Nonlinear Functional Analysis and Applications
Vol. 31, No. 3 (2026), pp. 1023-1039
ISSN: 1229-1595(print), 2466-0973(online)

https://doi.org/10.22771/nfaa.2026.31.03.20
http://nfaa.kyungnam.ac.kr/journal-nfaa

EXPLORING $\mathfrak{q}$-FIBONACCI NUMBERS IN GEOMETRIC FUNCTION THEORY: UNIVALENCE AND SHELL-LIKE CONVEX CURVES

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Abstract. This paper presents a new subclass of convex functions associated with $\mathfrak{q}$-Fibonacci numbers, which is closely connected to shell-like starlike curves. Using the principle of subordination and exploiting the inherent properties of $\mathfrak{q}$-Fibonacci sequences, we perform a detailed investigation of the geometric and analytic features of this subclass. In particular, we identify the ranges of univalence and non-univalence for certain functions belonging to this class and clarify their relationship with previously known families of convex functions.

[0]Received January 24, 2026. Revised March 24, 2026. Accepted March 31, 2026.
[0]2020 Mathematics Subject Classification: 46E35, 49K20, 76D05, 35Q30.
[0]Keywords: Analytic functions, univalent functions, starlike functions, convex functions, Fibonacci numbers, Fibonacci polynomials, Shell-like curves, quantum calculus.
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1. Introduction and Preliminaries

Let us denote the family of all analytic functions defined on the open unit disk $\mathscr{O}$, where $\mathscr{O}$ is the set of all complex numbers $\mathsf{z} = a + ib$ (with $a, b \in \mathbb{R}$) satisfying $|\mathsf{z}| < 1$. Geometrically, $\mathscr{O}$ represents the collection of all points in the complex plane that lie strictly inside the unit circle centered at the origin.

The function $f \in A$ is normalized to satisfy the following initial conditions:

$$f(0) = 0 \quad \text{and} \quad f'(0) = 1.$$

For every function $f \in \mathscr{A}$ the Taylor-Maclaurin series expansion can be expressed in the following form:

$$f(\mathsf{z}) = \mathsf{z} + \sum_{\mathsf{s}=2}^{\infty} \alpha_{\mathsf{s}}\, \mathsf{z}^{s}, \quad (\mathsf{z} \in \mathscr{O}). \tag{1.1}$$

The concept of subordination [12] has led to a lot of research on the class of starlike functions S^*. Ma and Minda [24] introduced the class

$$\mathsf{S}^*(\Omega) = \left\{ f \in \mathscr{A} : \frac{\mathsf{z}\,\mathsf{f}'(\mathsf{z})}{\mathsf{f}(\mathsf{z})} \prec \Omega(\mathsf{z}), \quad \text{where} \quad \Omega \in \mathsf{P} \;\; \text{and} \;\; \mathsf{z} \in \mathscr{O} \right\}.$$

Contemporary research produces several subclasses of starlike functions with the application of specific forms of Ω. Scholars demonstrate adaptability in their research employing several methodologies. Table 1 delineates these subfamilies and emphasizes the many starlike characteristics arising from the different choices Ω.

Table 1. Several subclasses of starlike functions defined via subordination.

Family of starlike functions	**Author**	**Ref.**
$\mathsf{S}^*\left(\frac{1+z}{1-z}\right) = \left\{ f \in \mathscr{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z} \right\}$	Janowski	[23]
$\mathsf{S}^*(\vartheta) = \left\{ f \in \mathscr{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\vartheta)z}{1-z},\ 0 \le \vartheta < 1 \right\}$	Robertson	[25]
$\mathsf{SL}(\vartheta) = \left\{ f \in \mathscr{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+\vartheta^2 z^2}{1-\vartheta z-\vartheta^2 z^2},\ \vartheta = \frac{1-\sqrt{5}}{2} \right\}$	Sokól	[27]
$\mathcal{SK}(\vartheta) = \left\{ f \in \mathscr{A} : \frac{zf'(z)}{f(z)} \prec \frac{3}{3+(\vartheta-3)z-\vartheta^2 z^2},\ \vartheta \in (-3,1] \right\}$	Sokól	[28]

Recently, quantum calculus, also known as $\mathfrak{q}$-calculus, has garnered significant interest in the fields of physics and mathematics. Its historical origins date back to the 19th century when Jackson [21, 22] introduced the $\mathfrak{q}$-difference operator and integral. Subsequently, Aral and Gupta [18, 19] expanded on this foundational research by examining $\mathfrak{q}$ analogs of many operators, particularly in the realm of geometric function theory. Quantum calculus, grounded in $\mathfrak{q}$-differences, extends classical calculus and offers a robust framework for exploring specific subclasses of analytic functions, such as starlike and convex functions. In this context, the parameter $\mathfrak{q}$ is anticipated to satisfy $0 < \mathfrak{q} < 1$, ensuring the necessary convergence and essential attributes necessary for these analyzes.

