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ON RELATION THEORETIC COMMON FIXED POINT THEOREMS FOR PAIR OF MAPPINGS ENDOWED WITH A WEAKLY CLOSED RELATION

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Abstract. The purpose of this paper is to first define the notion of $(\mathcal{G}, \mathcal{H})$-weakly closed relation, supported by non-trivial examples. Utilize this notion to common fixed point theorem for pair of mappings satisfying generalized contraction involving rational terms and auxiliary functions in the context of relation theoretic metric space, without imposing completeness of the space. Further, to guarantee the uniqueness of fixed point, we assume that the space $\mathbb{X}$ is $\mathcal{R}^S$-directed. Additionally, examples with graphical representations are provided to validate the main results.

1. Introduction

Fixed point theory is an important field of research with broad applications, especially in the sciences and engineering. Its wide applicability makes it a powerful component of Mathematical analysis. The Banach contraction principle [6] is widely recognized as a fundamental theorem in nonlinear analysis. Over the past 100 years, many researchers have built on this finding by making

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the class of underlying metric spaces better and lowering the conditions for contraction [4, 8, 9, 25].

Jungck [11] was the first person to put forward the idea of coincidence and common fixed point in a formal way. Jungck came up with the idea of commuting mappings, added a pair of self-maps to the Banach contraction principle, and proved the common fixed point theorem in a complete metric space. In subsequent year, numerous authors have developed several generalizations satisfying several kind of contractions. Few of them can be found in [12, 18, 27, 32] and references there in.

On the other side, there have been numerous exciting developments in the field of existence of fixed point for self-mappings in metric spaces endowed with arbitrary binary relation. To have a more thorough understanding, we encouraged the readers to explore the work of Turinici [29, 30, 31], Ran - Reurings [24], Nieto and Lopez [20, 21], Nashine and Samet [19], Samet and Turinici [26], Ben-El-Mechaiekh [7], Mani [15, 16, 17], Pingale et al. [5, 22] and the references there in.

Another new variant of Banach contraction principle was proposed by Alam and Imdad [2], in 2015, on complete metric space endowed with a binary relation. Such kind of fixed point results are commonly knows as relation theoretic metrical fixed point theorems. Contraction conditions proposed, extended or generalized in relation theoretic metric spaces are relatively weaker than the usual contractions, as it only needs to be satisfied for elements which are closed under the arbitrary binary relation. In 2021, Faruk et al. [13], coincidence point theorems are established for weak contraction type mappings by employing a novel auxiliary function, within the framework of a metric space equipped with an arbitrary binary relation in the setting of $(\mathcal{G}, \mathcal{H})$-closedness. In 2022, Sun and Liu [28] obtained some coincidence and common fixed point theorems for mappings via hybrid contractions and auxiliary functions in relation theoretic extended rectangular b-metric spaces. In last few years, numerous researchers have expanded and diversified the fixed point results equipped with binary relation in various manners. For further details, one can refer [1, 10, 23, 33, 34].

A metrical coincidence and common fixed point theorems are usually proved by utilizing the notions of $\mathcal{R}$-commutativity, $\mathcal{R}$-continuity, $\mathcal{R}$-completeness along with the contraction conditions. Our main purpose of this work is to relax these assumptions and derive some common fixed point theorems. In order to prove our result, we first define the notion of $(\mathcal{G}, \mathcal{H})$-weakly closed relation supported by illustrative examples, and utilized this notion to give the existence of common fixed point for pair of self mappings in relation theoretic metric spaces. To guarantee the uniqueness of the common fixed point, we further assumed that the space $\mathbb{X}$ is $\mathcal{R}^S$ - directed. Finally, in support of our findings, some illustrative examples along with graphical representations are presented.

2. Preliminaries

As abbreviations, throughout the paper, we will use the nonempty set $\mathbb{X}$, nonempty binary relation $\mathcal{R}$ on $\mathbb{X}$, set of nonzero positive integers $\mathbb{N}$ and set of whole numbers $\mathbb{N}_0$. Let us recall some notable notations and related findings on binary relations, which will be quite useful in proving our main result.

Definition 2.1. ([2]) Let $\mathbb{X}$ be a nonempty set. A subset $\mathcal{R}$ of $\mathbb{X}^2$ is said to be a binary relation on $\mathbb{X}$, if for each pair $\varrho, \varsigma \in \mathbb{X}$, the following conditions are satisfied:

(1) $(\varrho, \varsigma) \in \mathcal{R}$; means that ϱ is $\mathcal{R}$-related to ς or ϱ relates to ς under $\mathcal{R}$.
(2) $(\varrho, \varsigma) \notin \mathcal{R}$; means that ϱ is not $\mathcal{R}$-related to ς or ϱ does not relates to ς under $\mathcal{R}$.

Remark 2.2. Trivially, $\mathbb{X}^2$ and $\emptyset$ being subsets of $\mathbb{X}^2$ are binary relations on $\mathbb{X}$, which are respectively called the universal relation (or full relation) and empty relation.

Definition 2.3. ([2]) Let $\mathbb{X}$ be a nonempty set and $\mathcal{R}$ is a binary relation on $\mathbb{X}$.

(1) The inverse, transpose or dual relation of $\mathcal{R}$, denoted by $\mathcal{R}^{-1}$, is defined by
$$\mathcal{R}^{-1} = \left\{(\varrho, \varsigma) \in \mathbb{X}^2 : (\varsigma, \varrho) \in \mathcal{R}\right\}.$$
(2) The symmetric closure of $\mathcal{R}$, denoted by $\mathcal{R}^S$, is defined to be the set $\mathcal{R} \cup \mathcal{R}^{-1}$ (i.e., $\mathcal{R}^S = \mathcal{R} \cup \mathcal{R}^{-1}$). Indeed, $\mathcal{R}^S$ is the smallest symmetric relation on $\mathbb{X}$ containing $\mathcal{R}$.

Proposition 2.4. ([2]) *For a binary relation $\mathcal{R}$ defined on a nonempty set $\mathbb{X}$,*
$$(\varrho, \varsigma) \in \mathcal{R}^S \text{ if and only if } [\varrho, \varsigma] \in \mathcal{R}.$$

Definition 2.5. ([2]) Let $\mathbb{X}$ be a nonempty set and $\mathcal{R}$ be a binary relation on $\mathbb{X}$. A sequence $\{\varrho_n\} \subset \mathbb{X}$ is called $\mathcal{R}$-preserving, if
$$(\varrho_n, \varrho_{n+1}) \in \mathcal{R}, \ \forall n \in \mathbb{N}_0.$$

Definition 2.6. ([2]) Let $\mathbb{X}$ be a nonempty set and $\mathcal{R}$ be a binary relation on $\mathbb{X}$. Then a map $\mathcal{G} : \mathbb{X} \to \mathbb{X}$ is said to be $\mathcal{G}$-closed, if for each $\varrho, \varsigma \in \mathbb{X}$,
$$(\varrho, \varsigma) \in \mathcal{R} \quad \text{implies} \quad (\mathcal{G}\varrho, \mathcal{G}\varsigma) \in \mathcal{R}.$$

Definition 2.7. ([3]) Let $\mathbb{X}$ be a nonempty set and $\mathcal{R}$ be a binary relation on $\mathbb{X}$. Then the maps $\mathcal{G}, \mathcal{H} : \mathbb{X} \to \mathbb{X}$ are said to be $(\mathcal{G}, \mathcal{H})$-closed, if for each $\varrho, \varsigma \in \mathbb{X}$,

$$(\mathcal{H}\varrho, \mathcal{H}\varsigma) \in \mathcal{R} \quad \text{implies} \quad (\mathcal{G}\varrho, \mathcal{G}\varsigma) \in \mathcal{R}.$$

Definition 2.8. ([26]) Let $\mathbb{X}$ be a non empty set and $\mathcal{R}$ is a binary relation on $\mathbb{X}$. A subset $D \subseteq \mathbb{X}$ is called $\mathcal{R}^S$-directed if for each pair $\varrho, \varsigma \in D$, there exists $z \in \mathbb{X}$ such that $(\varrho, z) \in \mathcal{R}$ and $(\varsigma, z) \in \mathcal{R}$.

