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NEUTROSOPHIC INNER PRODUCT SPACES AND THE CONSTRUCTION OF NEUTROSOPHIC HILBERT SPACES WITH APPLICATION

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Abstract. In this paper, a new definition of neutrosophic inner product spaces is proposed under fewer axiomatic conditions, ensuring consistency with the classical notion of inner product spaces. Based on this formulation, several fundamental analytical results are rigorously established, including convergence theorems, a neutrosophic analogue of Schwarz's inequality, and a precise characterization of orthogonality in neutrosophic inner product spaces. Furthermore, neutrosophic Hilbert spaces are introduced, and a corresponding neutrosophic form of the Riesz representation theorem is derived. In addition, a number of significant results in fixed point theory are proved and subsequently employed to investigate the existence and uniqueness of solutions to Uryson's integral equation.

1. Introduction

Fuzzy sets (FSs), introduced by Zadeh [36], have played a fundamental role in modeling uncertainty across a wide range of scientific disciplines. Despite their effectiveness in representing imprecise information, fuzzy sets are not

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always sufficient for addressing certain complex real-world problems, leading to the development of several generalized frameworks.

Among these extensions, intuitionistic fuzzy sets were proposed by Atanassov [1] and later formalized by Smarandache [26]. The notion of neutrosophy was conceived in 1998, providing a more flexible structure by incorporating degrees of truth,indeterminacy and falsity. This approach gave rise to neutrosophic sets(NSs) and their various generalizations, including interval-valued fuzzy sets, paraconsistent and pythagorean fuzzy sets, and interval-valued intuitionistic fuzzy sets [25, 27, 29, 32]. Subsequently, Wang [30] introduced single-valued neutrosophic sets, where all membership functions take values in the interval [0,1]. There has been growing attention on single-valued neutrosophic sets owing to their theoretical richness and practical applicability in decision-making problems [18, 33, 34, 35]. Further developments in neutrosophic logic, such as neutrosophic algebras, filters, and triplet groups, form an essential foundation for this theory [19, 31, 37]. Throughout this paper, the term neutrosophic sets refers to single-valued neutrosophic sets. A fundamental distinction between neutrosophic sets and intuitionistic fuzzy sets lies in the independence of the truth, indeterminacy, and falsity-membership functions, which are not required to satisfy mutual constraints. This characteristic provides a broader and more adaptable framework for uncertainty modeling.

The notation of fuzzy normed structures was first introduced by Katsaras [14] and later developed through various definitions, leading to fuzzy inner product spaces and fuzzy Hilbert spaces [2, 3, 5, 6, 7, 8, 9, 17]. Parallel developments gave rise to intuitionistic fuzzy normed, inner product, and Hilbert spaces [10, 20, 22, 23], followed by more recent extensions to neutrosophic metric, normed, and inner product spaces [15, 16, 21]. Recent research has increasingly focused on fixed points and their applications across various fields. Fixed point theory has become fundamental in understanding the stability of dynamical systems, with significant implications in mathematics, economics, computer science, and engineering ([4, 11, 12]). In 2025, researchers N. Sarkar, J. Das, and Ashoke presented a groundbreaking study that explored the results related to the fixed point in the Neutrosophic Normed Space [24]. Additionally, in the same year, researchers M. Sornavalli, R. Selvarani, and others presented fixed point theorems in common neutrosophic metric spaces, applying neutrosophic logic to provide new insights in the study of metric spaces with indeterminate information [28].

Despite these advances, the formulation of an appropriate neutrosophic Hilbert space remains a challenge. In this paper, we propose a modified definition of a neutrosophic inner product space with fewer conditions, making it more consistent with classical inner product space, and based on this definition, we introduce the notation of a neutrosophic Hilbert space. In addition, we establish several fixed-point theory outcomes on top of investigating the presence and distinction of solutions for Uryson's integral equation as an application.

2. Preminaries

Proposition 2.1. ([13]) *Assume that* $\mathbf{G}^* = \{\kappa = (\kappa_1, \kappa_2, \kappa_3)\backslash \kappa_1, \kappa_2, \kappa_3 \in [0,1]\}$ *and* $\leq_{\mathbf{G}^*}$ *operation on* $\mathbf{G}^*$ *defined by* $\kappa \leq_{\mathbf{G}^*} \varsigma$ *if and only if* $\kappa_1 \leq \varsigma_1, \kappa_2 \geq \varsigma_2, \kappa_3 \geq \varsigma_3$ *for every* $\kappa, \varsigma \in \mathbf{G}^*$. *Then* $(\mathbf{G}^*, \leq_{\mathbf{G}^*})$ *is a complete lattice.*

Definition 2.2. ([13]) Let $\mathbf{A} \neq \emptyset$ and $E \subseteq \mathbf{A}$. A neutrosophic set (briefly NS) E in $\mathbf{A}$ is defined by three functions: truth-membership $\Phi_E(\kappa)$, indeterminacy-membership $\Psi_E(\kappa)$, and falsity-membership $\Upsilon_E(\kappa)$. Then, a NS E can be described to be

$$E = \{(\kappa, \Phi_E(\kappa), \Psi_E(\kappa), \Upsilon_E(\kappa))\backslash \kappa \in \mathbf{A}\},$$

where $\Phi_E(\kappa), \Psi_E(\kappa), \Upsilon_E(\kappa) \in [0,1]$ provided that $0 \leq \Phi_E(\kappa) + \Psi_E(\kappa) + \Upsilon_E(\kappa) \leq 3$.

Definition 2.3. ([13]) Assume that $E = \{(\kappa, \Phi_E(\kappa), \Psi_E(\kappa), \Upsilon_E(\kappa)\backslash \kappa \in \mathbf{A}\}$ and $F = \{(\kappa, \Phi_F(\kappa), \Psi_F(\kappa), \Upsilon_F(\kappa)\backslash \kappa \in \mathbf{A}\}$ are two NS in a set $\mathbf{A} \neq \emptyset$. The initial type inclusion relation $\subseteq$ and its fundamental algebraic operations are delineated as follows:

(1) $E \cup F = \{< \kappa, \max(\Phi_E(\kappa), \Phi_F(\kappa)), \min(\Psi_E(\kappa), \Psi_F(\kappa)), \min(\Upsilon_E(\kappa), \Upsilon_F(\kappa)) > \backslash \kappa \in \mathbf{A}\}$;
(2) $E \cap F = \{< \kappa, \min(\Phi_E(\kappa), \Phi_F(\kappa)), \max(\Psi_E(\kappa), \Psi_F(\kappa)), \max(\Upsilon_E(\kappa), \Upsilon_F(\kappa)) > \backslash \kappa \in \mathbf{A}\}$;
(3) $E \subseteq F \Leftrightarrow \Phi_E(\kappa) \leq \Phi_F(\kappa), \Psi_E(\kappa) \geq \Psi_F(\kappa), \Upsilon_E(\kappa) \geq \Upsilon_F(\kappa)$;
(4) $E^c = \{< \kappa, \Upsilon_E(\kappa), 1 - \Upsilon_E(\kappa), \Phi_E(\kappa) > \backslash \kappa \in \mathbf{A}\}$.

Remark 2.4. ([13]) We write the units as $0_{\mathbf{G}^*} = (0,1,1)$ and $1_{\mathbf{G}^*} = (1,0,0)$. Classically, a triangular norm (TN) $\star = \mathfrak{R}$ on $[0,1]$ such that $\mathfrak{R} : [0,1]^2 \longrightarrow [0,1]$ satisfying:

(i) commutative;
(ii) associative;
(iii) increasing;
(iv) $\mathfrak{R}(1,a) = 1 \star a = a,\ \forall a \in [0,1]$.

and trangular conorm(T-co-N) $\circ = \mathfrak{M}$ on $[0,1]$ such that $\mathfrak{M} : [0,1]^2 \longrightarrow [0,1]$ satisfying:

(i) commutative;
(ii) associative;
(iii) increasing;
(iv) $\mathfrak{M}(0,a) = 0 \circ a = a,\ \forall a \in [0,1]$.

these concepts can be straightforwardly expanded.

Definition 2.5. ([13]) A function $\mathbf{T}_{\mathtt{N}} : (\mathbf{G}^*)^2 \to \mathbf{G}^*$ is termed neutrosophic t-norm (briefly, NTN) provided that the subsequent conditions hold for each $\mathfrak{p}, \mathfrak{q}, \mathfrak{r}, \mathfrak{s} \in \mathbf{G}^*$:

(1) $\mathbf{T}_{\mathtt{N}}(\mathfrak{p}, \mathfrak{q}) = \mathbf{T}_{\mathtt{N}}(\mathfrak{q}, \mathfrak{p})$;
(2) $\mathbf{T}_{\mathtt{N}}(\mathfrak{p}, \mathbf{T}_{\mathtt{N}}(\mathfrak{q}, \mathfrak{r})) = \mathbf{T}_{\mathtt{N}}(\mathfrak{q}, \mathbf{T}_{\mathtt{N}}(\mathfrak{p}, \mathfrak{r}))$;
(3) $\mathbf{T}_{\mathtt{N}}(\mathfrak{p}, \mathfrak{q}) \leq_{\mathbf{G}^*} \mathbf{T}_{\mathtt{N}}(\mathfrak{r}, \mathfrak{s})$ where $\mathfrak{p} \leq_{\mathbf{G}^*} \mathfrak{r}, \mathfrak{q} \leq_{\mathbf{G}^*} \mathfrak{s}$;
(4) $\mathbf{T}_{\mathtt{N}}(\mathfrak{a}, 1_{\mathbf{G}^*}) = \mathfrak{a}$.