Definition 1.1. ([12]) The $\mathfrak{q}$-bracket $\lceil \kappa \rfloor_{\mathfrak{q}}$ is defined as follows:

$$\lceil \kappa \rfloor_{\mathfrak{q}} = \begin{cases} \frac{1-\mathfrak{q}^{\lambda}}{1-\mathfrak{q}}, & 0 < \mathfrak{q} < 1,\ \lambda \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}, \\ 1, & \mathfrak{q} \mapsto 0^+,\ \lambda \in \mathbb{C}^*, \\ \lambda, & \mathfrak{q} \mapsto 1^-,\ \lambda \in \mathbb{C}^*, \\ \mathfrak{q}^{\gamma-1} + \mathfrak{q}^{\gamma-2} + \cdots + \mathfrak{q} + 1 = \sum\limits_{\mathsf{s}=0}^{\gamma-1} \mathfrak{q}^{\mathsf{s}}, & 0 < \mathfrak{q} < 1,\ \lambda = \gamma \in \mathbb{N}. \end{cases}$$

Definition 1.2. ([12]) The $\mathfrak{q}-$derivative, also known as the $\mathfrak{q}$-difference operator, of a function f is defined by

$$\eth_{\mathfrak{q}}\langle \mathsf{f}(\mathsf{z}) \rangle = \begin{cases} (f(\mathsf{z}) - f(\mathfrak{q}\,\mathsf{z}))(\mathsf{z} - \mathfrak{q}\,\mathsf{z})^{-1}, & \text{for} \quad 0 < \mathfrak{q} < 1,\ \mathsf{z} \neq 0, \\ \mathsf{f}'(0), & \text{for} \quad \mathsf{z} = 0, \\ \mathsf{f}'(\mathsf{z}), & \text{for} \quad \mathfrak{q} \mapsto 1^-,\ \mathsf{z} \neq 0. \end{cases}$$

In accordance with the foundational principles governing the definition of the $\mathfrak{q}$-difference operator, the associated theoretical framework and corresponding results can be systematically derived. These derivations align with the established methodologies presented in the reference [17]:

(1) $\eth_{\mathfrak{q}}\langle \mathsf{z}^{\mathsf{s}} \rangle = \lceil \mathsf{s} \rfloor_{\mathfrak{q}} \mathsf{z}^{\mathsf{s}-1}$.

(2) $\eth_{\mathfrak{q}}\langle f(\mathsf{z}) \pm g(\mathsf{z}) \rangle = \eth_{\mathfrak{q}}\langle f(\mathsf{z}) \rangle \pm \eth_{\mathfrak{q}}\langle g(\mathsf{z}) \rangle$.

(3) $\eth_{\mathfrak{q}}\langle f(\mathsf{z}) \cdot g(\mathsf{z}) \rangle = f(\mathfrak{q}\mathsf{z}) \,.\, \eth_{\mathfrak{q}}\langle g(\mathsf{z}) \rangle + g(\mathsf{z}) \,.\, \eth_{\mathfrak{q}}\langle f(\mathfrak{q}\mathsf{z}) \rangle$.

(4) $\eth_{\mathfrak{q}}\left\langle \frac{f(\mathsf{z})}{g(\mathsf{z})} \right\rangle = \dfrac{\eth_{\mathfrak{q}}\langle f(\mathsf{z}) \rangle \,.\, g(\mathsf{z}) - f(\mathsf{z}) \,.\, \eth_{\mathfrak{q}}\langle g(\mathsf{z}) \rangle}{g(\mathsf{z}) g(\mathfrak{q}\mathsf{z})}$.

(5) $\eth_{\mathfrak{q}}\left\langle \log_{\mathfrak{q}} f(\mathsf{z}) \right\rangle = \frac{\log \mathfrak{q}}{\mathfrak{q}-1} \left(\dfrac{\eth_{\mathfrak{q}}\langle f(\mathsf{z}) \rangle}{f(\mathsf{z})} \right)$.

