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FIXED POINT RESULTS FOR MULTIVALUED F-CONTRACTIONS IN DISLOCATED B-METRIC SPACES WITH AN APPLICATION

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Abstract. This article aims to introduce a new multivalued F-contraction in order to establish some fixed point results involving $\alpha_{s,\varepsilon}$-admissibility in the context of dislocated b-metric spaces. An illustrate example is provided to support the main results. Finally, an application is furnished by proving the existence of a solution to integral inclusions.

1. Introduction

The study of fixed point theory has been a central topic in nonlinear analysis due to its wide range of applications in mathematics, computer science,

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optimization, and applied sciences. Classical fixed point results, such as the Banach contraction principle, have been extensively generalized to different settings in order to accommodate more complex structures and operators. In particular, the extension of fixed point results from single-valued to multivalued mappings has attracted significant attention, as such mappings naturally arise in control theory, game theory, differential inclusions, and optimization problems, see [3, 4, 7, 10, 12, 27, 29, 30].

The introduction of b-metric spaces by Czerwik [15] provided a fruitful generalization of metric spaces, where the triangular inequality is relaxed by an additional parameter. Later, various modifications and extensions of b-metric spaces have been proposed to handle specific analytical problems. Among them, the concept of dislocated b-metric spaces which introduced by Alghamdi et al. [1], where the authors combined the concept of dislocated metric spaces due to Amini-Herandi [6] (some results were obtained in this way, see [9, 13, 25] with b-metric spaces notion and it has proven to be a useful framework, since the self-distance of a point may not necessarily vanish, allowing a more flexible structure for convergence and continuity analysis, numerous results were established in this direction, see [19, 20, 22, 23, 26, 28, 32, 33, 35, 37, 40].

The concept of α-admissible mappings was first introduced in [43], where the authors established fixed point results for $\alpha - \psi$-contractive mappings. This idea was further developed by Sintunavarat [44], who introduced the admissibility of type S, and later extended to α_s-admissibility in [2]. Another significant development in the field was the notion of F-contractions, introduced by Wardowski [45]. This framework opened new possibilities for obtaining fixed point results beyond the scope of classical contractions. Subsequently, Altun et al. [5] examined multivalued F-contractions and established the existence of fixed points in the setting of metric spaces, thereby enriching the theory of multivalued mappings. These contributions illustrate the continuing effort to broaden fixed point theory by employing new contractive conditions and more general space structures.

Motivated by these developments, the present paper is devoted to introducing a new type of multivalued contractions, which we call $(\alpha_{s,\varepsilon}, F)$-contraction. We establish a fixed point existence theorem for this class of mappings in the context of dislocated b-metric spaces, a generalized framework that has recently attracted considerable interest due to its flexibility in dealing with nonstandard distance functions. As a consequence of our main result, we further derive a multivalued fixed point theorem in ordered dislocated b-metric spaces. To illustrate the applicability of our findings, we provide a concrete example and present an application to the existence of solutions for a Volterra integral inclusion, thereby demonstrating the usefulness of the proposed approach in addressing problems arising in integral equations.

2. Preliminaries

In this section, we give the definitions and lemmas for the main results.

Definition 2.1. ([15]) For a nonempty set X and a given real number with $s \geq 1$. A function $d_b : X \times X \to [0, \infty)$ is a b-metric on X if for all $x, y, z \in X$, the following conditions hold:
(b1) $d_b(x, y) = 0$ if and only if $x = y$,
(b2) $d_b(x, y) = d_b(y, x)$,
(b3) $d_b(x, z) \leq s[d_b(x, y) + d_b(y, z)]$.
A triple (X, d_b, s) is called a b-metric space.

Also, every metric space is a b-metric space but the converse is not necessarily true.

Definition 2.2. ([6]) Assume that X is a function and a nonempty set, $\sigma : X \times X \to \mathbb{R}_+$ is regarded as a dislocated metric (metric-like) on X if the subsequent criteria are met:
(1) $\sigma(x, y) = 0$ implies that $x = y$.
(2) $\sigma(x, y) = \sigma(y, x)$.
(3) $\sigma(x, z) \leq \sigma(x, y) + \sigma(y, z)$.
The pair (X, σ) is called a dislocated metric space.

