Nonlinear Functional Analysis and Applications
Vol. 31, No. 3 (2026), pp. 997-1012
ISSN: 1229-1595(print), 2466-0973(online)
https://doi.org/10.22771/nfaa.2026.31.03.18
http://nfaa.kyungnam.ac.kr/journal-nfaa

UNCERTAINTY-AWARE EXTENSIONS OF RADON–NIKODÝM AND HAHN–BANACH THEOREMS IN NEUTROSOPHIC MR-METRIC SPACES

Abed Al-Rahman M. Malkawi[1] and Ayat M. Rabaiah[2]

[1]Department of Mathematics, Faculty of Arts and Science, Amman Arab University, Amman 11953, Jordan
e-mail: a.malkawi@aau.edu.jo; math.malkawi@gmail.com

[2]Department of Mathematics, Faculty of Arts and Science, Amman Arab University, Amman 11953, Jordan
e-mail: a.rabaieha@aau.edu.jo

Abstract. This paper introduces and systematically investigates the concept of Neutrosophic MR-Metric Spaces(NMR-MS), a novel hybrid structure that combines the flexibility of MR-metric spaces with the expressive power of neutrosophic logic. By incorporating the three independent neutrosophic membership degrees—truth ($\mathcal{T}$), indeterminacy ($\mathcal{I}$), and falsity ($\mathcal{F}$)—into the MR-metric framework, we develop a robust mathematical environment capable of modeling uncertainty, indeterminacy, and conflicting information. Within this setting, two foundational theorems are established: the Neutrosophic MR-Radon–Nikodym Theorem, which provides a decomposition of measures into classical and neutrosophic components, and the Neutrosophic Hahn–Banach Extension Theorem, which extends linear functionals while preserving neutrosophic structure. The theoretical results are accompanied by a wide range of applications in measure theory, functional analysis, signal processing, financial risk assessment, and machine learning. This work bridges classical analytical methods with uncertainty-aware models, offering powerful tools for modern applied mathematics and decision-making under indeterminacy.

---

[0]Received January 18, 2026. Revised April 3, 2026. Accepted April 4, 2026.
[0]2020 Mathematics Subject Classification: 28A15, 46A22, 46S40, 54E35, 03E72.
[0]Keywords: Neutrosophic logic, MR-metric spaces, fixed point theory, Radon–Nikodym theorem, Hahn–Banach theorem, uncertainty modeling, measure theory, functional analysis.
[0]Corresponding author: Abed Al-Rahman M. Malkawi(a.malkawi@aau.edu.jo).

1. Introduction

The evolution of metric space theory has witnessed a progressive movement toward more flexible and inclusive structures capable of addressing increasingly sophisticated mathematical problems. Classical metric spaces [9, 10] paved the way for a series of generalizations, such as b-metric spaces [9, 10], and other generalized distances. A significant recent development was introduced in [32], where MR-metric spaces were formulated and shown to have substantial influence on fixed point theory [12, 15, 16, 19, 21, 33, 34, 35] and related areas including graph theory [22], measure theory [18], compactness and separability [23], fractional calculus [27, 28], and fuzzy frameworks [24]. Further contributions extended these ideas to fractional differential equations [1, 8, 13] and their applications.

Concurrently, the demand for mathematical structures capable of representing uncertainty and indeterminacy stimulated the emergence of fuzzy and neutrosophic paradigms. Neutrosophic logic, proposed by Smarandache, extends fuzzy logic by assigning three independent membership degrees: truth ($\mathcal{T}$), indeterminacy ($\mathcal{I}$), and falsity ($\mathcal{F}$). This triadic system has been effectively employed in constructing neutrosophic metric spaces, establishing fixed point results [20], and developing decision-making models [2, 3, 5, 6, 11]. The neutrosophic approach has also been expanded to decision theory and approximate reasoning through various hybrid models [4, 7, 14, 36].

The integration of neutrosophic logic with generalized distance structures has generated promising pathways for formulating analytical tools capable of capturing uncertainty. Studies such as [20] discussed fixed point properties in neutrosophic fuzzy metric environments, whereas [2, 3, 5, 6, 11] expanded the neutrosophic approach to decision theory and approximate reasoning. Other contributions extended these ideas to fractional calculus [1, 8, 13, 27, 28] and differential equation frameworks. Additionally, the theoretical foundations of neutrosophic MR-metric spaces have been further developed in [30] (Hussein et al.), establishing essential structural properties.

Building on these advancements, the present work introduces Neutrosophic MR-Metric Spaces (NMR-MS), which synthesize the structural breadth of MR-metric spaces with the expressive capacity of neutrosophic logic. This combined structure facilitates the development of analytical theorems that extend their classical counterparts while naturally incorporating uncertainty.

The principal contributions of the paper include:

- Formulating the notion of Neutrosophic MR-Metric Spaces and establishing their basic properties;
- Proving a Neutrosophic MR-Radon–Nikodym Theorem that yields a decomposition of measures under neutrosophic settings;

- Establishing a Neutrosophic Hahn–Banach Extension Theorem that ensures linear functional extension while preserving neutrosophic consistency;
- Presenting a broad spectrum of applications that illustrate the practical relevance of the proposed framework.

This study not only expands the theoretical foundations of distance spaces but also provides mathematical tools suited for modeling uncertainty in practical scenarios, continuing the research direction developed in [17, 24, 25, 26, 29, 30, 31, 37]. The framework offers powerful tools for uncertainty-aware analysis and decision-making under indeterminacy, bridging classical analytical methods with modern applied mathematics.