Additionally, Jackson [22] introduced the concept of the $\mathfrak{q}$-integral concerning any function f, characterizing it as

$$\int_0^{\mathsf{z}} f(\zeta)\ \eth_{\mathfrak{q}}\zeta = (1-\mathfrak{q})\,\mathsf{z}\sum_{\mathsf{s}=1}^{\infty} \mathfrak{q}^{\mathsf{s}-1}\ f(\mathfrak{q}^{\mathsf{s}-1}\mathsf{z}). \tag{1.2}$$

As a specific instance of the $\mathfrak{q}$-Jackson integral, we obtain the following equation:

$$\int_0^{\mathsf{z}} \frac{\eth_{\mathfrak{q}}\langle f(\zeta)\rangle}{f(\zeta)}\ \eth_{\mathfrak{q}}\zeta = \frac{\mathfrak{q}-1}{\log\mathfrak{q}}\log_{\mathfrak{q}} f(\mathsf{z}). \tag{1.3}$$

The $\mathfrak{q}$-Jackson difference operators have played a crucial role in extending classical function theory to the realm of $\mathfrak{q}$-calculus, by using the principle of subordination between analytic functions Uçar [29] introduce the starlike class $\mathsf{S}_{\mathfrak{q}}^{*}$ by

$$\mathsf{S}_{\mathfrak{q}}^{*} = \left\{ f \in \mathscr{A} : \quad \frac{\mathsf{z}\eth_{\mathfrak{q}}\langle f(\mathsf{z})\rangle}{f(\mathsf{z})} \prec \frac{1+\mathsf{z}}{1-\mathfrak{q}\mathsf{z}} \qquad (\mathsf{z}\in\mathscr{O}) \right\}, \tag{1.4}$$

Further, Wongsaijai and Sukantamala [30] introduce $\mathsf{S}_{\mathfrak{q}}^{*}(\alpha)$ the starlike class of order α where $0 \le \alpha < 1$ by

$$\mathsf{S}_{\mathfrak{q}}^{*}(\alpha) = \left\{ f \in \mathscr{A} : \quad \frac{\mathsf{z}\eth_{\mathfrak{q}}\langle f(\mathsf{z})\rangle}{f(\mathsf{z})} \prec \frac{1+(1-2\alpha)\mathsf{z}}{1-\mathfrak{q}\mathsf{z}} \qquad (\mathsf{z}\in\mathscr{O}) \right\}. \tag{1.5}$$

Ismail et al. [20] introduced the class $\mathsf{S}^{*}(A,B;\mathfrak{q})$ by

$$\mathsf{S}^{*}(A,B;\mathfrak{q}) = \left\{ \mathsf{f} \in \mathscr{A} : \quad \frac{\mathsf{z}\eth_{\mathfrak{q}}\langle \mathsf{f}(\mathsf{z})\rangle}{\mathsf{f}(\mathsf{z})} \prec \frac{1+A_{\mathfrak{q}}\mathsf{z}}{1+B_{\mathfrak{q}}\mathsf{z}} \qquad (\mathsf{z}\in\mathscr{O}) \right\}, \tag{1.6}$$

where

$$A_{\mathfrak{q}} = \frac{(A+1)+(A-1)\mathfrak{q}}{2} \quad \text{and} \quad B_{\mathfrak{q}} = \frac{(B+1)+(B-1)\mathfrak{q}}{2}.$$

In 2016, Seoudy and Aouf [26] defined the $\mathfrak{q}$-convex class $\mathsf{K}_{\mathfrak{q}}(\alpha)$ for $(0 \le \alpha < 1)$ by

$$\mathsf{K}_{\mathfrak{q}}(\alpha) = \left\{ f \in \mathscr{A} : \Re e\left[1 + \frac{\mathsf{z}\eth_{\mathfrak{q}}^{2}\langle f(\mathsf{z})\rangle}{\eth_{\mathfrak{q}}\langle f(\mathsf{z})\rangle}\right] > \alpha, \qquad \text{for all} \quad (\mathsf{z}\in\mathscr{O}) \right\}.$$

