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COMMON FIXED POINTS FOR A PAIR OF MAPPINGS IN PERTURBED METRIC SPACES

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Abstract. In the present paper, we study a pair of mappings satisfying a new type of contractive condition defined on perturbed metric spaces. We prove fixed point result for such mappings and give an example to illustrate that it is more general than previously established results. Furthermore, we extend the result to any finite number of such mappings.

1. Introduction and preliminaries

Fixed point theory and its results are an important tool in non-linear analysis. Existence and uniqueness theorems in fixed point theory are useful in solving various types of equations. A key result of this field is the Banach Contraction Principle [4]. Several attempts have been made to extend the theorem to more general forms in an attempt to improve its applicability. One approach involves broadening the contraction condition of the mapping. Dropping continuity condition inherent in Banach's contraction map may be an example. Prominent examples of discontinuous contractions include Kannan [14], Ciric [5], Rus [21] etc. Following this line of development, Din et al. [7] recently introduced the concept of generalized enriched Kannan mappings which extend the idea of enriched Kannan mappings to a three point analogue; merging these foundational studies to advance the theory of fixed

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point mappings. Furthermore, Anjum et al. [3] introduced and develop two novel classification of enriched (q, θ)-contraction, which can be regarded as a generalization of the enriched contraction and θ-contraction and establish some fixed point results for such new contractions.

Another direction of development focuses on the underlying ambient space by defining extensions of the metric structure such as b metric, partial metric, partially ordered sets etc. (see [6, 8, 13, 19, 20]). In another recent development, Din and Sintunavaral [9] investigated fixed point in interpolative metric space which is an extension of classical metric spaces. Further, Din and Sintunavaral [10] also introduced the concept of a Hausdorff interpolative metric over an interpolative metric space, along with fundamental notions such as open balls, limit points, closure of a set, boundedness, and compactness, and study some of their basic properties. Again, in 2026, Wang et al. [23] introduced a novel framework for operators in metric spaces equipped with asymptotically weak R-closed structures and investigated the existence and uniqueness of fixed points.

Since the contraction condition satisfied by the mapping involved is key to determining the existence and uniqueness of fixed points, which in turn helps us in determining solutions to equations, it becomes important to know whether fixed point results still hold when the contraction condition is satisfied through the experimental measurement of the distance, containing error. Jleli and Samet [22] investigated this question and introduced the concept of perturbed metric spaces. This paper studies fixed point theorems in the recently developed Perturbed metric spaces due to Jleli and Samet.

We recall the definition of Perturbed metric space [22] and various relevant notions:

Definition 1.1. ([22]) Let M be an arbitrary nonempty set. Let $\sigma, P : M \times M \to [0, \infty)$ be two mappings. We call σ a perturbed metric on M with respect to P if,

$$\begin{aligned} \sigma - P : M \times M &\to \mathbf{R}, \\ (x, y) &\longmapsto \sigma(x, y) - P(x, y) \end{aligned}$$

is a metric on M.

We call P a perturbed mapping, $\rho = \sigma - P$ an exact metric. (M, σ, P) is then called a perturbed metric space.

Example 1.2. Let $M = \mathbf{R}$ and $\sigma : M \times M \to [0, \infty)$ be the mapping defined by $\sigma(x, y) = |x - y| + 3x^2y^6$, $x, y \in \mathbf{R}$. Then, σ is a perturbed metric on $\mathbf{R}$ with the perturbed mapping $P(x, y) = 3x^2y^6$. The exact metric is given by

$\rho(x,y) = |x-y|$, $x, y \in \mathbf{R}$. Here, σ is not a metric since $\rho(1,1) = 3 \neq 0$. Thus, a perturbed metric space need not be a metric space.

Remark 1.3. It is easily seen that a metric space is also a perturbed metric space by considering the function $P \equiv 0$ for all $x, y \in \mathbf{R}$, to be the corresponding perturbed mapping

Furthermore, we recall relevant topological definitions:

Definition 1.4. ([22]) Let (M, σ, P) be a perturbed metric space, $\{x_n\}$ a sequence in M, and $F : M \to M$.

(i) We say that $\{x_n\}$ is a *perturbed convergent sequence* in (M, σ, P) if $\{x_n\}$ is convergent in the metric space (M, ρ), where $\rho = \sigma - P$ is the exact metric.
(ii) We say that $\{x_n\}$ is a *perturbed Cauchy sequence* in (M, σ, P) if $\{x_n\}$ is a Cauchy sequence in the metric space (M, ρ).
(iii) We say that (M, σ, P) is a *complete perturbed metric space* if (M, ρ) is a complete metric space, or equivalently, if every perturbed Cauchy sequence in (M, σ, P) is a perturbed convergent sequence in (M, σ, P).
(iv) We say that F is a *perturbed continuous mapping* if F is continuous with respect to the exact metric ρ.