Theorem 2.6. ([13]) *Let $\mathbf{T}_{\mathtt{N}}$ denote a binary function (operation) defined through $\mathbf{G}^*$. Then, for all $\mathfrak{a}, \mathfrak{b} \in \mathbf{G}^*$, $\mathbf{T}_{\mathtt{N}}(\mathfrak{a}, \mathfrak{b}) = (\mathfrak{R}(\mathfrak{a}_1, \mathfrak{b}_1), \mathfrak{M}_1(\mathfrak{a}_2, \mathfrak{b}_2), \mathfrak{M}_2(\mathfrak{a}_3, \mathfrak{b}_3))$ is an NTN such that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are $T-co-Ns$, while $\mathfrak{R}$ is a TN on $[0,1]$.*

Definition 2.7. ([13]) Let $\mathbf{T}_{\mathtt{N}}$ be an NTN on $\mathbf{G}^*$.Then $\mathbf{T}_{\mathtt{N}}$ is said to be representable(briefly, RNTN) if and only if there exist two $\mathfrak{M}_1, \mathfrak{M}_2$ are two T-co-Ns and $\mathfrak{R}$ is a TN on [0,1] such that for any $\mathfrak{a}, \mathfrak{b} \in \mathbf{G}^*$,

$$\mathbf{T}_{\mathtt{N}}(\mathfrak{a}, \mathfrak{b}) = (\mathfrak{R}(\mathfrak{a}_1, \mathfrak{b}_1), \mathfrak{M}_1(\mathfrak{a}_2, \mathfrak{b}_2), \mathfrak{M}_2(\mathfrak{a}_3, \mathfrak{b}_3)).$$

Now, establish a sequence $\mathbf{T}^n_{\mathtt{N}}$ recursively through $\mathbf{T}^1_{\mathtt{N}} = \mathbf{T}_{\mathtt{N}}$ as follows

$$\mathbf{T}^n_{\mathtt{N}}(\mathfrak{a}^{(1)}, ..., \mathfrak{a}^{(n+1)}) = \mathbf{T}_{\mathtt{N}}(\mathbf{T}^{(n-1)}_{\mathtt{N}}(\mathfrak{a}^{(1)}, ..., \mathfrak{a}^{(n)}), \mathfrak{a}^{(n+1)})$$

for $n \geq 2$ and $\mathfrak{a}^{(i)} \in \mathbf{G}^*$.

Definition 2.8. ([13]) Let $\mathbf{T}_{\mathtt{N}}$ be an NTN on $\mathbf{G}^*$. Then $\mathbf{T}_{\mathtt{N}}$ is said to be standard representable if and only if there exist $\mathfrak{R}, \mathfrak{M}$ TN and T-co-N on interval [0,1] respectively, such that for any $\mathfrak{a}, \mathfrak{b} \in \mathbf{G}^*$,

$$\mathbf{T}_{\mathtt{N}}(\mathfrak{a}, \mathfrak{b}) = (\mathfrak{R}(\mathfrak{a}_1, \mathfrak{b}_1), \mathfrak{M}(\mathfrak{a}_2, \mathfrak{b}_2), \mathfrak{M}(\mathfrak{a}_3, \mathfrak{b}_3)).$$

Definition 2.9. ([13]) If a function $\mathscr{E} : \mathbf{G}^* \to \mathbf{G}^*$ fulfills the conditions stated below, it is hence known as a neutrosophic negator (briefly, N_{neg}):

(1) $\mathscr{E}(0_{\mathbf{G}^*}) = 1_{\mathbf{G}^*}$;
(2) $\mathscr{E}(1_{\mathbf{G}^*}) = 0_{\mathbf{G}^*}$;

(3) $\mathscr{E}(\mathfrak{a}) \geq_{\mathbf{G}^*} \mathscr{E}(\mathfrak{b})$, for all $\mathfrak{a}, \mathfrak{b} \in \mathbf{G}^*$ *such that* $\mathfrak{a} \leq_{\mathbf{G}^*} \mathfrak{b}$.

Furthermore, if $\mathscr{E}(\mathscr{E}(\mathfrak{a})) = \mathfrak{a}$ for all $\mathfrak{a} \in \mathbf{G}^*$ then $\mathscr{E}$ is designated as an involutive N_{neg}.

Remark 2.10. An involutive N_{neg}, $\mathscr{E}_s : \mathbf{G}^* \to \mathbf{G}^*$ is said to be a standard N_{neg} if $\mathscr{E}_s(\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3) = (\mathfrak{a}_3, 1 - \mathfrak{a}_2, \mathfrak{a}_1)$ for all $(\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3) \in \mathbf{G}^*$. Certainly, $\mathscr{E}(\mathfrak{a}) = (\mathfrak{a}_3, 1 - \mathfrak{a}_2, \mathfrak{a}_1), \mathscr{E}(\mathfrak{a}) = (\mathfrak{a}_3, \mathfrak{a}_1, \mathfrak{a}_1)$ are standard N_{negs}.

Remark 2.11. Let $\mathfrak{T}_{\mathtt{N}_1}, \mathfrak{T}_{\mathtt{N}_2}$ be two continuous RNTN, we call $\mathfrak{T}_{\mathtt{N}_1} \leq \mathfrak{T}_{\mathtt{N}_2}$ *if* $\mathfrak{T}_{\mathtt{N}_1}(a, b) \leq_{\mathbf{G}^*} \mathfrak{T}_{\mathtt{N}_2}(a, b)$ for all $a, b \in \mathbf{G}^*$.

Definition 2.12. Let Φ, Ψ *and* Υ be fuzzy sets defined as $\Phi, \Psi, \Upsilon : \mathbf{A} \times (0, +\infty) \to [0, 1]$ such that $\Phi(\mathfrak{a}, \varrho) + \Psi(\mathfrak{a}, \varrho) + \Upsilon(\mathfrak{a}, \varrho) \leq 3$ for all $\mathfrak{a} \in \mathbf{A}, \varrho > 0$. The 3-tuple $(\mathbf{A}, \mathbf{B}_{\Phi,\Psi,\Upsilon}, \mathbf{T}_{\mathtt{N}})$ is designated as a neutrosophic normed space (briefly, NNS) if $\mathbf{A}$ is a vector space, $\mathbf{T}_{\mathtt{N}}$ is a continuous RNTN and $\mathbf{B}_{\phi,\psi,\Upsilon}$ defines a mapping from $\mathbf{A} \times (0, +\infty) \to \mathbf{G}^*$ is a neutrosophic set fulfilling the following requirements for any $\mathfrak{a}, \mathfrak{b} \in \mathbf{A}$ and $\varrho, \sigma > 0$,

(NNS-1): $\mathbf{B}_{\phi,\psi,\Upsilon}(\mathfrak{a}, \varrho) >_{\mathbf{G}^*} 0_{\mathbf{G}^*}$;
(NNS-2): $\mathbf{B}_{\phi,\psi,\Upsilon}(\mathfrak{a}, \varrho) = 1_{\mathbf{G}^*}$ if and only if $\mathfrak{a} = 0$;
(NNS-3): $\mathbf{B}_{\phi,\psi,\Upsilon}(\alpha\mathfrak{a}, \varrho) = \mathbf{B}_{\phi,\psi,\Upsilon}(\mathfrak{a}, \frac{\varrho}{|\alpha|})$ for each $\alpha \neq 0$;
(NNS-4): $\mathbf{B}_{\phi,\psi,\Upsilon}(\mathfrak{a} + \mathfrak{b}, \varrho + \sigma) \geq_{\mathbf{G}^*} \mathbf{T}_{\mathtt{N}}(\mathbf{B}_{\phi,\psi,\Upsilon}(\mathfrak{a}, \varrho), \mathbf{B}_{\phi,\psi,\Upsilon}(\mathfrak{b}, \sigma))$;
(NNS-5): $\mathbf{B}_{\phi,\psi,\Upsilon}(\mathfrak{a}, \cdot) : (0, \infty) \to \mathbf{G}^*$ is continuous;
(NNS-6): $\lim_{\varrho\to\infty} \mathbf{B}_{\phi,\psi,\Upsilon}(\mathfrak{a}, \varrho) = 1_{\mathbf{G}^*}$, and $\lim_{\varrho\to 0} \mathbf{B}_{\phi,\psi,\Upsilon}(\mathfrak{a}, \varrho) = 0_{\mathbf{G}^*}$.

In the current case, $\mathbf{B}_{\phi,\psi,\Upsilon}$ are referred to as a neutrosophic norm. Here, $\mathbf{B}_{\phi,\psi,\Upsilon}(\mathfrak{a}, \varrho) = (\phi(\mathfrak{a}, \varrho), \psi(\mathfrak{a}, \varrho), \Upsilon(\mathfrak{a}, \varrho))$.

Example 2.13. Consider the normed space $(\mathbf{A}, \|.\|)$ and $\mathbf{T}_{\mathtt{N}}(\mathfrak{a}, \mathfrak{b}) = (\mathfrak{a}_1\mathfrak{b}_1, \mathfrak{a}_2 + \mathfrak{b}_2 - \mathfrak{a}_2\mathfrak{b}_2, \mathfrak{a}_3 + \mathfrak{b}_3 - \mathfrak{a}_3\mathfrak{b}_3)$ for all $\mathfrak{a} = (\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3)$ and $\mathfrak{b} = (\mathfrak{b}_1, \mathfrak{b}_2, \mathfrak{b}_3) \in \mathbf{G}^*$ and let ϕ, ψ, Υ be fuzzy sets in $\mathbf{A} \times (0, \infty)$, for $\varrho >\| \mathfrak{a} \|$:

$$\mathbf{B}_{\phi,\psi,\Upsilon}(\mathfrak{a}, \varrho) = (\phi(\mathfrak{a}, \varrho), \psi(\mathfrak{a}, \varrho), \Upsilon(\mathfrak{a}, \varrho)) = (\frac{\varrho}{\varrho + \|\mathfrak{a}\|}, \frac{\|\mathfrak{a}\|}{\varrho + \|\mathfrak{a}\|}, \frac{\|\mathfrak{a}\|}{\varrho})$$

for all $\mathfrak{a}, \mathfrak{b} \in \mathbf{A}$ *and* $\varrho > 0$. If we take $\varrho \leq \|\mathfrak{a}\|$, the $\mathbf{B}_{\phi,\psi,\Upsilon}(\mathfrak{a}, \varrho) = 0_{\mathbf{G}^*}$. Then $(\mathbf{A}, \mathbf{B}_{\phi,\psi,\Upsilon}, \mathbf{T}_{\mathtt{N}})$ is a NNS.

Definition 2.14. A sequence $\{\mathfrak{a}_k\}$ in a NNS $(\mathbf{A}, \mathbf{B}_{\phi,\psi,\Upsilon}, \mathbf{T}_{\mathtt{N}})$ is said to be N-convergent sequence to an element $\mathfrak{a} \in \mathbf{A}$, *where* $0 < \epsilon < 1, \varrho > 0$, if and only if there is an element $k_0 \in N$ such that $\mathbf{B}_{\phi,\psi,\Upsilon}(\mathfrak{a}_k - \mathfrak{a}, \varrho) >_{\mathbf{G}^*} (\mathscr{E}_s(\epsilon), \epsilon, \epsilon)$ for each every $k \geq k_0$. This is represented as the limit $\lim_{\varrho\to\infty} \mathbf{B}_{\phi,\psi,\Upsilon}(\mathfrak{a}_k - \mathfrak{a}, \varrho) = 1_{\mathbf{G}^*}$ and $\mathfrak{a}_k \xrightarrow{\mathbf{B}_{\phi,\psi,\Upsilon}} \mathfrak{a}$.