More recently, Alsoboh et al. [4] introduced a significant class of functions, termed $\mathfrak{q}$-starlike functions and denoted by $\mathsf{SL}_{\mathfrak{q}}$. This class is formally defined as

$$\mathsf{SL}_{\mathfrak{q}} = \left\{ f \in \mathscr{A} : \quad \frac{\mathsf{z}\eth_{\mathfrak{q}}\langle f(\mathsf{z})\rangle}{f(\mathsf{z})} \prec \Upsilon(\mathsf{z};\mathfrak{q}) \quad (\mathsf{z}\in\mathscr{O}) \right\}, \tag{1.7}$$

where the function $\Upsilon(z; q)$ is explicitly given by

$$\Upsilon(z; q) = \frac{1 + q\vartheta_q^2 z^2}{1 - \vartheta_q z - q\vartheta_q^2 z^2} \tag{1.8}$$

and

$$\vartheta_q = \frac{1 - \sqrt{4q + 1}}{2q} \tag{1.9}$$

denotes the q-analogue of the Fibonacci numbers.

A fundamental property of $\Upsilon(z; q)$ is its non-univalence in the domain $\mathscr{O}$. Specifically, it attains the same value at two distinct points:

$$\Upsilon(0; q) = 1 \quad \text{and} \quad \Upsilon\left(-\frac{1}{2q\vartheta_q}; q\right) = 1.$$

Furthermore, Alsoboh et al. [4] established a fundamental link between the q-analogue of Fibonacci numbers, denoted by ϑ_q, and their associated Fibonacci polynomials $\varphi_s(q)$. In particular, they showed that if

$$\Upsilon(z; q) = 1 + \sum_{s=1}^{\infty} \widehat{p_s}\, z^s,$$

then the coefficients $\widehat{p_s}$ satisfy the recurrence relation:

$$\widehat{p_s} = \begin{cases} \vartheta_q, & \text{for } s = 1, \\ (2q + 1)\vartheta_q^2, & \text{for } s = 2, \\ (3q + 1)\vartheta_q^3, & \text{for } s = 3, \\ \left(\varphi_{s+1}(q) + q\varphi_{s-1}(q)\right)\vartheta_q^s, & \text{for } s \geq 4. \end{cases} \tag{1.10}$$

where the q-Fibonacci polynomials $\varphi_s(q)$ are defined as

$$\varphi_s(q) = \frac{(1 - q\vartheta_q)^s - (\vartheta_q)^s}{\sqrt{4q + 1}}, \quad s \in \mathbb{N}. \tag{1.11}$$

This result establishes a unified framework for investigating the intricate interplay between q-deformed Fibonacci numbers and their associated polynomial structures, thereby providing deeper insight into their analytical and geometric behavior within the unit disk. Building on this perspective, the emergence of q-calculus has played a pivotal role in advancing analytic function theory, as it enables the construction of new subclasses endowed with rich geometric and algebraic properties. Consequently, the integration of special functions, subordination principles, and quantum operators has become a central theme in modern geometric function theory.

In this direction, several recent contributions have focused on the development of analytic and bi-univalent function classes associated with special polynomials and q-operators. For instance, Alnajar et al. [3] introduced a subclass linked to Touchard polynomials, providing significant coefficient estimates, while Alsoboh et al. [5] investigated bi-univalent functions in leaf-like domains via subordination and quantum calculus, and later extended these results to bi-convex classes in [8]. These studies highlight the importance of geometrically defined domains in shaping the analytic behavior of such functions.

Furthermore, the role of special polynomials in coefficient problems has been systematically explored. In particular, the application of Faber polynomials in [9] led to sharp coefficient bounds for subclasses of bi-univalent functions. This line of research was further enriched by linking geometric structures with special sequences, as demonstrated in [7], where bi-starlike and bi-convex classes associated with shell-like curves and the q-analogue of Fibonacci numbers were introduced. In addition, the use of q-ultraspherical polynomials together with fractional integral operators in [2], as well as subclasses subordinate to Gegenbauer polynomials studied in [1], further illustrate the effectiveness of special functions in capturing geometric properties.

Parallel to these developments, recent works have also emphasized the synthesis between subordination theory and quantum calculus techniques. Amourah et al. [15] constructed a bi-starlike class in a leaf-like domain using q-calculus, offering a novel geometric viewpoint, whereas Alsoboh et al. [10] derived Fekete–Szegö inequalities for subclasses associated with q-ultraspherical polynomials. Moreover, the incorporation of probabilistic structures into geometric function theory was explored in [14, 16], where subclasses defined via the bell distribution and subordinated to Gegenbauer polynomials were analyzed.