In [22], Jleli and Samet, found that when the contraction condition is satisfied by the mapping in terms of the perturbed metric, containing experimental error, along with certain continuity condition, Banach's [4] fixed point result still holds.

Jleli and Samet proved the following result.

Theorem 1.5. ([22]) *Let (M, σ, P) be a complete perturbed metric space and $F : M \to M$ be a given mapping. Assume that the following conditions hold:*

(i) *F is a perturbed continuous mapping;*
(ii) *There exists $\alpha \in (0,1)$ such that*
$$\sigma(Fu, Fv) \leq \alpha\sigma(u, v)$$
for all $u, v \in M$.

Then, F admits one and only one fixed point.

Maria Nutu, Cristina Maria Pacurar [18] removed the continuity requirement from the mapping by introducing Kannan mappings in the setting of perturbed metric spaces.They proved an analogue of Kannan' [14] fixed point theorem in the setting of perturbed metric spaces and showed that such mappings are not necessarily continuous. Karapinar et al. investigated rational contraction and interpolative contraction on perturbed metric space(see

[2, 16]) and further introduced the concept of perturbed b-metric spaces [15]. This was followed by the introduction of perturbed extended b-metric space by A. Moussaoui [17] and the notion of C^*-algebra-valued perturbed metric space by Ahmad et al. [1].

Results on common fixed points for two or several mappings (see, [11], [12], [24] helps us deal with the difficulties that arise when we try to adapt existing fixed point results in solving systems of equations.

Motivated by the works mentioned above, in this paper we study a pair of mappings satisfying a new type of contractive condition defined on perturbed metric spaces. We shall consider the case where the contraction condition is satisfied by the perturbed metric, containing experimental error. We prove fixed point result for such mappings and give example to illustrate our result. Furthermore, we extend the result to any finite number of such mappings. We also show that earlier results of Jleli and Samet [22] can be obtained as a corollary of our main theorem. We will retain the continuity conditions of the mappings involved.

2. Main results

Theorem 2.1. *Let $F, G : M \to M$ be a pair of mappings on a complete perturbed metric space (M, σ, P) satisfying the following conditions:*

(i) *F and G are perturbed continuous mappings.*

(ii) *There exists $\alpha \in (0,1)$ such that*

$$\sigma(Gx, Fy) \leq \alpha\sigma(x, y), \tag{2.1}$$

$$\sigma(Fx, Gy) \leq \alpha\sigma(x, y) \tag{2.2}$$

for all $x, y \in M$.

Then, F and G admit a unique common fixed point.

Proof. Fix $x_0 \in M$. Define a sequence $\{x_n\} \in M$ by

$$\begin{aligned} x_1 = Gx_0,\ x_2 &= Fx_1, \\ x_{2n-1} &= Gx_{2n-2}, \\ x_{2n} &= Fx_{2n-1}, \qquad n = 0, 1, 2, \cdots . \end{aligned}$$

Taking $x = x_0$, $y = x_1$ in, (2.1) we obtain

$$\sigma(Gx_0, Fx_1) \leq \alpha\sigma(x_0, x_1),$$

that is,

$$\sigma(x_1, x_2) \leq \alpha\sigma(x_0, x_1). \tag{2.3}$$

Similarly, taking $x = x_1$, $y = x_2$ in (2.2), we obtain

$$\sigma(Fx_1, Gx_2) \leq \alpha\sigma(x_1, x_2),$$

which gives

$$\sigma(x_2, x_3) \leq \alpha\sigma(x_1, x_2).$$

This with (2.3) gives

$$\sigma(x_2, x_3) \leq \alpha^2\sigma(x_0, x_1).$$

By induction,

$$\sigma(x_n, x_{n+1}) \leq \alpha^n\theta, \qquad n = 0, 1, 2, \ldots, \tag{2.4}$$

where $\theta = \sigma(x_0, x_1)$. Since $\rho = \sigma - P$ and $P \geq 0$, we have

$$\rho(u, v) \leq \sigma(u, v)$$

for all $u, v \in M$. Hence, for every $p \in \mathbb{N}$,

$$\rho(x_n, x_{n+p}) \leq \sum_{m=n}^{n+p-1} \rho(x_m, x_{m+1}) \leq \sum_{m=n}^{n+p-1} \sigma(x_m, x_{m+1}) \leq \frac{\alpha^n}{1-\alpha}\theta.$$

