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CHARACTERIZATIONS OF OPERATORS INVOLVING THE GENERALIZED IMAGINARY ERROR FUNCTION IN ANALYTIC FUNCTION FAMILIES

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Abstract. This paper presents novel characterization results for a function, a linear operator, and an integral operator constructed using the generalized normalized imaginary error function, with emphasis on their inclusion in two distinct subfamilies of analytic functions defined on the open unit disk.

1. Introduction

Special functions are central in pure and applied mathematics, with significant applications in areas such as probability theory, statistics, and quantum

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mechanics, particularly in geometric function theory (GFT). Among these functions, the classical error function and its various extensions have received considerable attention due to their rich analytical properties and practical effectiveness in modeling diverse phenomena.

Recently, the generalized imaginary error function has emerged as a valuable tool in defining new subfamilies of analytic functions on the open unit disk $\complement := \{\wp \in \mathbb{C} : |\wp| < 1\}$. Within the framework of GFT, such operators have been instrumental in characterizing significant function families, including starlike, convex, and close-to-convex functions of complex order. Through the convolution of the generalized imaginary error function with established families of analytic functions, researchers have derived sufficient conditions for inclusion in certain subfamilies.

These developments extend foundational work by classical GFT scholars such as Pommerenke and Goodman and align with ongoing advancements in special function theory [7, 22]. A growing body of literature explores the role of special functions in establishing inclusion criteria for subfamilies of analytic functions, for examples, [3, 14, 15, 16, 24, 25]. Notably, Abramowitz and Stegun [1] introduced the classical error function $erh(\wp)$, defined by

$$erh\,(\wp) = \frac{2}{\sqrt{\pi}} \int_0^{\wp} e^{-t^2} dt = \frac{2}{\sqrt{\pi}} \sum_{\eth=0}^{\infty} \frac{(-1)^{\eth} \wp^{2\eth+1}}{(2\eth+1)\eth!}, \quad \wp \in \mathbb{C}.$$

Its counterpart, the imaginary error function, is defined as

$$erhi\,(\wp) = \frac{2}{\sqrt{\pi}} \int_0^{\wp} e^{t^2} dt = \frac{2}{\sqrt{\pi}} \sum_{\eth=0}^{\infty} \frac{\wp^{2\eth+1}}{(2\eth+1)\eth!}, \quad \wp \in \mathbb{C}.$$

Generalizations of these functions introduce a positive integer parameter $s \in \mathbb{N}_0$ and are defined, respectively, as

$$erh_s\,(\wp) = \frac{s!}{\sqrt{\pi}} \int_0^{\wp} e^{-t^s} dt = \frac{s!}{\sqrt{\pi}} \sum_{\eth=0}^{\infty} \frac{(-1)^{\eth} \wp^{s\eth+1}}{(s\eth+1)\eth!}, \quad \wp \in \mathbb{C}, \tag{1.1}$$

$$erhi_s\,(\wp) = \frac{s!}{\sqrt{\pi}} \int_0^{\wp} e^{t^s} dt = \frac{s!}{\sqrt{\pi}} \sum_{\eth=0}^{\infty} \frac{\wp^{s\eth+1}}{(s\eth+1)\eth!}, \quad \wp \in \mathbb{C}. \tag{1.2}$$

For specific values of s, these generalizations recover classical forms. In particular,

$$erh_0(\wp) = \tfrac{\wp}{e\sqrt{\pi}}, \quad erh_1(\wp) = \tfrac{1-e^{\wp}}{\sqrt{\pi}} = -erhi_1(\wp), \quad erh_2(\wp) = erh(\wp),$$
$$\text{and} \quad erhi_2(\wp) = erhi(\wp).$$

However, the functions $erh_s(\wp)$ and $erhi_s(\wp)$ do not, in general, belong to the family of normalized analytic functions. Consequently, Frasin [12] introduced the transformed functions:

$$\begin{aligned}\mu_s(\wp) &= \frac{\sqrt{\pi}}{s!}\wp^{1-\frac{1}{s}} erh\left(\wp^{1/s}\right)\\ &= \wp + \sum_{\eth=2}^{\infty} \frac{(-1)^{\eth-1}}{((\eth-1)s+1)(\eth-1)!}\wp^{\eth}, \quad s\in\mathbb{N},\end{aligned} \tag{1.3}$$

$$\begin{aligned}\mu i_s(\wp) &= \frac{\sqrt{\pi}}{s!}\wp^{1-\frac{1}{s}} erhi_s\left(\wp^{1/s}\right)\\ &= \wp + \sum_{\eth=2}^{\infty} \frac{1}{((\eth-1)s+1)(\eth-1)!}\wp^{\eth}, \quad s\in\mathbb{N}.\end{aligned} \tag{1.4}$$

These operators have normalized expansions suitable for function-theoretic analysis. For instance:

$$\mu_1(\wp) = \sqrt{\pi}\cdot erh_1(\wp) = 1 - e^{\wp}, \quad \mu i_1(\wp) = \sqrt{\pi}\cdot erhi_1(\wp) = e^{\wp} - 1,$$
$$\mu_2(\wp) = \frac{\sqrt{\pi\wp}}{2}\cdot erh\left(\sqrt{\wp}\right), \quad \mu i_2(\wp) = \frac{\sqrt{\pi\wp}}{2}\cdot erhi\left(\sqrt{\wp}\right).$$

In the next section, we establish a foundational framework to facilitate our investigation. This includes recalling essential subfamilies of analytic functions along with their generalized forms involving complex parameters. We also introduce a function, a linear convolution operator, and an integral operator defined via the generalized imaginary error function, which plays a central role in deriving our main results. Several supporting lemmas and inequalities are presented to provide the necessary analytical tools for characterizing membership in the targeted function families.

2. Preliminaries

Let $\digamma$ denote the family of analytic functions of the form

$$h(\wp) = \wp - \sum_{\eth=2}^{\infty} a_{\eth}\wp^{\eth}, \quad a_{\eth}\geq 0, \quad \wp\in\complement, \tag{2.1}$$

and $U\digamma\subset\digamma$ represent the subfamily of functions that are univalent in $\complement$, [8].