Definition 2.9. ([14]) A function $\alpha : [0, \infty) \to [0, \infty)$ is known as altering distance function, if the following properties are satisfied:

(1) α is monotone increasing and continuous,
(2) $\alpha(t) = 0$ implies $t = 0$.

3. $(\mathcal{G}, \mathcal{H})$-weakly closed relation: definition and examples

We start this section by presenting the definition of weakly closed relation followed by examples with graphical representation.

Definition 3.1. Let $\mathbb{X}$ be a nonempty set and $\mathcal{G}, \mathcal{H} : \mathbb{X} \to \mathbb{X}$ are maps. Then a relation $\mathcal{R}$ defined on $\mathbb{X}$ is said to be $(\mathcal{G}, \mathcal{H})$-weakly closed, if for each $\varrho \in \mathbb{X}$,

$$(\mathcal{G}\varrho, \mathcal{H}\mathcal{G}\varrho) \in \mathcal{R} \quad \text{and} \quad (\mathcal{H}\varrho, \mathcal{G}\mathcal{H}\varrho) \in \mathcal{R}.$$

The following are some examples of $(\mathcal{G}, \mathcal{H})$-weakly closed relations.

Example 3.2. Let $\mathbb{X} = [1, \infty)$. Define mappings $\mathcal{G}, \mathcal{H} : \mathbb{X} \to \mathbb{X}$ by

$$\mathcal{G}\varrho = 2\varrho \;\text{ and } \mathcal{H}\varrho = \varrho^3 \text{ for all } \varrho \in \mathbb{X}.$$

Further, define a binary relation $\mathcal{R}$ on $\mathbb{X}$ by

$$\mathcal{R} = \{(\varrho, \varsigma) \in \mathbb{X}^2 : \varrho < \varsigma\}.$$

Then for each $\varrho \in \mathbb{X}$,

$$\mathcal{H}\mathcal{G}\varrho = \mathcal{H}(\mathcal{G}\varrho) = 8\varrho^3 > 2\varrho = \mathcal{G}\varrho,$$

implies that $(\mathcal{G}\varrho, \mathcal{H}\mathcal{G}\varrho) \in \mathcal{R}$.
Again,

$$\mathcal{G}\mathcal{H}\varrho = \mathcal{G}(\mathcal{H}\varrho) = 2\varrho^3 > \varrho^3 = \mathcal{H}\varrho,$$

implies that $(\mathcal{H}\varrho, \mathcal{G}\mathcal{H}\varrho) \in \mathcal{R}$. Thus the relation $\mathcal{R}$ defined on $\mathbb{X}$ is $(\mathcal{G}, \mathcal{H})$-weakly closed.

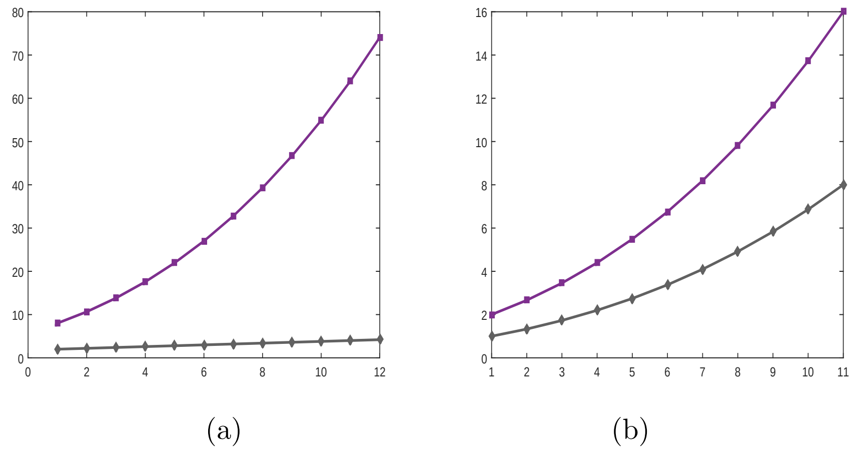

FIGURE 1. (a) Graphical Behaviour of inequality $\mathcal{G}\varrho < \mathcal{H}\mathcal{G}\varrho$, and (b) Graphical Behaviour of inequality $\mathcal{H}\varrho < \mathcal{G}\mathcal{H}\varrho$ in the context of Example 3.2. Then, the maps $\mathcal{G}$ and $\mathcal{H}$ are weakly-closed relation.

Example 3.3. Let $\mathbb{X} = [0, 30]$. Define mappings $\mathcal{G}, \mathcal{H} : \mathbb{X} \to \mathbb{X}$ by

$$\mathcal{G}\varrho = \varrho + 1 \ \text{ and } \mathcal{H}\varrho = e^{\varrho} \text{ for all } \varrho \in \mathbb{X}.$$

Further, define a binary relation $\mathcal{R}$ on $\mathbb{X}$ by

$$\mathcal{R} = \{(\varrho, \varsigma) \in \mathbb{X}^2 : \varrho < \varsigma\}.$$

Then for each $\varrho \in \mathbb{X}$,

$$\mathcal{H}\mathcal{G}\varrho = \mathcal{H}(\mathcal{G}\varrho) = e^{\varrho+1} > \varrho + 1 = \mathcal{G}\varrho,$$

implies that $(\mathcal{G}\varrho, \mathcal{H}\mathcal{G}\varrho) \in \mathcal{R}$.
Again,

$$\mathcal{G}\mathcal{H}\varrho = \mathcal{G}(\mathcal{H}\varrho) = e^{\varrho} + 1 > e^{\varrho} = \mathcal{H}\varrho,$$

implies that $(\mathcal{H}\varrho, \mathcal{G}\mathcal{H}\varrho) \in \mathcal{R}$. Therefore, the relation $\mathcal{R}$ defined on $\mathbb{X}$ is $(\mathcal{G}, \mathcal{H})$-weakly closed.

Example 3.4. Let $\mathbb{X} = (0, 2)$. Define mappings $\mathcal{G}, \mathcal{H} : \mathbb{X} \to \mathbb{X}$ by

$$\mathcal{G}\varrho = \frac{\varrho}{2} \ \text{ and } \mathcal{H}\varrho = \varrho^2 \text{ for all } \varrho \in \mathbb{X}.$$

Further, define a binary relation $\mathcal{R}$ on $\mathbb{X}$ by

$$\mathcal{R} = \{(\varrho, \varsigma) \in \mathbb{X}^2 : \varrho > \varsigma\}.$$

Then for each $\varrho \in \mathbb{X}$,

$$\mathcal{H}\mathcal{G}\varrho = \mathcal{H}(\mathcal{G}\varrho) = \frac{\varrho^2}{4} < \frac{\varrho}{2} = \mathcal{G}\varrho,$$

implies that $(\mathcal{G}\varrho, \mathcal{H}\mathcal{G}\varrho) \in \mathcal{R}$.
Again,

$$\mathcal{G}\mathcal{H}\varrho = \mathcal{G}(\mathcal{H}\varrho) = \frac{\varrho^2}{2} < \varrho^2 = \mathcal{H}\varrho,$$

implies that $(\mathcal{H}\varrho, \mathcal{G}\mathcal{H}\varrho) \in \mathcal{R}$. Then for each $\varrho \in \mathbb{X}$, the relation $\mathcal{R}$ defined on $\mathbb{X}$ is $(\mathcal{G}, \mathcal{H})$-weakly closed.

