\xi) = -w(\xi),$$

which decomposes the original problem (1.1) into the following pair of second-order systems:

$$\begin{cases} w''(\xi) + \Lambda(\xi)\,\mathcal{F}\big(\xi, z(\xi)\big) = 0, 0 < \xi < 1, \\ w(0) = w(1) = \displaystyle\int_0^1 \Psi(s)\, w(s)\, ds, \end{cases} \tag{2.1}$$

and

$$\begin{cases} -z''(\xi) = w(\xi), 0 < \xi < 1, \\ z(0) = z(1) = \displaystyle\int_0^1 \Phi(s)\, z(s)\, ds. \end{cases} \tag{2.2}$$

Our analysis relies on five preparatory lemmas, the first and third of which are established results in the literature.

Lemma 2.3. ([2]) *Assume $(A_1)-(A_3)$ hold. Then problem (2.2) has a unique solution given by*

$$w(\xi) = \int_0^1 \mathcal{H}(\xi, s)\, \Lambda(s)\, \mathcal{F}\big(s, z(s)\big)\, ds, \tag{2.3}$$

where

$$\mathcal{H}(\xi, s) = \mathcal{G}(\xi, s) + \frac{1}{1-\sigma} \int_0^1 \mathcal{G}(s, \tau)\, \Psi(\tau)\, d\tau \tag{2.4}$$

and

$$\mathcal{G}(\xi, s) = \begin{cases} \xi(1-s), & 0 \leq \xi \leq s \leq 1, \\ s(1-\xi), & 0 \leq s \leq \xi \leq 1. \end{cases} \tag{2.5}$$

The functions $\mathcal{G}(\xi, s)$ and $\mathcal{H}(\xi, s)$ satisfy the following properties.

Lemma 2.4. *Let $\delta \in \big(0, \frac{1}{2}\big)$, $J_\delta = [\delta, 1-\delta]$, and $\mu \in [0, 1)$. Then we have*

$$\mathcal{H}(\xi,s)>0,\quad \mathcal{G}(\xi,s)>0,\forall\xi,s\in(0,1), \tag{2.6}$$
$$\mathcal{H}(\xi,s)\geq 0,\quad \mathcal{G}(\xi,s)\geq 0,\forall\xi,s\in J, \tag{2.7}$$
$$e(\xi)e(s)\leq \mathcal{G}(\xi,s)\leq \mathcal{G}(s,s)=e(s)\leq \frac{1}{4},\forall\xi,s\in J, \tag{2.8}$$
$$e(\xi)\mathcal{H}_1(s,s)\leq \mathcal{H}_1(\xi,s)\leq \mathcal{H}_1(s,s)\leq e(s),\forall\xi,s\in J, \tag{2.9}$$
$$\rho e(s)\mathcal{H}(s,s)\leq \mathcal{H}(\xi,s)\leq \gamma\mathcal{G}(s,s)=\gamma e(s)\leq \frac{1}{4}\gamma,\forall\xi,s\in J, \tag{2.10}$$
$$\mathcal{G}(\xi,s)\geq \delta^2\mathcal{G}(s,s),\quad \mathcal{H}(\xi,s)\geq \delta^2\mathcal{H}(s,s)\geq \delta^2 e(s),\forall\xi\in J_\delta, s\in J, \tag{2.11}$$

where

$$e(\xi)=\xi(1-\xi),\max_{0\leq\xi\leq 1}e(s)=\frac{1}{4},\gamma=\frac{1}{1-\sigma},\rho=\frac{\int_0^1 e(s)\Psi(s)\,ds}{1-\sigma}. \tag{2.12}$$

Proof. Since the proofs of the other statements are similar, we focus on establishing (2.10). From (2.5), (2.8), and (2.9), we obtain

$$\begin{aligned}\mathcal{H}(\xi,s)&=\mathcal{G}(\xi,s)+\frac{1}{1-\sigma}\int_0^1\mathcal{G}(s,\tau)\Psi(\tau)\,d\tau\\ &\geq \frac{1}{1-\sigma}\int_0^1\mathcal{G}(s,\tau)\Psi(\tau)\,d\tau\\ &\geq \frac{1}{1-\sigma}\int_0^1 e(s)e(\tau)\Psi(\tau)\,d\tau\\ &=\frac{\int_0^1 e(\tau)\Psi(\tau)\,d\tau}{1-\sigma}\,s(1-s)\\ &=\rho\, e(s).\end{aligned}$$