We recall several geometric subfamilies of $\digamma$ defined via differential subordination involving a complex order $\aleph\in\mathbb{C}^* = \mathbb{C}\setminus\{0\}$:

(1) A function $h\in\digamma$ is said to be *starlike of complex order* $\aleph$ if

$$\operatorname{Re}\left\{1 + \frac{1}{\aleph}\left(\frac{\wp h'(\wp)}{h(\wp)} - 1\right)\right\} > 0, \quad \wp\in\complement.$$

This family, denoted by $S(\aleph)$, was introduced by Nasr and Aouf [18] (see also [19]).

(2) A function $h \in \digamma$ is *convex of complex order* $\aleph$ if

$$\operatorname{Re}\left\{1+\frac{1}{\aleph}\cdot\frac{\wp h''(\wp)}{h'(\wp)}\right\}>0, \quad \wp \in \mathsf{C}.$$

The family of such functions, $C(\aleph)$, was introduced by Wiatrowski [23] (see also [9, 17]).

(3) A function $h \in \digamma$ is *close-to-convex of complex order* $\aleph$ if

$$\operatorname{Re}\left\{1+\frac{1}{\aleph}(h'(\wp)-1)\right\}>0, \quad \wp \in \mathsf{C}.$$

This family, denoted by $R(\aleph)$, was introduced by Halim [13] and Owa [20] (see also Aouf and Mostafa [10]).

More general subfamilies involving parameters $\ell_1, \ell_2 \in [0,1]$ are defined as follows:

- The family $S(\aleph, \ell_1, \ell_2)$ consists of functions $h \in \digamma$ satisfying

$$\left|\frac{1}{\aleph}\left(\frac{\wp h'(\wp)+\ell_1\wp^2 h''(\wp)}{(1-\ell_1)h(\wp)+\ell_1 h'(\wp)}-1\right)\right|<\ell_2. \tag{2.2}$$

- The family $R(\aleph, \ell_1, \ell_2)$ is defined by

$$\left|\frac{1}{\aleph}\left(h'(\wp)+\ell_1\wp h''(\wp)-1\right)\right|<\ell_2. \tag{2.3}$$

These families, introduced and studied by Altintas et al. [5], generalize $S(\aleph)$, $C(\aleph)$, and $R(\aleph)$. Their relationships are summarized in the following remark.

Remark 2.1. The following inclusions and equivalences hold:

(1) $S(\aleph,0,1) \subset S(\aleph), \quad S(\aleph,1,1) \subset C(\aleph), \quad R(\aleph,0,1) \subset R(\aleph).$

(2) $S(\aleph,0,\ell_2) = S(\aleph,\ell_2) = \left\{h \in \digamma : \left|\frac{1}{\aleph}\left(\frac{\wp h'(\wp)}{h(\wp)}-1\right)\right| < \ell_2\right\}.$

(3) $S(\aleph,1,\ell_2) = C(\aleph,\ell_2) = \left\{h \in \digamma : \left|\frac{1}{\aleph}\left(\frac{\wp h''(\wp)}{h'(\wp)}-1\right)\right| < \ell_2\right\}.$

(4) $R(\aleph,0,\ell_2) = R(\aleph,\ell_2) = \left\{h \in \digamma : \left|\frac{1}{\aleph}(h'(\wp)-1)\right| < \ell_2\right\}.$

We also consider the family $G^\tau(Q_1,Q_2)$, introduced by Dixit and Pal [11], defined as follows.

Definition 2.2. ([11]) Let $\tau \in \mathbb{C}\setminus\{0\}$, $-1 \le Q_2 < Q_1 \le 1$. A function $h \in \digamma$ belongs to $G^\tau(Q_1,Q_2)$ if

$$\left|\frac{h'(\wp)-1}{(Q_1-Q_2)\tau - Q_2(h'(\wp)-1)}\right|<1, \quad \wp \in \mathsf{C}.$$

Setting $\tau = 1$, $Q_1 = \varrho$, and $Q_2 = -\varrho$, yields the subfamily of functions satisfying

$$\left|\frac{h'(\wp)-1}{h'(\wp)+1}\right| < \varrho, \quad \wp \in \mathsf{C}, \quad 0 < \varrho \leq 1,$$

which was studied by Padmanabhan [21].

Now, define the function $\Upsilon i_s(\wp)$ as

$$\Upsilon i_s(\wp) = 2\wp - \mu i_s(\wp) = \wp - \sum_{\eth=2}^{\infty} \frac{1}{((\eth-1)s+1)(\eth-1)!}\wp^{\eth}, \quad \wp \in \mathsf{C}, \qquad (2.4)$$

and introduce the linear convolution operator $\mathcal{I}i_s : \digamma \to \digamma$, defined by

$$\mathcal{I}i_s(h)(\wp) = \mu i_s(\wp) * h(\wp) = \wp + \sum_{\eth=2}^{\infty} \frac{1}{((\eth-1)s+1)(\eth-1)!} a_{\eth}\wp^{\eth}. \qquad (2.5)$$

Moreover, we define the Alexander transform (see, [2, 4]) Ki_s of $\Upsilon i_s(\wp)$ by

$$Ki_s(\wp) := \int_0^{\wp} \frac{\Upsilon i_s(t)}{t} dt, \ \wp \in \mathsf{C}.$$

We now state several lemmas that are instrumental for proving the main results of this study.