Example 3.5. Let $\mathbb{X} = (0, 1)$. Define mappings $\mathcal{G}, \mathcal{H} : \mathbb{X} \to \mathbb{X}$ by

$$\mathcal{G}\varrho = \frac{1}{\varrho} \text{ and } \mathcal{H}\varrho = \frac{1}{\sqrt{\varrho}} \text{ for all } \varrho \in \mathbb{X}.$$

Further, define a binary relation $\mathcal{R}$ on $\mathbb{X}$ by

$$\mathcal{R} = \{(\varrho, \varsigma) \in \mathbb{X}^2 : \varrho > \varsigma\}.$$

Then for each $\varrho \in \mathbb{X}$,

$$\mathcal{H}\mathcal{G}\varrho = \mathcal{H}(\mathcal{G}\varrho) = \sqrt{\varrho} < \frac{1}{\varrho} = \mathcal{G}\varrho,$$

implies that $(\mathcal{G}\varrho, \mathcal{H}\mathcal{G}\varrho) \in \mathcal{R}$.
Again,

$$\mathcal{G}\mathcal{H}\varrho = \mathcal{G}(\mathcal{H}\varrho) = \sqrt{\varrho} < \frac{1}{\sqrt{\varrho}} = \mathcal{H}\varrho,$$

implies that $(\mathcal{H}\varrho, \mathcal{G}\mathcal{H}\varrho) \in \mathcal{R}$. Then for each $\varrho \in \mathbb{X}$, the relation $\mathcal{R}$ defined on $\mathbb{X}$ is $(\mathcal{G}, \mathcal{H})$-weakly closed.

Remark 3.6. It is important to note that the mappings and relation defined in Example 3.5, are neither $\mathcal{G}$-closed nor $\mathcal{H}$-closed. However, the pair of mappings $(\mathcal{G}, \mathcal{H})$-weakly closed.

Indeed, let $\varrho = \frac{1}{2}$ and $\varsigma = \frac{1}{4}$ are such that $\frac{1}{2} > \frac{1}{4}$. This implies that $(\frac{1}{2}, \frac{1}{4}) \in \mathcal{R}$. But

$$\mathcal{G}(\frac{1}{2}) = 2 \not> 4 = \mathcal{G}\big(\frac{1}{4}\big) = 4 \text{ implies } \Big(\mathcal{G}(\frac{1}{2}), \mathcal{G}(\frac{1}{4})\Big) \notin \mathcal{R}.$$

Similarly,

$$\mathcal{H}(\frac{1}{2}) = 1.414214 \not> \mathcal{H}(\frac{1}{4}) = 2 \text{ implies } \Big(\mathcal{H}(\frac{1}{2}), \mathcal{H}(\frac{1}{4})\Big) \notin \mathcal{R}.$$

4. Common fixed point results under $(\mathcal{G}, \mathcal{H})$-weakly closed relations

In this section, we utilize the definition of $(\mathcal{G}, \mathcal{H})$-weakly closed relation to prove the existence of common fixed point for pair of self mappings in relation theoretic metric spaces. Later, we prove another result that guarantees the uniqueness of common fixed point for maps under certain assumptions on the space $\mathbb{X}$.

For a given binary relation $\mathcal{R}$, and self-mappings $\mathcal{G}$ and $\mathcal{H}$ on a nonempty set $\mathbb{X}$, we use the following notations:

(1) $\mathbb{X}(\mathcal{G}, \mathcal{R}) = \{\varrho_0 \in \mathbb{X} : (\varrho_0, \mathcal{G}\varrho_0) \in \mathcal{R}\}$,
(2) $\mathbb{X}(\mathcal{G}, \mathcal{H}, \mathcal{R}) = \{\varrho_0 \in \mathbb{X} : (\mathcal{G}\varrho_0, \mathcal{H}\varrho_0) \in \mathcal{R}\}$.

Theorem 4.1. *Let $\mathcal{R}$ be a arbitrary binary relation defined on a nonempty set $\mathbb{X}$, and let d be a metric defined on $\mathbb{X}$ such that $(\mathbb{X}, d)$ is a relation theoretic metric space. Define self-maps $\mathcal{G}, \mathcal{H} : \mathbb{X} \to \mathbb{X}$ such that the following conditions hold:*

(1) $\mathbb{X}(\mathcal{G}, \mathcal{H}, \mathcal{R}) \neq \emptyset$,
(2) $\mathcal{R}$ *is* $(\mathcal{G}, \mathcal{H})$*-weakly closed,*
(3) for all $\varrho, \varsigma \in \mathbb{X}$ *with* $(\varrho, \varsigma) \in \mathcal{R}$*, the mappings* $\mathcal{G}$ *and* $\mathcal{H}$ *satisfy*

$$\alpha(d(\mathcal{G}\varrho, \mathcal{H}\varsigma)) \leq \beta(\mathcal{M}(\varrho, \varsigma)), \tag{4.1}$$

where

$$\mathcal{M}(\varrho, \varsigma) \in \left\{ \frac{d(\varsigma, \mathcal{H}\varsigma) d(\varrho, \mathcal{G}\varrho)}{[1 + d(\varrho, \varsigma)]}, \frac{d(\varsigma, \mathcal{H}\varsigma)[1 + d(\varrho, \mathcal{G}\varrho)]}{[1 + d(\varrho, \varsigma)]} \right\}, \tag{4.2}$$

$\alpha : [0, \infty) \to [0, \infty)$ *is an altering distance function and* $\beta : [0, \infty) \to [0, \infty)$ *is a continuous function such that*

$$0 < \beta(s) < \alpha(s), \ \forall \ s > 0. \tag{4.3}$$

Then $\mathcal{G}$ *and* $\mathcal{H}$ *have a common fixed point in* $\mathbb{X}$*.*

Proof. Since $\mathbb{X}(\mathcal{G}, \mathcal{H}, \mathcal{R})$ is nonempty, therefore, there exists $\varrho_0 \in \mathbb{X}$ such that $(\mathcal{G}\varrho_0, \mathcal{H}\varrho_0) \in \mathcal{R}$. If $\mathcal{G}\varrho_0 = \mathcal{H}\varrho_0 = \varrho_0$, then ϱ_0 is a common fixed point of maps $\mathcal{G}$ and $\mathcal{H}$. This established the proof.