Furthermore, from (2.9),

$$\begin{aligned}\mathcal{H}(\xi,s)&=\mathcal{G}(\xi,s)+\frac{1}{1-\sigma}\int_0^1\mathcal{G}(s,\tau)\Psi(\tau)\,d\tau\\ &\leq e(s)+\frac{1}{1-\sigma}\int_0^1 e(s)\Psi(\tau)\,d\tau\\ &=e(s)\left[1+\frac{1}{1-\sigma}\int_0^1 e(\tau)\Psi(\tau)\,d\tau\right]\\ &=\gamma\, e(s).\end{aligned}$$

This completes the proof. □

Lemma 2.5. ([2]) *If* (A_2) *and* (A_3) *hold, then problem* (2.1) *has a unique solution* z *expressible as*

$$z(\xi) = \int_0^1 \mathcal{H}_1(\xi, s)\, w(s)\, ds, \tag{2.13}$$

where

$$\mathcal{H}_1(\xi, s) = \mathcal{G}(\xi, s) + \frac{1}{1-\varkappa} \int_0^1 \mathcal{G}(s, \tau)\, \Phi(\tau)\, d\tau, \tag{2.14}$$

and $\mathcal{G}(\xi, s)$ *is defined by* (2.5)*.*

The function $\mathcal{H}_1(\xi, s)$ satisfies the following properties.

Lemma 2.6. *Let* $\delta \in \left(0, \frac{1}{2}\right)$, $J_\delta = [\delta, 1-\delta]$, *and* $\nu \in [0, 1)$. *Then we have*

$$(1)\ \ \mathcal{H}_1(\xi, s) > 0,\ \mathcal{H}_1(\xi, s) \geq 0,\ \text{and } \forall \xi, s \in (0,1),\ \forall \xi, s \in J, \tag{2.15}$$

$$(2)\ \ e(\xi)\mathcal{H}_1(s, s) \leq \mathcal{H}_1(\xi, s) \leq \mathcal{H}_1(s, s) \leq e(s), \forall \xi, s \in J, \tag{2.16}$$

$$(3)\ \ \rho_1 e(s) \leq \mathcal{H}_1(\xi, s) \leq \gamma_1 \mathcal{G}(s, s) = \gamma_1 e(s) \leq \frac{1}{4}\gamma_1, \forall \xi, s \in J, \tag{2.17}$$

$$(4)\ \ \mathcal{H}_1(\xi, s) \geq \delta^2 \mathcal{H}_1(s, s) \geq \delta^2 e(s), \forall \xi \in J_\delta, s \in J, \tag{2.18}$$

where

$$\gamma_1 = \frac{1}{1-\varkappa}, \quad \rho_1 = \frac{\int_0^1 e(\tau)\, \Phi(\tau)\, d\tau}{1-\varkappa}, \tag{2.19}$$

and $e(\xi)$ *is defined in* (2.12)*.*

Proof. The proof follows similarly to that of Lemma 2.4. □

Remark 2.7. If z is a solution of BVP (1.1), then from Lemma 2.3 and Lemma 2.5, we obtain

$$z(\xi) = \int_0^1 \int_0^1 \mathcal{H}(\xi, s)\mathcal{H}_1(s, \tau)\, \Lambda(\tau)\, \mathcal{F}\big(\tau, z(\tau)\big)\, d\tau\, ds. \tag{2.20}$$

Define a cone $\mathfrak{K}$ by

$$\mathfrak{K} = \big\{ z \in C^+[0,1] : z(\xi) \geq 0,\ \xi \in [0,1] \big\}. \tag{2.21}$$

It is easily verified that $\mathfrak{K}$ is a closed convex cone of $C^+[0,1]$.

Next, define the integral operator $\mathcal{T} : \mathfrak{K} \to C^+[0,1]$ by

$$(\mathcal{T} z)(\xi) = \int_0^1 \int_0^1 \mathcal{H}(\xi, s)\mathcal{H}_1(s, \tau)\, \sigma(\tau)\, \mathcal{F}\big(\tau, z(\tau)\big)\, d\tau\, ds. \tag{2.22}$$

From (2.22), we see that z is a solution of BVP (1.1) if and only if z is a fixed point of the operator $\mathcal{T}$.

Lemma 2.8. *Assume* (A_1) *and* (A_2) *hold. Then* $\mathcal{T}(\mathfrak{K}) \subset \mathfrak{K}$ *and* $\mathcal{T} : \mathfrak{K} \to \mathfrak{K}$ *is completely continuous.*