Lemma 2.3. ([6], with $\eth = 1$) *A sufficient condition for a function h of the form (2.1) to belong to $S(\aleph, \ell_1, \ell_2)$ is*

$$\sum_{\eth=2}^{\infty} \left[\ell_1\eth^2 + (1 + \ell_1(\ell_2|\aleph| - 2))\,\eth + (\ell_2|\aleph| - 1)(1 - \ell_1)\right] |a_{\eth}| \leq \ell_2|\aleph|.$$

Lemma 2.4. ([6], with $\eth = 1$) *A sufficient condition for a function h of the form (2.1) to belong to $R(\aleph, \ell_1, \ell_2)$ is*

$$\sum_{\eth=2}^{\infty} \left[\ell_1\eth + (1 - \ell_1)\right] \eth |a_{\eth}| \leq \ell_2|\aleph|.$$

Lemma 2.5. ([11]) *If $h \in G^{\tau}(Q_1, Q_2)$, then for $\eth \geq 2$,*

$$|a_{\eth}| \leq \frac{(Q_1 - Q_2)|\tau|}{\eth}.$$

This bound is sharp for the function

$$h(\wp) = \int_0^{\wp} \left(1 + \frac{(Q_1 - Q_2)\tau t^{\eth-1}}{1 + Q_2 t^{\eth-1}}\right) dt, \quad \wp \in \mathsf{C}.$$

We will also use the following well-known summation identities:

$$\sum_{\eth=2}^{\infty} \frac{1}{(\eth-1)2^{\eth}} = \frac{1}{2}\ln 2, \tag{2.6}$$

$$\sum_{\eth=3}^{\infty} \frac{1}{(\eth-1)2^{\eth}} = \frac{1}{2}\ln 2 - \frac{1}{4}, \tag{2.7}$$

$$\sum_{\eth=u}^{\infty} \frac{1}{(\eth-1)2^{\eth}} = \frac{1}{2}\ln 2 - \sum_{\eth=2}^{u-1} \frac{1}{(\eth-1)2^{\eth}}, \quad u = 3, 4, \ldots. \tag{2.8}$$

Alongside these, we shall frequently apply the inequalities:

$$(\eth-1)s + 1 > (\eth-1)s, \quad \eth, s \in \mathbb{N}, \tag{2.9}$$

$$\eth! \geq 2^{\eth-1}, \quad \eth \in \mathbb{N}. \tag{2.10}$$

In this study, we aim to establish sufficient conditions for a function, a linear convolution operator, and an Alexander transform constructed using the generalized normalized imaginary error function to belong to the subfamilies $S(\aleph, \ell_1, \ell_2)$ and $R(\aleph, \ell_1, \ell_2)$.

3. Main results

We start this section with the sufficient conditions for the function Υi_s to be in the subfamilies $S(\aleph, \ell_1, \ell_2)$ and $R(\aleph, \ell_1, \ell_2)$.

Theorem 3.1. *If $s \in \mathbb{N}$, then sufficient conditions for the function $\Upsilon i_s(\wp)$ to belong to the family $S(\aleph, \ell_1, \ell_2)$ are*

$$[4\ell_1(\ell_2|\aleph| + 3) + 2(\ell_2|\aleph| + 2)]\ln 2 - 4\ell_1 \leq s\ell_2|\aleph| \tag{3.1}$$

and

$$[(\ell_1+1)(\ell_2|\aleph| + 1) + \ell_1].e \leq \ell_2(s+2)|\aleph| - \ell_1. \tag{3.2}$$

Proof. In light of Lemma 2.3, it suffices to demonstrate that $L_1(\ell_1, \ell_2) \leq \ell_2|\aleph|$, where

$$L_1(\ell_1,\ell_2) = \sum_{\eth=2}^{\infty} \left[\ell_1\eth^2 + (1+\ell_1(\ell_2|\aleph|-2))\eth + (\ell_2|\aleph|-1)(1-\ell_1)\right] \times \frac{1}{((\eth-1)s+1)(\eth-1)!}.$$

Putting

$$\eth = (\eth-1)+1 \tag{3.3}$$

and

$$\eth^2 = (\eth-1)(\eth-2) + 3(\eth-1) + 1, \tag{3.4}$$

we get

$$
\begin{aligned}
L_1(\ell_1,\ell_2) &= \ell_1 \sum_{\eth=2}^{\infty} \frac{(\eth-1)(\eth-2)}{((\eth-1)\,s+1)\,(\eth-1)!} \\
&\quad + (\ell_1\,(\ell_2\,|\aleph|+1)+1) \sum_{\eth=2}^{\infty} \frac{\eth-1}{((\eth-1)\,s+1)\,(\eth-1)!} \\
&\quad + \ell_2\,|\aleph| \sum_{\eth=2}^{\infty} \frac{1}{((\eth-1)\,s+1)\,(\eth-1)!} \\
&= \ell_1 \sum_{\eth=3}^{\infty} \frac{1}{((\eth-1)\,s+1)\,(\eth-3)!} \\
&\quad + (\ell_1\,(\ell_2\,|\aleph|+1)+1) \sum_{\eth=2}^{\infty} \frac{1}{((\eth-1)\,s+1)\,(\eth-2)!} \\
&\quad + \ell_2\,|\aleph| \sum_{\eth=2}^{\infty} \frac{1}{((\eth-1)\,s+1)\,(\eth-1)!}.
\end{aligned}
$$

By (2.9), we get

$$
\begin{aligned}
L_1(\ell_1,\ell_2) \leq \frac{\ell_1}{s} \sum_{\eth=3}^{\infty} \frac{1}{(\eth-1)\,(\eth-3)!} &+ \frac{\ell_1\,(\ell_2\,|\aleph|+1)+1}{s} \sum_{\eth=2}^{\infty} \frac{1}{(\eth-1)\,(\eth-2)!} \\
&+ \frac{\ell_2\,|\aleph|}{s} \sum_{\eth=2}^{\infty} \frac{1}{(\eth-1)\,(\eth-1)!}.
\end{aligned}
$$

From equation (2.10), we get

$$
\begin{aligned}
L_1(\ell_1,\ell_2) \leq \frac{16\ell_1}{s} \sum_{\eth=3}^{\infty} \frac{1}{(\eth-1)\,2^{\eth}} &+ \frac{8(\ell_1\,(\ell_2\,|\aleph|+1)+1)}{s} \sum_{\eth=2}^{\infty} \frac{1}{(\eth-1)\,2^{\eth}} \\
&+ \frac{4\ell_2\,|\aleph|}{s} \sum_{\eth=2}^{\infty} \frac{1}{(\eth-1)\,2^{\eth}}.
\end{aligned}
$$

Using the series sums (2.6) and (2.7), we get

$$\begin{aligned} L_1(\ell_1,\ell_2) &\leq \frac{\ell_1}{s}\,(8\ln 2-4)+\frac{\ell_1\,(\ell_2\,|\aleph|+1)+1}{s}\,(4\ln 2)+\frac{\ell_2\,|\aleph|}{s}(2\ln 2)\\ &= \frac{4\ell_1(\ell_2\,|\aleph|+3)+2(\ell_2\,|\aleph|+2)}{s}\ln 2-\frac{4\ell_1}{s}. \end{aligned}$$

But $\ell_2\,|\aleph|$ is the bound above the final expression if (3.1) holds.