Assume that $\mathcal{G}\varrho_0 \neq \mathcal{H}\varrho_0 \neq \varrho_0$. Therefore there exist some other points, say $\varrho_1, \varrho_2 \in \mathbb{X}$, and a sequence $\{\varsigma_i\}$ such that

$$\varsigma_0 = \mathcal{G}\varrho_0 = \varrho_1 \text{ and } \varsigma_1 = \mathcal{H}\varrho_1 = \varrho_2.$$

Since $\varrho_0, \varrho_1, \varrho_2 \in \mathbb{X}$, and the relation $\mathcal{R}$ is $(\mathcal{G}, \mathcal{H})$-weakly closed, then, we have

$$(\mathcal{G}\varrho_0, \mathcal{H}\mathcal{G}\varrho_0) \in \mathcal{R} \quad \text{and} \quad (\mathcal{H}\varrho_1, \mathcal{G}\mathcal{H}\varrho_1) \in \mathcal{R},$$

implies

$$(\varrho_1, \mathcal{H}\varrho_1) \in \mathcal{R} \quad \text{and} \quad (\varrho_2, \mathcal{G}\varrho_2) \in \mathcal{R},$$

further, implies

$$(\varrho_1, \varrho_2) \in \mathcal{R} \quad \text{and} \quad (\varrho_2, \varrho_3) \in \mathcal{R}.$$

Proceeding in this manner we can, in a general sense, choose two sequences $\{\varrho_{2i}\}, \{\varrho_{2i+1}\} \subset \mathbb{X}$, and a sequence $\{\varsigma_{2i}\}$ such that

$$\begin{aligned} &\varrho_{2i+1} = \mathcal{G}\varrho_{2i} = \varsigma_{2i}, \\ &\varrho_{2i+2} = \mathcal{H}\varrho_{2i+1} = \varsigma_{2i+1}, \quad \forall\, i \in \mathbb{N}. \end{aligned} \tag{4.4}$$

Further, on using the fact that relation $\mathcal{R}$ is $(\mathcal{G}, \mathcal{H})$-weakly closed, we have

$$(\mathcal{G}\varrho_{2i}, \mathcal{H}\mathcal{G}\varrho_{2i}) \in \mathcal{R} \quad \text{and} \quad (\mathcal{H}\varrho_{2i+1}, \mathcal{G}\mathcal{H}\varrho_{2i+1}) \in \mathcal{R},$$

implies

$$(\varrho_{2i+1}, \mathcal{H}\varrho_{2i+1}) \in \mathcal{R} \quad \text{and} \quad (\varrho_{2i+2}, \mathcal{G}\varrho_{2i+2}) \in \mathcal{R},$$

this gives that

$$(\varrho_{2i+1}, \varrho_{2i+2}) \in \mathcal{R} \quad \text{and} \quad (\varrho_{2i+2}, \varrho_{2i+3}) \in \mathcal{R}$$

and so,

$$(\varsigma_{2i}, \varsigma_{2i+1}) \in \mathcal{R} \quad \text{and} \quad (\varsigma_{2i+1}, \varsigma_{2i+2}) \in \mathcal{R}, \text{ for all } i \in \mathbb{N},$$

which yields that the sequence $\{\varsigma_i\}$ is $\mathcal{R}$-preserving.

Assume that

$$\varsigma_{2i} = \varsigma_{2i+1} \text{ for } i \notin \mathbb{N}. \tag{4.5}$$

Since ς_{2i} and ς_{2i+1} are $\mathcal{R}$-preserving. Therefore, for all $i \in \mathbb{N}$, on substituting $\varrho = \varrho_{2i}$ and $\varsigma = \varrho_{2i+1}$, and making use of Eq.(4.4) in Eq.(4.1), we get

$$\begin{aligned} \alpha(d(\varsigma_{2i}, \varsigma_{2i+1})) &= \alpha(d(\mathcal{G}\varrho_{2i}, \mathcal{H}\varrho_{2i+1})) \\ &\leq \beta(\mathcal{M}(\varrho_{2i}, \varrho_{2i+1})), \end{aligned} \tag{4.6}$$

where

$$\mathcal{M}(\varrho_{2i},\varrho_{2i+1}) \in \left\{ \begin{array}{c} \dfrac{d(\varrho_{2i+1},\mathcal{H}\varrho_{2i+1})\,d(\varrho_{2i},\mathcal{G}\varrho_{2i})}{1+d(\varrho_{2i},\varrho_{2i+1})} \\ \dfrac{d(\varrho_{2i+1},\mathcal{H}\varrho_{2i+1})\,[1+d(\varrho_{2i},\mathcal{G}\varrho_{2i})]}{1+d(\varrho_{2i},\varrho_{2i+1})} \end{array} \right\}$$

$$\subseteq \left\{ \begin{array}{c} \dfrac{d(\varrho_{2i+1},\varrho_{2i+2})\,d(\varrho_{2i},\varrho_{2i+1})}{1+d(\varrho_{2i},\varrho_{2i+1})} \\ \dfrac{d(\varrho_{2i+1},\varrho_{2i+2})\,[1+d(\varrho_{2i},\varrho_{2i+1})]}{1+d(\varrho_{2i},\varrho_{2i+1})} \end{array} \right\}$$

$$\subseteq \left\{ \frac{d(\varrho_{2i+1},\varrho_{2i+2})\,d(\varrho_{2i},\varrho_{2i+1})}{1+d(\varrho_{2i},\varrho_{2i+1})},\ d(\varrho_{2i+1},\varrho_{2i+2}) \right\}$$

$$\subseteq \left\{ \frac{d(\varsigma_{2i},\varsigma_{2i+1})\,d(\varsigma_{2i-1},\varsigma_{2i})}{1+d(\varsigma_{2i-1},\varsigma_{2i})},\ d(\varsigma_{2i},\varsigma_{2i+1}) \right\}. \tag{4.7}$$

This implies that: Either

$$\mathcal{M}(\varrho_{2i},\varrho_{2i+1}) = \frac{d(\varsigma_{2i},\varsigma_{2i+1})d(\varsigma_{2i-1},\varsigma_{2i})}{1+d(\varsigma_{2i-1},\varsigma_{2i})}$$

or

$$\mathcal{M}(\varrho_{2i},\varrho_{2i+1}) = d(\varsigma_{2i},\varsigma_{2i+1}).$$

Suppose, $\mathcal{M}(\varrho_{2i},\varrho_{2i+1}) = \dfrac{d(\varsigma_{2i},\varsigma_{2i+1})d(\varsigma_{2i-1},\varsigma_{2i})}{1+d(\varsigma_{2i-1},\varsigma_{2i})}$, then from (4.6), we get

$$\alpha(d(\varsigma_{2i},\varsigma_{2i+1})) = \alpha(d(\mathcal{G}\varrho_{2i},\mathcal{H}\varrho_{2i+1})) \le \beta\left(\frac{d(\varsigma_{2i},\varsigma_{2i+1})d(\varsigma_{2i-1},\varsigma_{2i})}{1+d(\varsigma_{2i-1},\varsigma_{2i})}\right). \tag{4.8}$$

Since $\dfrac{d(\varsigma_{2i},\varsigma_{2i+1})d(\varsigma_{2i-1},\varsigma_{2i})}{1+d(\varsigma_{2i-1},\varsigma_{2i})} > 0$, then on using the condition (4.3), we get

$$\beta\left(\frac{d(\varsigma_{2i},\varsigma_{2i+1})d(\varsigma_{2i-1},\varsigma_{2i})}{1+d(\varsigma_{2i-1},\varsigma_{2i})}\right) < \alpha\left(\frac{d(\varsigma_{2i},\varsigma_{2i+1})d(\varsigma_{2i-1},\varsigma_{2i})}{1+d(\varsigma_{2i-1},\varsigma_{2i})}\right). \tag{4.9}$$

Since $0 < \dfrac{d(\varsigma_{2i-1},\varsigma_{2i})}{1+d(\varsigma_{2i-1},\varsigma_{2i})} < 1$ for all $i \in \mathbb{N}$, we have

$$\left(\frac{d(\varsigma_{2i},\varsigma_{2i+1})d(\varsigma_{2i-1},\varsigma_{2i})}{1+d(\varsigma_{2i-1},\varsigma_{2i})}\right) < d(\varsigma_{2i},\varsigma_{2i+1}),$$

and hence, by using the monotonic property of α, we have

$$\alpha\left(\frac{d(\varsigma_{2i},\varsigma_{2i+1})d(\varsigma_{2i-1},\varsigma_{2i})}{1+d(\varsigma_{2i-1},\varsigma_{2i})}\right) < \alpha\Big(d(\varsigma_{2i},\varsigma_{2i+1})\Big). \tag{4.10}$$