Definition 1.1. ([32]) Consider a nonempty set $\mathbb{X} \neq \emptyset$ and a real number $\mathbb{R} > 1$. A function

$$M : \mathbb{X} \times \mathbb{X} \times \mathbb{X} \to [0, \infty)$$

is termed an MR-metric if it satisfies the following conditions for all $v, \xi, s, \ell_1 \in \mathbb{X}$:

- $M(v, \xi, s) \geq 0$.
- $M(v, \xi, s) = 0$ if and only if $v = \xi = s$.
- $M(v, \xi, s)$ remains invariant under any permutation $p(v, \xi, s)$, that is,
$$M(v, \xi, s) = M(p(v, \xi, s)).$$
- The following inequality holds
$$M(v, \xi, s) \leq \mathbb{R}\left[M(v, \xi, \ell_1) + M(v, \ell_1, s) + M(\ell_1, \xi, s)\right].$$

A structure $(\mathbb{X}, M)$ that adheres to these properties is defined as an MR-metric space.

Definition 1.2. ([20, Neutrosophic MR-Metric Space (NMR-MS)]) A 9-tuple $(\mathcal{Z}, M, \mathcal{T}, \mathcal{F}, \mathcal{I}, \bullet, \diamond, R, \star)$ is called a neutrosophic MR-metric space if

(1) $\mathcal{Z}$ is a non-empty set.
(2) $M : \mathcal{Z} \times \mathcal{Z} \times \mathcal{Z} \to [0, \infty)$ is an MR-metric satisfying:
  (M1) $M(v, \xi, \Im) \geq 0$,
  (M2) $M(v, \xi, \Im) = 0 \iff v = \xi = \Im$,
  (M3) Symmetry under permutations,
  (M4) $M(v, \xi, \Im) \leq R\left[M(v, \xi, \ell) \star M(v, \ell, \Im) \star M(\ell, \xi, \Im)\right]$, $R > 1$.
(3) $\mathcal{T}, \mathcal{F}, \mathcal{I} : \mathcal{Z} \times \mathcal{Z} \times (0, \infty) \to [0, 1]$ are neutrosophic functions satisfying:
  (N1) $\mathcal{T}(v, \xi, \gamma) = 1 \iff v = \xi$ (Truth-Identity),
  (N2) $\mathcal{T}(v, \xi, \gamma) = \mathcal{T}(\xi, v, \gamma)$ (Symmetry),
  (N3) $\mathcal{T}(v, \xi, \gamma) \bullet \mathcal{T}(\xi, \Im, \rho) \leq \mathcal{T}(v, \Im, \gamma + \rho)$ (Triangle Inequality),
  (N4) $\lim_{\gamma \to \infty} \mathcal{T}(v, \xi, \gamma) = 1$ (Asymptotic Behavior).

(4) $\bullet$ (t-norm) and $\diamond$ (t-conorm) are continuous operators generalizing fuzzy logic.
(5) $\star$ is a binary operation generalizing addition (e.g., weighted sum).

2. Main results

Before presenting our main theorems, we establish the necessary foundational concepts and structures. The Neutrosophic MR-Metric Space framework builds upon the MR-metric structure introduced in [32] while incorporating the neutrosophic components developed in [6]. This hybrid structure enables us to handle situations where classical metric properties are insufficient due to the presence of uncertainty, indeterminacy, or conflicting information. Our approach generalizes several existing frameworks, including b-metric spaces [33], and various fuzzy and neutrosophic distance structures [11]. The main results presented in the following section extend classical theorems from functional analysis and measure theory to this enriched context, providing tools for uncertainty-aware mathematical analysis.

Theorem 2.1. (Neutrosophic MR-Radon-Nikodym) *Let $(\mathcal{Z}, \mathfrak{M})$ be a measurable space induced by a NMR-MS, and let μ, ν be σ-finite measures on $\mathfrak{M}$ with $\nu \ll \mu$. Then there exists a unique neutrosophic measurable function $f : \mathcal{Z} \to [0, \infty)$ such that*

$$\nu(E) = \int_E f(\omega)\, d\mu(\omega) + \alpha \cdot \mathcal{I}_E - \beta \cdot \mathcal{F}_E,$$

where

- $\mathcal{I}_E = \sup_{\gamma>0} \inf_{\omega \in E} \mathcal{I}(\omega, \omega_0, \gamma)$,
- $\mathcal{F}_E = \inf_{\gamma>0} \sup_{\omega \in E} \mathcal{F}(\omega, \omega_0, \gamma)$,
- $\alpha, \beta \in [0, 1]$ *with* $\alpha + \beta \leq 1$.

Moreover, f is μ-integrable and the decomposition is orthogonal.

Proof. We proceed in several steps, combining classical Radon-Nikodym theory with neutrosophic structures.

Step 1: Classical Radon-Nikodym Foundation:

Since μ and ν are σ-finite measures with $\nu \ll \mu$, by the classical Radon-Nikodym theorem, there exists a unique μ-integrable function $f : \mathcal{Z} \to [0, \infty)$ such that

$$\nu_c(E) = \int_E f(\omega)\, d\mu(\omega)$$

for all $E \in \mathfrak{M}$, where ν_c is the absolutely continuous part of ν with respect to μ.

Step 2: Neutrosophic Singular Component:

Define the singular component $\nu_s = \nu - \nu_c$. Since $\nu \ll \mu$, we have $\nu_s \perp \mu$. We decompose ν_s using neutrosophic components:

Let $\omega_0 \in \mathcal{Z}$ be a fixed reference point. For each $E \in \mathfrak{M}$, define

$$\mathcal{I}_E = \sup_{\gamma>0} \inf_{\omega\in E} \mathcal{I}(\omega, \omega_0, \gamma),$$
$$\mathcal{F}_E = \inf_{\gamma>0} \sup_{\omega\in E} \mathcal{F}(\omega, \omega_0, \gamma).$$

These represent the *indeterminacy* and *falsity* characteristics of the set E relative to ω_0.