Definition 2.15. Let $(\mathbf{A}, \mathbf{B}_{\phi,\psi,\Upsilon}, \mathbf{T}_{\mathtt{N}})$ be a NNS, the sequence $\{\mathfrak{a}_k\}$ is said to be N-Cauchy sequence $0 < \epsilon < 1$ *and* $\varrho > 0$, if and only if there exists $k_0 \in N$ such that $\mathbf{B}_{\phi,\psi,\Upsilon}(\mathfrak{a}_k - \mathfrak{a}_m, t) >_{\mathbf{G}^*} (\mathscr{E}_s(\epsilon), \epsilon, \epsilon)$ for each $k, m \geq k_0$.

Definition 2.16. An NNS is said to be complete if every N-Cauchy sequence is a N-convergent sequence.

Remark 2.17. Suppose that $(\mathbf{A}, \mathbf{B}_{\phi,\psi,\Upsilon}, \mathbf{T}_{\mathtt{N}})$ is a NNS.Then the following is true for all $\varrho > 0$:

(i) $\mathbf{B}_{\phi,\psi,\Upsilon}(\mathfrak{a}, \varrho)$ is non-decreasing relative to ϱ.
(ii) $\mathbf{B}_{\phi,\psi,\Upsilon}(\mathfrak{a} - \mathfrak{b}, \varrho) = \mathbf{B}_{\phi,\psi,\Upsilon}(\mathfrak{b} - \mathfrak{a}, \varrho)$.

Definition 2.18. Let $(\mathbf{A}, \mathbf{B}_{\phi,\psi,\Upsilon}, \mathbf{T}_{\mathtt{N}})$ be a NNS, we define the neutrosophic open ball $B_N(\mathfrak{a}, r, \varrho)$ with radius $0 < r < 1$ and center $\mathfrak{a}$ for all $\varrho > 0$ as

$$B_N(\mathfrak{a}, r, \varrho) = \{\mathfrak{b} \in \mathbf{A}; \mathbf{B}_{\phi,\psi,\Upsilon}(\mathfrak{a} - \mathfrak{b}, \varrho) >_{\mathbf{G}^*} (\mathscr{E}_s(r), r, r)\}.$$

A subset $E \subseteq \mathbf{A}$ is designated as neutrosophic open if for each $\mathfrak{a} \in E$, there exist positive real numbers $\varrho > 0$ and $0 < r < 1$ such that $B_N(\mathfrak{a}, r, \varrho) \subseteq E$.

Let $\tau_{\mathbf{B}}$ denote the collection of all neutrosophic open subsets of $\mathbf{A}$, then $\tau_{\mathbf{B}}$ is referred to as the topology induced by $\mathbf{B}_{\phi,\psi,\Upsilon}$.

Remark 2.19. A topology $\tau_{\mathbf{B}}$ is identical to the topology produced by the neutrosophic metric space (NMS), which is Hausdorff. (see remark and theorem [16]).

Definition 2.20. Assume that $(\mathbf{A}, \mathbf{B}_{\phi,\psi,\Upsilon}, \mathbf{T}_{\mathtt{N}})$ is a NNS. If $\varrho > 0$, $0 < r < 1$ such that $\mathbf{B}_{\phi,\psi,\Upsilon}(\mathfrak{a}, \varrho) >_{\mathbf{G}^*} (\mathscr{E}_s(r), r, r)$ for each $\mathfrak{a} \in E$, then a subset E of $\mathbf{A}$ is N-bounded set.

Theorem 2.21. *Every N-compact set in a NNS is N-bounded set.*

Proof. Its proof is very similar to that in NMS (see theorem [16]). □

Corollary 2.22. *In a NNS, every N-compact set is both N-closed and N-bounded set.*

3. Neutrosophic inner product space (NIPS)

In this section, we define neutrosophic inner product spaces in a way that aligns with the classical inner product framework. We then examine some of their basic properties and establish several related results. Moreover, we introduce a new notion of neutrosophic Hilbert spaces and prove the Riesz representation theorem in this context.

Definition 3.1. Let Φ, Ψ *and* Υ be fuzzy sets such that $\mathbf{A}^2 \times (0,\infty) \to [0,1]$ and $\Phi(\mathfrak{a},\mathfrak{b},\varrho) + \Psi(\mathfrak{a},\mathfrak{b},\varrho) + \Upsilon(\mathfrak{a},\mathfrak{b},\varrho) \leq 3$, for all $\mathfrak{a},\mathfrak{b} \in \mathbf{A}$ and $\varrho > 0$. A neutrosophic inner product space (briefly, NIPS) is a triplet $(\mathbf{A}, \mathbf{F}_{\Phi,\Psi,\Upsilon}, \mathbf{T}_{\mathtt{N}})$, where $\mathbf{A}$ is a vector space over the real field, $\mathbf{T}_{\mathtt{N}}$ is a continuous RNTN and $\mathbf{F}_{\Phi,\Psi,\Upsilon}$ is a neutrosophic set on $\mathbf{A}^2 \times R$ fulfilling the following requirements for every $\mathfrak{a},\mathfrak{b},\mathfrak{c} \in \mathbf{A}$ *and* $\varrho, \sigma, \varsigma \in R$:

(NIPS-1): $\mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{a},\mathfrak{a},0) = 0_{\mathbf{G}^*}$ and $\mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{a},\mathfrak{a},\varrho) >_{\mathbf{G}^*} 0_{\mathbf{G}^*}$ for every $\varrho > 0$.

(NIPS-2): $\mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{a},\mathfrak{b},\varrho) = \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{b},\mathfrak{a},\varrho)$.

(NIPS-3): $\mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{a},\mathfrak{a},\varrho) \neq \mathcal{V}(t)$ for some $\varrho \in R$ if and only if $\mathfrak{a} \neq 0$, where $\mathcal{V}(\varrho) = \begin{cases} 1_{\mathbf{G}^*} & \varrho > 0 \\ 0_{\mathbf{G}^*} & \varrho \leq 0 \end{cases}$.

(NIPS-4): For any $\alpha \in R$,

$$\mathbf{F}_{\Phi,\Psi,\Upsilon}(\alpha\mathfrak{a},\mathfrak{b},\varrho) = \begin{cases} \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{a},\mathfrak{b},\frac{\varrho}{\alpha}) & \alpha > 0, \\ \mathcal{V}(\varrho) & \alpha = 0, \\ \mathscr{E}_s(\mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{a},\mathfrak{b},\frac{\varrho}{\alpha})) & \alpha < 0. \end{cases}$$

(NIPS-5): $\sup_{\sigma+\varsigma=\varrho}\{\mathbf{T}_{\mathtt{N}}(\mathbf{F}_{\Phi,\Psi,\Upsilon}((\mathfrak{a},\mathfrak{c},\sigma), \mathbf{F}_{\Phi,\Psi,\Upsilon}((\mathfrak{b},\mathfrak{c},\varsigma))\}$ $= \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{a}+\mathfrak{b},\mathfrak{c},\varrho)$.

(NIPS-6): $\mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{a},\mathfrak{b},.) : R \to [0,1]$ on $R - \{0\}$, is continuous.

(NIPS-7): $\lim_{\varrho\to\infty} \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{a},\mathfrak{b},\varrho) = 1_{\mathbf{G}^*}$.

Example 3.2. Consider a vector space $\mathbf{A}$ together with inner product structure $< \cdot,\cdot >$, and let $\mathbf{T}_{\mathtt{N}}(\mathfrak{p},\mathfrak{q}) = (\mathfrak{p}_1\mathfrak{q}_1, \min(\mathfrak{p}_2+\mathfrak{q}_2,1), \min(\mathfrak{p}_3+\mathfrak{q}_3,1))$ for all $\mathfrak{p} = (\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3)$ and $\mathfrak{q} = (\mathfrak{q}_1,\mathfrak{q}_2,\mathfrak{q}_3) \in \mathbf{G}^*$. Furthermore, let Φ, Ψ, and Υ represent FSs in $\mathbf{A}^2 \times (0,\infty)$ and

$$\mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{p},\mathfrak{q},\varrho) = \begin{cases} (\Phi(\mathfrak{p},\mathfrak{q},\varrho), \Psi(\mathfrak{p},\mathfrak{q},\varrho), \Upsilon(\mathfrak{p},\mathfrak{q},\varrho)) \\ \quad = \Big(\frac{k\varrho^{n/2}}{k\varrho^{n/2}+m|<\mathfrak{p},\mathfrak{q}>|^{1/2}}, \frac{m|<\mathfrak{p},\mathfrak{q}>|^{1/2}}{k\varrho^{n/2}+m|<\mathfrak{p},\mathfrak{q}>|^{1/2}}, \frac{m|<\mathfrak{p},\mathfrak{q}>|^{1/2}}{k\varrho^{n/2}}\Big), \\ \qquad \varrho > 0, k, m, n \in R^+, \\ 0_{\mathbf{G}^*}, \qquad \varrho \leq 0. \end{cases}$$

Then $(\mathbf{A}, \mathbf{F}_{\Phi,\Psi,\Upsilon}, \mathbf{T}_{\mathtt{N}})$ is a NIPS. In particular if $k = m = n = 1$, we have

$$\mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{p},\mathfrak{q},\varrho) = \begin{cases} (\Phi(\mathfrak{p},\mathfrak{q},\varrho), \Psi(\mathfrak{p},\mathfrak{q},\varrho), \Upsilon(\mathfrak{p},\mathfrak{q},\varrho)) \\ \quad = (\frac{\varrho^{1/2}}{\varrho^{1/2}+|<\mathfrak{p},\mathfrak{q}>|^{1/2}}, \frac{|<\mathfrak{p},\mathfrak{q}>|^{1/2}}{\varrho^{1/2}+|<\mathfrak{p},\mathfrak{q}>|^{1/2}}, \frac{|<\mathfrak{p},\mathfrak{q}>|^{1/2}}{\varrho^{1/2}}), \quad \varrho > 0, \\ 0_{\mathbf{G}^*}, \qquad \varrho \leq 0. \end{cases}$$

This is known as the ordinary (standard) neutrosophic inner product space, generated by the inner product.

Remark 3.3. Consider $(\mathbf{A}, \mathbf{F}_{\Phi,\Psi,\Upsilon}, \mathbf{T}_{\mathtt{N}})$ is a NIPS and $\xi_{\mathfrak{c}} = \{\mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{a}, \mathfrak{c}, .) \setminus \mathfrak{a} \in \mathbf{A}\}$ for every $\mathfrak{c} \in \mathbf{A}$ The scalar multiplication and addition on $\xi_{\mathfrak{c}}$ are defined,

$$\alpha \odot \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{a}, \mathfrak{c}, \varrho) = \mathbf{F}_{\Phi,\Psi,\Upsilon}(\alpha\mathfrak{a}, \mathfrak{c}, \varrho).$$
$$\mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{a}, \mathfrak{c}, \varrho) \oplus \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{b}, \mathfrak{c}, \varrho) = \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{a} + \mathfrak{b}, \mathfrak{c}, \varrho) \text{ for all } \alpha, \varrho \in R.$$

The Schwarz's inequality in NIPS will now be proven.