Combining Eq. (4.8), Eq.(4.9) and Eq.(4.10), we get

$$\alpha(d(\varsigma_{2i},\varsigma_{2i+1})) < \alpha(d(\varsigma_{2i},\varsigma_{2i+1})).$$

If $\mathcal{M}(\varrho_{2i}, \varrho_{2i+1}) = d(\varsigma_{2i}, \varsigma_{2i+1})$, then again from (4.6), we get

$$\begin{aligned} \alpha(d(\varsigma_{2i}, \varsigma_{2i+1})) &= \alpha(d(\mathcal{G}\varrho_{2i}, \mathcal{H}\varrho_{2i+1})) \\ &\leq \beta(d(\varsigma_{2i}, \varsigma_{2i+1})) \\ &< \alpha(d(\varsigma_{2i}, \varsigma_{2i+1})). \end{aligned} \tag{4.11}$$

Thus, we arrive at a contradiction due to the positivity of $\alpha(d(\varsigma_{2i}, \varsigma_{2i+1}))$ when $d(\varsigma_{2i}, \varsigma_{2i+1}) > 0$, since α is an altering distance function. So our assertion in (4.5) is false. Therefore, $\varsigma_{2i} = \varsigma_{2i+1}$ for some $i \in \mathbb{N}$. That is, for some $i = k$, we have $\varsigma_{2k} = \varsigma_{2k+1}$. Hence, from Eq. (4.3), we obtain

$$\mathcal{H}\varrho_{2i+1} = \varrho_{2i+2} = \varrho_{2i+1} = \nu \text{ (say)}.$$

Consequently, we have $\mathcal{H}\nu = \nu$.

Next, we claim that ν is also fixed point of map $\mathcal{G}$. Since $\nu \in \mathbb{X}(\mathcal{G}, \mathcal{H}, \mathcal{R})$. Thus, we have $(\mathcal{G}\nu, \mathcal{H}\nu) \in \mathcal{R}$. Substitute, $\varrho = \varsigma = \nu$ in Eq. (4.1), we have

$$\alpha(d(\mathcal{G}\nu, \nu)) = \alpha(d(\mathcal{G}\nu, \mathcal{H}\nu)) \leq \beta(\mathcal{M}(\nu, \nu)), \tag{4.12}$$

where

$$\begin{aligned} \mathcal{M}(\nu, \nu) &\in \left\{ \frac{d(\nu, \mathcal{H}\nu)\, d(\nu, \mathcal{G}\nu)}{1 + d(\nu, \nu)}, \frac{d(\nu, \mathcal{H}\nu)\, [1 + d(\nu, \mathcal{G}\nu)]}{1 + d(\nu, \nu)} \right\} \\ &\subseteq \left\{ \frac{d(\nu, \nu)\, d(\nu, \nu)}{1 + d(\nu, \nu)}, \frac{d(\nu, \nu)\, [1 + d(\nu, \nu)]}{1 + d(\nu, \nu)} \right\} \\ &= \{0\}. \end{aligned} \tag{4.13}$$

In view of Eq.(4.12), we have

$$\alpha(d(\mathcal{G}\nu, \nu)) = \alpha(d(\mathcal{G}\nu, \mathcal{H}\nu)) = \beta(0).$$

On using Eq.(4.3) and the fact that α and β are continuous at 0, taking $s \to 0^+$ gives $\beta(0) = 0$. Thus, $d(\mathcal{G}\nu, \nu) = 0$. that is, $\mathcal{G}\nu = \nu$. Therefore, we get $\mathcal{G}\nu = \mathcal{H}\nu = \nu$. This proves that ν is the common fixed point of maps $\mathcal{G}$ and $\mathcal{H}$. □

Remark 4.2. From Eq.(4.1) it can be observed that if l is a fixed point of $\mathcal{H}$, then $\mathcal{G}\nu = l \;\forall\; (\nu, l) \in \mathcal{R}$. This implies that every fixed point of $\mathcal{H}$ is also a fixed point of $\mathcal{G}$.

In the subsequent result, we build upon the foundation established in Theorem 4.1 by extending its scope to guarantee the uniqueness of a common fixed point, under the assumption that the space $\mathbb{X}$ is $\mathcal{R}^S$ - directed.

Theorem 4.3. *Under the assumptions of Theorem* 4.1*, further suppose that the space* $\mathbb{X}$ *is* $\mathcal{R}^S$*-directed. Then the mappings* $\mathcal{G}$ *and* $\mathcal{H}$ *admits a unique common fixed point.*

Proof. Assume on contrary that maps $\mathcal{G}$ and $\mathcal{H}$ have two common fixed points, say ν, μ with $\nu \neq \mu$ such that

$$\mathcal{G}\nu = \mathcal{H}\nu = \nu \quad \text{and} \quad \mathcal{G}\mu = \mathcal{H}\mu = \mu. \tag{4.14}$$

Since $\nu \neq \mu$, then $d(\nu, \mu) > 0$. We are required to show that $\nu = \mu$.

To prove this, we will consider two different cases:

Case-1: If ν is related to μ i.e., $(\nu, \mu) \in \mathcal{R}$. Then, from Eq.(4.14), $\nu = \mathcal{G}\nu$ is related to $\mu = \mathcal{H}\mu$, and so $(\mathcal{G}\nu, \mathcal{H}\mu) \in \mathcal{R}$. Further, from Remark 4.2, we have $\mathcal{G}\nu = \mu$. Thus from Eq. (4.1), on substituting $\varrho = \nu$ and $\varsigma = \mu$, we have

$$\alpha(d(\nu, \mu)) = \alpha(d(\mathcal{G}\nu, \mathcal{H}\mu)) \leq \beta(\mathcal{M}(\nu, \mu)), \tag{4.15}$$

where

$$\begin{aligned}\mathcal{M}(\nu, \mu) &\in \left\{\frac{d(\mu, \mathcal{H}\mu)d(\nu, \mathcal{G}\nu)}{[1 + d(\nu, \mu)]}, \frac{d(\mu, \mathcal{H}\mu)[1 + d(\nu, \mathcal{G}\nu)]}{[1 + d(\nu, \mu)]}\right\} \\ &\subseteq \left\{\frac{d(\mu, \mu)d(\nu, \nu)}{[1 + d(\nu, \mu)]}, \frac{d(\mu, \mu)[1 + d(\nu, \nu)]}{[1 + d(\nu, \mu)]}\right\} \\ &= \{0\}. \end{aligned} \tag{4.16}$$

Eq. (4.15) gives that

$$\alpha(d(\nu, \mu)) \leq \beta(0). \tag{4.17}$$

On using Eq.(4.3) and the fact that α and β are continuous imply that $\beta(0) = 0$. Therefore, $d(\nu, \mu) = 0$. This is a contradiction to our assumption that $d(\nu, \mu) \geq 0$. So, our assumption is wrong and hence $\nu = \mu$.