From an operator-theoretic perspective, the differential sandwich approach introduced by Yousef et al. [31] provides a powerful framework for studying p-valent functions involving generalized differential and integral operators. This approach complements the operator-based developments in quantum calculus, such as the study of harmonic functions associated with q-operators in [6] and the introduction of new subclasses defined by q-differential operators with respect to k-symmetric points in [11, 13].

Overall, these contributions collectively demonstrate that the interaction between special functions, geometric domains, and q-calculus forms a rich and evolving framework for the study of analytic and bi-univalent functions, thereby motivating further investigation into more generalized subclasses and their associated coefficient problems.

2. Definition and example

Motivated by $\mathfrak{q}$-Fibonacci numbers, this section will now look at a novel subclass of analytic functions. This subclass unifies the conventional theory of analytic functions and highlights other geometric and analytic features revealed by such functions by means of $\mathfrak{q}$ Fibonacci numbers. This subclass develops new ideas for comprehending the connections among geometric analytic functions, Fibonacci-type sequences, and $\mathfrak{q}$-calculus and enhances the previous theory.

Definition 2.1. The function f belongs to the class $\mathsf{KSL}_{\mathfrak{q}}$ if and only if

$$1+\frac{\mathsf{z}\eth_{\mathfrak{q}}^{2}\langle f(\mathsf{z})\rangle}{\eth_{\mathfrak{q}}\langle f(\mathsf{z})\rangle}\prec\Upsilon(\mathsf{z};\mathfrak{q})\qquad(\mathsf{z}\in\mathscr{O}),\tag{2.1}$$

where $\Upsilon(\mathsf{z};\mathfrak{q})$ is given by (1.8).

Remark 2.2. If $f\in\mathsf{KSL}_{\mathfrak{q}}$, then by (2.1)

$$\mathsf{KSL}_{\mathfrak{q}}\subset\mathsf{K}_{\mathfrak{q}}(\sigma),\tag{2.2}$$

where $\sigma=\frac{\sqrt{4\mathfrak{q}+1}}{2(4\mathfrak{q}+1)}$ which means that if $f\in\mathsf{KSL}_{\mathfrak{q}}$, then it is convex of order σ and hence univalent on the unit disk $\mathscr{O}$.

Example 2.3. If $\mathfrak{q}\to1^{-}$ in Definition 2.1, we obtain the class $\mathsf{KSL}=\lim\limits_{\mathfrak{q}\to1^{-}}\mathsf{KSL}_{\mathfrak{q}}$ defined as follows: A function f belongs to the class KSL if and only if

$$1+\frac{\mathsf{z}f''(\mathsf{z})}{f'(\mathsf{z})}\prec\Upsilon(\mathsf{z}),$$

where

$$\Upsilon(\mathsf{z})=\frac{1+\vartheta^{2}\mathsf{z}^{2}}{1-\vartheta\mathsf{z}-\vartheta^{2}\mathsf{z}^{2}},\qquad\vartheta=\frac{1-\sqrt{5}}{2}.\tag{2.3}$$

3. Preliminary lemmas

For an in-depth understanding of the class $\mathsf{KSL}_{\mathfrak{q}}$, it would be worthwhile here to find the shape of the curve $\Omega_{\mathfrak{q}}=\Upsilon(e^{i\theta};\mathfrak{q}),\quad\theta\in[0,2\pi)\setminus\{\pi\}$. We begin our study by noting that

$$\Upsilon(0;\mathfrak{q})=\Upsilon\left(\frac{-1}{2\mathfrak{q}\vartheta_{\mathfrak{q}}};\mathfrak{q}\right)=1\quad\text{and}\quad\Upsilon(1;\mathfrak{q})=\Upsilon\left(\mathfrak{q}^{4}\vartheta_{\mathfrak{q}}^{4};\mathfrak{q}\right)=\frac{\sqrt{1+4q}}{2}.$$

Moreover, the curve $\Omega_{\mathfrak{q}}$ has the following cases:

(1) In the range of $\mathfrak{q}$ where $\mathfrak{q} \in \left(\frac{1}{4}, 1\right)$, we observe that

$$\Upsilon\left(e^{\pm i \text{ arccos}(1/4\mathfrak{q})}\right) = \frac{\sqrt{4\mathfrak{q}+1}}{4\mathfrak{q}+1},$$

imply that the curve $\Omega_{\mathfrak{q}}$ intersects itself on the real axis at the point $\frac{\sqrt{4\mathfrak{q}+1}}{4\mathfrak{q}+1}$. Consequently, $\Omega_{\mathfrak{q}}$ forms a loop that intersects the real axis at two points, $e_1 = \frac{\sqrt{4\mathfrak{q}+1}}{2}$ and $e_2 = \frac{\sqrt{4\mathfrak{q}+1}}{4\mathfrak{q}+1}$, see Fig 1.