Definition 2.3. ([1]) Let X be a nonempty set, and $s \geq 1$ be a constant. A function: $\sigma_b : X \times X \to \mathbb{R}_+$ is a b-metric-like if for all $x, y, z \in X$, the following conditions are satisfied:
(1) $\sigma_b(x, y) = 0$ implies $x = y$;
(2) $\sigma_b(x, y) = \sigma_b(y, x))$;
(3) $\sigma_b(x, z) \leq s[\sigma_b(x, y) + \sigma_b(y, z)$.
The triple (X, σ_b, s) is a dislocated $b-$metric space with parameter $s \geq 1$.

Let (X, d_b) be a $b-$metric space. Let $CB(X)$ denote the family of all bounded and closed subsets of X. For $\zeta \in X$ and $A, B \in CB(X)$, we define

$$D(x, A) = \inf_{a \in A} d_b(x, a) \quad and \quad D(A, B) = \sup_{a \in A} D(a, B).$$

Define a mapping $H_b : CB(X) \times CB(X) \to [0, \infty)$ by

$$H_b(A, B) = \max \left\{ \sup_{x \in A} D(x, B), \sup_{y \in B} D(y, A) \right\},$$

for every $A, B \in CB(X)$. Then, the mapping H_b forms a $b-$metric and it is called a Hausdorff $b-$metric [16] induced by a $b-$metric space (X, d_b).

Lemma 2.4. ([16]) *Let (X, d, s) be a $b-$metric space and $A, B \in CL(X)$ with $H_b(A, B) > 0$. Then, for each $b \in B$, there exists $a = a(b) \in A$ such that*

$$d_b(a, b) \leq sH_b(A, B).$$

Definition 2.5. ([8]) The function $H_\sigma : CB(X) \times CB(X) \to \mathbb{R}_+$, defined by

$$H_\sigma(A, B) = \max\{\sup_{x\in A} \sigma(x, B), sup_{y\in B}\sigma(A, y)\}$$

is called dislocated Hausdorff metric on $CB(X)$.

Lemma 2.6. ([8]) *Let (X, σ) be a dislocated metric space and A be a subset of X. If $\{x_n\}$ a sequence in (X, σ) such that $\{x_n\}$ is convergent to x, then we have*

$$|\sigma(x_n, A) - \sigma(x, A| = \sigma(x, x).$$

From the previous definitions, the dislocated Hausdorff $b-$metric is a function $H_{\sigma,b} : CB(X) \times CB(X) \to \mathbb{R}_+$ such that

$$H_{\sigma,b}(A, B) = \max\{\sup_{x\in A} \sigma_b(x, B), \sup_{y\in B} \sigma_b(A, y)\}$$

for all $A, B \in CB(X)$.

Definition 2.7. ([44]) Let X be a nonempty set and let $T : X \to X$, $\alpha : X \times X \to [0, \infty)$ be two mappings. For a given real number $s \geq 1$, T is weak α-admissible of type s if for $x \in X$ and $\alpha(x, Tx) \geq s$, then $\alpha(Tx, TTx) \geq s$.

Definition 2.8. ([2]) Let (X, d_b, s) be a b-metric space. For a given function $\alpha\colon X \times X \to [0, +\infty)$ a multivalued mapping $T\colon X \to CL(X)$ is said to be

(1) α_s-admissible, if for each $x_1 \in X$ and $x_2 \in Tx_1$ with $\alpha(x_1, x_2) \geq s^2$, we have $\alpha(x_2, x_3) \geq s^2$ for each $x_3 \in Tx_2$.