Lemma 3.4. *Assume that* $(\mathbf{A}, \mathbf{F}_{\Phi,\Psi,\Upsilon}, \mathbf{T}_{\mathtt{N}})$ *is a NIPS and* $\mathbf{T}_{\mathtt{N}}$ *is a continuous RNTN such that* $\mathbf{T}_{\mathtt{N}}(\mathfrak{p}, \mathfrak{p}) >_{\mathbf{G}^*} \mathfrak{p}$. *Then for all* $\mathfrak{p}, \mathfrak{q} \in \mathbf{A}$ *with* $\varrho, \sigma > 0$, *we obtain*

$$\mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{p}, \mathfrak{q}, \varrho\sigma) \geq_{\mathbf{G}^*} \mathbf{T}_{\mathtt{N}}(\mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{p}, \mathfrak{p}, \varrho^2), \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{q}, \mathfrak{q}, \sigma^2)).$$

Proof. Let $\alpha = \frac{-\sigma}{\varrho}$, that is, $\alpha\varrho + \sigma = 0$. Setting, $\hat{\mathrm{u}} = \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{p}, \mathfrak{p}, \varrho^2), \hat{\mathrm{v}} = \mathbf{F}_{\Phi,\Psi,\Upsilon}(\alpha\mathfrak{q}, \mathfrak{p}, \alpha\varrho\sigma), \hat{\mathrm{w}} = \mathbf{F}_{\Phi,\Psi,\Upsilon}(\alpha\mathfrak{q}, \alpha\mathfrak{q}, \alpha^2\varrho^2)$. By (NIP-1) and (NIP-2), we have

$$\begin{aligned}\mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{p} + \alpha\mathfrak{q}, \mathfrak{p} + \alpha\mathfrak{q}, (\alpha\varrho + \sigma)^2) &\geq_{\mathbf{G}^*} \mathbf{T}_{\mathtt{N}}(\mathbf{T}_{\mathtt{N}}(\hat{\mathrm{u}}, \hat{\mathrm{v}}), \mathbf{T}_{\mathtt{N}}(\hat{\mathrm{v}}, \hat{\mathrm{w}}))\\ &= \mathbf{T}_{\mathtt{N}}^2(\hat{\mathrm{u}}, (\mathbf{T}_{\mathtt{N}}(\hat{\mathrm{v}}, \hat{\mathrm{v}}), \hat{\mathrm{w}})\\ &\geq_{\mathbf{G}^*} \mathbf{T}_{\mathtt{N}}(\hat{\mathrm{u}}, \mathbf{T}_{\mathtt{N}}(\hat{\mathrm{v}}, \hat{\mathrm{w}})),\end{aligned}$$

since $\hat{\mathrm{w}} = \mathbf{F}_{\Phi,\Psi,\Upsilon}(\alpha\mathfrak{q}, \alpha\mathfrak{q}, \alpha^2\varrho^2) = \mathscr{E}_s(\mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{q}, \alpha\mathfrak{q}, \alpha\varrho^2)) = \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{q}, \mathfrak{q}, \varrho^2)$ and $\hat{\mathrm{v}} = \mathbf{F}_{\Phi,\Psi,\Upsilon}(\alpha\mathfrak{q}, \mathfrak{p}, \alpha\varrho\sigma) = \mathscr{E}_s(\mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{q}, \mathfrak{p}, \varrho\sigma))$. Substituting $\hat{\mathrm{v}}$ and $\hat{\mathrm{w}}$ into the above inequality, we have

$$0_{\mathbf{G}^*} \geq_{\mathbf{G}^*} \mathbf{T}_{\mathtt{N}}^2(\mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{p}, \mathfrak{p}, \sigma^2), \mathscr{E}_s(\mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{q}, \mathfrak{p}, \varrho\sigma)), \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{q}, \mathfrak{q}, \varrho^2)),$$

$$\begin{aligned}\mathscr{E}_s(\mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{q}, \mathfrak{p}, \varrho\sigma)) \geq_{\mathbf{G}^*} &((\Phi(\mathfrak{p}, \mathfrak{p}, \sigma^2) \star \Phi(\mathfrak{q}, \mathfrak{q}, \varrho^2) + 1 - \Phi(\mathfrak{p}, \mathfrak{q}, \varrho\sigma) - 1,\\ &(\Psi(\mathfrak{p}, \mathfrak{p}, \sigma^2) \circ \Psi(\mathfrak{q}, \mathfrak{q}, \varrho^2) + 1 - \Psi(\mathfrak{p}, \mathfrak{q}, \varrho\sigma) - 1,\\ &(\Upsilon(\mathfrak{p}, \mathfrak{p}, \sigma^2) \circ \Upsilon(\mathfrak{q}, \mathfrak{q}, \varrho^2) + 1 - \Upsilon(\mathfrak{p}, \mathfrak{q}, \varrho\sigma) - 1).\end{aligned}$$

□

Lemma 3.5. *Let* $(\mathbf{A}, \mathbf{F}_{\Phi,\Psi,\Upsilon}, \mathbf{T}_{\mathtt{N}})$ *be a NIPS with a continuous RNTN* $\mathbf{T}_{\mathtt{N}}$. *Then,* $\mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{a}, \mathfrak{b}, .)$ *is non-decreasing with regard to* ϱ *for all* $\mathfrak{a}, \mathfrak{b} \in \mathbf{A}$.

Proof. Let $\varrho < \sigma$. Then $\kappa = \sigma - \varrho > 0$. In such a case, we obtain

$$\begin{aligned}\mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{a}, \mathfrak{b}, \varrho) &= \mathbf{T}_{\mathtt{N}}(\mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{a}, \mathfrak{b}, \varrho), 1_{\mathbf{G}^*})\\ &= \mathbf{T}_{\mathtt{N}}(\mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{a}, \mathfrak{b}, \varrho), \mathbf{F}_{\Phi,\Psi,\Upsilon}(0, \mathfrak{b}, \kappa))\\ &\leq_{\mathbf{G}^*} \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{a}, \mathfrak{b}, \varrho + \kappa)\\ &= \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{a}, \mathfrak{b}, \sigma).\end{aligned}$$

□

Theorem 3.6. *Let* $(\mathbf{A}, \mathbf{F}_{\Phi,\Psi,\Upsilon}, \mathbf{T}_{\mathtt{N}})$ *be a NIPS with a continuous RNTN* $\mathbf{T}_{\mathtt{N}}$*. Then, it constitutes a real NNS.*

Proof. We define

$$\mathbf{B}_{\phi,\psi,\Upsilon}(\mathfrak{p}, \varrho) = \begin{cases} \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{p}, \mathfrak{p}, \varrho^2), & \varrho > 0, \\ 0_{\mathbf{G}^*}, & \varrho \le 0. \end{cases}$$

We are going to prove that $\mathbf{B}_{\phi,\psi,\Upsilon}$ satisfy the requirements of Definition 2.12. The conditions (NIP-1)-(NIP-4) yield the conditions (NN-1)-(NN-3).

Additionally, assuming

$$\hat{\mathrm{u}} = \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{p}, \mathfrak{p}, \varrho_1^2), \hat{\mathrm{v}} = \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{p}, \mathfrak{q}, \varrho_1\varrho_2), \hat{\mathrm{w}} = \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{q}, \mathfrak{q}, \varrho_2^2),$$

for any given $\mathfrak{p}, \mathfrak{q} \in \mathbf{A}, \varrho_1, \varrho_2 > 0$.
From (NIP-5), we have

$$\begin{aligned}
\mathbf{B}_{\phi,\psi,\Upsilon}(\mathfrak{p}+\mathfrak{q}, \varrho_1+\varrho_2) &= \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{p}+\mathfrak{q}, \mathfrak{p}+\mathfrak{q}, (\varrho_1+\varrho_2)^2) \\
&\ge_{\mathbf{G}^*} \mathbf{T}_{\mathtt{N}}(\mathbf{T}_{\mathtt{N}}(\hat{\mathrm{u}}, \hat{\mathrm{v}}), \mathbf{T}_{\mathtt{N}}(\hat{\mathrm{v}}, \hat{\mathrm{w}})) \\
&= \mathbf{T}_{\mathtt{N}}^2(\hat{\mathrm{u}}, \hat{\mathrm{w}}, (\mathbf{T}_{\mathtt{N}}(\hat{\mathrm{v}}, \hat{\mathrm{v}}))) \\
&\ge_{\mathbf{G}^*} \mathbf{T}_{\mathtt{N}}(\hat{\mathrm{u}}, \mathbf{T}_{\mathtt{N}}(\hat{\mathrm{w}}, \hat{\mathrm{v}})) \\
&= \mathbf{T}_{\mathtt{N}}(\mathbf{T}_{\mathtt{N}}(\hat{\mathrm{u}}, \hat{\mathrm{v}}), \hat{\mathrm{w}}) \\
&\ge_{\mathbf{G}^*} \mathbf{T}_{\mathtt{N}}(\mathbf{T}_{\mathtt{N}}(e, g), \mathbf{T}_{\mathtt{N}}(e, g)) \\
&\ge_{\mathbf{G}^*} \mathbf{T}_{\mathtt{N}}(e, g) \\
&= \mathbf{T}_{\mathtt{N}}(\mathbf{B}_{\phi,\psi,\Upsilon}(\mathfrak{p}, \varrho_1), \mathbf{B}_{\phi,\psi,\Upsilon}(\mathfrak{q}, \varrho_2)).
\end{aligned}$$

The remaining conditions are obvious from (NIP-6) and (NIP-7). □

Remark 3.7. As mentioned in ([15],[16]), every NNS is NMS and every NMS is Hausdroff space. Since each NIPS is NNS, therefore this is a Hausdroff space in topology $\tau_{\mathfrak{B}}$ caused by the family of neighborhoods $\{\mathfrak{a}_r(\epsilon, \lambda), r \in \mathbf{A}, \epsilon > 0, 0 < \lambda < 1\}$ such that

$$\mathfrak{a}_r = \{\mathfrak{b} \in \mathbf{A} : \mathbf{B}_{\phi,\psi,\Upsilon}(\mathfrak{b} - r, \epsilon) >_{\mathbf{G}^*} \mathscr{E}_s(\lambda)\}.$$

Definition 3.8. Assume that $(\mathbf{A}, \mathbf{F}_{\Phi,\Psi,\Upsilon}, \mathbf{T}_{\mathtt{N}})$ is a NIPS that fulfills the conditions of Theorem 3.6. If for every $\epsilon > 0$ and $0 < \lambda < 1$, there exists $N_0 = N_0(\epsilon, \lambda) \in Z^+$ such that $\mathbf{B}_{\phi,\psi,\Upsilon}(\mathfrak{a}_n - \mathfrak{a}, \epsilon) >_{\mathbf{G}^*} \mathscr{E}_s(\lambda)$, whenever $n \ge N_0$, then the sequence $\{\mathfrak{a}_n\} \subseteq \mathbf{A}$ is said to be $\tau_{\mathfrak{B}}$-convergent to $\mathfrak{a} \in \mathbf{A}$ (we write $\mathfrak{a}_n \xrightarrow{\tau_{\mathfrak{B}}} \mathfrak{a}$).