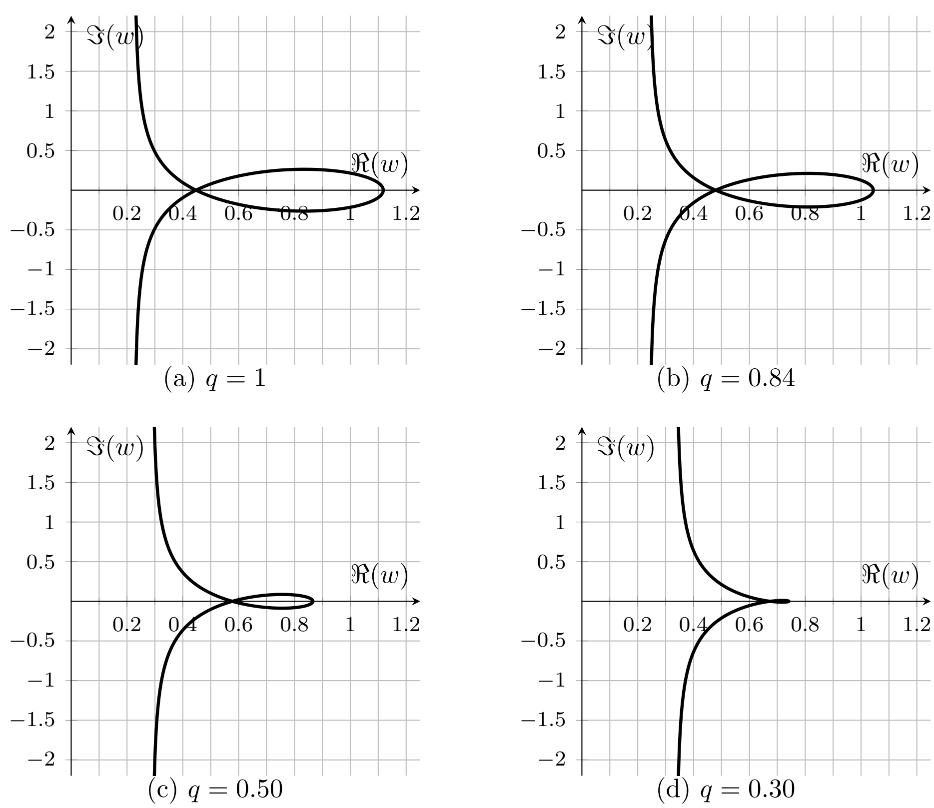


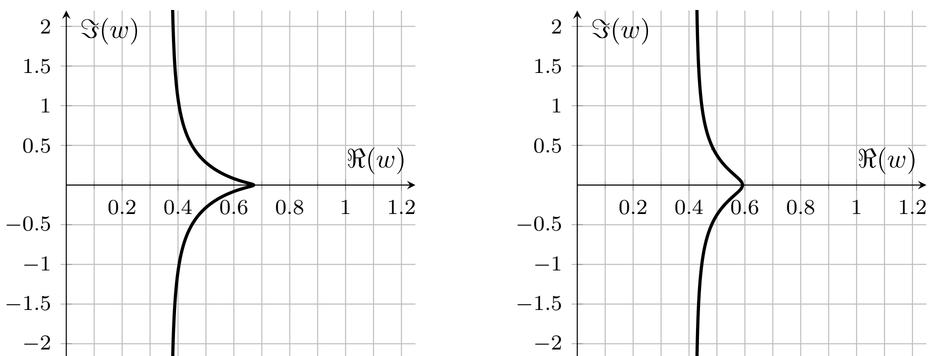
FIGURE 1. Boundary curves $\Omega_q = \Upsilon(e^{i\theta}; q)$ for selected values of q, where $\theta \in [0, 2\pi) \setminus \{\pi\}$. The plots illustrate the variation in the geometry of Ω_q as the parameter q decreases from 1 to 0.30.

q	ϑ_q	$e_1 = \dfrac{\sqrt{4q+1}}{2}$	$e_2 = \dfrac{1}{\sqrt{4q+1}}$
1.00	-0.6180	1.1180	0.4472
0.84	-0.6477	1.0440	0.4789
0.50	-0.7321	0.8660	0.5774
0.30	-0.8054	0.7416	0.6742