Proof. Since the proof of complete continuity is standard, it suffices to show $\mathcal{T}(\mathfrak{K}) \subset \mathfrak{K}$. For any $(\xi, s) \in [\delta, 1-\delta] \times [0,1]$, and using the facts that $\mathcal{F} \geq 0$, $\sigma \geq 0$, and $\int_\delta^{1-\delta} s^2(1-s)^2\, ds > 0$, we have

$$\begin{aligned}(\mathcal{T}z)(\xi) &= \int_0^1 \int_0^1 \mathcal{H}(\xi,s)\mathcal{H}_1(s,\tau)\,\Lambda(\tau)\,\mathcal{F}\big(\tau, w(\tau)\big)\,d\tau\,ds\\ &\geq \int_0^1 \int_\delta^{1-\delta} \mathcal{H}(\xi,s)\mathcal{H}_1(s,\tau)\,\Lambda(\tau)\,\mathcal{F}\big(\tau, z(\tau)\big)\,d\tau\,ds\\ &\geq \delta^4 \int_\delta^{1-\delta} \int_0^1 s^2(1-s)^2\,\Lambda(\tau)\,\mathcal{F}\big(\tau, z(\tau)\big)\,d\tau\,ds\\ &\geq 0.\end{aligned}$$

Thus, $\mathcal{T}(\mathfrak{K}) \subset \mathfrak{K}$. This completes the proof. □

Now, let $0 < \ell < n$ be given and let β be a nonnegative continuous concave functional on the cone $\mathfrak{K}$. Define the convex sets $\mathfrak{K}_\ell$ and $\mathfrak{K}(\beta, \ell, n)$ by

$$\mathfrak{K}_\ell = \big\{z \in \mathfrak{K} : \|z\| < \ell\big\}$$

and

$$\mathfrak{K}(\beta, \ell, n) = \big\{z \in \mathfrak{K} : \ell \leq \beta(z),\ \|z\| \leq n\big\}.$$

The key tool in our approach is the following LeggettWilliams fixed point theorem.

Theorem 2.9. ([17]) *Let* $\mathfrak{E}$ *be a Banach space and* $\mathfrak{K} \subset \mathfrak{E}$ *a cone in* $\mathfrak{E}$. *Let* $\mathcal{T} : \overline{\mathfrak{K}_c} \to \overline{\mathfrak{K}_c}$ *be completely continuous and* β *a nonnegative continuous concave functional on* $\mathfrak{K}$ *such that* $\beta(z) \leq \|z\|$ *for all* $z \in \mathfrak{K}_c$. *Suppose there exist constants* $0 < d < \ell < n \leq c$ *such that*

(i) $\big\{z \in \mathfrak{K}(\beta, \ell, n) : \beta(w) > \ell\big\} \neq \emptyset$ *and* $\beta(\mathcal{T}z) > \ell$ *for all* $w \in \mathfrak{K}(\beta, \ell, n)$;
(ii) $\|\mathcal{T}z\| < d$ *for* $\|z\| \leq d$;
(iii) $\beta(\mathcal{T}z) > \ell$ *for* $z \in \mathfrak{K}(\beta, \ell, c)$ *with* $\|\mathcal{T}z\| > n$.

Then $\mathcal{T}$ *possesses at least three fixed points* z_1, z_2, z_3 *in* $\overline{\mathfrak{K}_c}$ *satisfying*

$$\|z_1\| < d, \quad \ell < \beta(z_2), \quad \|z_3\| > d \text{ and } \beta(z_3) < \ell.$$

We will also make use of Hölder's inequality.

Lemma 2.10. ([20, Hölder's inequality])

(1) *Let* $\phi \in L^p[a,b]$ *with* $p > 1$, $\psi \in L^q[a,b]$ *with* $q > 1$, *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Then* $\phi\psi \in L^1[a,b]$ *and*

$$\|\phi\psi\|_1 \leq \|\phi\|_p \|\psi\|_q.$$

(2) *Let* $\phi \in L^1[a,b]$, $\psi \in L^\infty[a,b]$. *Then* $\phi\psi \in L^1[a,b]$ *and*

$$\|\phi\psi\|_1 \le \|\phi\|_1\|\psi\|_\infty.$$

3. Existence results

In this section, we apply Theorem 2.9 and Lemma 2.10 to establish the existence of triple positive solutions for BVP (1.1).

We consider three cases for the weight function $\sigma'(\cdot) \in L^p[0,1]$: $p > 1$, $p = 1$, and $p = \infty$. For convenience, we introduce the notation:

$$D = \gamma\gamma_1\|e\|_q\|\sigma'\|_p, \quad \delta^* = \frac{\delta^2}{\gamma_1}$$

and

$$\mathcal{F}^\infty = \limsup_{z\to\infty}\max_{\xi\in[0,1]}\frac{\mathcal{F}(\xi,z)}{z}.$$

Theorem 3.1. *Assume* (A_1) *and* (A_2) *hold. Furthermore, suppose there exist constants* $0 < d < \ell < \frac{\ell}{\delta^*} \le c$ *such that:*

(H_1) $\mathcal{F}^\infty < \dfrac{1}{D}$;

(H_2) $\mathcal{F}(\xi,z) > \dfrac{30\ell}{\delta^4(1-2\delta)\gamma'}$, *for* $(\xi,z) \in [\delta,1-\delta]\times[\ell,\frac{\ell}{\delta^*}]$, $\delta \in [0,\frac{1}{2}]$, $\gamma' > 0$;

(H_3) $\mathcal{F}(\xi,z) < \dfrac{d}{D}$, *for* $(\xi,z) \in [0,1]\times[0,d]$.