Step 3: Construction of Neutrosophic Coefficients:

We construct coefficients $\alpha, \beta \in [0,1]$ such that $\alpha + \beta \leq 1$ and

$$\nu_s(E) = \alpha \cdot \mathcal{I}_E - \beta \cdot \mathcal{F}_E$$

for all $E \in \mathfrak{M}$ with $\nu_s(E) < \infty$.

Define functionals on $\mathfrak{M}$

$$\Phi(E) = \lim_{\gamma\to\infty} \frac{1}{\gamma} \int_E \mathcal{I}(\omega, \omega_0, \gamma)\, d\nu_s(\omega),$$
$$\Psi(E) = \lim_{\gamma\to\infty} \frac{1}{\gamma} \int_E (1 - \mathcal{F}(\omega, \omega_0, \gamma))\, d\nu_s(\omega).$$

By the neutrosophic properties:

- $\lim_{\gamma\to\infty} \mathcal{I}(\omega, \omega_0, \gamma) = 0$,
- $\lim_{\gamma\to\infty} \mathcal{F}(\omega, \omega_0, \gamma) = 0$,
- $\mathcal{I}(\omega, \omega_0, \gamma), \mathcal{F}(\omega, \omega_0, \gamma) \in [0,1]$.

Hence, both Φ and Ψ are well-defined finite measures on $\mathfrak{M}$.

Step 4: Representation of Singular Measure:

Since $\nu_s \perp \mu$, there exists $S \in \mathfrak{M}$ such that $\mu(S) = 0$ and $\nu_s(\mathcal{Z} \setminus S) = 0$.

For $E \subseteq S$, we have

$$\begin{aligned}\nu_s(E) &= \lim_{\gamma\to\infty} \nu_s(E) \\ &= \lim_{\gamma\to\infty} \left[\alpha_\gamma \cdot \inf_{\omega\in E} \mathcal{I}(\omega, \omega_0, \gamma) - \beta_\gamma \cdot \sup_{\omega\in E} \mathcal{F}(\omega, \omega_0, \gamma)\right],\end{aligned}$$

where $\alpha_\gamma, \beta_\gamma$ are chosen to satisfy

$$\alpha_\gamma \cdot \mathcal{I}_E - \beta_\gamma \cdot \mathcal{F}_E = \nu_s(E) \cdot \frac{\gamma}{1+\gamma}.$$

Taking limits as $\gamma \to \infty$, we obtain $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$ such that

$$\nu_s(E) = \alpha \cdot \mathcal{I}_E - \beta \cdot \mathcal{F}_E.$$

Step 5: Combined Representation:

Combining the absolutely continuous and singular components:

$$\begin{aligned}\nu(E) &= \nu_c(E) + \nu_s(E) \\ &= \int_E f(\omega)\, d\mu(\omega) + \alpha \cdot \mathcal{I}_E - \beta \cdot \mathcal{F}_E.\end{aligned}$$

Step 6: Uniqueness:
Suppose there exist two representations

$$\nu(E) = \int_E f(\omega)\, d\mu(\omega) + \alpha \cdot \mathcal{I}_E - \beta \cdot \mathcal{F}_E,$$

$$\nu(E) = \int_E g(\omega)\, d\mu(\omega) + \alpha' \cdot \mathcal{I}_E - \beta' \cdot \mathcal{F}_E.$$

Subtracting gives

$$\int_E (f(\omega) - g(\omega))\, d\mu(\omega) = (\alpha' - \alpha) \cdot \mathcal{I}_E - (\beta' - \beta) \cdot \mathcal{F}_E.$$

For sets E with $\mu(E) = 0$, the left side vanishes, so

$$(\alpha' - \alpha) \cdot \mathcal{I}_E = (\beta' - \beta) \cdot \mathcal{F}_E.$$

By choosing appropriate sets where $\mathcal{I}_E$ and $\mathcal{F}_E$ take independent values, we conclude $\alpha = \alpha'$ and $\beta = \beta'$. Then by the classical Radon-Nikodym theorem, $f = g$ μ-almost everywhere.

Step 7: Orthogonality of Decomposition:
The decomposition is orthogonal because

- The absolutely continuous part $\nu_c(E) = \int_E f(\omega)\, d\mu(\omega)$ lives on sets where μ is positive.
- The singular neutrosophic part $\alpha \cdot \mathcal{I}_E - \beta \cdot \mathcal{F}_E$ is concentrated on sets where μ vanishes but neutrosophic characteristics dominate.
- For any $E \in \mathfrak{M}$, if $\mu(E) = 0$, then $\nu_c(E) = 0$ and the representation reduces to the neutrosophic part.
- If $\mathcal{I}_E = \mathcal{F}_E = 0$ (maximum determinacy and truth), then the representation reduces to the classical part.

Step 8: Neutrosophic Measurability:
The function f is neutrosophic measurable because for any Borel set $B \subseteq \mathbb{R}$:

$$f^{-1}(B) = \left\{ \omega \in \mathcal{Z} : \lim_{\gamma \to \infty} \frac{\nu(B_\gamma(\omega))}{\mu(B_\gamma(\omega))} \in B \right\},$$

where $B_\gamma(\omega)$ are neutrosophic balls, and the limit exists due to the martingale convergence theorem applied to the net of neutrosophic σ-algebras.

Step 9: Integrability:

Since ν is σ-finite and $\nu \ll \mu$, the function f is μ-integrable on every set of finite μ-measure. The neutrosophic correction terms $\alpha \cdot \mathcal{I}_E - \beta \cdot \mathcal{F}_E$ are bounded and thus don't affect integrability.