Case-2: If ν is not related to μ, that is, $(\nu, \mu) \notin \mathcal{R}$. Since $\mathcal{R}^S$-directed, therefore there exist a $\theta \in \mathbb{X}$, different from ν and μ, such that θ is related to ν and θ is related to μ. That is,

$$(\nu, \theta) \in \mathcal{R} \quad \text{and} \quad (\mu, \theta) \in \mathcal{R}.$$

Conversely, on using Eq.(4.14), we have

$$(\mathcal{G}\nu, \mathcal{H}\theta) \in \mathcal{R} \quad \text{and} \quad (\mathcal{G}\mu, \mathcal{H}\theta) \in \mathcal{R}.$$

It implies that

$$(\nu, \mathcal{H}\theta) \in \mathcal{R} \quad \text{and} \quad (\mu, \mathcal{H}\theta) \in \mathcal{R}.$$

(1) First assume that if $(\nu, \mathcal{H}\theta) \in \mathcal{R}$. Then from Eq. (4.1), on substituting $\varrho = \nu$ and $\varsigma = \theta$, we have

$$\alpha(d(\nu, \mathcal{H}\theta)) = \alpha(d(\mathcal{G}\nu, \mathcal{H}\theta)) \leq \beta(\mathcal{M}(\nu, \theta)), \tag{4.18}$$

where

$$\begin{aligned}\mathcal{M}(\nu,\theta) &\in \left\{\frac{d(\theta,\mathcal{H}\theta)d(\nu,\mathcal{G}\nu)}{[1+d(\nu,\theta)]}, \frac{d(\theta,\mathcal{H}\theta)[1+d(\nu,\mathcal{G}\nu)]}{[1+d(\nu,\theta)]}\right\} \\ &\subseteq \left\{\frac{d(\theta,\theta)d(\nu,\nu)}{[1+d(\nu,\theta)]}, \frac{d(\theta,\theta)[1+d(\nu,\nu)]}{[1+d(\nu,\theta)]}\right\} \\ &= \{0\}. \end{aligned} \tag{4.19}$$

Eq. (4.18) gives that

$$\alpha(d(\nu,\mathcal{H}\theta)) \leq \beta(0). \tag{4.20}$$

Again, on using Eq.(4.3) and the fact that α and β are continuous gives that $\beta(0) = 0$. Therefore, $d(\nu,\mathcal{H}\theta) = 0$, i.e., $\mathcal{H}\theta = \nu$.

(2) Secondly, if $(\mu,\mathcal{H}\theta) \in \mathcal{R}$, then again from Eq. (4.1), on substituting $\varrho = \mu$ and $\varsigma = \theta$, we have

$$\alpha(d(\mu,\mathcal{H}\theta)) = \alpha(d(\mathcal{G}\mu,\mathcal{H}\theta)) \leq \beta(\mathcal{M}(\mu,\theta)), \tag{4.21}$$

where

$$\begin{aligned}\mathcal{M}(\mu,\theta) &\in \left\{\frac{d(\theta,\mathcal{H}\theta)d(\mu,\mathcal{G}\mu)}{[1+d(\mu,\theta)]}, \frac{d(\theta,\mathcal{H}\theta)[1+d(\mu,\mathcal{G}\mu)]}{[1+d(\mu,\theta)]}\right\} \\ &\subseteq \left\{\frac{d(\theta,\theta)d(\mu,\mu)}{[1+d(\mu,\theta)]}, \frac{d(\theta,\theta)[1+d(\mu,\mu)]}{[1+d(\mu,\theta)]}\right\} \\ &= \{0\}. \end{aligned} \tag{4.22}$$

Eq.(4.21) gives that

$$\alpha(d(\mu,\mathcal{H}\theta)) \leq \beta(0). \tag{4.23}$$

Again, on using Eq.(4.3) and the fact that α and β are continuous gives that $\beta(0) = 0$. Therefore, $d(\mu,\mathcal{H}\theta) = 0$. Thus $\mathcal{H}\theta = \mu$.

As the function $\mathcal{H}$ is well-defined, we get $\mathcal{H}\theta = \mu = \nu$. This competes the proof of the Theorem 4.3. □

Remark 4.4. In proving our main result, $\mathcal{R}$-continuity of maps, and $\mathcal{R}$-completeness of the space not necessary required to guarantee the existence and uniqueness of common fixed points of maps.

Remark 4.5. If we let $\mathcal{G} = \mathcal{H}$, in above theorems (Theorem 4.1 and Theorem 4.3), is it possible to get fixed point of the map without $\mathcal{R}$-continuity, and without $\mathcal{R}$-completeness of the space. Here we are leaving this problem for future research.

5. Corollaries and numerical illustration

In this section we first give some consequence finding of our main result, some of them are generalizations of existing results but some of them are noval in nature.

On letting $\alpha(s) = s$ in the Theorem 4.1 and in Theorem 4.3, we have the following results:

Corollary 5.1. *Let $\mathcal{R}$ be a arbitrary binary relation defined on a non-empty set $\mathbb{X}$, and let d be a metric defined on $\mathbb{X}$ such that $(\mathbb{X}, d)$ is a relation theoretic metric space. Define self-maps $\mathcal{G}, \mathcal{H} : \mathbb{X} \to \mathbb{X}$ such that the following conditions hold:*

(1) $\mathbb{X}(\mathcal{G}, \mathcal{H}, \mathcal{R}) \neq \emptyset$,
(2) $\mathcal{R}$ *is* $(\mathcal{G}, \mathcal{H})$*-weakly closed,*
(3) for all $\varrho, \varsigma \in \mathbb{X}$ *with* $(\varrho, \varsigma) \in \mathcal{R}$*, maps* $\mathcal{G}$ *and* $\mathcal{H}$ *satisfies,*

$$d(\mathcal{G}\varrho, \mathcal{H}\varsigma) \leq \beta(\mathcal{M}(\varrho, \varsigma)), \tag{5.1}$$

where

$$\mathcal{M}(\varrho, \varsigma) \in \left\{ \frac{d(\varsigma, \mathcal{H}\varsigma)d(\varrho, \mathcal{G}\varrho)}{[1 + d(\varrho, \varsigma)]}, \frac{d(\varsigma, \mathcal{H}\varsigma)[1 + d(\varrho, \mathcal{G}\varrho)]}{[1 + d(\varrho, \varsigma)]} \right\}$$

and $\beta : [0, \infty) \to [0, \infty)$ *is a continuous function such that*

$$0 < \beta(s) < s, \ \forall \ s > 0. \tag{5.2}$$

Then $\mathcal{G}$ and $\mathcal{H}$ have a common fixed point in $\mathbb{X}$. Further, suppose that the space $\mathbb{X}$ is $\mathcal{R}^S$-directed. Then the mappings $\mathcal{G}$ and $\mathcal{H}$ have a unique common fixed point.

Lets recall Eq. (4.2) as

$$\mathcal{M}(\varrho, \varsigma) \in \left\{ \frac{d(\varsigma, \mathcal{H}\varsigma)d(\varrho, \mathcal{G}\varrho)}{[1 + d(\varrho, \varsigma)]}, \frac{d(\varsigma, \mathcal{H}\varsigma)[1 + d(\varrho, \mathcal{G}\varrho)]}{[1 + d(\varrho, \varsigma)]} \right\}.$$

Then we have following possibilities for $\mathcal{M}(\varrho, \varsigma)$: Either

$$\mathcal{M}(\varrho, \varsigma) = \max \left\{ \frac{d(\varsigma, \mathcal{H}\varsigma)d(\varrho, \mathcal{G}\varrho)}{[1 + d(\varrho, \varsigma)]}, \frac{d(\varsigma, \mathcal{H}\varsigma)[1 + d(\varrho, \mathcal{G}\varrho)]}{[1 + d(\varrho, \varsigma)]} \right\},$$

$$\mathcal{M}(\varrho, \varsigma) = \min \left\{ \frac{d(\varsigma, \mathcal{H}\varsigma)d(\varrho, \mathcal{G}\varrho)}{[1 + d(\varrho, \varsigma)]}, \frac{d(\varsigma, \mathcal{H}\varsigma)[1 + d(\varrho, \mathcal{G}\varrho)]}{[1 + d(\varrho, \varsigma)]} \right\},$$

$$\mathcal{M}(\varrho, \varsigma) = \frac{d(\varsigma, \mathcal{H}\varsigma)d(\varrho, \mathcal{G}\varrho)}{[1 + d(\varrho, \varsigma)]}$$

or

$$\mathcal{M}(\varrho, \varsigma) = \frac{d(\varsigma, \mathcal{H}\varsigma)[1 + d(\varrho, \mathcal{G}\varrho)]}{[1 + d(\varrho, \varsigma)]}.$$

On substituting any one of the above possibilities, our next corollary is a sharpened version of our main result (Theorem 4.1 and Theorem 4.3).