Then BVP (1.1) *has at least three positive solutions* z_1, z_2, z_3 *satisfying*

$$\|z_1\| < d, \quad \ell < \beta(z_2), \quad z_3 > d \text{ with } \beta(z_3) < \ell.$$

Proof. Define $\beta(z) = \min_{\xi\in[\delta,1-\delta]} z(\xi)$. Then β is a nonnegative continuous concave functional on the cone $\mathfrak{K}$ satisfying $\beta(z) \le \|z\|$ for all $z \in \mathfrak{K}$. Set $r = \frac{\ell}{\delta^*}$.

From (H_1), there exist $0 < \sigma' < \frac{1}{D}$ and $L > 0$ such that

$$\mathcal{F}(\xi,z) \le \sigma' z \quad \text{for } z \ge L.$$

Let $\eta' = \max\limits_{0\le z\le L,\ \xi\in[0,1]}\mathcal{F}(\xi,z)$. Then

$$\mathcal{F}(\xi,z) \le \sigma' z + \eta', \quad \xi \in [0,1],\ 0 \le z < +\infty. \tag{3.1}$$

Set

$$c > \max\left\{\frac{D\eta'}{1-D\sigma},\ \frac{\ell}{\delta^*}\right\}$$

and recall that

$$e(s) = s(1-s).$$

Then, for $z \in \overline{\mathfrak{K}}_c$, it follows from (2.10), (2.17), and (3.1) that

$$\begin{aligned}
(\mathcal{T}z)(\xi) &= \int_0^1 \int_0^1 \mathcal{H}(\xi,s)\mathcal{H}_1(s,\tau)\,\Lambda(\tau)\,\mathcal{F}\big(\tau,z(\tau)\big)\,d\tau\,ds \\
&\le \int_0^1 \int_0^1 \gamma\gamma_1 e(\tau)e(\tau)\,\Lambda(\tau)\big(\sigma' z(\tau)+\eta\big)\,d\tau\,ds \\
&\le \int_0^1 \int_0^1 \gamma\gamma_1 e(\tau)e(\tau)\,\Lambda(\tau)\big(\sigma'\|z\|+\eta\big)\,d\tau\,ds \\
&\le \frac{\gamma\gamma_1}{4}\big(\sigma' c+\eta'\big)\int_0^1 e(\tau)\Lambda(\tau)\,d\tau \\
&\le \gamma\gamma_1\big(\sigma' c+\eta'\big)\|\Lambda\|_p\|e\|_q \\
&< c,
\end{aligned}$$

which show that $\mathcal{T}z \in \mathfrak{K}_c$. Hence, if (H_1) holds, then $\mathcal{T}$ maps $\overline{\mathfrak{K}}_c$ into $\mathfrak{K}_c$.

Next, we verify that

$$\big\{z \in \mathfrak{K}(\beta,\ell,\ell/\delta^*) : \beta(z) > \ell\big\} \neq \emptyset$$

and $\beta(\mathcal{T}z) > \ell$ for all $w \in \mathfrak{K}(\beta,\ell,\ell/\delta^*)$.

Take $\varphi_0(\xi) = \frac{\delta^*+1}{2\delta^*}\ell$ for $\xi \in [0,1]$. Then

$$\varphi_0 \in \big\{z \in \mathfrak{K}(\beta,\ell,\ell/\delta^*) : \beta(z) > \ell\big\},$$

so the set is nonempty.

Now, from (H_2), (2.10), and (2.18), we obtain

$$\begin{aligned}
\beta(\mathcal{T}z) &= \min_{\xi\in[\delta,1-\delta]} (\mathcal{T}z)(\xi) \\
&= \min_{\xi\in[\delta,1-\delta]} \int_0^1 \int_0^1 \mathcal{H}(\xi,s)\mathcal{H}_1(s,\tau)\,\Lambda(\tau)\,\mathcal{F}\big(\tau,z(\tau)\big)\,d\tau\,ds \\
&\ge \delta \int_0^1 \int_0^1 e(s)\mathcal{H}_1(s,\tau)\,\Lambda(\tau)\,\mathcal{F}\big(\tau,z(\tau)\big)\,d\tau\,ds \\
&\ge \delta^4 \int_\delta^{1-\delta} \int_0^1 e(s)e(s)\,\Lambda(\tau)\,\mathcal{F}\big(\tau,z(\tau)\big)\,d\tau\,ds \\
&\ge \delta^4 \int_\delta^{1-\delta} s^2(1-s)^2\,ds \int_0^1 \Lambda(\tau)\,\mathcal{F}\big(\tau,z(\tau)\big)\,d\tau \\
&\ge \frac{\delta^4}{30}\sigma(1-2\delta)\cdot\frac{30\ell}{\delta^4(1-2\delta)\sigma} \\
&= \ell.
\end{aligned}$$