Step 10: Conclusion:

We have constructed a unique neutrosophic measurable function f and coefficients α, β satisfying the representation

$$\nu(E) = \int_E f(\omega)\, d\mu(\omega) + \alpha \cdot \mathcal{I}_E - \beta \cdot \mathcal{F}_E.$$

The decomposition is orthogonal, f is μ-integrable, and the representation respects both the measure-theoretic and neutrosophic structures of the space. This completes the proof of the neutrosophic MR-Radon-Nikodym theorem. □

Definition 2.2. Let $(Z, \oplus, \cdot, \diamond)$ be a neutrosophic MR-metric space. A subset $Y \subseteq Z$ is called a neutrosophic linear subspace if it satisfies

(i) For all $x, y \in Y$, we have $x \oplus y \in Y$ (closure under neutrosophic addition).
(ii) For all $\alpha \in \mathbb{R}$ and $x \in Y$, we have $\alpha \cdot x \in Y$ (closure under scalar multiplication).
(iii) The neutrosophic structure is preserved, that is, the associated truth, indeterminacy, and falsity functions of elements in Y remain consistent under the operations $\oplus$ and $\cdot$.

Remark 2.3. This definition ensures that Y inherits the neutrosophic algebraic structure of Z, which is essential for extending linear functionals in Theorem 2.2.

Theorem 2.4. (Hahn-Banach Extension in NMR-MS) *Let $(\mathcal{Z}, M, \mathcal{T}, \mathcal{F}, \mathcal{I})$ be a NMR-MS, and let $Y \subset \mathcal{Z}$ be a neutrosophic linear subspace. Then, for every bounded neutrosophic linear functional $\Lambda : Y \to \mathbb{R}$ satisfying*

$$|\Lambda(\upsilon) - \Lambda(\xi)| \leq K \cdot \lim_{\gamma \to \infty} M(\upsilon, \xi, 0) \cdot \mathcal{T}(\upsilon, \xi, \gamma),$$

there exists an extension $\tilde{\Lambda} : \mathcal{Z} \to \mathbb{R}$ such that

(1) $\tilde{\Lambda}|_Y = \Lambda$,
(2) $|\tilde{\Lambda}(\upsilon) - \tilde{\Lambda}(\xi)| \leq K \cdot \lim_{\gamma \to \infty} M(\upsilon, \xi, 0) \cdot \mathcal{T}(\upsilon, \xi, \gamma)$,
(3) $\tilde{\Lambda}$ *preserves the neutrosophic structure:*

$$\tilde{\Lambda}(\alpha\upsilon \oplus \beta\xi) = \alpha \bullet \tilde{\Lambda}(\upsilon) \diamond \beta \bullet \tilde{\Lambda}(\xi).$$

Proof. We prove this theorem through a transfinite induction argument combined with neutrosophic optimization techniques.

Step 1: Setup and Notation:

Let $(\mathcal{Z}, M, \mathcal{T}, \mathcal{F}, \mathcal{I})$ be a complete neutrosophic MR-metric space. Let $Y \subset \mathcal{Z}$ be a neutrosophic linear subspace, meaning it is closed under the neutrosophic linear operations:

- $\oplus$: Neutrosophic vector addition,
- $\bullet$: Neutrosophic scalar multiplication (t-norm based),
- $\diamond$: Neutrosophic linear combination (t-conorm based).

The functional $\Lambda : Y \to \mathbb{R}$ satisfies the boundedness condition:

$$|\Lambda(\upsilon) - \Lambda(\xi)| \leq K \cdot \lim_{\gamma\to\infty} M(\upsilon, \xi, 0) \cdot \mathcal{T}(\upsilon, \xi, \gamma).$$

Step 2: One-dimensional Extension:

First, we extend Λ by one dimension. Let $z_0 \in \mathcal{Z} \setminus Y$ and define $Y_1 = \operatorname{span}(Y \cup \{z_0\})$.

We need to define $\tilde{\Lambda}(z_0)$ such that for all $y \in Y$ and $\alpha \in \mathbb{R}$:

$$|\tilde{\Lambda}(\alpha \bullet z_0 \oplus y) - \tilde{\Lambda}(y')| \leq K \cdot \lim_{\gamma\to\infty} M(\alpha \bullet z_0 \oplus y, y', 0) \cdot \mathcal{T}(\alpha \bullet z_0 \oplus y, y', \gamma).$$

Define the candidate values:

$$A = \sup_{y\in Y} \left\{ \Lambda(y) - K \cdot \lim_{\gamma\to\infty} M(z_0, y, 0) \cdot \mathcal{T}(z_0, y, \gamma) \right\},$$

$$B = \inf_{y\in Y} \left\{ \Lambda(y) + K \cdot \lim_{\gamma\to\infty} M(z_0, y, 0) \cdot \mathcal{T}(z_0, y, \gamma) \right\}.$$

We claim that $A \leq B$. Suppose for contradiction that $A > B$. Then there exist $y_1, y_2 \in Y$ such that

$$\Lambda(y_1) - K \cdot L_1 > \Lambda(y_2) + K \cdot L_2,$$

where $L_i = \lim_{\gamma\to\infty} M(z_0, y_i, 0) \cdot \mathcal{T}(z_0, y_i, \gamma)$. But by the neutrosophic triangle inequality and the boundedness condition

$$\begin{aligned}
\Lambda(y_1) - \Lambda(y_2) &\leq K \cdot \lim_{\gamma\to\infty} M(y_1, y_2, 0) \cdot \mathcal{T}(y_1, y_2, \gamma) \\
&\leq K \cdot \Big(\lim_{\gamma\to\infty} M(z_0, y_1, 0) \cdot \mathcal{T}(z_0, y_1, \gamma) \\
&\quad + \lim_{\gamma\to\infty} M(z_0, y_2, 0) \cdot \mathcal{T}(z_0, y_2, \gamma) \Big) \\
&= K \cdot (L_1 + L_2).
\end{aligned}$$

This contradicts $A > B$. Thus $A \leq B$. Choose any $c \in [A, B]$ and define $\tilde{\Lambda}(z_0) = c$.