(2) α_s^*-admissible, if for $x_1, x_2 \in X$ with $\alpha(x_1, x_2) \geq s^2$, we have $\alpha^*(Tx_1, Tx_2) \geq s^2$, where

$$\alpha^*(Tx_1, Tx_2) = \inf\{\alpha(a, b) : \; a \in Tx_1, b \in Tx_2\}.$$

Definition 2.9. ([45]) Let $F : (0, +\infty) \to \mathbb{R}$ be a function satisfying:

(F_1) F is strictly increasing,

(F_2) for each sequence $\{\alpha_n\}$ in X, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$,

(F_3) there exists $k \in (0, 1)$ satisfying $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

In [31], the authors have replaced the property (F_3) by the following property:

(F_3') F is continuous on $[0,\infty)$.

In [14] the authors introduce the following condition:

(F_4) if $(\alpha_n) \subset \mathbb{N}$ is a sequence such that $\tau + F(s\alpha_n) \leq F(\alpha_{n-1})$, for all $n \in \mathbb{N}$, $s \geq 1$ and for some $\tau > 0$, then
$$\tau + F(s^n \alpha_n) \leq F(s^{n-1}\alpha_{n-1})$$
for all $n \in \mathbb{N}$.

Denote by $\mathcal{F}$ the set of all functions $F : (0,+\infty) \to \mathbb{R}$ satisfying the conditions $(F_1), (F_2), (F_3')$ and (F_4).

Example 2.10. Let $F_i : (0,+\infty) \to \mathbb{R},\ i \in \{1,2,3\}$, defined by

(1) $F_1(t) = \ln t$,
(2) $F_2(t) = t + \ln t$,
(3) $F_3(t) = -\frac{1}{\sqrt{t}}$.

Then $F_i \in \mathcal{F}$, for each $i \in \{1,2,3\}$.

3. Main results

Definition 3.1. Let (X,σ_b) be a dislocated $b-$metric space, $s \geq 1$ and $\varepsilon \geq 2$. A mapping $T \to CL$ is called a multivalued $(F,\alpha_{s,\varepsilon})$ contraction if there are a function $F \in \mathcal{F}$ and $\tau > 0$ with $\alpha(x,y) \geq s$ such that
$$\tau + F\left(s^{\varepsilon}\alpha(x,y)H_{\sigma,s}(Tx,Ty)\right) \leq F(\sigma_b(M_b(x,y)), \tag{3.1}$$
where
$$M_b(x,y) = \max\left\{\sigma_b(x,y), D_\sigma(x,Tx), D_\sigma(y,Ty), \frac{D_\sigma(x,Ty)+D_\sigma(y,Tx)}{2s}\right\}.$$

Theorem 3.2. *Let (X,σ_b) be a complete dislocated $b-$metric space with $s \geq 1$ and $T : X \to CB(X)$ be a multivalued $(F_{,s,\varepsilon})$ contraction. Suppose that:*

(1) *T is continuous;*
(2) *T is an α_s-admissible mapping;*
(3) *there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \geq s^2$.*

Then T has a fixed point.

Proof. From hypothesis, there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \geq s$. If $x_0 = x_1$ or $x_1 \in Tx_1$, then x_1 is a fixed point of T and so the proof is completed. Assume that $x_0 \neq x_1$ and $x_1 \notin Tx_1$. From Lemma 2.4, there is $x_2 \in Tx_1$ such that
$$\sigma_b(x_1,x_2) \leq sH_{\sigma,b}(Tx_0,Tx_1),$$

which implies that

$$s^{\varepsilon-1}\sigma_b(x_1,x_2) \le \alpha(x_0,x_1)s^{\varepsilon}H_{\sigma,b}(Tx_0,Tx_1).$$

Using (3.1),

$$\begin{aligned}\tau + F(s^{\varepsilon-1}\sigma_b(x_1,x_2)) &\le F(\alpha(x_0,x_1)s^{\varepsilon}H(Tx_0,Tx_1))\\ &\le F(M(x_0,x_1)) - \tau,\end{aligned}$$

where

$$M_b(x_0,x_1) = \max\left\{\sigma_b(x_0,x_1), \sigma_b(x_0,x_1), \sigma_b(x_1,x_2), \frac{\sigma_b(x_0,x_1)+\sigma_b(x_1,x_2)}{2s}\right\}.$$