At these intersections, along with the presence of the asymptote and loop, we have

$$\lim_{\xi\to\pi^-} \Im m\left(\Upsilon(e^{i\xi});\mathfrak{q}\right) = -\infty \quad \text{and} \quad \lim_{\xi\to\pi^+} \Im m\left(\Upsilon(e^{i\xi});\mathfrak{q}\right) = \infty.$$

(2) In the range of $\mathfrak{q}$ where $\mathfrak{q} \in \left(0, \frac{1}{4}\right)$, the curve takes a similar appearance of a conchoid without any loops. See Fig. **??** for visual representations of $\Omega_{\mathfrak{q}}$ corresponding to various $\mathfrak{q}$ values.



(a) $q = 0.20$ (b) $q = 0.10$

FIGURE 2. Boundary curves $\Omega_q = \Upsilon(e^{i\theta}; q)$ for $q = 0.20$ and $q = 0.10$, where $\theta \in [0, 2\pi) \setminus \{\pi\}$.

If we denote

$$\Re e\left\{\Upsilon(e^{i\xi};\mathfrak{q})\right\} = \varkappa \quad \text{and} \quad \Im m\left\{\Upsilon(e^{i\xi};\mathfrak{q})\right\} = \mathsf{y}, \qquad \theta \in [0, 2\pi) \setminus \{\pi\},$$

then after simple calculations, we get the following.

$$\varkappa = \frac{\sqrt{4\mathfrak{q}+1}}{2(1+2\mathfrak{q}-2\mathfrak{q}\cos\theta)} \quad \text{and} \quad \mathsf{y} = \frac{\frac{\sin\theta}{2(1+\cos\theta)}(4\mathfrak{q}\cos\theta - 1)}{(1+2\mathfrak{q}-2\mathfrak{q}\cos\theta)}, \ \left(\theta \in [0, 2\pi) \setminus \{\pi\}\right). \tag{3.1}$$

It is useful here to use (2.1) to find the corresponding Cartesian equation of the curve $\Omega_{\mathfrak{q}}$. This curve is described by the equation

$$\left[2(1+4\mathfrak{q})\varkappa - \sqrt{1+4\mathfrak{q}}\right]\mathsf{y}^2 = \left(\sqrt{4\mathfrak{q}+1} - 2\varkappa\right)\left(\sqrt{4\mathfrak{q}+1}, \varkappa - 1\right)^2. \tag{3.2}$$

The curve known as the conchoid of Sluze is described by the equation:

$$\varrho(\varkappa-\varrho)\left(\varkappa^2+\mathsf{y}^2\right)+\eta^2\varkappa^2=0, \quad (\varrho,\eta>0). \tag{3.3}$$

It is worthy to point out that for $\eta = 2\varrho$, the equation for the conchoid of Sluze (3.3) becomes the trisectrix of Maclaurin

$$\varkappa^3+(\varkappa-\varrho)\mathsf{y}^2+3\varrho\varkappa^2=0, \tag{3.4}$$

If we express equation (3.2) in the following form:

$$\begin{aligned}&\left(\frac{\sqrt{4\mathsf{q}+1}}{4\mathsf{q}+1}-\varkappa\right)^3+\frac{(4\mathsf{q}-1)\sqrt{4\mathsf{q}+1}}{2(4\mathsf{q}+1)}\left(\frac{\sqrt{4\mathsf{q}+1}}{4\mathsf{q}+1}-\varkappa\right)^2\\&+\left[\left(\frac{\sqrt{4\mathsf{q}+1}}{4\mathsf{q}+1}-\varkappa\right)-\frac{\sqrt{4\mathsf{q}+1}}{2(4\mathsf{q}+1)}\right]\mathsf{y}^2=0,\end{aligned} \tag{3.5}$$

The image of the unit circle under the function $\Upsilon(\mathsf{z};\mathsf{q})$ is therefore converted into a curve specified by Eq. (3.4), where

$$\varrho=\frac{1-2\mathsf{q}\vartheta_q}{2(4\mathsf{q}+1)}=\frac{\sqrt{4\mathsf{q}+1}}{2(4\mathsf{q}+1)}.$$

Therefore the curve Ω_q displays a shell-like configuration and is symmetric about the real axis, as illustrated in Fig. 1 and Fig. 2.