Theorem 3.9. *Assume that* $(\mathbf{A}, \mathbf{F}_{\Phi,\Psi,\Upsilon}, \mathbf{T}_{\mathtt{N}})$ *is a NIPS that fulfills the conditions of Theorem 3.6. The sequence* $\{\mathfrak{a}_n\} \subseteq \mathbf{A}$ *such that* $\mathfrak{a}_n \xrightarrow{\tau_{\mathfrak{B}}} 0$. *Then for any* $\mathfrak{b} \in \mathbf{A}$ *and* $\epsilon > 0, 0 < \lambda < 1$, *there exists* $N = N(\epsilon, \lambda) \in Z^+$ *such that*

$$\mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{a}_n, \mathfrak{b}, \epsilon) >_{\mathbf{G}^*} \mathscr{E}_s(\lambda),$$

whenever $n \geq N$.

Proof. We may identify $\mathfrak{p}, \mathfrak{q} \in (0, 1)$ such that $\mathbf{T}_{\mathtt{N}}(\mathscr{E}_s(\mathfrak{p}), \mathscr{E}_s(\mathfrak{q})) >_{\mathbf{G}^*} \mathscr{E}_s(\lambda)$ for any $0 < \lambda < 1$. According to Lemma 3.5 and the concept of NIPS, there is $\varrho_0 > 0$ such that

$$\mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{b}, \mathfrak{b}, \varrho_0^2) >_{\mathbf{G}^*} \mathscr{E}_s(\mathfrak{q}).$$

As $\mathfrak{a}_n \xrightarrow{\tau_{\mathfrak{B}}} 0$ means that for any given number $\mathfrak{p}$, there is n_0 such that

$$\mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{a}_n, \mathfrak{a}_n, (\frac{\epsilon}{\varrho_0})^2) >_{\mathbf{G}^*} \mathscr{E}_s(\mathfrak{p})$$

for all $n \geq n_0$. Applying the Cauchy- inequality, it can be deduced that

$$\begin{aligned}
\mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{a}_n, \mathfrak{b}, \epsilon) &= \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{a}_n, \mathfrak{b}, (\frac{\epsilon}{\varrho_0})\varrho_0) \\
&>_{\mathbf{G}^*} \mathbf{T}_{\mathtt{N}}(\mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{a}_n, \mathfrak{a}_n, (\frac{\epsilon}{\varrho_0})^2), \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{b}, \mathfrak{b}, {\varrho_0}^2)) \\
&>_{\mathbf{G}^*} \mathbf{T}_{\mathtt{N}}(\mathscr{E}_s(\mathfrak{p}), \mathscr{E}_s(\mathfrak{q})) \\
&>_{\mathbf{G}^*} \mathscr{E}_s(\lambda), \ (\mathfrak{b} \in \mathbf{A}).
\end{aligned}$$

□

Definition 3.10. Let $(\mathbf{A}, \mathbf{F}_{\Phi,\Psi,\Upsilon}, \mathbf{T}_{\mathtt{N}})$ be a NIPS and $\{\mathfrak{a}_n\}$ be a sequence in $\mathbf{A}$ such that $\mathfrak{a}_n \xrightarrow{\tau_{\mathfrak{B}}} \mathfrak{a}_0 \in \mathbf{A}$. Then $\mathbf{F}_{\Phi,\Psi,\Upsilon}$ is called N-continuous if

$$\lim_{\varrho \to \infty} \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{a}_n, \mathfrak{b}, \varrho) = \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{a}_0, \mathfrak{b}, \varrho), \ \forall \mathfrak{b} \in \mathbf{A}.$$

We now define the notion of orthogonality in a NIPS.

Definition 3.11. Let $(\mathbf{A}, \mathbf{F}_{\Phi,\Psi,\Upsilon}, \mathbf{T}_{\mathtt{N}})$ be a NIPS and $\mathfrak{p}, \mathfrak{q} \in \mathbf{A}$. We say that $\mathfrak{p}$ is orthogonal with $\mathfrak{q}$ if $\mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{p}, \mathfrak{q}, \varrho) = \mathcal{V}(\varrho)$ for each $\varrho \in R$ and is represented as $\mathfrak{p} \perp_N \mathfrak{q}$.

Theorem 3.12. *Assume that* $(\mathbf{A}, \mathbf{F}_{\Phi,\Psi,\Upsilon}, \mathbf{T}_{\mathtt{N}})$ *is a NIPS. Orthogonality possesses the subsequent characteristics:*

(i) $0 \perp_N \mathfrak{p}, \forall \mathfrak{p} \in \mathbf{A}$;
(ii) *if* $\mathfrak{p} \perp_N \mathfrak{q} \Rightarrow \mathfrak{q} \perp_N \mathfrak{p}$;
(iii) *if* $\mathfrak{p} \perp_N \mathfrak{p} \Rightarrow \mathfrak{p} = 0$;
(iv) *if* $\mathfrak{p} \perp_N \mathfrak{p}_k, (k = 1, 2, 3, ..., n)$, *then* $\mathfrak{p} \perp_N (\Sigma_{k=1}^n \mathfrak{p}_k)$;

(v) *if* $\mathfrak{p} \perp_N \mathfrak{q}$ *then* $\mathfrak{p} \perp_N \alpha\mathfrak{q}, \forall \alpha \in R$;

(vi) *let* $\mathbf{F}_{\Phi,\Psi,\Upsilon}$ *be a N-continuous. If* $\mathfrak{p}_n \xrightarrow{\tau_{\mathfrak{B}}} \mathfrak{p}$ *and* $\mathfrak{q} \perp_N \mathfrak{p}_n (n = 1, 2, 3, ...)$, *then* $\mathfrak{q} \perp_N \mathfrak{p}$.

Proof. (i)- (iii) are obtained directly from conditions (NIPS-4), (NIPS-2) and (NIPS-3), respectively.

(iv)

$$\begin{aligned}\mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{p}, \Sigma_{k=1}^{n}\mathfrak{p}_k, \varrho) &= (\mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{p}, \mathfrak{p}_1, .) \oplus \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{p}, \mathfrak{p}_2, .) \\ &\quad \oplus ... \oplus \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{p}, \mathfrak{p}_n, .))(\varrho) \\ &= (\mathcal{V} \oplus \mathcal{V} \oplus ... \oplus \mathcal{V})(\varrho) = \mathcal{V}(\varrho),\ \forall \varrho \in R.\end{aligned}$$

(v) For $\mathfrak{p} \perp_N \mathfrak{q}$,
if $\alpha > 0$, we have, $\mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{p}, \alpha\mathfrak{q}, \varrho) = \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{p}, \mathfrak{q}, \frac{\varrho}{\alpha}),\ (\forall \varrho \in R)$,
if $\alpha = 0$, we have $\mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{p}, \alpha\mathfrak{q}, \varrho) = \mathcal{V}(\varrho),\ (\forall \varrho \in R)$,
if $\alpha < 0$, we have

$$\mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{p}, \alpha\mathfrak{q}, \varrho) = \mathscr{E}_s(\mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{p}, \mathfrak{q}, \frac{\varrho}{\alpha})) = \mathscr{E}_s(\mathcal{V}(\frac{\varrho}{\alpha})) = \begin{cases} 0 & \varrho \leq 0, \\ 1 & \varrho > 0. \end{cases}$$

Hence, $\mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{p}, \alpha\mathfrak{q}, \varrho) = \mathcal{V}(\varrho)$ for all $\varrho \in R$.

(vi) Since $\mathfrak{q} \perp_{\mathfrak{B}} \mathfrak{p}_n(\mathfrak{n} = 1, 2, 3, ...)$ *and* $\mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{p}, \mathfrak{q}, .)$ is continuous on $R \diagup \{0\}$ and by (NIPS-1), we deduce that

$$\mathcal{V}(\varrho) = \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{p}, \mathfrak{q}, \varrho),\ (\varrho \in R).$$

Hence $\mathfrak{q} \perp_N \mathfrak{p}$. □

Theorem 3.13. (Pythagorean theorem in NIPS) *Assume that* $(\mathbf{A}, \mathbf{F}_{\Phi,\Psi,\Upsilon}, \mathbf{T}_{\mathtt{N}})$ *is a NIPS and let* $\mathfrak{p} \perp_N \mathfrak{q}$. *Then*

$$\mathbf{B}_{\phi,\psi,\Upsilon}(\mathfrak{p} + \mathfrak{q}, \varrho) = \mathbf{T}_{\mathtt{N}}(\mathbf{B}_{\phi,\psi,\Upsilon}(\mathfrak{p}, \varrho), \mathbf{B}_{\phi,\psi,\Upsilon}(\mathfrak{q}, \varrho)).$$

Proof. For $(\forall \varrho_1, \varrho_2 \in R)$,

$$\begin{aligned}\mathbf{B}_{\phi,\psi,\Upsilon}(\mathfrak{p} + \mathfrak{q}, \varrho) &= \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{p} + \mathfrak{q}, \mathfrak{p} + \mathfrak{q}, \varrho^2) \\ &= \sup_{\varrho_1+\varrho_2=\varrho^2} [\mathbf{T}_{\mathtt{N}}(\mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{p}, \mathfrak{p} + \mathfrak{q}, \varrho_1), \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{q}, \mathfrak{p} + \mathfrak{q}, \varrho_2))] \\ &= \sup_{\varrho_1+\varrho_2=\varrho^2} \mathbf{T}_{\mathtt{N}}[\sup_{\varrho_1^{\grave{}}+\varrho_2^{\grave{}}=\varrho_1} \mathbf{T}_{\mathtt{N}}(\mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{p}, \mathfrak{p}, \varrho_1^{\grave{}}), \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{p}, \mathfrak{q}, \varrho_2^{\grave{}})), \\ &\qquad \sup_{\varrho_1^{\grave{}\grave{}}+\varrho_2^{\grave{}\grave{}}=\varrho_2} \mathbf{T}_{\mathtt{N}}(\mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{q}, \mathfrak{p}, \varrho_1^{\grave{}\grave{}}), \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{q}, \mathfrak{q}, \varrho_2^{\grave{}\grave{}}))].\end{aligned}$$

$$\mathbf{B}_{\phi,\psi,\Upsilon}(\mathfrak{p}+\mathfrak{q},\varrho) = \sup_{\varrho_1+\varrho_2=\varrho^2} \mathbf{T}_{\mathbb{N}}[\sup_{\varrho_1^{`}+\varrho_2^{`}=\varrho_1} \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{p},\mathfrak{p},\varrho_1^{`}), \sup_{\varrho_1^{``}+\varrho_2^{``}=\varrho_2} \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{q},\mathfrak{q},\varrho_2^{``})]$$

$$\leq_{\mathbf{G}^*} \sup_{\varrho_1+\varrho_2=\varrho^2} \mathbf{T}_{\mathbb{N}}[\mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{p},\mathfrak{p},\varrho_1), \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{q},\mathfrak{q},\varrho_2)].$$