If $\sigma_b(x_0,x_1) \le \sigma_b(x_1,x_2)$, we get

$$F(s^{\varepsilon}\sigma_b(x_1,x_2)) \le F(\sigma_b(x_1,x_2) - \tau,$$

which is a contradiction, then

$$F(s^{\varepsilon-1}\sigma_b(x_1,x_2)) \le F(\sigma_b(x_0,x_1)) - \tau.$$

By the same process we obtain

$$F(s^{\varepsilon-1}\sigma_b(x_2,x_3)) \le F(\sigma_b(x_1,x_2)) - \tau.$$

Continuing in this manner, for all $n \in \mathbb{N}$ we get

$$F(s^{\varepsilon-1}\sigma_b(x_n,x_{n+1})) \le F(\sigma_b(x_{n-1},x_n)) - \tau.$$

From the property (F_4), we get

$$F(s^{n(\varepsilon-1)}\sigma_b(x_n,x_{n+1})) \le F(s^{(n-1)(\varepsilon-1)}\sigma_b(x_{n-1},x_n)) - \tau,$$

which gives that

$$F(s^{n(\varepsilon-1)}\sigma_b(x_n,x_{n+1})) \le F(\sigma_b(x_0,x_1)) - n\tau.$$

Passing to the limit in the last inequality, we get

$$\lim_{n\to\infty} F(s^{n(\varepsilon-1)}\sigma_b(x_n,x_{n+1})) = -\infty.$$

(F_2) gives

$$\lim_{n\to\infty} s^{n(\varepsilon-1)}\sigma_b(x_n,x_{n+1}) = 0.$$

For some $k \in (0,1)$, we have

$$\begin{aligned}&\left(s^{\varepsilon-1}\sigma_b(x_n,x_{n+1})\right)^k [F(s^{\varepsilon-1}\sigma_b(x_n,x_{n+1})) - F(\sigma_b(x_0,x_1))]\\ &\le -\left(s^{\varepsilon-1}\sigma_b(x_n,x_{n+1})\right)^k \tau.\end{aligned}$$

Passing to the limit and by using (F_3), we get

$$\lim_{n\to\infty}\left(s^{n(\varepsilon-1)}\sigma_b(x_n,x_{n+1})\right)^k n = 0.$$

By the definition of the limit, there exits $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have

$$\left(s^{n(\varepsilon-1)}\sigma_b(x_n, x_{n+1}\right)^k n < 1,$$

which implies that

$$s^{n(\varepsilon-1)}\sigma_b(x_n, x_{n+1} < \frac{1}{n^{\frac{1}{k}}}.$$

For $n, m \in \mathbb{N}$, we have

$$\sigma_b(x_n, x_m) \leq \sum_{i=n}^{m-1} s^{i-n+1}\sigma_b(x_i, x_{i+1}) \leq \sum_{i=n}^{m-1} s^i \sigma_b(x_i, x_{i+1}),$$

which implies that

$$\sigma_b(x_n, x_m) \leq s^{i(2-\varepsilon)}(s^{i(\varepsilon-1)}\sigma_b(x_i, x_{i+1}) \leq \sum_{i=n}^{m-1} s^{i(2-\varepsilon)}\frac{1}{n^{\frac{1}{k}}}.$$

When m tends to infinity the right side is the rest of a convergence series, so it tends to 0. Then $\sigma_b(x_n, x_m) \to 0$ as $n, m \to \infty$, which implies that $\{x_n\}$ is a Cauchy sequence. Since the space (X, σ_b) is complete, the sequence $\{x_n\}$ converges to some $x \in X$, that is, $\lim\limits_{n\to\infty} \sigma_b(x_n, x) = \sigma_b(x, x) = 0$.