Next, since the function

$$\eth \;\mapsto\; \frac{1}{(\eth-1)s+1}$$

is decreasing on $(1,\infty)$ for each $s>0$, we obtain

$$\begin{aligned} L_1(\ell_1,\ell_2) &= \ell_1\sum_{\eth=3}^{\infty}\frac{1}{((\eth-1)s+1)(\eth-3)!}\\ &\quad+\big(\ell_1(\ell_2|\aleph|+1)+1\big)\sum_{\eth=2}^{\infty}\frac{1}{((\eth-1)s+1)(\eth-2)!}\\ &\quad+\ell_2|\aleph|\sum_{\eth=2}^{\infty}\frac{1}{((\eth-1)s+1)(\eth-1)!}\\ &\leq \frac{e\ell_1}{2s+1}+\big(\ell_1(\ell_2|\aleph|+1)+1\big)\frac{e}{s+1}+\ell_2|\aleph|\frac{e-1}{s+1}\\ &< \frac{e\ell_1}{s+1}+\big(\ell_1(\ell_2|\aleph|+1)+1\big)\frac{e}{s+1}+\ell_2|\aleph|\frac{e-1}{s+1}\\ &= \frac{\big[(\ell_1+1)(\ell_2|\aleph|+1)+\ell_1\big]\cdot e+\ell_1-\ell_2|\aleph|}{s+1}, \end{aligned}$$

and the final expression is bounded above by $\ell_2|\aleph|$ provided that condition (3.2) holds. □

Theorem 3.2. *If $s\in\mathbb{N}$, then sufficient conditions for the function $\Upsilon i_s(\wp)$ to belong to the family $R(\aleph,\ell_1,\ell_2)$ are*

$$[22\ell_2+6(1-\ell_1)]\ln 2-4\ell_1\leq s\ell_2\,|\aleph| \tag{3.5}$$

and

$$(3\ell_1+2)e\leq \ell_2(s+1)\,|\aleph|+1. \tag{3.6}$$

Proof. Since $\Upsilon i_s(\wp)$ is given by (2.4) and due to Lemma 2.4, it suffices to show that $L_2(\ell_1, \ell_2) \leq \ell_2 |\aleph|$, where

$$\begin{aligned} L_2(\ell_1, \ell_2) &= \sum_{\eth=2}^{\infty} [(\ell_1 \eth + (1-\ell_1))\eth] \frac{1}{((\eth-1)s+1)(\eth-1)!} \\ &= \sum_{\eth=2}^{\infty} \left[(\ell_1 \eth^2 + (1-\ell_1)\eth\right] \frac{1}{((\eth-1)s+1)(\eth-1)!}. \end{aligned}$$

Writing $\eth$ and $\eth^2$ as given in (3.3) and (3.4), and by a similar method of Theorem 3.1, we realize that

$$\eth i_s(\wp) \in R(\aleph, \ell_1, \ell_2)$$

if (3.5) holds.

Next, since the function

$$\eth \mapsto \frac{1}{(\eth-1)s+1}$$

is decreasing on $(1, \infty)$ for each $s > 0$, we obtain

$$\begin{aligned} L_2(\ell_1, \ell_2) = \ell_1 &\sum_{\eth=3}^{\infty} \frac{1}{((\eth-1)s+1)(\eth-3)!} \\ &+ \left(2\ell_1+1\right) \sum_{\eth=2}^{\infty} \frac{1}{((\eth-1)s+1)(\eth-2)!} \\ &+ \sum_{\eth=2}^{\infty} \frac{1}{((\eth-1)s+1)(\eth-1)!} \\ &\leq \frac{e\ell_1}{2s+1} + \left(2\ell_1+1\right)\frac{e}{s+1} + \frac{e-1}{s+1} \\ &< \frac{e\ell_1}{s+1} + \left(2\ell_1+1\right)\frac{e}{s+1} + \frac{e-1}{s+1} \\ &= \frac{(3\ell_1+2)\cdot e - 1}{s+1}, \end{aligned}$$

and the final expression is bounded above by $\ell_2 |\aleph|$ provided that condition (3.6) holds. $\square$

Figure 1 demonstrates the graphical representations of the sufficient conditions for the function $\Upsilon i_s(\wp)$ to be in the family $S(\aleph)$ and the family $C(\aleph)$.

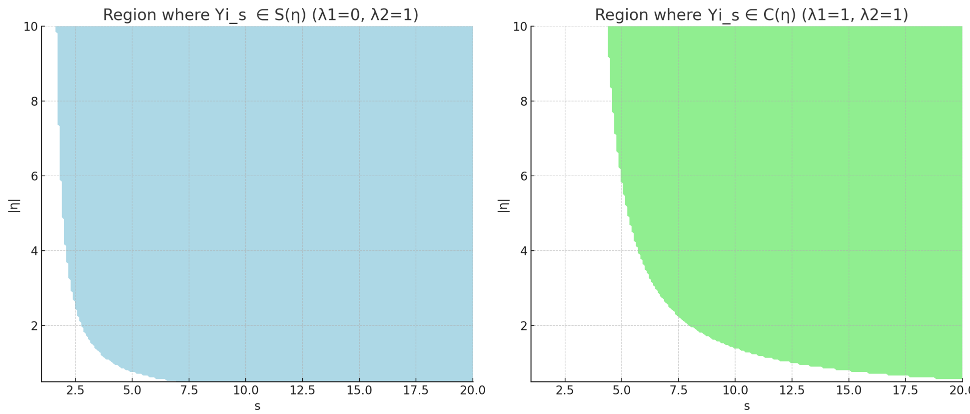

Figure 1. Admissible regions in the $(s, |\aleph|)$-plane where the sufficient conditions for $\Upsilon i_s(\wp) \in S(\aleph)$ hold in the particular case $(\ell_1, \ell_2) = (0, 1)$ (left figure) and for $\Upsilon i_s(\wp) \in C(\aleph)$ hold in the particular case $(\ell_1, \ell_2) = (1, 1)$ (right figure). The shaded area indicates the pairs $(s, |\aleph|)$ for which the inequalities are satisfied.