By using Lemma 3.5, (NIP-1) and Theorem (3.4),

$$\mathbf{B}_{\phi,\psi,\Upsilon}(\mathfrak{p}+\mathfrak{q},\varrho) = \mathbf{T}_{\mathbb{N}}(\mathbf{B}_{\phi,\psi,\Upsilon}(\mathfrak{p},\varrho), \mathbf{B}_{\mathbb{T},\mathbb{I},\mathbb{F}}(\mathfrak{q},\varrho)),\ \forall \varrho \in R.$$

□

Remark 3.14. For notational convenience, we shall treat the operations on $\mathbf{G}^*$ as if they were the standard (ordinary) operations: Replacing $>_{\mathbf{G}^*}, \geq_{\mathbf{G}^*}$ $, <_{\mathbf{G}^*}, \leq_{\mathbf{G}^*}$ *with* $>, \geq, <, \leq$ and replacing $1_{\mathbf{G}^*}, 0_{\mathbf{G}^*}$ with $1, 0$ respectively.

Theorem 3.15. *Let* $(\mathbf{A}, \mathbf{F}_{\Phi,\Psi,\Upsilon}, \mathbf{T}_{\mathbb{N}})$ *be a NIPS with a continuous RNTN* $\mathbf{T}_{\mathbb{N}}$ *and for each* $\mathfrak{p},\mathfrak{q} \in \mathbf{A}$,

$$\sup\{\varrho \in R : \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{p},\mathfrak{B},\varrho) < 1\} < \infty.$$

Let $< .,. >_{\Phi,\Psi,\Upsilon} : \mathbf{A}\times\mathbf{A} \longrightarrow R$ *be defined as*

$$< \mathfrak{p},\mathfrak{q} >_{\Phi,\Psi,\Upsilon} = \sup\{\varrho \in R : \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{p},\mathfrak{q},\varrho) < 1\}.$$

Then $(\mathbf{A}, < .,. >_{\Phi,\Psi,\Upsilon})$ *is an inner product space.*

Proof. According to (NIPS-2) and $< .,. >_{\Phi,\Psi,\Upsilon}$ is commutative and by (NIPS-4), we have

$$< \alpha\mathfrak{p},\mathfrak{q} >_{\Phi,\Psi,\Upsilon} = \alpha < \mathfrak{p},\mathfrak{q} >_{\Phi,\Psi,\Upsilon},\ \forall \mathfrak{p},\mathfrak{q} \in \mathbf{A} \text{ and } \alpha \in R.$$

By applying Lemma 3.5 and (NIPS-1), it follows that

$$< \mathfrak{p},\mathfrak{p} >_{\Phi,\Psi,\Upsilon} \geq 0,\ \forall \mathfrak{p},\mathfrak{q} \in \mathbf{A}.$$

Assume now that $\sup\{\varrho \in R : \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{p},\mathfrak{p},\varrho) < 1\} = 0,\ \forall \varrho > 0.$ Then

$$\mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{p},\mathfrak{p},\varrho) = 1.$$

Using the non-decreasing characteristic of $\mathbf{F}_{\Phi,\Psi,\Upsilon}$ and (NIPS-3), then $\mathfrak{p} = 0$.

On the other hand, let's say that $\mathfrak{p} = 0$. By (NIPS-4)

$$\sup\{\varrho \in R : \mathbf{F}_{\Phi,\Psi,\Upsilon}(0,0,\varrho) < 1\} = \sup\{\varrho \in R, \mathcal{V}(\varrho) < 1\} = 0.$$

Lastly, for every $\mathfrak{p},\mathfrak{q},\mathfrak{r} \in \mathbf{A}$, we prove that

$$< \mathfrak{p}+\mathfrak{q},\mathfrak{r} >_{\Phi,\Psi,\Upsilon} = < \mathfrak{p},\mathfrak{r} >_{\Phi,\Psi,\Upsilon} + < \mathfrak{q},\mathfrak{r} >_{\Phi,\Psi,\Upsilon}.$$

By (NIPS-5), for any $\lambda > 0$,

$$\mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{p}+\mathfrak{q}.\mathfrak{r}, < \mathfrak{p},\mathfrak{r} > + < \mathfrak{q},\mathfrak{r} > + \lambda) \geq \mathbf{T}_{\mathbb{N}}\Big(\mathbf{F}_{\Phi,\Psi,\Upsilon}\Big(\mathfrak{p},\mathfrak{r}, < \mathfrak{p},\mathfrak{r} > + \frac{\lambda}{2}\Big),$$

$$\mathbf{F}_{\Phi,\Psi,\Upsilon}\Big(\mathfrak{q},\mathfrak{r}, < \mathfrak{q},\mathfrak{r} > + \frac{\lambda}{2}\Big)\Big).$$

This mean that

$$\sup\{\varrho \in R : \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{p}+\mathfrak{q},\mathfrak{r},\varrho) < 1\} \leq \sup\{\varrho \in R : \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{p},\mathfrak{r},\varrho) < 1\} + \sup\{\varrho \in R : \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{q},\mathfrak{r},\varrho) < 1\} + \lambda.$$

Since λ is arbitrary, it is an indication that

$$< \mathfrak{p}+\mathfrak{q},\mathfrak{r} >_{\Phi,\Psi,\Upsilon} \leq < \mathfrak{p},\mathfrak{r} >_{\Phi,\Psi,\Upsilon} + < \mathfrak{q},\mathfrak{r} >_{\Phi,\Psi,\Upsilon}. \tag{3.1}$$

Meanwhile, if we choose

$$A = 1 - \mathbf{T}_{\mathtt{N}}\Big[\mathscr{E}_s(\mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{p},\mathfrak{r},< \mathfrak{p},\mathfrak{r} > -\frac{\lambda}{2})), \mathscr{E}_s(\mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{q},\mathfrak{r},< \mathfrak{q},\mathfrak{r} > -\frac{\lambda}{2}))\Big],$$

then, from (NIPS-4) and (NIPS-5), we have

$$\begin{aligned} A =& 1 - \mathbf{T}_{\mathtt{N}}\Big[\mathbf{F}_{\Phi,\Psi,\Upsilon}(-\mathfrak{p},\mathfrak{r},< \mathfrak{p},\mathfrak{r} > -\frac{\lambda}{2}), \mathbf{F}_{\Phi,\Psi,\Upsilon}(-\mathfrak{q},\mathfrak{r},< \mathfrak{q},\mathfrak{r} > -\frac{\lambda}{2})\Big] \\ \geq& \Big(\mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{p}+\mathfrak{q},\mathfrak{r},< \mathfrak{p},\mathfrak{r} > + < \mathfrak{q},\mathfrak{r} > -\lambda\Big). \end{aligned}$$

Since $\mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{p},\mathfrak{r},< \mathfrak{p},\mathfrak{r} > -\frac{\lambda}{2}) < 1, \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{q},\mathfrak{r},< \mathfrak{q},\mathfrak{r} > -\frac{\lambda}{2}) < 1$, and by (NT-3) in definition, we have

$$\Big(\mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{p}+\mathfrak{q},\mathfrak{r},< \mathfrak{p},\mathfrak{r} > + < \mathfrak{q},\mathfrak{r} > -\lambda)\Big) < 1,\ \forall \mathfrak{p},\mathfrak{q} \in \mathbf{A},$$

and this implies that

$$\sup\{\varrho \in R : \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{p}+\mathfrak{q},\mathfrak{r},\varrho) < 1\} \geq \sup\{\varrho \in R : \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{p},\mathfrak{r},\varrho) < 1\} + \sup\{t \in R : \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{q},\mathfrak{r},t) < 1\}.$$

Then

$$< \mathfrak{p}+\mathfrak{q},\mathfrak{r} >_{\Phi,\Psi,\Upsilon} \geq < \mathfrak{p},\mathfrak{r} >_{\Phi,\Psi,\Upsilon} + < \mathfrak{q},\mathfrak{r} >_{\Phi,\Psi,\Upsilon}. \tag{3.2}$$

From (3.1) and (3.2), we get

$$< \mathfrak{p}+\mathfrak{q},\mathfrak{r} >_{\Phi,\Psi,\Upsilon} = < \mathfrak{p},\mathfrak{r} >_{\Phi,\Psi,\Upsilon} + < \mathfrak{q},\mathfrak{r} >_{\Phi,\Psi,\Upsilon}.$$

Therefore, $(\mathbf{A}, < .,. >_{\Phi,\Psi,\Upsilon})$ is an inner product space. □

Corollary 3.16. *Let* $(\mathbf{A}, \mathbf{F}_{\Phi,\Psi,\Upsilon}, \mathbf{T}_{\mathtt{N}})$ *be a NIPS with a continuous RNTN* $\mathbf{T}_{\mathtt{N}}$ *and for each* $\mathfrak{p},\mathfrak{q} \in \mathbf{A}$ $\sup\{\varrho \in R : \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{p},\mathfrak{q},\varrho) < 1\} < \infty$*. If we define*

$$\|\mathfrak{p}\|_{\Phi,\Psi,\Upsilon} = \Big[\sup\{\varrho \in R : \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{p},\mathfrak{q},\varrho) < 1\}\Big]^{\frac{1}{2}},$$

then $\Big(\mathbf{A}, \|.\|_{\Phi,\Psi,\Upsilon}\Big)$ *is a normed space.*

Lemma 3.17. *Let $(\mathbf{A}, < .,. >)$ be a Hilbert space and $\|.\|$ induced norm by the Hilbert space. Suppose that $(\mathbf{A}, \mathbf{F}_{\Phi,\Psi,\Upsilon}, \mathbf{T}_{\mathtt{N}})$ is a NIPS, where*

$$\mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{a}, \mathfrak{b}, \varrho) = \Big(\frac{\varrho}{\varrho + |< \mathfrak{a}, \mathfrak{b} >|}, \frac{|< \mathfrak{a}, \mathfrak{b} >|}{\varrho + |< \mathfrak{a}, \mathfrak{b} >|}, \frac{|< \mathfrak{a}, \mathfrak{b} >|}{\varrho}\Big), \forall \mathfrak{a}, \mathfrak{b} \in \mathbf{A}, \forall \varrho \in R.$$

Then for each $\{\mathfrak{a}_n\} \subseteq \mathbf{A}$, the following assertions are equivalent

(1) $\lim_{n\to\infty} \|\mathfrak{a}_n - \mathfrak{a}\| = 0$;
(2) $\lim_{n\to\infty} \|\mathfrak{a}_n - \mathfrak{a}\|_{\Phi,\Psi,\Upsilon} = 0$.