We show that $\sigma_b(x, Tx) = 0$. Suppose $\sigma_b(x \in Tx) > 0$, then there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have $\sigma_b(x_n \in Tx) = 0$. Recall that $\alpha(x_n, x) \geq s^2$, so

$$\begin{aligned}\tau + F(\sigma_b(x_{n+1}, Tx)) &\leq \tau + F(\alpha(x_n, x)s^{\varepsilon} H_\sigma(Tx_n, Tx)) \\ &\leq F(M(x_n, x),\end{aligned} \tag{3.2}$$

where

$$\begin{aligned}M_b(x_n, x) &= \max\left\{\sigma_b(x_n, x), \sigma_b(x_n, Tx_n), \sigma_b(x, Tx), \frac{\sigma_b(x_n, Tx) + \sigma_b(x, Tx_n)}{2s}\right\} \\ &\leq \max\Big\{\sigma_b(x_n, x), \sigma_b(x_n, x_{n+1}), \sigma_b(x, Tx), \\ &\qquad \frac{s(\sigma_b(x_n, x) + \sigma_b(x, Tx)) + \sigma_b(x, x_{n+1})}{2s}\Big\}.\end{aligned}$$

Passing to the limit, we get

$$\lim_{n\to\infty} M_b(x_n, x) \leq \sigma_b(x, Tx).$$

On other hand, from Lemma 2.6, we get

$$\lim_{n\to\infty} \sigma_b(x, Tx_n) = \sigma_b(x_n, Tx) = \sigma_b(x, Tx).$$

Hence, return to inequality (3.2) and passing to the limit, we get

$$\begin{aligned}\lim_{n\to\infty} F(\sigma_b(x_{n+1}, Tx)) &= \tau + F(\sigma_b(x, Tx)) \\ &\leq \lim_{n\to\infty} M_b(x_n, x) = \sigma_b(x, Tx),\end{aligned}$$

which is a contradiction, then $\sigma_b(x, Tx) = 0$ and $x \in Tx$. □

Remark 3.3. Theorem 3.2 is a generalization of the Theorem 2.2 in [42] in the context of Wordowski contractive conditions, it improves also some results as in [2, 14, 22, 44].

Since each α_s^*-admissible mapping is also α_s-admissible, we obtain following result.

Corollary 3.4. *Let (X, d, s) be a complete dislocated $b-$metric space, $\alpha\colon X \times X \to [0, +\infty)$ be a function and $T\colon X \to CB(X)$ be a multivalued mapping. Assume that the following conditions hold:*

(1) *T is an α_s^*-admissible;*
(2) *there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha\,(x_0, x_1) \geq s^2$;*
(3) *for every sequence $\{x_n\}$ in X such that $\{x_n\}$ converges to x in X and $\alpha^*(x_n, x_{n+1}) \geq s^2$ for all $n \in N$. Then $\alpha^*(x_n, x) \geq s^2$ for all $n \in \mathbb{N}$;*
(4) *there exist $F \in \mathcal{F}$ and $\tau > 0$ such that*

$$\tau + F(\alpha(x, y)s^{\varepsilon} H_{\sigma,b}(Tx, Ty)) \leq F(\psi(M_b(x, y)))$$

where,

$$M_b(x; y) = \max\left\{\sigma_b(x, y), \sigma_b(x, Tx), \sigma_b(y, Ty), \tfrac{D\sigma(x,Ty)+D_\sigma(y,Tx)}{2s}\right\}.$$

Then T has a fixed point.

Theorem 3.5. *Let $(X, \sigma_b, \preceq)$ be a complete partially ordered dislocated $b-$metric space and $T : X \to CB(X)$ be a $(F, \alpha_{s,\varepsilon})$- contraction with respect to $\preceq$. Suppose that the following conditions hold:*

(1) *T is continuous;*
(2) *T is an α_s-admissible mapping;*
(3) *there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $x_0 \preceq x_1$.*

Then T has a fixed point.