As direct consequences, we have the corollaries below.

Corollary 3.3. *For $(\ell_1, \ell_2) = (0, 1)$, the function $\Upsilon i_s(\wp)$ belongs to the family $S(\aleph)$ if*

$$2(\,|\aleph| + 2)\ln 2 \;\le\; s|\aleph|, \qquad (\,|\aleph| + 1)e \;\le\; (s + 2)|\aleph|.$$

Similarly, $\Upsilon i_s(\wp)$ belongs to the family $R(\aleph)$ if

$$28 \ln 2 \;\le\; s|\aleph|, \qquad 2e \;\le\; (s + 1)|\aleph| + 1.$$

Corollary 3.4. *For $(\ell_1, \ell_2) = (1, 1)$, the function $\Upsilon i_s(\wp)$ belongs to the family $C(\aleph)$ provided*

$$(6|\aleph| + 16)\ln 2 - 4 \;\le\; s|\aleph|, \qquad (2|\aleph| + 3)e \;\le\; (s + 2)|\aleph| - 1.$$

Next, we shall determine the action of the linear convolution operator $\mathcal{I}i_s(\wp)$ on the subfamilies $S(\aleph, \ell_1, \ell_2)$ and $R(\aleph, \ell_1, \ell_2)$.

Theorem 3.5. *Let $s \in \mathbb{N}$. If $h \in G^{\tau}(Q_1, Q_2)$. Then the sufficient conditions for $\mathcal{I}i_s(\wp)$ to belong to the family $S(\aleph, \ell_1, \ell_2)$ are*

$$(\ell_2\,|\aleph|\,(\ell_1 + 1)) + 3\ell_1 + 1)\ln 2 \le \frac{s\ell_2\,|\aleph|}{(Q_1 - Q_2)|\tau|} \tag{3.7}$$

and

$$(\ell_1 + \ell_2\,|\aleph|)\,e + (\ell_1 - 2)\,(\ell_2\,|\aleph| - 1) - 1 \le \frac{\ell_2(s + 1)\,|\aleph|}{(Q_1 - Q_2)|\tau|}. \tag{3.8}$$

Proof. In light of Lemma 2.3 it suffices to demonstrate that

$$\begin{aligned} M_1(\ell_1,\ell_2) &= \sum_{\eth=2}^{\infty}\left[\ell_1\eth^2+(1+\ell_1(\ell_2|\aleph|-2))\eth+(\ell_2|\aleph|-1)(1-\ell_1)\right] \\ &\quad\times\frac{1}{((\eth-1)s+1)(\eth-1)!}|a_\eth| \\ &\le \ell_2|\aleph|. \end{aligned}$$

Since $h\in G^{\tau}(Q_1,Q_2)$, then by Lemma 2.5, we have

$$\begin{aligned} M_1(\ell_1,\ell_2) \le (Q_1-Q_2)|\tau| &\left(\sum_{\eth=2}^{\infty}\frac{\ell_1\eth}{((\eth-1)s+1)(\eth-1)!}\right. \\ &\left.+\sum_{\eth=2}^{\infty}\frac{1+\ell_1(\ell_2|\aleph|-2)}{((\eth-1)s+1)(\eth-1)!}+\sum_{\eth=2}^{\infty}\frac{(\ell_2|\aleph|-1)(1-\ell_1)}{((\eth-1)s+1)\eth!}\right). \end{aligned}$$

By (3.3), we get

$$\begin{aligned} M_1(\ell_1,\ell_2) &\le (Q_1-Q_2)|\tau|\left(\sum_{\eth=2}^{\infty}\frac{\ell_1(\eth-1)}{((\eth-1)s+1)(\eth-1)!}\right. \\ &\quad\left.+\sum_{\eth=2}^{\infty}\frac{1+\ell_1(\ell_2|\aleph|-1)}{((\eth-1)s+1)(\eth-1)!}+\sum_{\eth=2}^{\infty}\frac{(\ell_2|\aleph|-1)(1-\ell_1)}{((\eth-1)s+1)\eth!}\right) \\ &= (Q_1-Q_2)|\tau|\left(\sum_{\eth=2}^{\infty}\frac{\ell_1}{((\eth-1)s+1)(\eth-2)!}\right. \\ &\quad\left.+\sum_{\eth=2}^{\infty}\frac{1+\ell_1(\ell_2|\aleph|-1)}{((\eth-1)s+1)(\eth-1)!}+\sum_{\eth=2}^{\infty}\frac{(\ell_2|\aleph|-1)(1-\ell_1)}{((\eth-1)s+1)\eth!}\right). \end{aligned}$$

By (2.9) and (2.10), we get

$$\begin{aligned} M_1(\ell_1,\ell_2) \quad\le\quad \frac{(Q_1-Q_2)|\tau|}{s}&\left(8\sum_{\eth=2}^{\infty}\frac{\ell_1}{(\eth-1)2^{\eth}}+4\sum_{\eth=2}^{\infty}\frac{1+\ell_1(\ell_2|\aleph|-1)}{(\eth-1)2^{\eth}}\right. \\ &\left.+2\sum_{\eth=2}^{\infty}\frac{(\ell_2|\aleph|-1)(1-\ell_1)}{(\eth-1)2^{\eth}}\right). \end{aligned}$$

By (2.6), we get

$$M_1(\ell_1,\ell_2)\le\frac{(Q_1-Q_2)|\tau|}{s}(\ell_2|\aleph|(\ell_1+1))+3\ell_1+1)\ln 2.$$

But $\ell_2|\aleph|$ is the bound above the final expression if (3.7) holds.