It follows that $\{x_n\}$ is a Cauchy sequence in (M, ρ), equivalently, a perturbed Cauchy sequence in (M, σ, P). Since (M, σ, P) is complete, there exists $x^* \in M$ such that

$$\lim_{n\to\infty} \rho(x_n, x^*) = 0.$$

We claim that x^* is a common fixed point of F and G. Since

$$\lim_{n\to\infty} \rho(x_n, x^*) = 0.$$

We have

$$\lim_{n\to\infty} \rho(x_{2n-1}, x^*) = 0$$

and

$$\lim_{n\to\infty} \rho(x_{2n}, x^*) = 0.$$

Since, F is perturbed continuous,

$$\lim_{n\to\infty} \rho(Fx_{2n-1}, Fx^*) = 0,$$

that is,

$$\lim_{n\to\infty} \rho(x_{2n}, Fx^*) = 0.$$

Since, the limit is unique, $Fx^* = x^*$.

Similarly, perturbed continuity of G gives

$$\lim_{n\to\infty} \rho(Gx_{2n-2}, Gx^*) = 0,$$

that is,

$$\lim_{n\to\infty} \rho(x_{2n-1}, Gx^*) = 0,$$

which gives $Gx^* = x^*$.

To prove uniqueness, suppose that x^* and y^* are two common fixed points of F and G. Then by (2.1), we have

$$\sigma(x^*, y^*) = \sigma(Gx^*, Fy^*) \leq \alpha\sigma(x^*, y^*).$$

If $x^* \neq y^*$, then $\rho(x^*, y^*) > 0$, and therefore $\sigma(x^*, y^*) = \rho(x^*, y^*) + P(x^*, y^*) > 0$. Dividing by $\sigma(x^*, y^*)$ yields $1 \leq \alpha$, a contradiction to $\alpha \in (0, 1)$. Hence $x^* = y^*$. Therefore the common fixed point is unique. □

Remark 2.2. We can obtain the result of Jleli and Samet from Theorem 2.1 by taking $F = G$.

Corollary 2.3. *Let $F : M \to M$ be a mapping on a complete perturbed metric space (M, σ, P) satisfying the following conditions:*

(i) *F is a perturbed continuous mapping.*

(ii) *There exists $\alpha \in (0, 1)$ such that*

$$\sigma(Fx, Fy) \leq \alpha\sigma(x, y)$$

for all $x, y \in M$.

Then, F admits a unique fixed point.

Proof. Taking $F = G$ in Theorem 2.1, we see that the pair (F, F) satisfies both conditions (i) and (ii) of the theorem, so that F admits a unique fixed point. □

Example 2.4. Let $M = [0, \frac{1}{2}]$ and define the perturbed metric σ on M by,

$$\sigma(x, y) = \max\{x, y\}$$

with the perturbation,

$$P(x, y) = \begin{cases} 0, & \text{if } x \neq y, \\ x, & \text{if } x = y. \end{cases}$$

Then, $\rho(x, y) = \sigma(x, y) - P(x, y)$ is an exact metric. Define the mappings $F, G : M \to M$ as

$$Fx = \frac{x}{2},$$
$$Gx = x^2.$$

Then, F and G are perturbed continuous functions with,

$$\sigma(Fx, Gy) \leq \frac{1}{2}\sigma(x, y),$$
$$\sigma(Gx, Fy) \leq \frac{1}{2}\sigma(x, y).$$

Then, F and G have the unique common fixed point $x^* = 0$, and can be approximated by the sequence,

$$x_{2n-1} = Gx_{2n-2} = x_{2n-2}^2,$$
$$x_{2n} = Fx_{2n-1} = \frac{x_{2n-1}}{2}.$$

For instance, for $x_0 = \frac{1}{2}$ we get the sequence whose first five terms are,

$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{64}, \frac{1}{128}, \cdots$

It can be easily verified that $\{x_n\}$ converges to the common fixed point 0.