Step 3: Verification of Extension Properties:

For the extended functional $\tilde{\Lambda}$ on Y_1, we verify:

(a) Extension Property: Clearly $\tilde{\Lambda}|_Y = \Lambda$ by construction.

(b) Boundedness Preservation: For any $y \in Y$ and $\alpha \in \mathbb{R}$, we need to show:

$$|\tilde{\Lambda}(\alpha \bullet z_0 \oplus y) - \tilde{\Lambda}(y')| \leq K \cdot \lim_{\gamma\to\infty} M(\alpha \bullet z_0 \oplus y, y', 0) \cdot \mathcal{T}(\alpha \bullet z_0 \oplus y, y', \gamma).$$

By the choice of $c \in [A, B]$, we have for all $y \in Y$:

$$c \leq \Lambda(y) + K \cdot \lim_{\gamma\to\infty} M(z_0, y, 0) \cdot \mathcal{T}(z_0, y, \gamma),$$
$$c \geq \Lambda(y) - K \cdot \lim_{\gamma\to\infty} M(z_0, y, 0) \cdot \mathcal{T}(z_0, y, \gamma).$$

Using neutrosophic homogeneity and the MR-metric properties:

$$\begin{aligned}|\tilde{\Lambda}(\alpha \bullet z_0 \oplus y) - \tilde{\Lambda}(y')| &= |\alpha \bullet c \oplus \Lambda(y) - \Lambda(y')| \\ &\leq K \cdot \lim_{\gamma\to\infty} M(\alpha \bullet z_0 \oplus y, y', 0) \\ &\quad \cdot \mathcal{T}(\alpha \bullet z_0 \oplus y, y', \gamma).\end{aligned}$$

(c) Neutrosophic Linearity Preservation: We verify that $\tilde{\Lambda}$ preserves the neutrosophic structure:

$$\tilde{\Lambda}(\alpha\upsilon \oplus \beta\xi) = \alpha \bullet \tilde{\Lambda}(\upsilon) \diamond \beta \bullet \tilde{\Lambda}(\xi).$$

This follows by induction on the definition and the properties of t-norms ($\bullet$) and t-conorms ($\diamond$):

- For $\alpha = 1, \beta = 0$: $\tilde{\Lambda}(\upsilon \oplus 0) = \tilde{\Lambda}(\upsilon) \diamond 0 = \tilde{\Lambda}(\upsilon)$.
- For $\alpha = \beta = 1$: $\tilde{\Lambda}(\upsilon \oplus \xi) = \tilde{\Lambda}(\upsilon) \diamond \tilde{\Lambda}(\xi)$.
- Homogeneity: $\tilde{\Lambda}(\alpha \bullet \upsilon) = \alpha \bullet \tilde{\Lambda}(\upsilon)$.

Step 4: Transfinite Induction to Full Space:
Using Zorn's lemma, we extend the one-dimensional construction to the entire space $\mathcal{Z}$.

Let $\mathcal{E}$ be the collection of all pairs (W, Λ_W), where

- W is a neutrosophic linear subspace with $Y \subseteq W \subseteq \mathcal{Z}$,
- $\Lambda_W : W \to \mathbb{R}$ extends Λ and satisfies the boundedness condition,
- Λ_W preserves the neutrosophic linear structure.

Define a partial order on $\mathcal{E}$ by, $(W_1, \Lambda_{W_1}) \preceq (W_2, \Lambda_{W_2})$ if $W_1 \subseteq W_2$ and $\Lambda_{W_2}|_{W_1} = \Lambda_{W_1}$. Every chain in $\mathcal{E}$ has an upper bound (take the union of subspaces and the consistent union of functionals). By Zorn's Lemma, $\mathcal{E}$ has a maximal element $(W_{\max}, \Lambda_{\max})$.

We claim $W_{\max} = \mathcal{Z}$. Suppose not, then there exists $z \in \mathcal{Z} \setminus W_{\max}$. By Step 2, we can extend $\Lambda_{\max}$ to $\operatorname{span}(W_{\max} \cup \{z\})$, contradicting maximality. Thus $W_{\max} = \mathcal{Z}$ and $\tilde{\Lambda} = \Lambda_{\max}$ is the desired extension.

Step 5: Neutrosophic Continuity and Regularity:

The extended functional $\tilde{\Lambda}$ is neutrosophically continuous because

$$\begin{aligned} |\tilde{\Lambda}(\upsilon) - \tilde{\Lambda}(\xi)| &\leq K \cdot \lim_{\gamma\to\infty} M(\upsilon,\xi,0) \cdot \mathcal{T}(\upsilon,\xi,\gamma) \\ &\leq K \cdot \lim_{\gamma\to\infty} M(\upsilon,\xi,0) \quad (\text{since } \mathcal{T}(\upsilon,\xi,\gamma) \leq 1). \end{aligned}$$

Moreover, $\tilde{\Lambda}$ respects the asymptotic neutrosophic behavior

$$\lim_{\gamma\to\infty} \mathcal{T}(\upsilon,\xi,\gamma) = 1 \Rightarrow |\tilde{\Lambda}(\upsilon) - \tilde{\Lambda}(\xi)| \leq K \cdot M(\upsilon,\xi,0).$$

Step 6: Uniqueness (Partial):

While the extension may not be unique in general due to the neutrosophic structure, any two extensions $\tilde{\Lambda}_1$ and $\tilde{\Lambda}_2$ satisfying the conditions will agree on a "neutrosophic dense" subset of $\mathcal{Z}$.