Next, since the function

$$\eth \ \mapsto \ \frac{1}{(\eth-1)s+1}$$

is decreasing on $(1,\infty)$ for each $s>0$, we obtain

$$\begin{aligned}
M_1(\ell_1,\ell_2) &\leq (Q_1-Q_2)|\tau| \left(\sum_{\eth=2}^{\infty} \frac{\ell_1}{((\eth-1)\,s+1)\,(\eth-2)!} \right. \\
&\quad \left. + \sum_{\eth=2}^{\infty} \frac{1+\ell_1\,(\ell_2\,|\aleph|-1)}{((\eth-1)\,s+1)\,(\eth-1)!} + \sum_{\eth=2}^{\infty} \frac{(\ell_2\,|\aleph|-1)\,(1-\ell_1)}{((\eth-1)\,s+1)\eth!} \right) \\
&\leq (Q_1-Q_2)|\tau| \Big(\frac{e\ell_1}{s+1} \ + \frac{(1+\ell_1\,(\ell_2\,|\aleph|-1))\,(e-1)}{s+1} \\
&\quad + \frac{(\ell_2\,|\aleph|-1)\,(1-\ell_1)\,(\,e-2)}{s+1} \Big) \\
&= (Q_1-Q_2)|\tau| \left(\frac{(\ell_1+\ell_2\,|\aleph|)\,e + (\ell_1-2)\,(\ell_2\,|\aleph|-1)-1}{s+1} \right),
\end{aligned}$$

and the final expression is bounded by $\ell_2\,|\aleph|$ provided that condition (3.8) holds. □

Theorem 3.6. *Let $s \in \mathbb{N}$. If $h \in G^\tau(Q_1,Q_2)$, then the sufficient conditions for $\mathcal{I}i_s(\wp)$ to belong to the family $R(\aleph,\ell_1,\ell_2)$ are*

$$2\,(2\ell_1+1)\ln 2 \leq \frac{s\ell_2\,|\aleph|}{(Q_1-Q_2)|\tau|} \tag{3.9}$$

and

$$(\ell_1+1)e-1 \leq \frac{\ell_2(s+1)\,|\aleph|}{(Q_1-Q_2)|\tau|}. \tag{3.10}$$

Proof. In light of Lemma 2.4 it suffices to demonstrate that

$$\begin{aligned}
M_2(\ell_1,\ell_2) &= \sum_{\eth=2}^{\infty} [(\ell_1\eth+(1-\ell_1))\eth]\ \frac{1}{((\eth-1)\,s+1)\,(\eth-1)!}\,|a_\eth| \\
&\leq \ell_2\,|\aleph|\,.
\end{aligned}$$

Since $h \in G^\tau(Q_1,Q_2)$, then by virtue of Lemma 2.5, we have

$$M_2(\ell_1,\ell_2) \leq (Q_1-Q_2)|\tau| \sum_{\eth=2}^{\infty} (\ell_1\eth+(1-\ell_1))\,\frac{1}{((\eth-1)\,s+1)\,(\eth-1)!}.$$

By (3.3), we get

$$\begin{aligned} M_2(\ell_1,\ell_2) &\le (Q_1-Q_2)|\tau|\Big(\sum_{\eth=2}^{\infty}\frac{\ell_1(\eth-1)}{((\eth-1)\,s+1)\,(\eth-1)!} \\ &\quad+\sum_{\eth=2}^{\infty}\frac{1}{((\eth-1)\,s+1)\,(\eth-1)!}\Big) \\ &= (Q_1-Q_2)|\tau|\Big(\sum_{\eth=2}^{\infty}\frac{\ell_1}{((\eth-1)\,s+1)\,(\eth-2)!} \\ &\quad+\sum_{\eth=2}^{\infty}\frac{1}{((\eth-1)\,s+1)\,(\eth-1)!}\Big). \end{aligned}$$

By (2.9) and (2.10), we get

$$M_2(\ell_1,\ell_2) \le \frac{(Q_1-Q_2)|\tau|}{s}\left(8\sum_{\eth=2}^{\infty}\frac{\ell_1}{(\eth-1)\,2^{\eth}}+4\sum_{\eth=2}^{\infty}\frac{1}{(\eth-1)\,2^{\eth}}\right).$$

By (2.6), we get

$$M_2(\ell_1,\ell_2) \le \frac{(Q_1-Q_2)|\tau|}{s}\,(4\ell_1+2)\ln 2.$$

But $\ell_2\,|\aleph|$ is the bound above the final expression if (3.9) holds.

Next, since the function

$$\eth \;\mapsto\; \frac{1}{(\eth-1)s+1}$$

is decreasing on $(1,\infty)$ for each $s>0$, we obtain

$$\begin{aligned} M_2(\ell_1,\ell_2) &\le (Q_1-Q_2)|\tau|\Big(\sum_{\eth=2}^{\infty}\frac{\ell_1}{((\eth-1)\,s+1)\,(\eth-2)!} \\ &\quad+\sum_{\eth=2}^{\infty}\frac{1}{((\eth-1)\,s+1)\,(\eth-1)!}\Big) \\ &\le (Q_1-Q_2)|\tau|\left(\frac{e\ell_1}{s+1}+\frac{e-1}{s+1}\right) \\ &= (Q_1-Q_2)|\tau|\left(\frac{(\ell_1+1)\,e-1}{s+1}\right), \end{aligned}$$

and the last expression is bounded by $\ell_2\,|\aleph|$, if (3.10) holds. □

Finally, we investigate the action of the Alexander transform $Ki_s(\wp)$ on the subfamilies $S(\aleph,\ell_1,\ell_2)$ and $R(\aleph,\ell_1,\ell_2)$.