Corollary 5.2. *Let $\mathcal{R}$ be a arbitrary binary relation defined on a non-empty set $\mathbb{X}$, and let d be a metric defined on $\mathbb{X}$ such that $(\mathbb{X}, d)$ is a relation theoretic metric space. Define self-maps $\mathcal{G}, \mathcal{H} : \mathbb{X} \to \mathbb{X}$ such that the following conditions hold:*

(1) $\mathbb{X}(\mathcal{G}, \mathcal{H}, \mathcal{R}) \neq \emptyset$,
(2) $\mathcal{R}$ *is* $(\mathcal{G}, \mathcal{H})$*-weakly closed,*
(3) for all $\varrho, \varsigma \in \mathbb{X}$ *with* $(\varrho, \varsigma) \in \mathcal{R}$*, maps* $\mathcal{G}$ *and* $\mathcal{H}$ *satisfies,*

$$\alpha(d(\mathcal{G}\varrho, \mathcal{H}\varsigma)) \leq \beta(\mathcal{M}(\varrho, \varsigma)), \tag{5.3}$$

where α is an altering distance function and $\beta : [0, \infty) \to [0, \infty)$ is a continuous function such that

$$0 < \beta(s) < \alpha(s), \ \forall \ s > 0. \tag{5.4}$$

Then $\mathcal{G}$ and $\mathcal{H}$ have a common fixed point in $\mathbb{X}$. Further, suppose that the space $\mathbb{X}$ is $\mathcal{R}^S$-directed. Then the mappings $\mathcal{G}$ and $\mathcal{H}$ have a unique common fixed point.

Next, we provide an example of a metric space equipped with a binary relation where the mappings are not necessarily continuous, but relation $\mathcal{R}$ is $(\mathcal{G}, \mathcal{H})$-weakly closed.

Example 5.3. Let $\mathbb{X} = (0, 0.7]$ and let d be a metric defined on $\mathbb{X}$ by

$$d(\varrho, \varsigma) = \frac{|\varrho - \varsigma|}{1 + |\varrho - \varsigma|}.$$

Define a binary relation $\mathcal{R}$ on $\mathbb{X}$ by

$$\mathcal{R} = \left\{(\varrho, \varsigma) \in \mathbb{X}^2 : \varrho \mathcal{R} \varsigma \text{ implies } |\varrho - \varsigma| \geq 0\right\}.$$

Then $(\mathbb{X}, d, \mathcal{R})$ is a relation theoretic metric space.

Define mappings $\mathcal{G}, \mathcal{H} : \mathbb{X} \to \mathbb{X}$ by

$$\mathcal{G}\varrho = \begin{cases} \frac{\varrho}{4}, \text{ if } \varrho \in (0, 0.7) \\ 0.7, \text{ if } \varrho = 0.7 \end{cases} \quad \text{and} \quad \mathcal{H}\varrho = \begin{cases} \frac{\varrho}{8}, \text{ if } \varrho \in (0, 0.7) \\ 0.7, \text{ if } \varrho = 0.7 \end{cases}.$$

Then, clearly, relation $\mathcal{R}$ is $(\mathcal{G}, \mathcal{H})$-weakly closed.

TABLE 1. Behaviour of the inequality (4.1) in context of Example 5.3 for random values of ϱ and ς in $\mathbb{X}$.

ϱ	ς	$\mathcal{G}\varrho$	$\mathcal{G}\varsigma$	$\mathcal{H}\varrho$	$\mathcal{H}\varsigma$	L.H.S	R.H.S	
							R_1	R_2
0.100	0.690	0.025	0.172	0.012	0.086	0.003	0.019	0.227
0.110	0.100	0.028	0.025	0.014	0.012	0.000	0.006	0.079
0.150	0.200	0.038	0.050	0.019	0.025	0.000	0.014	0.135
0.200	0.180	0.050	0.045	0.025	0.022	0.001	0.017	0.131
0.250	0.660	0.062	0.165	0.031	0.082	0.000	0.043	0.247
0.300	0.680	0.075	0.170	0.038	0.085	0.000	0.051	0.257
0.350	0.210	0.088	0.052	0.044	0.026	0.003	0.028	0.143
0.400	0.130	0.100	0.032	0.050	0.016	0.006	0.019	0.094
0.450	0.310	0.112	0.078	0.056	0.039	0.005	0.046	0.192
0.500	0.590	0.125	0.148	0.062	0.074	0.002	0.079	0.286
0.550	0.460	0.138	0.115	0.069	0.058	0.005	0.072	0.255
0.600	0.670	0.150	0.168	0.075	0.084	0.004	0.097	0.312
0.650	0.640	0.162	0.160	0.081	0.080	0.006	0.104	0.321
0.690	0.140	0.172	0.035	0.086	0.018	0.018	0.027	0.097
0.700	0.700	0.700	0.700	0.700	0.700	0.000	0.000	0.000

Further, define two mappings $\beta, \alpha : [0, \infty) \to [0, \infty)$ by

$$\beta(\varrho) = \frac{\varrho}{1+\varrho} \quad \text{and } \alpha(\varrho) = \varrho^2 \text{ for all } \varrho > 0.$$

On referring Table 1, we can see that for all $\varrho, \varsigma \in \mathbb{X}$ with $(\varrho, \varsigma) \in \mathcal{R}$, we have

$$\alpha(d(\mathcal{G}\varrho, \mathcal{H}\varsigma)) \leq \beta(\mathcal{M}(\varrho, \varsigma)),$$

where

$$\mathcal{M}(\varrho, \varsigma) \in \{R_1, R_2\}$$

with

$$R_1 = \frac{d(\varsigma, \mathcal{H}\varsigma)d(\varrho, \mathcal{G}\varrho)}{[1 + d(\varrho, \varsigma)]} \qquad \text{and} \qquad R_2 = \frac{d(\varsigma, \mathcal{H}\varsigma)[1 + d(\varrho, \mathcal{G}\varrho)]}{[1 + d(\varrho, \varsigma)]}.$$

Thus, all the assumptions of Theorem 4.1 are satisfied. Moreover, the maps $\mathcal{G}$ and $\mathcal{H}$ have a unique common fixed points $0.7 \in \mathbb{X}$.

6. Conclusion

This article presents the concept of a weakly closed relation for two mappings and employs this concept to establish common fixed point theorems in relation-theoretic metric spaces. The findings were obtained without enforcing the conventional assumption of continuity on the mappings or imposing the completeness property of the underlying space. We also present several remarks and illustrative examples to support and validate the significance of our findings. Graphical representations were provided to illustrate and validate the obtained results.

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References

[1] A. Alam, R. George and M. Imdad, *Refinements to relation-theoretic contraction principle,* Axioms, **11**(7) (2022): 316, doi:10.3390/axioms11070316.