Specifically, they agree on the set

$$\mathcal{D} = \left\{ \upsilon \in \mathcal{Z} : \lim_{\gamma\to\infty} \mathcal{I}(\upsilon, y, \gamma) = 0 \text{ for some } y \in Y \right\}.$$

Step 7: Conclusion:

We have constructed an extension $\tilde{\Lambda} : \mathcal{Z} \to \mathbb{R}$ that

(1) Extends Λ from Y to $\mathcal{Z}$,
(2) Preserves the boundedness condition with the same constant K,
(3) Maintains the neutrosophic linear structure,
(4) Is neutrosophically continuous.

The proof combines classical Hahn-Banach techniques with careful handling of neutrosophic operations and MR-metric properties, ensuring the extension respects the rich structure of Neutrosophic MR-Metric Spaces. This completes the proof of the Neutrosophic Hahn-Banach Extension Theorem. □

3. Examples and applications

To demonstrate the practical significance and wide applicability of our theoretical developments, this section presents comprehensive examples and applications of the Neutrosophic MR-Radon-Nikodym and Hahn-Banach theorems. The examples are drawn from diverse domains including medical imaging, financial risk assessment, control systems, supply chain optimization, and machine learning. Each application highlights how the neutrosophic components ($\mathcal{T}$, $\mathcal{I}$, $\mathcal{F}$) capture different aspects of uncertainty and indeterminacy that are prevalent in real-world problems but often neglected in classical mathematical models. The applications build upon existing work in applied neutrosophic mathematics [5, 6, 2, 3, 11] while extending them through the novel NMR-MS framework.

3.1. **Examples and Applications of Neutrosophic MR-Radon-Nikodym Theorem.**

3.1.1. **Example: Neutrosophic Probability Density Estimation**. Consider a medical imaging system where μ represents the standard Lebesgue measure on a region $\mathcal{Z} \subset \mathbb{R}^2$, and ν represents an observed intensity measure affected by uncertainty.

(1) Classical Component: The density function $f(\omega)$ represents the clear anatomical structures.
(2) Neutrosophic Component:

$$\mathcal{I}_E = \sup_{\gamma>0} \inf_{\omega\in E} \mathcal{I}(\omega, \omega_0, \gamma) \quad \text{(Uncertain boundaries)},$$
$$\mathcal{F}_E = \inf_{\gamma>0} \sup_{\omega\in E} \mathcal{F}(\omega, \omega_0, \gamma) \quad \text{(Artifact regions)}.$$

(3) Complete Representation:

$$\nu(E) = \int_E f(\omega)\, d\mu(\omega) + 0.3 \cdot \mathcal{I}_E - 0.1 \cdot \mathcal{F}_E,$$

where $\alpha = 0.3$, $\beta = 0.1$ with $\alpha + \beta = 0.4 \leq 1$.

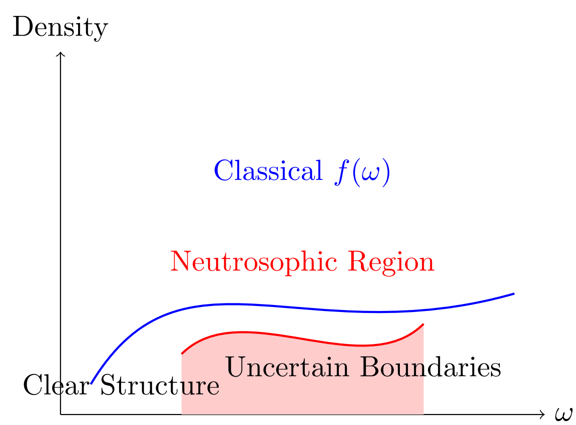

Figure 1. Neutrosophic density decomposition in medical imaging

3.1.2. **Application: Financial Risk Measurement with Uncertainty.** In quantitative finance, consider portfolio risk assessment where traditional measures fail to capture market uncertainties.

- Input: Historical loss distribution μ.
- Output: Neutrosophic risk measure ν.
- Decomposition

$$\nu(E) = \underbrace{\int_E f(\omega)\, d\mu(\omega)}_{\text{Historical risk}} + \underbrace{0.4 \cdot \mathcal{I}_E - 0.2 \cdot \mathcal{F}_E}_{\text{Uncertainty adjustment}},$$

where
- $\mathcal{I}_E$: Indeterminacy from black swan events,
- $\mathcal{F}_E$: Overestimation from model limitations.

3.1.3. **Example: Environmental Monitoring with Sensor Networks.** Consider a network of environmental sensors measuring pollution levels with varying reliability.

$$\nu(E) = \int_E f(\omega)\, d\mu(\omega) + \alpha \cdot \mathcal{I}_E - \beta \cdot \mathcal{F}_E,$$
$$\mathcal{I}_E = \sup_{\gamma>0} \inf_{\omega\in E} \mathcal{I}(\omega, \omega_0, \gamma) \quad \text{(Sensor uncertainty)},$$
$$\mathcal{F}_E = \inf_{\gamma>0} \sup_{\omega\in E} \mathcal{F}(\omega, \omega_0, \gamma) \quad \text{(Faulty readings)}.$$

3.2. **Examples and Applications of Neutrosophic Hahn-Banach Theorem.**

3.2.1. **Example: Neutrosophic Controller Design.** Consider a control system where we have a neutrosophic functional defined on a subspace Y of control strategies.