Theorem 3.7. *Let $s \in \mathbb{N}$. The sufficient conditions for the Alexander transform $Ki_s(\wp)$ to belong to the family $S(\aleph, \ell_1, \ell_2)$ are*

$$(\ell_2 |\aleph| (\ell_1 + 1)) + 3\ell_1 + 1) \ln 2 \leq s\ell_2 |\aleph| \tag{3.11}$$

and

$$(\ell_2 |\aleph| + \ell_1)e + (\ell_2 |\aleph| - 1)(\ell_1 - 2) \leq \ell_2(s + 1) |\aleph| + 1. \tag{3.12}$$

Proof. According to (2.4) it follows that

$$Ki_s(\wp) = \wp - \sum_{\eth=2}^{\infty} \frac{1}{((\eth - 1) s + 1) (\eth - 1)!} \frac{\wp^{\eth}}{\eth}, \ \wp \in \mathsf{C}. \tag{3.13}$$

Using Lemma 2.3, the Alexander transform $Ki_s(\wp)$ belongs to $S(\aleph, \ell_1, \ell_2)$ if

$$\begin{aligned}
&\sum_{\eth=2}^{\infty} \left[\ell_1 \eth^2 + (1 + \ell_1 (\ell_2 |\aleph| - 2))\eth + (\ell_2 |\aleph| - 1) (1 - \ell_1)\right] \\
&\qquad \times \frac{1}{\eth((\eth - 1) s + 1) (\eth - 1)!} \\
&\quad = \sum_{\eth=2}^{\infty} \frac{\ell_1 \eth}{((\eth - 1) s + 1) (\eth - 1)!} + \sum_{\eth=2}^{\infty} \frac{1 + \ell_1 (\ell_2 |\aleph| - 2)}{((\eth - 1) s + 1) (\eth - 1)!} \\
&\qquad + \sum_{\eth=2}^{\infty} \frac{(\ell_2 |\aleph| - 1) (1 - \ell_1)}{((\eth - 1) s + 1)\eth!} \\
&\quad \leq \ell_2 |\aleph| .
\end{aligned}$$

By a similar proof of Theorem 3.5 we get that $Ki_s(\wp) \in S(\aleph, \ell_1, \ell_2)$ if (3.11) holds.

Next, since the function

$$\eth \ \mapsto \ \frac{1}{(\eth - 1)s + 1}$$

is decreasing on $(1, \infty)$ for each $s > 0$, we obtain

$$\begin{aligned}
&\sum_{\eth=2}^{\infty} \frac{\ell_1 \eth}{((\eth - 1) s + 1) (\eth - 1)!} + \sum_{\eth=2}^{\infty} \frac{1 + \ell_1 (\ell_2 |\aleph| - 2)}{((\eth - 1) s + 1) (\eth - 1)!} \\
&\qquad + \sum_{\eth=2}^{\infty} \frac{(\ell_2 |\aleph| - 1) (1 - \ell_1)}{((\eth - 1) s + 1)\eth!}
\end{aligned}$$

$$= \sum_{\eth=2}^{\infty} \frac{\ell_1}{((\eth-1)\,s+1)\,(\eth-2)!} + \sum_{\eth=2}^{\infty} \frac{1+\ell_1\,(\ell_2\,|\aleph|-1)}{((\eth-1)\,s+1)\,(\eth-1)!}$$

$$+ \sum_{\eth=2}^{\infty} \frac{(\ell_2\,|\aleph|-1)\,(1-\ell_1)}{((\eth-1)\,s+1)\eth!}$$

$$\leq \frac{e\ell_1}{s+1} + \frac{(1+\ell_1\,(\ell_2\,|\aleph|-1))(e-1)}{s+1} + \frac{(\ell_2\,|\aleph|-1)\,(1-\ell_1)\,(e-2)}{s+1}$$

$$= \frac{(\ell_2\,|\aleph|+\ell_1)e+(\ell_2\,|\aleph|-1)(\ell_1-2)-1}{s+1},$$

and $\ell_2\,|\aleph|$ is the bound above the final expression if (3.12) holds. □

Theorem 3.8. *Let $s \in \mathbb{N}$. The sufficient conditions for the Alexander transform $Ki_s(\wp)$ to belong to the family $R(\aleph, \ell_1, \ell_2)$ are*

$$2\,(2\ell_1+1)\ln 2 \leq s\ell_2\,|\aleph| \tag{3.14}$$

and

$$(\ell_1+1)e+\ell_1 \leq \ell_2(s+1)\,|\aleph|+2. \tag{3.15}$$

Proof. Since $Ki_s(\wp)$ is given by (3.13) and by Lemma 2.4, the Alexander transform $Ki_s(\wp)$ belongs to $R(\aleph, \ell_1, \ell_2)$ if

$$\sum_{\eth=2}^{\infty} [(\ell_1\eth+(1-\ell_1))\eth] \quad \frac{1}{\eth((\eth-1)\,s+1)\,(\eth-1)!}$$

$$= \sum_{\eth=2}^{\infty} \frac{\ell_1\eth}{((\eth-1)\,s+1)\,(\eth-1)!} + \sum_{\eth=2}^{\infty} \frac{1-\ell_1}{((\eth-1)\,s+1)\eth!}$$

$$\leq \ell_2\,|\aleph|\,.$$

By a similar proof of Theorem 3.6 we get that $Ki_s(\wp) \in R(\aleph, \ell_1, \ell_2)$ if and only if (3.14) holds.

Next, since the function

$$\eth \;\mapsto\; \frac{1}{(\eth-1)s+1}$$

is decreasing on $(1,\infty)$ for each $s>0$, we obtain

$$\begin{aligned}
&\sum_{\eth=2}^{\infty}\frac{\ell_1\eth}{((\eth-1)\,s+1)\,(\eth-1)!}+\sum_{\eth=2}^{\infty}\frac{1-\ell_1}{((\eth-1)\,s+1)\eth!}\\
&\quad=\sum_{\eth=2}^{\infty}\frac{\ell_1}{((\eth-1)\,s+1)\,(\eth-2)!}+\sum_{\eth=2}^{\infty}\frac{\ell_1}{((\eth-1)\,s+1)\,(\eth-1)!}\\
&\qquad+\sum_{\eth=2}^{\infty}\frac{1}{((\eth-1)\,s+1)\eth!}\\
&\quad\le\frac{e\ell_1}{s+1}+\frac{\ell_1(e-1)}{s+1}+\frac{(1-\ell_1)(e-2)}{s+1}\\
&\quad=\frac{(\ell_1+1)e+\ell_1-2}{s+1},
\end{aligned}$$

and $\ell_2\,|\aleph|$ is the bound above the final expression if (3.15) holds. □

Remark 3.9. By specializing the parameters $\aleph$, ℓ_1, and ℓ_2 in our theorems, we obtain several results related to the subfamilies $S(\aleph,\ell_1,\ell_2)$ and $R(\aleph,\ell_1,\ell_2)$. For example, from Remark 2.1, we get some results for the subfamilies $S(\aleph,\ell_2)$, $C(\aleph,\ell_2)$, $R(\aleph,\ell_2)$, $S(\aleph), C(\aleph)$, and $R(\aleph)$. These particular cases not only simplify the general inequalities but also demonstrate the flexibility of our approach in capturing well-known geometric function families.