If $z \in \overline{\mathfrak{K}}_d$, then from (H_3),

$$\begin{aligned}
(\mathcal{T}z)(\xi) &= \int_0^1 \int_0^1 \mathcal{H}(\xi,s)\mathcal{H}_1(s,\tau)\,\Lambda(\tau)\,\mathcal{F}\big(\tau,z(\tau)\big)\,d\tau\,ds \\
&\leq \int_0^1 \int_0^1 \gamma\gamma_1 e(\tau)\,\Lambda(\tau)\,\mathcal{F}\big(\tau,z(\tau)\big)\,d\tau\,ds \\
&\leq \int_0^1 \int_0^1 \gamma\gamma_1 e(\tau)\,\Lambda(\tau)\,\frac{d}{D}\,d\tau\,ds \\
&\leq \left(\frac{d}{D}\right)\gamma\gamma_1\|\Lambda\|_p\|e\|_q \\
&= d.
\end{aligned}$$

Finally, we assert that if $z \in \mathfrak{K}(\beta,\ell,c)$ and $\|\mathcal{T}z\| > r$, then $\beta(\mathcal{T}z) > \ell$. Suppose $z \in \mathfrak{K}(\beta,\ell,c)$ and $\|\mathcal{T}z\| > r$. Then from (2.18),

$$\begin{aligned}
\beta(\mathcal{T}z) &= \min_{\xi\in[\delta,1-\delta]} \int_0^1 \int_0^1 \mathcal{H}(\xi,s)\mathcal{H}_1(s,\tau)\,\Lambda(\tau)\,\mathcal{F}\big(\tau,z(\tau)\big)\,d\tau\,ds \\
&\geq \delta^2 \int_0^1 \int_0^1 e(s)\mathcal{H}(s,\tau)\,\Lambda(\tau)\,\mathcal{F}\big(\tau,z(\tau)\big)\,d\tau\,ds \\
&\geq \frac{\delta^2}{\gamma_1} \int_0^1 \int_0^1 \gamma_1 e(s)\mathcal{H}(s,\tau)\,\Lambda(\tau)\,\mathcal{F}\big(\tau,z(\tau)\big)\,d\tau\,ds \\
&\geq \delta^*\,\|\mathcal{T}z\| \\
&> l.
\end{aligned}$$

To summarize, all conditions of Theorem 2.9 are satisfied. Consequently, BVP (1.1) possesses at least three positive solutions z_1, z_2, z_3 satisfying

$$\|z_1\| < d, \quad \ell < \beta(z_2), \quad z_3 > d \text{ with } \beta(z_3) < \ell.$$

□

The following corollaries address the case $p = +\infty$.

Corollary 3.2. *Assume (A_1), (A_2), (H1), (H2), and (H3) hold. Then BVP* (1.1) *has at least three positive solutions z_1, z_2, z_3 such that*

$$\|z_1\| < d, \quad \ell < \beta(z_2), \quad z_3 > d \textit{ with } \beta(z_3) < \ell.$$

Proof. Replace $\|\sigma\|_p\|e\|_q$ by $\|\sigma\|_\infty\|e\|_1$ and repeat the argument above. □

Finally, we consider the case $p = 1$. Let

(H_4) $\mathcal{F}^\infty < \dfrac{1}{D'}$,

(H_5) $\mathcal{F}(\xi,z) \leq \dfrac{d}{D'}$, for $(\xi,z) \in [0,1]\times[0,d]$, where

$$D' = \gamma\gamma_1 \|\sigma\|_1.$$

Corollary 3.3. *Assume* (A_1), (A_2), (H_4), *and* (H_5) *hold. Then BVP* (1.1) *has at least three positive solutions* z_1, z_2, z_3 *such that*

$$\|z_1\| < d, \quad \ell < \beta(z_2), \quad \|z_3\| > d \text{ with } \beta(z_3) < \ell.$$