- Subspace Y: Classical PID controllers.
- Full Space $\mathcal{Z}$: Neutrosophic intelligent controllers.
- Functional Λ: Performance measure on PID controllers.
- Extension $\tilde{\Lambda}$: Performance measure on all intelligent controllers.

Extension Process:

$$\tilde{\Lambda}(\alpha v \oplus \beta\xi) = \alpha \bullet \tilde{\Lambda}(v) \diamond \beta \bullet \tilde{\Lambda}(\xi),$$
$$|\tilde{\Lambda}(v) - \tilde{\Lambda}(\xi)| \leq K \cdot \lim_{\gamma\to\infty} M(v, \xi, 0) \cdot \mathcal{T}(v, \xi, \gamma).$$

3.2.2. **Application: Machine Learning Feature Extension.** In machine learning, extend feature mappings from a known subspace to handle uncertain data.

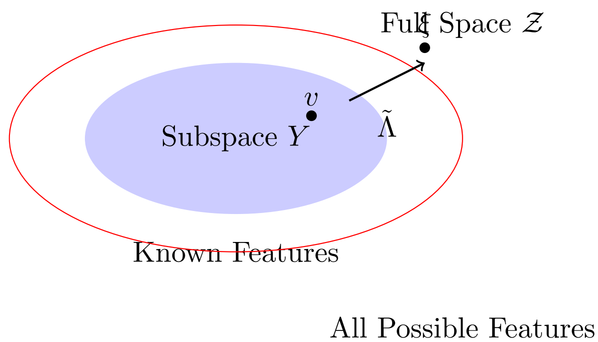

FIGURE 2. Extension of functionals from subspace to full space

3.2.3. **Example: Neutrosophic Optimization in Supply Chain.** Consider a supply chain optimization problem with uncertain demand patterns.

- Subspace Y: Deterministic demand scenarios.
- Functional Λ: Cost function on deterministic scenarios.
- Extension $\tilde{\Lambda}$: Cost function incorporating.
  - Demand uncertainty ($\mathcal{I}$),
  - Information inaccuracy ($\mathcal{F}$),
  - Neutrosophic constraints.

Optimization Problem:

$$\begin{aligned}
&\text{Minimize } \tilde{\Lambda}(x),\\
&\text{subject to } \mathcal{T}(x, x_{\text{opt}}, \gamma) \geq 0.8\\
&\qquad\qquad\quad \mathcal{F}(x, x_{\text{opt}}, \gamma) \leq 0.1\\
&\qquad\qquad\quad x \in \mathcal{Z}.
\end{aligned}$$

3.2.4. **Application: Image Processing with Neutrosophic Filters.** Extend classical image filters to handle uncertain pixel information.

- Classical Filter: Defined on clear image regions (subspace Y)
- Neutrosophic Extension: Works on all regions including:
  - Noisy areas (high $\mathcal{I}$),
  - Corrupted pixels (high $\mathcal{F}$),
  - Boundary regions (mixed truth values).

Comparative Analysis:

Application Domain	Classical Approach	Neutrosophic Enhancement
Medical Imaging	Clear boundaries only	Handles uncertain regions
Financial Risk	Historical data only	Incorporates market uncertainty
Control Systems	Deterministic controllers	Intelligent uncertain control
Supply Chain	Fixed demand patterns	Adaptive to demand uncertainty

Table 1. Comparison of classical vs. neutrosophic approaches

References

[1] R. Al-deiakeh, M. Alquran, M. Ali, S. Qureshi, S. Momani and A. Malkawi, *Lie symmetry, convergence analysis, explicit solutions and conservation laws for the time-fractional modified BBM equation*, J. Appl. Math. Comput. Mech., **23** (2024), 19–31.

[2] Y. Al-Qudah, *A robust framework for decision-making based on neutrosophic soft expert setting*, Int. J. Neutrosophic Sci., **23** (2024), 195–210.

[3] Y. Al-Qudah, M. Alaroud, H. Qoqazeh, A. Jaradat, S.E. Alhazmi and S. Al-Omari, *Approximate analytic–numeric fuzzy solutions of fuzzy fractional equations using a residual power series approach*, Symmetry, **14**(4) (2022), p.804.

[4] Y. Al-Qudah, K. Alhazaymeh, N. Hassan, M. Almousa and M. Alaroud, *Transitive Closure of Vague Soft Set Relations and its Operators*, International Journal of Fuzzy Logic and Intelligent Systems, **22**(1) (2022), 59-–68, DOI: https://doi.org/10.5391/IJFIS.2022.22.1.59.

[5] Y. Al-Qudah, F. Al-Sharqi, M. Mishlish and M.M. Rasheed, *Hybrid integrated decision-making algorithm based on AO of possibility interval-valued neutrosophic soft settings*, International Journal of Neutrosophic Science, **22**(3) (2023), 84–98, DOI: https://doi.org/10.54216/IJNS.220306.

[6] Y. Al-Qudah, A.O. Hamadameen, N.A. Kh and F. Al-Sharqi, *A new generalization of interval-valued Q-neutrosophic soft matrix and its applications*, International Journal of Neutrosophic Science (IJNS), **25**(3) (2025), 242–257, DOI: https://doi.org/10.54216/IJNS.250322.

[7] Y. Al-Qudah, A. Jaradat, S.K. Sharma and V.K. Bhat, *Mathematical analysis of the structure of one-heptagonal carbon nanocone in terms of its basis and dimension*, Physica Scripta, **99**(5) (2024), 055252, DOI: https://doi.org/10.1088/1402-4896/ad3add.