Theorem 2.5. *Let (M, σ, P) be a complete perturbed metric space, and let $F_1, F_2, F_3, \ldots, F_k : M \to M$ be mappings satisfying*

$$\sigma(F_i x, F_j y) \leq \alpha\sigma(x, y) \tag{2.5}$$

for all $x, y \in M$ and all $i, j \in \{1, 2, \ldots, k\}$, where $\alpha \in (0, 1)$. Further, assume that the mappings F_i are perturbed continuous for every $i \in \{1, 2, \ldots, k\}$. Then the mappings $F_1, F_2, \ldots, F_k$ admit a unique common fixed point.

Proof. Fix $x_0 \in M$. Define a cyclic index map

$$r(n) = 1 + (n \bmod k), \qquad n = 0, 1, 2, \ldots,$$

so that $r(n) \in \{1, 2, \ldots, k\}$ and $r(n+k) = r(n)$ for every $n \geq 0$. Equivalently, we use the convention $F_{k+1} = F_1$. Define the iterative sequence $\{x_n\}$ by

$$x_{n+1} = F_{r(n)} x_n, \qquad n = 0, 1, 2, \ldots. \tag{2.6}$$

Thus

$$x_1 = F_1 x_0, \quad x_2 = F_2 x_1, \quad \ldots, \quad x_k = F_k x_{k-1}, \quad x_{k+1} = F_1 x_k,$$

and the same cyclic pattern continues.

For $n \geq 1$, using (2.6) and the contractive condition (2.5) with $i = r(n)$ and $j = r(n-1)$, we obtain

$$\sigma(x_{n+1}, x_n) = \sigma(F_{r(n)} x_n, F_{r(n-1)} x_{n-1}) \leq \alpha\sigma(x_n, x_{n-1}).$$

By induction,

$$\sigma(x_{n+1}, x_n) \leq \alpha^n \theta, \qquad n = 0, 1, 2, \ldots, \tag{2.7}$$

where $\theta = \sigma(x_1, x_0)$. Since $\rho = \sigma - P$ and $P \geq 0$, we have $\rho(u,v) \leq \sigma(u,v)$ for all $u, v \in M$. Hence, for every $p \in \mathbb{N}$,

$$\begin{aligned}\rho(x_n, x_{n+p}) &\leq \sum_{m=n}^{n+p-1} \rho(x_m, x_{m+1}) \\ &\leq \sum_{m=n}^{n+p-1} \sigma(x_m, x_{m+1}) \\ &\leq \frac{\alpha^n}{1-\alpha}\theta.\end{aligned}$$

It follows that $\{x_n\}$ is a Cauchy sequence in (M, ρ), equivalently, a perturbed Cauchy sequence in (M, σ, P). Since (M, σ, P) is complete, there exists $x^* \in M$ such that

$$\lim_{n\to\infty} \rho(x_n, x^*) = 0.$$

We now show that x^* is a common fixed point of all the mappings. Fix $i \in \{1, 2, \ldots, k\}$. There are infinitely many integers n with $r(n) = i$; along this subsequence, (2.6) gives $x_{n+1} = F_i x_n$. Since every subsequence of a convergent sequence has the same limit, we have $x_n \to x^*$ and $x_{n+1} \to x^*$ in the exact metric ρ along this subsequence. By perturbed continuity of F_i,

$$F_i x^* = \lim_{\substack{n\to\infty \\ r(n)=i}} F_i x_n = \lim_{\substack{n\to\infty \\ r(n)=i}} x_{n+1} = x^*.$$

Since i was arbitrary, x^* is a common fixed point of $F_1, F_2, \ldots, F_k$.

To prove uniqueness, suppose that x^* and y^* are two common fixed points of $F_1, F_2, \ldots, F_k$. For any $i, j \in \{1, 2, \ldots, k\}$, the contractive condition gives

$$\sigma(x^*, y^*) = \sigma(F_i x^*, F_j y^*) \leq \alpha\sigma(x^*, y^*).$$

If $x^* \neq y^*$, then $\rho(x^*, y^*) > 0$, and therefore $\sigma(x^*, y^*) = \rho(x^*, y^*) + P(x^*, y^*) > 0$. Dividing by $\sigma(x^*, y^*)$ yields $1 \leq \alpha$, a contradiction to $\alpha \in (0, 1)$. Hence $x^* = y^*$. Therefore the common fixed point is unique. □

3. CONCLUSION

This article provides an extension of the existing one map version to the two and higher map version of the Banach Contraction principle in the setting of perturbed metric spaces. Our goal was to give a result that can be applied in solving systems of equations where the existing theorems feels inadequate. However, we have retained the continuity condition of the mappings involved since it not automatically implied by the contraction involved. Further improvement can be made by determining conditions that allows us to drop the continuity conditions.

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