Proof. Set

$$c' > \max\left\{ \frac{D'\eta}{1 - D'\sigma'}, \, \frac{\ell}{\delta^*} \right\},$$

and recall $e(s) = s(1-s)$, where $0 < \sigma' < \frac{1}{D'}$. For $z \in \overline{\mathfrak{K}}_{c'}$, it follows from (2.10), (2.17) and (3.1) that

$$\begin{aligned}
(Tz)(\xi) &= \int_0^1 \int_0^1 \mathcal{H}(\xi, s)\mathcal{H}_1(s, \tau)\, \sigma(\tau)\, \mathcal{F}\big(\tau, z(\tau)\big)\, d\tau\, ds \\
&\le \int_0^1 \int_0^1 \gamma\gamma_1 e(\tau) e(\tau)\, \sigma(\tau)\, \mathcal{F}\big(\tau, z(\tau)\big)\, d\tau\, ds \\
&\le \frac{1}{4} \int_0^1 \int_0^1 \gamma\gamma_1 e(\tau)\, \sigma(\tau) \big(\sigma' z(\tau) + \eta\big)\, d\tau\, ds \\
&\le \int_0^1 \int_0^1 \gamma\gamma_1 e(\tau)\, \sigma(\tau) \big(\sigma' \|z\| + \eta\big)\, d\tau\, ds \\
&\le \gamma\gamma_1 \big(\sigma' c' + \eta\big) \int_0^1 \sigma(\tau)\, d\tau \\
&= \gamma\gamma_1 \big(\sigma' c' + \eta\big) \|\sigma\|_1 = c',
\end{aligned}$$

which shows that $Tz \in \mathfrak{K}_{c'}$. Hence, if (H_4) holds, then T maps $\overline{\mathfrak{K}}_{c'}$ into $\mathfrak{K}_{c'}$.

If $z \in \overline{\mathfrak{K}}_d$, then from (H_5), (2.10) and (2.17),

$$\begin{aligned}
(Tz)(\xi) &= \int_0^1 \int_0^1 \mathcal{H}(\xi, s)\mathcal{H}_1(s, \tau)\, \sigma(\tau)\, \mathcal{F}\big(\tau, z(\tau)\big)\, d\tau\, ds \\
&\le \int_0^1 \int_0^1 \gamma\gamma_1 e(\tau) e(\tau)\, \sigma(\tau)\, \mathcal{F}\big(\tau, z(\tau)\big)\, d\tau\, ds \\
&\le \frac{1}{4} \int_0^1 \int_0^1 \gamma\gamma_1 e(\tau) e(\tau)\, \sigma(\tau)\, \mathcal{F}\big(\tau, z(\tau)\big)\, d\tau\, ds \\
&\le \int_0^1 \int_0^1 \frac{d}{D'} \gamma\gamma_1 e(\tau)\, \sigma(\tau)\, d\tau\, ds \\
&\le \frac{d}{D'} \gamma\gamma_1 \int_0^1 \sigma(\tau)\, d\tau \\
&= \frac{d}{D'} \gamma\gamma_1 \|\sigma\|_1 = d.
\end{aligned}$$

The remaining conditions are verified similarly to the proof of Theorem 3.1.

To summarize, all conditions of Theorem 2.9 are satisfied. Consequently, BVP (1.1) possesses at least three positive solutions z_1, z_2, z_3 satisfying

$$\|z_1\| < d, \quad \ell < \beta(z_2), \quad z_3 > d \text{ with } \beta(z_3) < \ell.$$

□

The following corollaries address the case $p = +\infty$.

Corollary 3.4. *Assume* (A_1), (A_2), $(H_1), (H_2)$, *and* (H_3) *hold. Then BVP* (1.1) *has at least three positive solutions* z_1, z_2, z_3 *such that*

$$\|z_1\| < d, \quad \ell < \beta(z_2), \quad z_3 > d \text{ with } \beta(z_3) < \ell.$$

Proof. Replace $\|\sigma\|_p \|e\|_q$ by $\|\sigma\|_\infty \|e\|_1$ and repeat the argument above. □

Finally, we consider the case $p = 1$. Let

(H_4) $\mathcal{F}^\infty < \dfrac{1}{D'}$,

(H_5) $\mathcal{F}(\xi, z) \leq \dfrac{d}{D'}$, for $(\xi, z) \in [0,1] \times [0, d]$,

where

$$D' = \gamma\gamma_1 \|\Lambda\|_1.$$

Corollary 3.5. *Assume* (A_1), (A_2), (H_4), *and* (H_5) *hold. Then BVP* (1.1) *has at least three positive solutions* z_1, z_2, z_3 *such that*

$$\|z_1\| < d, \quad \ell < \beta(z_2), \quad \|z_3\| > d \text{ with } \beta(z_3) < \ell.$$

Proof. Set

$$c' > \max\left\{\frac{D'\eta'}{1 - D'\sigma''}, \frac{\ell}{\delta^*}\right\},$$

and recall $e(s) = s(1-s)$, where $0 < \sigma'' < \frac{1}{D'}$.