Proof. The result follows from Theorem 3.2 by considering the function α defined as follows:

$$\alpha(x, y) = \begin{cases} s^2, & x \preceq y, \\ 0, & \text{otherwise.} \end{cases}$$

□

Example 3.6. Let $X = \{1, 2^2, 2^4\}$, $\varepsilon = 3$ and $d(x, y) = |x + y|^2$. Define $T\colon X \to CB(X)$ and $\alpha : X \times X \to [0;\infty)$ by

$$Tx = \begin{cases} \{1, 2^2\}, & x \in \{1, 2^4\}, \\ \{1, 2^4\}, & x = 4 \end{cases}$$

and

$$\alpha(x, y) = \begin{cases} 4, & (x, y) \in \{(1, 2^4), (2^4, 1), (2^2, 2^4), (2^4, 2^2)\}, \\ 0, & \text{otherwise.} \end{cases}$$

We claim that T is an $(\alpha_{s,\varepsilon}, F)$-contraction, by taking $F(x) = \ln x$ and for any $\tau \in (0, 161]$ we get:

Note that $H_{\sigma,b}(Tx, Ty) > 0$ and $\alpha(x, y) \geq 4$ if and only if

$$(x, y) \in \{(1, 2^4), (2^4, 1), (2^2, 2^4), (2^2, 2^4)\}.$$

(1) For $x = 1$ and $y = 2^4$ we have $H_{\sigma,b}(T1, T16) = (1 + 1)^2 = 2^2$ and $\sigma_b(1, 16) = (1 + 16)^2 = 17^2$. So we get

$$2^3\alpha(1, 16)H_{\sigma,b}(T1, T16) = 2^7 < \sigma_b(1, 16) = 17^2.$$

Then for any $\tau \in (0, 161]$, we get

$$\tau + \ln(2^3\alpha(1, 16)H_{\sigma,b}(T1, T16)) \leq \ln(\sigma_b(1, 16)) \leq \ln(M_b(x, y).$$

(2) For $x = 4$ and $y = 16$, we have $H_{\sigma,b}(T4, T16) = (1 + 1)^2 = 4$ and $\sigma_b(2^2, 2^4) = (4 + 16)^2 = 400$. So we get

$$2^3\alpha(2^2, 2^4)H_{\sigma,b}(T4, T16) = 2^7 < \sigma_b(4, 16) = 400.$$

Then for any $\tau \in (0, 161]$,

$$\tau + \ln(2^3\alpha(4, 16)H_{\sigma,b}(T4, T16)) \leq \ln(\sigma_b(4, 16)) \leq \ln(M_b(x, y).$$

Hence, T is $(\alpha_{s,\varepsilon}, F)$- contraction. It is clear that T is α_s-admissible and there exist $x_0 = 1$ and $x_1 = 16 \in T1$ such that $\alpha(x_0, x_1) = 4$. Also, it is obvious that T is continuous. Consequently, all conditions of Theorem 3.2 are satisfied. Then T has a fixed point which is 1.

4. Applications to fractional integral inclusions

Fractional integral inclusions provide a flexible framework for modeling systems affected by memory or nonlocal interactions, where classical fractional differential equations prove inadequate. This section explores key applications of this framework, including feedback control systems and constitutive modeling in materials science. By integrating multivalued analysis with fixed point theory, we demonstrate how existence results translate into physically meaningful predictions, see [11, 21, 24, 34, 36, 37, 38, 39, 41].

Consider the following problem,

$$x(t) \in \int_0^t K(t,s,x(s))ds, \quad s,t \in I = [0,1], \tag{4.1}$$

where $K : I \times I \times \mathbb{R} \to \mathcal{K}(\mathbb{R})$ is given continuous multivalued function. Then, clearly, $X = C(I)$ is a complete dislocated $b-$metric space with the dislocated $b-$metric

$$\sigma_b(x,y) = \sup_{0 \le t \le 1} |x(t) + y(t)|^2.$$

We introduce the following hypotheses:

(A_1) For each $x \in X$, the multivalued function $K_x : (t,s) \to K(t,s,x(s))$ is lower semi-continuous.

(A_2) There exists $\varphi : I \times I \to \mathbb{R}_+$ such that for all $x, y \in \mathfrak{X}$ and for each $s,t \in I$;

$$|k_x(t,s) + k_y(t,s)| \le \varphi(t,s)|x+y|,$$

for all $x, y \in X$, $k_x \in K_x$ and $k_y \in K_y$.

(A_3) There exists $\tau > 0$ such that $\sup\limits_{0 \le t \le 1} \displaystyle\int_0^t \varphi(t,s)ds \le \left(\frac{e^{-}\tau}{16}\right)^{\frac{1}{2}}$.