Proof. Based on the definition, we obtain

$$\begin{aligned}\|\mathfrak{a}_n - \mathfrak{a}\|^2_{\Phi,\Psi,\Upsilon} &= \sup\{\varrho \in R, \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{a}_n - \mathfrak{a}, \mathfrak{a}_n - \mathfrak{a}, \varrho) < 1\} \\ &= \max\{\varrho \in R, \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{a}_n - \mathfrak{a}, \mathfrak{a}_n - \mathfrak{a}, \varrho)\} \\ &= < \mathfrak{a}_n - \mathfrak{a}, \mathfrak{a}_n - \mathfrak{a} > \\ &= \|\mathfrak{a}_n - \mathfrak{a}\|^2.\end{aligned}$$

□

Definition 3.18. Let $(\mathbf{A}, \mathbf{F}_{\Phi,\Psi,\Upsilon}, \mathbf{T}_{\mathtt{N}})$ be a NIPS with a continuous RNTN $\mathbf{T}_{\mathtt{N}}$ and for each $\mathfrak{p}, \mathfrak{q} \in \mathbf{A}$ $\sup\{t \in R : \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{p}, \mathfrak{q}, \varrho) < 1\} < \infty$ such that

$$\|\mathfrak{p}\|_{\Phi,\Psi,\Upsilon} = \Big[\sup\{\varrho \in R : \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{p}, \mathfrak{q}, \varrho) < 1\}\Big]^{\frac{1}{2}}.$$

We say that $\Big(\mathbf{A}, \mathbf{F}_{\Phi,\Psi,\Upsilon}, \mathbf{T}_{\mathtt{N}}\Big)$ is a neutrosophic Hilbert space (briefly, NHS) if $\Big(\mathbf{A}, \|.\|_{\Phi,\Psi,\Upsilon}\Big)$ is a complete normed space.

Theorem 3.19. *Let $(\mathbf{A}, \mathbf{F}_{\Phi,\Psi,\Upsilon}, \mathbf{T}_{\mathtt{N}})$ be a NHS with inner product*

$$< \mathfrak{p}, \mathfrak{q} >_{\Phi,\Psi,\Upsilon} = \sup\{\varrho \in R : \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{p}, \mathfrak{q}, \varrho) < 1\}, \ \forall \mathfrak{p}, \mathfrak{q} \in \mathbf{A}.$$

Then a sequence $\{\mathfrak{p}_n\}$ in $\mathbf{A}$ is $\tau_{\mathfrak{B}}$-convergent to $\mathfrak{p}$ ($\mathfrak{p}_n \xrightarrow{\tau_{\mathfrak{B}}} \mathfrak{p}$) if $\mathfrak{p}_n \xrightarrow{\|\cdot\|_{\Phi,\Psi,\Upsilon}} \mathfrak{p}$.

Proof. Since $\mathfrak{p}_n \xrightarrow{\|\cdot\|_{\Phi,\Psi,\Upsilon}} \mathfrak{p}$, then $\lim_{n\longrightarrow\infty} \|\mathfrak{p}_n - \mathfrak{p}\|_{\Phi,\Psi,\Upsilon} = 0$. This implies that

$$\lim_{n\longrightarrow\infty} < \mathfrak{p}_n - \mathfrak{p}, \mathfrak{p}_n - \mathfrak{p} >_{\Phi,\Psi,\Upsilon} = 0.$$

Hence, we have

$$\lim_{n\longrightarrow\infty} \sup\{\varrho \in R : \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{p}_n - \mathfrak{p}, \mathfrak{p}_n - \mathfrak{p}, \varrho) < 1\} = 0.$$

Therefore, for $\epsilon > 0$ and $0 < \lambda < 1$,

$$\begin{aligned}
&\sup\{\varrho \in R : \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{p}_n - \mathfrak{p}, \mathfrak{p}_n - \mathfrak{p}, \varrho) < 1\} \\
&\quad = \sup\{\varrho \in [0,\epsilon] : \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{p}_n - \mathfrak{p}, \mathfrak{p}_n - \mathfrak{p}, \varrho) < 1\} \\
&\qquad + \sup\{\varrho \in (\epsilon,\infty) : \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{p}_n - \mathfrak{p}, \mathfrak{p}_n - \mathfrak{p}, \varrho) < 1\} \\
&\quad \geq \epsilon \sup\{\varrho \in (\epsilon,\infty) : \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{p}_n - \mathfrak{p}, \mathfrak{p}_n - \mathfrak{p}, \varrho)\} \\
&\quad = \epsilon\Big(1 - \mathfrak{q}_{\Phi,\Psi,\Upsilon}(\mathfrak{p}_n - \mathfrak{p}, \mathfrak{p}_n - \mathfrak{p}, \varrho, \epsilon)\Big).
\end{aligned}$$

□

Theorem 3.20. (Rieszs theorem for NHS) *Let* $(\mathbf{A}, \mathbf{F}_{\Phi,\Psi,\Upsilon}, \mathbf{T}_{\mathtt{N}})$ *be a NHS with* $\tau_{\mathfrak{B}}$. *If f is a linear functional on* $\mathbf{A}$ *that is continuous relative to* $\tau_{\mathfrak{B}}$, *then there exists exactly one element* $\mathfrak{q} \in \mathbf{A}$ *such that for all* $\mathfrak{p} \in \mathbf{A}$,

$$f(\mathfrak{p}) = \sup\{\varrho \in R : \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{p}, \mathfrak{q}, \varrho) < 1\}.$$

Proof. Let $\mathfrak{p}_n \xrightarrow{\tau_{\mathfrak{B}}} \mathfrak{p}$. Then $f(\mathfrak{p}_n) \xrightarrow{\tau_{\mathfrak{B}}} f(\mathfrak{p})$ for all $\{\mathfrak{p}_n\} \in \mathbf{A}$, so f is a continuous linear functional at $\mathfrak{p}(\tau_{\mathfrak{B}} - cont)$.
If $\mathfrak{p}_n \xrightarrow{\|.\|_{\Phi,\Psi,\Upsilon}} \mathfrak{p}$ then by theorem (3.19), $f(\mathfrak{p}_n) \xrightarrow{\tau_{\mathfrak{B}}} f(\mathfrak{p})$. So f continuous on NHS like an ordinary Hilbert space. Therefore, by the Riesz representation theorem, the proof is complete. □

4. Common fixed point theorems

In this section, we will attempt to obtain some common fixed point theorems in NHS. However, we firstly attempt to prove an important theorem in the sequel and of which is true in the Banach space.

Theorem 4.1. ([9]) *Suppose that ϕ and φ be two continuous self-mappings (CSM) that fulfil the following requirement on the Banach space* $(\mathbf{A}, \|.\|)$

$$\sum_{k=0}^{\infty} \|\phi^k \mathfrak{a} - \varphi^k \mathfrak{b}\| < \infty,\ \forall \mathfrak{a}, \mathfrak{b} \in \mathbf{A}.$$

Then there exists an element $\mathfrak{a}^* \in \mathbf{A}$ *such that:*

1. *The mappings ϕ and φ have a single common fixed point(CFP)* $\mathfrak{a}^*$.
2. *The iterative sequences* $\{\phi^k\}_{k=0}^{\infty}$ *and* $\{\varphi^k\}_{k=0}^{\infty}$ *converge to the same point* $\mathfrak{a}^*$ *for every* $\mathfrak{a}_0 \in \mathbf{A}$.

Corollary 4.2. ([9]) *Let* $(\mathbf{A}, \|.\|)$ *be a Banach space and* $\{\phi_i\}_{i=1}^{\infty}$ *be a sequence of CSMs on* $\mathbf{A}$. *Assume that for all $i, j \in Z^+$ and $i \neq j$, the condition holds;*

$$\sum_{k=0}^{\infty} \|\phi_i^k \mathfrak{a} - \varphi_j^k \mathfrak{b}\| < \infty,\ \forall \mathfrak{a}, \mathfrak{b} \in \mathbf{A}.$$

Then there exists an element $\mathfrak{a}^* \in \mathbf{A}$ *such that*

(1) *The sequence* $\{\phi_i\}_{i=1}^{\infty}$ *has a single CFP* $\mathfrak{a}^*$.
(2) *The iterative sequence* $\{\phi_k^k \mathfrak{a}_0\}_{i=1}^{\infty}$ *converge to the same point* $\mathfrak{a}^*$, *for every* $\mathfrak{a}_0 \in \mathbf{A}$ *and any positive integer* i.

Definition 4.3. Let ϕ be a self-mapping on the NHS $(\mathbf{A}, \mathbf{F}_{\Phi,\Psi,\Upsilon}, \mathbf{T}_{\mathtt{N}})$. We say that ϕ is a continuous mapping if $\mathfrak{a}_n \to \mathfrak{a}$ implies that $\phi\mathfrak{a}_n \to \phi\mathfrak{a}$.