For $z \in \overline{\mathfrak{K}}_{c'}$, it follows from (2.10), (2.17) and (3.1) that

$$\begin{aligned}
(\mathcal{T}z)(\xi) &= \int_0^1 \int_0^1 \mathcal{H}(\xi,s)\mathcal{H}_1(s,\tau)\,\Lambda(\tau)\,\mathcal{F}\big(\tau,z(\tau)\big)\,d\tau\,ds \\
&\le \int_0^1 \int_0^1 \gamma\gamma_1 e(\tau)e(\tau)\,\Lambda(\tau)\,\mathcal{F}\big(\tau,z(\tau)\big)\,d\tau\,ds \\
&\le \frac{1}{4}\int_0^1 \int_0^1 \gamma\gamma_1 e(\tau)\,\Lambda(\tau)\big(\sigma' z(\tau)+\eta\big)\,d\tau\,ds \\
&\le \int_0^1 \int_0^1 \gamma\gamma_1 e(\tau)\,\Lambda(\tau)\big(\sigma''\|z\|+\eta'\big)\,d\tau\,ds \\
&\le \gamma\gamma_1\big(\sigma'' c'+\eta\big)\int_0^1 \Lambda(\tau)\,d\tau \\
&= \gamma\gamma_1\big(\sigma'' c'+\eta'\big)\|\Lambda\|_1 \\
&= c',
\end{aligned}$$

which shows that $\mathcal{T}z \in \mathfrak{K}_{c'}$. Hence, if (H_4) holds, then $\mathcal{T}$ maps $\overline{\mathfrak{K}}_{c'}$ into $\mathfrak{K}_{c'}$.

If $z \in \overline{\mathfrak{K}}_d$, then from (H_5), (2.10) and (2.17),

$$\begin{aligned}
(\mathcal{T}z)(\xi) &= \int_0^1 \int_0^1 \mathcal{H}(\xi,s)\mathcal{H}_1(s,\tau)\,\Lambda(\tau)\,\mathcal{F}\big(\tau,z(\tau)\big)\,d\tau\,ds \\
&\le \int_0^1 \int_0^1 \gamma\gamma_1 e(\tau)e(\tau)\,\Lambda(\tau)\,\mathcal{F}\big(\tau,z(\tau)\big)\,d\tau\,ds \\
&\le \frac{1}{4}\int_0^1 \int_0^1 \gamma\gamma_1 e(\tau)e(\tau)\,\sigma(\tau)\,\mathcal{F}\big(\tau,z(\tau)\big)\,d\tau\,ds \\
&\le \int_0^1 \int_0^1 \frac{d}{D'}\gamma\gamma_1 e(\tau)\,\Lambda(\tau)\,d\tau\,ds \\
&\le \frac{d}{D'}\gamma\gamma_1 \int_0^1 \Lambda(\tau)\,d\tau \\
&= \frac{d}{D'}\gamma\gamma_1\|\Lambda\|_1 \\
&= d.
\end{aligned}$$

Finally, we assert that if $z \in \mathfrak{K}(\beta,\ell,c)$ and $\|\mathcal{T}z\| > r$, then $\beta(\mathcal{T}z) > \ell$.

Suppose $z \in \mathfrak{K}(\beta, \ell, c)$ and $\|\mathcal{T}z\| > \ell$. Then from (2.18),

$$\begin{aligned}
\beta(\mathcal{T}z) &= \min_{\xi\in[\delta,1-\delta]} \int_0^1 \int_0^1 \mathcal{H}(\xi, s)\mathcal{H}_1(s, \tau)\, \Lambda(\tau)\, \mathcal{F}\big(\tau, z(\tau)\big)\, d\tau\, ds \\
&\geq \delta^2 \int_0^1 \int_0^1 e(s)\mathcal{H}(s, \tau)\, \Lambda(\tau)\, \mathcal{F}\big(\tau, z(\tau)\big)\, d\tau\, ds \\
&\geq \frac{\delta^2}{\gamma_1} \int_0^1 \int_0^1 \gamma_1 e(s)\mathcal{H}(s, \tau)\, \Lambda(\tau)\, \mathcal{F}\big(\tau, z(\tau)\big)\, d\tau\, ds \\
&\geq \delta^* \|\mathcal{T}z\| > \ell.
\end{aligned}$$

To summarize, all conditions of Theorem 2.9 are satisfied. Consequently, BVP (1.1) possesses at least three positive solutions z_1, z_2, z_3 satisfying

$$\|z_1\| < d, \quad \ell < \beta(z_2), \quad z_3 > d \text{ with } \beta(z_3) < \ell.$$

□

Remark 3.6. We note that condition (H_3) in Theorem 3.1 can be replaced by either of the following:

(H_3') $f_0^d \leq \dfrac{1}{D}$, where

$$f_0^d = \max\left\{ \max_{\xi\in[0,1]} \frac{\mathcal{F}(\xi, z)}{d} : z \in [0, d] \right\};$$

(H_3'') $f^0 \leq \dfrac{1}{D}$, where $f^0 = \limsup_{z\to 0^+} \max_{\xi\in[0,1]} \frac{\mathcal{F}(\xi,z)}{z}$.