Theorem 4.1. *Under* $(A_1) - (A_3)$*, the problem* (4.1) *has a solution in* $X = C(I)$*.*

Proof. Define an operator

$$\mathcal{T}x(t) = \{z \in X, z \in \int_0^t K(t,s,x(s))ds \text{ for all } s,t \in I\}.$$

It is clear that the problem (4.1) has a solution if and only if T has a fixed point.

The set-valued operator $K_x(t,s) : I \times I \to \mathcal{K}(R)$ is lower semi-continuous, then from Michael's selection theorem, for $x \in X$ there exists a continuous function $k_x : I \times I \to \mathbb{R}$ such that $k_x(t,s) \in K_x(t,s)$ for all $t,s \in I$. It follows that

$$\int_0^t k_x(t,s)ds \in Tx,$$

so Tx is nonempty for all $x \in X$. Since k_x is continuous on I^2, its range is bounded and closed, hence Tx is bounded, that is, $T : X \to \mathcal{K}(\mathbb{R})$.

Let $x, y \in X$ and let $v \in Tx$. Then $v(t) \in \int_0^t K(t,s,x(s))ds, t \in I$. It follows that there exists $k_x \in K_x(t,s)$ such that

$$v(t) = \int_0^t k_x(t,s)ds, (t,s) \in I \times I.$$

From (A_2), there is $k_y(t,s) \in K_y(t,s)$ such that

$$|k_x(t,s) + k_y(t,s)| \leq \varphi(t,s)|x+y|$$

for all $(t,s) \in I \times I$.

Let P be a multi valued operator defined by

$$P(t,s) = K_y(t,s) \cap \{z \in \mathbb{R}, |k_x(t,s)+z|^2 \leq |x(t)+y(t)|^2$$

for all $(t,s) \in I \times I\}$. From (A_1), P is lower semi-continuous, then there exists a continuous function $k_y(t,s) \in P(t,s)$. Hence, we have

$$u(t) = \int_0^t k_y(t,s)ds \in \int_0^t K(t,s,y(s))ds; t \in I$$

and

$$\begin{aligned} |v(t)+u(t)| &\leq \int_0^t |k_x(t,s)+k_y(t,s))|ds \\ &\leq \int_0^t \varphi(t,s)|x(s)+y(s)|ds, \end{aligned}$$

which implies that

$$\begin{aligned} \sigma_b(v,Ty) &\leq |v(t)+u(t)|^2 \\ &\leq \sigma_b(x,y) \sup_{0\leq t\leq 1} \int_0^t \varphi(t,s)ds \\ &\leq \frac{e^{-\tau}}{16}\sigma_b(x,y). \end{aligned}$$

Thus,

$$16\sigma_b(v,Ty) \leq e^{-\tau}\sigma_b(x,y).$$

Interchanging the role of x and y, we get

$$16\sigma_b(Tx,u) \leq e^{-\tau}\sigma_b(x,y).$$

Consequently, we get

$$16H_{\sigma,b}(Tx,Ty) \leq e^{-\tau}\sigma_b(x,y),$$

then

$$\tau + \ln(2^4 H_{\sigma,b}(Tx,Ty)) \leq \ln(\sigma_b(x,y)) \leq \ln(M_b(x,y)).$$

All the hypotheses of Theorem 3.2 are satisfied with $F(t) = \ln t$, $s = 2$, $\alpha(x,y) = 4$ and $\varepsilon = 2$. So $\mathcal{T}$ has a fixed point, which is a solution of the problem (4.1). $\square$

5. Conclusion

In this study, we have extended and generalized several results within the framework of a novel class of multivalued $(F, \alpha_{s,\varepsilon})$-contractions. For this class, we have established a fixed point theorem by employing a combination of relevant concepts. The validity of our results is illustrated through an example. Consequently, we present new fixed point theorems in the setting of complete dislocated $b-$metric spaces endowed with a partial order. Finally, we applied our results to demonstrate the existence of solutions to certain integral inclusions.

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