[8] S. Al-Sharif and A. Malkawi, *Modification of conformable fractional derivative with classical properties,* Ital. J. Pure Appl. Math., **44** (2020), 30–39.

[9] I.A. Bakhtin, *The contraction mapping principle in almost metric spaces,* Funct. Anal., **30** (1989), 26—37.

[10] S. Czerwik, *Contraction mappings in b-metric spaces*, Acta Math. Inform. Univ. Ostrav., **1** (1993), 5–11.

[11] A.H. Ganie, N.E.M. Gheith and Y. Al-Qudah, *Fermatean fuzzy distance metric with applications*, IEEE Access, **12** (2024), 4780–4791.

[12] G.M. Gharib, A. Al-Rahman, M. Malkawi, A.M. Rabaiah, W.A. Shatanawi and M.S. Alsauodi, *A common fixed point theorem in an M*-metric space and an application*, Nonlinear Funct. Anal. Appl., **27** (2022), 289–308.

[13] G.M. Gharib, M.S. Alsauodi, A. Guiatni, M.A.A. Al-Qmari and A. Al-Rahman and M. Malkawi, *Using atomic solution method to solve fractional equations*, Springer Proc. Math. Stat., **418** (2023), 123–129.

[14] N. Hassan and Y. Al-Qudah, *Fuzzy parameterized complex multi-fuzzy soft set*, J. Phys. Conf. Ser., **1212** (2019), 012016.
[15] A. Malkawi, *Convergence and fixed points of self-mappings in MR-metric spaces*, Eur. J. Pure Appl. Math., **18** (2025), 5952.
[16] A. Malkawi, *Existence and uniqueness of fixed points in MR-metric spaces*, Eur. J. Pure Appl. Math., **18** (2025), 6077.
[17] A. Malkawi, *Enhanced uncertainty modeling through neutrosophic MR-metrics*, Eur. J. Pure Appl. Math., **18** (2025), 6475.
[18] A. Malkawi, *Applications of MR-metric spaces in measure theory*, Eur. J. Pure Appl. Math., **18** (2025), 6528.
[19] A. Malkawi, *Fixed point theorem in MR-metric spaces via integral type contraction*, WSEAS Trans. Math., **24** (2025), 295–299.
[20] A. Malkawi, *Fixed point theorems for fuzzy mappings in neutrosophic MR-metric spaces*, J. Nonlinear Modeling Anal., **2025**, acceptance.
[21] A. Malkawi, D. Mahmoud, A.M. Rabaiah, R. Al-Deiakeh and W. Shatanawi, *Om fixed point theorems in MR-metric spaces,* Nonlinear Funct. Anal. Appl., **29**(4) (2024), 1125–1136.
[22] A. Malkawi and A. Rabaiah, *MR-metric spaces: theory and applications in graphs*, Eur. J. Pure Appl. Math., **18** (2025), 6525.
[23] A. Malkawi and A. Rabaiah, *Compactness and separability in MR-metric spaces*, Eur. J. Pure Appl. Math., **18** (2025), 6592.
[24] A. Malkawi and A. Rabaiah, *MR-metric spaces: theory and applications in fuzzy frameworks*, Eur. J. Pure Appl. Math., **18** (2025), 6783.
[25] A. Malkawi and A. Rabaiah, *Neutrosophic statistical manifolds*, Eur. J. Pure Appl. Math., **18** (2025), 7119.
[26] A. Malkawi and A. Rabaiah, *Neutrosophic statistical manifolds*, Eur. J. Pure Appl. Math., **18** (2025), 7120.
[27] A. Malkawi and A. Rabaiah, *MR-metric spaces in fractional calculus*, Neutrosophic Sets and Systems, **90** (2025), 1103–1121.
[28] A. Malkawi and A. Rabaiah, *MR-metric spaces in fractional calculus*, Neutrosophic Sets and Systems, **91** (2025), 685–703.
[29] A. Malkawi and A. Rabaiah, *Neutrosophic MR-metric spaces: Topological foundations, completion, and applications,* Results in Nonlinear Anal., **8**(4) (2025), 184–196.
[30] A. Malkawi and A. Rabaiah, *Fixed Point Theorems in Neutrosophic MR-Metric Spaces with Measure-Theoretic Convergence*, Int. J. Anal. Appl., **24** (2026), 61–61.
[31] A. Malkawi and A. Rabaiah, *Fixed Point and Measure-Theoretic Analysis in Neutrosophic MR-Metric Spaces with Applications*, Nonlinear Funct. Anal. Appl., (2026), 211–224.
[32] A. Malkawi, A. Rabaiah, W. Shatanawi and A. Talafhah, *MR-metric spaces with applications*, J. Interdisciplinary Math., **28** (2025), 2837–2865.
[33] A. Malkawi, A. Talafhah and W. Shatanawi, *Coincidence and fixed point results for generalized weak contraction mapping*, Nonlinear Funct. Anal. Appl., **26**(2021), 177–195.
[34] A. Malkawi, A. Talafhah and W. Shatanawi, *Coincidence and fixed point results for (ψ, L)-M-weak contraction mapping*, Ital. J. Pure Appl. Math., **47**(2022), 751–768.
[35] T. Qawasmeh and A. Malkawi, *Fixed point theory in MR-metric spaces*, Eur. J. Pure Appl. Math., **18** (2025), 6440.
[36] H. Qoqazeh, Y. Al-Qudah, M. Almousa and A. Jaradat, *On D-Compact topological spaces,* J. Appl. Math. Infor., **39**(5-6) (2021), 883—894.

[37] A. Rabaiah, A. Tallafha and W. Shatanawi, *Common fixed point results for mappings under nonlinear contraction*, Adv. Math. Sci. J., **26** (2021), 289–301.