4. Conclusion

In this study, we established sufficient conditions for the function Υi_s, defined via the generalized normalized imaginary error function, to be in the subfamilies of analytic functions $S(\aleph,\ell_1,\ell_2)$ and $R(\aleph,\ell_1,\ell_2)$ defined on the open unit disk C.

Furthermore, we investigated the action of the linear convolution operator $\mathcal{I}i_s(\wp)$ and the Alexander transform $Ki_s(\wp)$ on these subfamilies.

The results obtained in this work may motivate further research aimed at developing new sufficient conditions for functions and operators involving the generalized normalized imaginary error function to belong to other subfamilies on C.

References

[1] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions: with Formulas, Graphs, and Mathematical Tables*; Dover Publications Inc., New York, 1965.

[2] J.W. Alexander, *Functions which map the interior of the unit circle upon simple regions*, Ann. of Math., **17** (1915), 12-22.

[3] T. Al-Hawary, I. Aldawish, B.A. Frasin, O. Alkam and F. Yousef, *Necessary and sufficient conditions for normalized Wright functions to be in certain classes of analytic functions*, Mathematics, **10**(24) (2022), Article 4693.

[4] T. Al-Hawary, M. Illafe and F. Yousef, *Analytic subclass conditions for generalized normalized imaginary error functions*, Scientific African, **31** (2026), e03280.

[5] O. Altintas, *On a subclass of certain starlike functions with negative coefficients*, Math. Japon., **36**(2) (1991), 489–495.

[6] O. Altintas, O. Ozkan and H. M. Srivastava, *Neighborhoods of a class of analytic functions with negative coefficients*, Appl. Math. Letters, **13** (2000), 63–67.

[7] H. Alzer, *Error function inequalities*, Advances in Computational Mathematics, **33** (2010), 349–379.

[8] A.I. Al-Zoubi, T. Al-Hawary, S. Hasan and F. Yousef, *Characterizations for a Wright distribution series on certain subfamilies of Univalent functions*, International Journal of Analysis and Applications, **24** (2026), 75–75.

[9] M.K. Aouf, *p-Valent classes related to convex functions of complex order*, Rocky Mountain J. Math., **15**(4) (1985), 855–865.

[10] M.K. Aouf and A.O. Mostafa, *Certain bounded functions of complex order*, Math. Vesnik, **62**(2) (2010), 109–116.

[11] K.K. Dixit and S.K. Pal, *On a class of univalent functions related to complex order*, Indian J. Pure Appl. Math., **26**(9) (1995), 889–896.

[12] B.A. Frasin, *Some lower bounds for the quotients of normalized error function and their partial sums*, Archivum Mathematicum, **61**(2) (2025), 73–83.

[13] A. Halim, *On a class of functions of complex order*, Tamkang J. Math., **30**(2) (1999), 147–153.

[14] A. Hussen, *An application of the Mittag-Leffler-type Borel distribution and Gegenbauer polynomials on a certain subclass of bi-univalent functions*, Heliyon, **10**(10) (2024).

[15] A. Hussen and A. Zeyani, *Coefficients and FeketeSzeg functional estimations of bi-univalent subclasses based on Gegenbauer polynomials*, Mathematics, **11**(13) (2023), 2852.

[16] T. Janani and G. Murugusundaramoorthy, *Inclusion results on subclasses of starlike and convex functions associated with Struve functions*, Italian J. Pure Appl. Math., **32** (2014), 467–476.

[17] M.A. Nasr and M.K. Aouf, *On convex functions of complex order*, Bull. Fac. Sci. Mansoura Univ., **9** (1982), 565–582.

[18] M.A. Nasr and M.K. Aouf, *Starlike function of complex order*, J. Natar Sci. Math., **25**(1) (1985), 1–12.

[19] M.A. Nasr and M.K. Aouf, *Bounded starlike functions of complex order*, Proc. Indian Acad. Sci. (Math. Sci.), **92**(2) (1983), 97–102.

[20] S. Owa, *Notes on starlike, convex, and close-to-convex functions of complex order*, Univalent Functions, Fractional Calculus, and Their Applications, **199** (1989), 218.

[21] K.S. Padmanabhan, *On sufficient conditions for starlikeness*, Indian J. Pure Appl. Math., **32** (2001), 543–550.

[22] S. Sivasubramanian and J. Sokol, *Hypergeometric transforms in certain classes of analytic functions*, Mathematical and Computer Modelling, **54**(11-12) (2011), 3076-3082.

[23] P. Wiatrowski, *The sufficient of a certain family holomorphic functions*, Zeszyty Naukowe Uniwesytetu Lodzkiego, Seria: Nauki Matematyzno-Przyrodnicze. J. Appl. Math. Phys., **39** (1971), 75–85.

[24] F. Yousef, T. Al-Hawary, B. Frasin, J. Salah and M. Illafe, *Comprehensive subclasses of bi-univalent functions specified by LiouvilleCaputo-type fractional derivatives and Euler polynomials*, Appl. Math, **20**(2) (2026), 497–502.

[25] F. Yousef, T. Al-Hawary, M. El-Ityan and I. Aldawish, *Novel bi-univalent subclasses generated by the q-analogue of the Ruscheweyh operator and Hermite polynomials*, Mathematics, **14**(2) (2026), 382.