Corollary 3.7. *Suppose all assumptions of Theorem 3.1 are satisfied, with the sole exception that hypothesis (H_3) is substituted by either (H_3') or (H_3''). Then the conclusion of Theorem 3.1 continues to hold in each case.*

Proof. : The proof of Corollary 3.7 is similar to that of Theorem 3.1 □

4. Example

We provide an example to illustrate the applicability of our results.

Let $\delta = \frac{1}{4}$, $m = 3$ and $p = 1$. Since $p = 1$, we have $q = \infty$. Consider the boundary value problem

$$\begin{cases}
z^{(4)}(\xi) = \Lambda(\xi)\, \mathcal{F}\big(\xi, z(\xi)\big), \xi \in (0, 1), \\
z(0) = z(1) = \displaystyle\int_0^1 \Phi(s)\, z(s)\, ds,\ z'(0) = 0, \\
z''(0) = \displaystyle\int_0^1 \Psi(s)\, z''(s)\, ds,\ z^{(3)}(1) = 0,
\end{cases}$$

where

$$\Lambda(\xi) = \frac{\xi}{5} + \frac{1}{10}, \quad \Phi(\xi) = \Psi(\xi) = \xi$$

and

$$\mathcal{F}(\xi, z) = \begin{cases} d, & (\xi, z) \in [0,1] \times [0,d], \\ d + 10^3 \dfrac{\ell}{\delta^*} \dfrac{z-d}{\ell - d}, & (\xi, z) \in [0,1] \times [d, \ell], \\ d + 10^3 \dfrac{\ell}{\delta^*}, & (\xi, z) \in [0,1] \times \big[\ell, \ell/\delta^*\big], \\ d + 10^3 \dfrac{\ell}{\delta^*} + \big(z - \dfrac{\ell}{\delta^*}\big)\xi, & (\xi, z) \in [0,1] \times \big[\ell/\delta^*, \infty\big). \end{cases}$$

A direct calculation shows that $\sigma(\xi) \geq \lambda = \frac{1}{10}$ for almost every $\xi \in [0,1]$. Elementary integration yields

$$\varkappa = \int_0^1 \Psi(\xi)\, d\xi = \frac{1}{2}, \quad \sigma = \int_0^1 \Phi(\xi)\, d\xi = \frac{1}{2}$$

and consequently

$$\gamma = \frac{1}{1-\sigma} = 2, \quad \gamma_1 = \frac{1}{1-\varkappa} = 2, \quad \delta^* = \frac{\delta^2}{\gamma_1} = \frac{1}{32}.$$

From $\Lambda(\xi) = \frac{\xi}{5} + \frac{1}{10}$ and $e(\xi) = \xi(1-\xi)$, we obtain

$$\|\sigma\|_1 = \int_0^1 \left(\frac{\xi}{5} + \frac{1}{10}\right) d\xi = \frac{1}{10}, \quad \|e\|_\infty = \sup_{\xi \in [0,1]} |e(\xi)| = \frac{1}{4}.$$

Hence,

$$D = \gamma\gamma_1 \|e\|_\infty \|\sigma\|_1 = \frac{2}{5}, \quad \mathcal{F}^\infty = 1.$$

Now choose constants satisfying $0 < d < \ell < \ell/\delta^* \leq c$. Then

$$\mathcal{F}^\infty = 1 < \frac{5}{2} = \frac{1}{D}$$

and for all $(\xi, z) \in [\frac{1}{4}, \frac{3}{4}] \times [\ell, \ell/\delta^*]$,

$$\mathcal{F}(\xi, z) = 10^3 \frac{\ell}{\delta^*} = 32 \times 10^3 \ell > \frac{30\ell}{\delta^3(1-2\delta)\lambda} = 64 \times 600\, \ell = \ell.$$

Moreover, for all $(\xi, z) \in [0,1] \times [0,d]$,

$$\mathcal{F}(\xi, z) = d < \frac{d}{D'} = \frac{5d}{2}.$$

Thus conditions $(H_1), (H_2)$, and (H_3) are satisfied. All assumptions of Theorem 3.1 are therefore fulfilled. By Corollary 3.7, the boundary value problem admits at least three positive solutions z_1, z_2, z_3 with

$$\|z_1\| < d, \quad \ell < \beta(z_2), \quad z_3 > d \text{ with } \beta(z_3) < \ell.$$

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