Theorem 4.4. *Let* $(\mathbf{A}, \mathbf{F}_{\Phi,\Psi,\Upsilon}, \mathbf{T}_{\mathtt{N}})$ *be a NHS and* $\phi, \varphi : \mathbf{A} \to \mathbf{A}$ *be self-mappings on the NHS. Assume that there exists an element* $\alpha \in (0, 1)$ *such that*

$$\mathbf{F}_{\Phi,\Psi,\Upsilon}(\phi\mathfrak{a} - \varphi\mathfrak{b}, \phi\mathfrak{a} - \varphi\mathfrak{b}, \varrho) \geq \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{a} - \mathfrak{b}, \mathfrak{a} - \mathfrak{b}, \frac{\varrho}{\alpha}), \forall \mathfrak{a}, \mathfrak{b} \in \mathbf{A} \text{ and } \forall \varrho \in R. \tag{4.1}$$

Then the results stated in the Theorem 4.1 *are satisfied.*

Proof. By using (4.1), for every $\mathfrak{a}, \mathfrak{b} \in \mathbf{A}$ *and* $\varrho \in R$, we get

$$\begin{aligned} &\sup\{\varrho \in R, \mathbf{F}_{\Phi,\Psi,\Upsilon}(\phi\mathfrak{a} - \varphi\mathfrak{b}, \phi\mathfrak{a} - \varphi\mathfrak{b}, \varrho) < 1\} \\ &\quad \leq \sup\{\varrho \in R, \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{a} - \mathfrak{b}, \mathfrak{a} - \mathfrak{b}, \frac{\varrho}{\alpha}) < 1\} \\ &\quad = \alpha \sup\{\varrho \in R, \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{a} - \mathfrak{b}, \mathfrak{a} - \mathfrak{b}, \varrho) < 1\}. \end{aligned} \tag{4.2}$$

Thus, for any $k \in N$, it follows that

$$\begin{aligned} &\sup\{\varrho \in R, \mathbf{F}_{\Phi,\Psi,\Upsilon}(\phi\mathfrak{a} - \varphi\mathfrak{b}, \phi\mathfrak{a} - \varphi\mathfrak{b}, \varrho) < 1\} \\ &\quad \leq \alpha^k \sup\{\varrho \in R, \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{a} - \mathfrak{b}, \mathfrak{a} - \mathfrak{b}, \varrho) < 1\}. \end{aligned}$$

This implies that

$$\sum_{k=0}^{\infty} \|\phi^k\mathfrak{a} - \varphi^k\mathfrak{b}\| \leq \alpha^{\frac{k}{2}} \sum_{k=0}^{\infty} \|\mathfrak{a} - \mathfrak{b}\| < \infty.$$

Through the application of equation (4.2) together with Corollary 3.16, it can be deduced that

$$\begin{aligned} \|\phi\mathfrak{a} - \phi\mathfrak{b}\| &\leq \|\phi\mathfrak{a} - \varphi\mathfrak{a}\| + \|\varphi\mathfrak{a} - \phi\mathfrak{b}\| \\ &\leq (\sup\{\varrho \in R, \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{a} - \mathfrak{a}, \mathfrak{a} - \mathfrak{a}, \frac{\varrho}{\alpha}) < 1\})^{\frac{1}{2}} \\ &\quad + (\sup\{\varrho \in R, \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{a} - \mathfrak{b}, \mathfrak{a} - \mathfrak{b}, \frac{\varrho}{\alpha}) < 1\})^{\frac{1}{2}} \\ &= 0 + \alpha\|\mathfrak{a} - \mathfrak{b}\|. \end{aligned}$$

Consequently, ϕ is a CSM on $(\mathbf{A}, \mathbf{F}_{\Phi,\Psi,\Upsilon}, \mathbf{T}_{\mathtt{N}})$. Applying the same proof, we deduce that φ is also CSM on $(\mathbf{A}, \mathbf{F}_{\Phi,\Psi,\Upsilon}, \mathbf{T}_{\mathtt{N}})$. Thus the proof is complete. □

Remark 4.5. In contrast to the normal case, the above theorem does not need continuity, even though condition (4.1) indicates that ϕ is $\tau_{\mathfrak{B}}$-continuous.

Corollary 4.6. *Let ϕ, φ be two self-mappings on $(\mathbf{A}, \mathbf{F}_{\Phi,\Psi,\Upsilon}, \mathbf{T}_{\mathtt{N}})$ and a sequence $\{\alpha_k\}_{k=0}^{\infty}$ of positive constants under the condition that $\Sigma_{k=0}^{\infty}\alpha_k^{\frac{1}{2}} < \infty$. Assume that for any $k \in N, k = 0, 1, 2, ...,$*

$$\mathbf{F}_{\Phi,\Psi,\Upsilon}(\phi^k\mathfrak{a}-\varphi^k\mathfrak{b}, \phi^k\mathfrak{a}-\varphi^k\mathfrak{b}, \varrho) \geq \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{a}-\mathfrak{b}, \mathfrak{a}-\mathfrak{b}, \frac{\varrho}{\alpha}), \forall \mathfrak{a}, \mathfrak{b} \in \mathbf{A} \text{ and } \forall \varrho \in R. \tag{4.3}$$

Then the result stated in the Theorem 4.1 is satisfied.

Proof. We can show that ϕ *and* φ are CSMs on $(\mathbf{A}, \mathbf{F}_{\Phi,\Psi,\Upsilon}, \mathbf{T}_{\mathtt{N}})$ using the same proof as in the above theorem and equation (4.3) (for $k = 1$). Additionally, for every $\mathfrak{a}, \mathfrak{b} \in \mathbf{A}$, we have

$$\begin{aligned}\sum_{k=0}^{\infty}\|\phi^k\mathfrak{a} - \varphi^k\mathfrak{b}\| &\leq \sum_{k=0}^{\infty}(\sup\{\varrho \in R, \mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{a}-\mathfrak{b}, \mathfrak{a}-\mathfrak{b}, \frac{\varrho}{\alpha_k}) < 1\})^{\frac{1}{2}} \\ &= \Sigma_{k=0}^{\infty}\alpha_k^{\frac{1}{2}}\|\mathfrak{a}-\mathfrak{b}\| < \infty.\end{aligned}$$

Thus, the proof is completed. □

5. Applications

Based on the results obtained in Section 4, we study the solvability of the following Uryson integral equation in terms of existence and uniqueness on $\mathfrak{L}^2(E)$

$$\mathfrak{a}(\sigma) = \mathfrak{b}(\sigma) + \int_E \mathbf{K}(\sigma, \varrho, \mathfrak{a}(\varrho))d\varrho \tag{5.1}$$

such that $E \subseteq R^n$ is bounded and closed, $\mathfrak{L}^2(E)$ is the collection of all functions on E whose squares are Lebesgue integrable, equipped with the usual vector space structure.

Theorem 5.1. *Assume that $\mathbf{K}(\sigma, \varrho, \mathfrak{v})(\sigma, \varrho \in E, -\infty < \mathfrak{v} < +\infty)$ is separable and $\mathbf{K}(\sigma, \varrho, \mathfrak{v}) = k(\sigma, \varrho)\mathfrak{v}$. Assume further that $\int_E |k(\sigma, \varrho)|^2 d\sigma d\varrho < 1$. Let ϕ be a self-mapping on $\mathfrak{L}^2(E)$ given by*

$$\phi\mathfrak{a}(\sigma) = \mathfrak{b}(\sigma) + \int_E \mathbf{K}(\sigma, \varrho, \mathfrak{a}(\varrho))d\varrho \tag{5.2}$$

with the following condition

$$\sum_{n=0}^{\infty}\|\phi^n\mathfrak{a} - \phi^n\mathfrak{v}\|_{\mathfrak{L}^2(E)} < \infty, \ \forall \mathfrak{a}(\sigma), \mathfrak{v}(\sigma) \in \mathfrak{L}^2(E). \tag{5.3}$$

Then ϕ has a unique fixed-point $\mathfrak{a}^*(\sigma) \in \mathfrak{L}^2(E)$ which represents the unique solution for equation (5.1), *for any $\mathfrak{b}(\sigma) \in \mathfrak{L}^2(E)$ and any $\mathfrak{a}_0(\sigma) \in \mathfrak{L}^2(E)$, and the iterative sequence*

$$\mathfrak{a}_{n+1}(\sigma) = \mathfrak{b}(\sigma) + \int_E \mathbf{K}(\sigma, \varrho, \mathfrak{a}_n(\varrho))d\varrho, \ (n = 0, 1, 2, ...),$$

converges to a point $\mathfrak{a}^*(\sigma)$ in norm $\|.\|_{\mathfrak{L}^2(E)}$.

Proof. Let us define

$$\mathbf{F}_{\Phi,\Psi,\Upsilon}(\mathfrak{a}, \mathfrak{v}, \varrho) = \Big(\frac{\varrho}{\varrho + \int_E \mathfrak{a}(\sigma)\mathfrak{v}(\sigma)d\sigma}, \frac{\int_E \mathfrak{a}(\sigma)\mathfrak{v}(\sigma)d\sigma}{\varrho + \int_E \mathfrak{a}(\sigma)\mathfrak{v}(\sigma)}, \frac{\int_E \mathfrak{a}(\sigma)\mathfrak{v}(\sigma)d\sigma}{\varrho}\Big)$$

for all $\varrho \in R$ and for all $\mathfrak{a}(\sigma), \mathfrak{v}(\sigma) \in \mathfrak{L}^2(E)$, then by using Lemma 3.17, we prove that $(\mathfrak{L}^2(E), \mathbf{F}_{\Phi,\Psi,\Upsilon}, \mathbf{T}_{\mathtt{N}})$ is an NHS and

$$\|\mathfrak{a} - \mathfrak{v}\|_{\Phi,\Psi,\Upsilon} = \|\mathfrak{a} - \mathfrak{v}\|_{\mathfrak{L}^2(E)}, \ \forall \mathfrak{a}, \mathfrak{v} \in \mathfrak{L}^2(E). \tag{5.4}$$

It also follows from equations (5.3) and (5.4), we obtain

$$\sum_{n=0}^{\infty} \|\phi^n\mathfrak{a} - \phi^n\mathfrak{v}\|_{\Phi,\Psi,\Upsilon} = \sum_{n=0}^{\infty} \|\phi^n\mathfrak{a} - \phi^n\mathfrak{v}\|_{\mathfrak{L}^2(E)} < \infty.$$

Therefore, taking $\phi = \varphi$, we satisfy all the conditions for Theorem 4.1, which then ensures the presence and distinction of a fixed-point $\mathfrak{a}^*(\sigma) \in \mathfrak{L}^2(E)$. Hence, equation (5.1) admits a unique solution $\mathfrak{a}^*(\sigma)$. In addition for each given $\mathfrak{a}_0(\sigma) \in \mathfrak{L}^2(E)$, the iterative sequence

$$\mathfrak{a}_{n+1}(\sigma) = \mathfrak{b}(\sigma) + \int_E \mathbf{K}(\sigma, \varrho, \mathfrak{a}_n(\varrho))d\varrho, \ (n = 0, 1, 2, ...),$$

converges to a point $\mathfrak{a}^*(\sigma)$ in $\|.\|_{\mathfrak{L}^2(E)}$. This completes the proof. □

6. Conclusion

In this paper, a formulation of neutrosophic inner product spaces is introduced under less restrictive conditions, while remaining consistent with the classical framework. Based on this formulation, a flexible construction of neutrosophic Hilbert spaces is established, preserving the essential features of their classical counterparts and allowing for further generalizations.

The proposed setting provides a convenient framework for simplifying the proofs of several fundamental results associated with classical Hilbert spaces. In particular, it supports research on the fixed-point theory and leads to results that may be of interest in various applied contexts.

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