[2] A. Alam and M. Imdad, *Relational-theoretic contraction principle,* J. Fixed Point Theory Appl., **17**(4) (2015), 693–702, doi:10.1007/s11784-015-0247-y.

[3] A. Alam and M. Imdad, *Relation-theoretic metrical coincidence theorems,* Filomat, **31**(14) (2017), 4421–4439, doi:10.2298/FIL174421A.

[4] Y.I. Albert and S. Guerre-Delabriere, *Principle of weakly contractive maps in Hilbert spaces, in: New Results in Operator Theory and its Applications,* Oper. Theory Adv. Appl. Birkhuser, Basel, **98** (1997), 7–22.

[5] Anjana, N. Mani, A. Sharma and M. Pingale, *Fixed points theorems for generalized C_ψ^β-rational contraction mapping in relational metric space,* J. Interdisciplinary Math., **27**(8) (2024), 1787–1795, doi:10.47974/JIM-2042.

[6] S. Banach, *Sur les oprations dans les ensembles abstraits et leur application aux quations intgrales,* Fund. Math., **3** (1922), 133–181, doi:10.4064/fm-3-1-133-181.

[7] H. Ben-El-Mechaiekh, *The Ran-Reurings fixed point theorem without partial order: a simple proof,* J. Fixed Point Theory Appl., **16**(1-2) (2015), 373–383, doi:10.1007/s11784-015-0218-3.

[8] A. Branciari, *A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces,* Publ. Math. Debrecen, **57**(1-2) (2000), 31–37, doi:10.5486/pmd.2000.2133.

[9] P.N. Dutta and B.S. Choudhury, *A generalisation of contraction principle in metric spaces,* Fixed Point Theory Appl., (2008) Art. ID 406368, doi:10.1155/2008/406368.

[10] K.S. Eke, B. Davvaz and J.G. Oghonyon, *Relation-theoretic common fixed point theorems for a pair of implicit contractive maps in metric spaces,* Commu. Math. Appl., **10**(1)(2019), 159–168, doi:10.26713/cma.v10i1.873.

[11] G. Jungck, *Commuting mappings and fixed points,* Amer. Math. Monthly, **83**(4) (1976), 261–263, doi:10.2307/2318216.

[12] G. Jungck, *Compatible mappings and common fixed points,* Int. J. Math. Math. Sci., **9**(4) (1986), 771–779, doi:10.1155/S0161171286000935.

[13] F.A. Khan, F. Sk. M.G. Alshehri, Q.H. Khan and A. Alam, *Relational Meir-Keeler contractions and common fixed point theorems,* J. Funct. Spaces, **2022**(1) (2022), 1–9.

[14] M.S. Khan, M. Swaleh and S. Sessa, *Fixed point theorems by altering distances between the points,* Bull. Aust. Math. Soc., **30**(1) (1984), 1–9, doi:10.1017/S0004972700001659.

[15] N. Mani, *Generalized C_{ψ}^{β}-rational contraction and fixed point theorem with application to second order differential equation,* Math. Morav., **22**(1) (2018), 43–54, doi:10.5937/matmor1801043m.

[16] N. Mani, Anjana and R. Shukla, *Generalized $\mathcal{R}$-contractions and fixed point theorems with an application to boundary value problem governing transverse oscillation in homogeneous bar,* J. Inequa. Appl., **2025** (2025), 1–27, doi:10.1186/s13660-025-03298-3.

[17] N. Mani, R. Bhardwaj, S. Chauhan and R. Shukla, *Some fixed point theorems of generalized contractions with application to boundary value problem: theory, methods and integrative approaches, in: Recent Developments in Fixed-Point TheoryTheoretical Foundations and Real-World Applications,* Ind. Appl. Math., Springer, Singapore, 2024, pp. 139–157, doi:10.1007/978-981-99-9546-2_4.

[18] J.R. Morales, E.M. Rojas and R.K. Bisht, *Common fixed points for pairs of mappings with variable contractive parameters,* Abstr. Appl. Anal., (2014) Art. ID 209234, doi:10.1155/2014/209234.

[19] S. Nashine and B. Samet, *Fixed point results for mappings satisfying (ψ, ϕ)-weakly contractive condition in partially ordered metric spaces,* Nonlinear Anal., **74**(6) (2011), 2201–2209, doi:10.1016/j.na.2010.11.024.

[20] J.J. Nieto and R. Rodríguez-López, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations,* Order, **22**(3) (2005), 223–239, doi:10.1007/s11083-005-9018-5.

[21] J.J. Nieto and R. Rodríguez-López, *Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations,* Acta Math. Sin. (Engl. Ser.)**23**(12) (2007), 2205–2212, doi:10.1007/s10114-005-0769-0.

[22] M. Pingale, R. Pathak, N. Mani and R. Shukla, *Study of fixed point theorems using c-class function in partially ordered b-metric spaces,* J. Interdisciplinary Math., **27** (2024), 1787–1795, doi:10.47974/JIM-2039.

[23] M. Raji, A. Rajpoot, L. Rathour, L. Mishra and V. Mishra, *Coincidence point results for relation-theoretic contraction mappings in metric spaces with applications,* Open J. Math. Anal., **8** (2024), 1–17, doi:10.30538/psrp-oma2024.0131.

[24] A.M. Ran and M.C.B. Reurings, *A fixed point theorem in partially ordered sets and some applications to matrix equations,* Proc. Amer. Math. Soc., **132**(5) (2004), 1435–1443, doi:10.1090/S0002-9939-03-07220-4.

[25] B.E. Rhoades, *Some theorems on weakly contractive maps,* Nonlinear Anal., **47**(4) (2001), 2683–2693, doi:10.1016/S0362-546X(01)00388-1.

[26] B. Samet and M. Turinici, *Fixed point theorems on a metric space endowed with an arbitrary binary relation and applications,* Commun. Math. Anal., **13**(2) (2012), 82–97, doi:10.1163/15685539x00162.

[27] S. Sessa, *On a weak commutativity condition of mappings in fixed point considerations,* Publ. Inst. Math. (Beograd) (N.S.) **32**(46) (1982), 149–153.

[28] Y. Sun, X. Liu and L. Liu, *Relation-theoretic coincidence and common fixed point results in extended rectangular b-metric spaces with applications,* Symmetry, **14**(8) (2022), doi:10.3390/sym14081588.

[29] M. Turinici, *Abstract comparison principles and multivariable Gronwall-Bellman inequalities,* J. Math. Anal. Appl., **117**(1) (1986), 100–127, doi:10.1016/0022-247X(86)90251-9.

[30] M. Turinici, *Ran-Reurings fixed point results in ordered metric spaces,* Libertas Math., **31** (2011), 49–55.

[31] M. Turinici, *Nieto-Lopez theorems in ordered metric spaces,* Math. Student, **81**(1-4) (2012), 219–229.

[32] I. Wangwe and S. Kumar, *A common fixed point theorem for generalised F-Kannan mapping in metric space with applications,* Abstr. Appl. Anal., (2021) Art. ID 6619877, doi:10.1155/2021/6619877.

[33] I. Wangwe and S. Kumar, *Common fixed point theorems under implicit contractive condition using α property on metric-like spaces employing an arbitrary binary relation with some application,* Int. J. Nonlinear Anal. Appl., **13**(2) (2022), 2325–2346, doi:10.22075/ijnaa.2021.23494.2548.

[34] M.B. Zada and M. Sarwar, *Common fixed point theorems for rational $F_{\mathcal{R}}$-contractive pairs of mappings with applications,* J. Inequal. Appl., (2019), 1–14, doi:10.1186/s13660-018-1952-z.