Lemma 3.1. ([4]) *The function* $\Upsilon(\mathsf{z};\mathsf{q})$ *is univalent within the disk*

$$\mathbb{D}_\mathsf{q}=\left\{\mathsf{z}:|\mathsf{z}|<\mathsf{r}_\mathsf{q}=\frac{1+\mathsf{q}-\sqrt{(1+\mathsf{q})^2+1}}{\mathsf{q}\vartheta_\mathsf{q}}\right\} \tag{3.6}$$

and is not univalent in any disk with a radius equal to or greater than r_q.

Lemma 3.2. ([4]) *Let* f *be given by* (1.1) *and belong to the class* SL_q. *Then, the coefficients satisfy the bound*

$$|\alpha_\mathsf{s}|\leq|\vartheta_\mathsf{q}|^{\mathsf{s}-1}\varphi_\mathsf{s}(\mathsf{q}), \qquad \mathsf{s}=2,3,4,\cdots, \tag{3.7}$$

where $\vartheta_\mathsf{q}=\frac{1-\sqrt{4q+1}}{2\mathsf{q}}$ *and* $\varphi_\mathsf{s}(\mathsf{q})$ *is given by* (1.10). *Moreover, this bound is sharp and is attained by the function*

$$\widetilde{\Upsilon}(\mathsf{z};\mathsf{q})=\frac{\mathsf{z}}{1-\vartheta_\mathsf{q}\mathsf{z}-\mathsf{q}\vartheta_\mathsf{q}^2\mathsf{z}^2}. \tag{3.8}$$

4. Main results

In the first theorem, we determine the sharp upper bound for the coefficients of functions boling to the class $\mathsf{KSL}_{\mathfrak{q}}$.

Theorem 4.1. *Let f be given by* (1.1) *and belong to the class* $\mathsf{KSL}_{\mathfrak{q}}$*. Then, the coefficients satisfy the bound*

$$|\alpha_{\mathsf{s}}| \leq \frac{|\vartheta_{\mathfrak{q}}|^{\mathsf{s}-1}\varphi_{\mathsf{s}}(\mathfrak{q})}{\lceil \mathsf{s} \rfloor_{\mathfrak{q}}}, \qquad \mathsf{s} = 2, 3, 4, \cdots, \tag{4.1}$$

where $\vartheta_{\mathfrak{q}} = \frac{1-\sqrt{4q+1}}{2\mathfrak{q}}$ *and* $\varphi_{\mathsf{s}}(\mathfrak{q})$ *is given by* (1.10)*. Moreover, this bound is sharp and is attained by the function*

$$\widetilde{f}(\mathsf{z}) = \frac{\mathfrak{q}-1}{(\log \mathfrak{q})\left(1+\mathfrak{q}\vartheta_{\mathfrak{q}}^2\right)} \log_{\mathfrak{q}}\left(\frac{1+\mathsf{z}}{1-\mathfrak{q}\vartheta_{\mathfrak{q}}^2\mathsf{z}}\right). \tag{4.2}$$

Proof. A function f is in the class $\mathsf{KSL}_{\mathfrak{q}}$ if and only if there exists a function $\mathsf{g} \in \mathsf{SL}_{\mathfrak{q}}$ such that

$$\mathsf{g}(z) = \mathsf{z}\,\eth_{\mathfrak{q}}\langle f(\mathsf{z})\rangle \qquad \text{for all } (\mathsf{z} \in \mathscr{O}), \tag{4.3}$$

or equivalently

$$f(\mathsf{z}) = \int_0^{\mathsf{z}} \frac{\mathsf{g}(\zeta)}{\zeta}\,\eth_{\mathfrak{q}}\zeta \qquad \text{for all } (\mathsf{z} \in \mathscr{O}). \tag{4.4}$$

The relations (4.3) and (4.4) follow directly from (1.7) and (2.1). Therefore, if

$$\mathsf{z}\,\eth_{\mathfrak{q}}\langle f(\mathsf{z})\rangle = \mathsf{z} + \sum_{\mathsf{s}=2}^{\infty} \lceil \mathsf{s} \rfloor_{\mathfrak{q}} \alpha_{\mathsf{s}}\,\mathsf{z}^s \qquad (\mathsf{z} \in \mathscr{O}), \tag{4.5}$$

belongs to the class $\mathsf{SL}_{\mathfrak{q}}$, then by Lemma 3.2, we conclude that

$$\lceil \mathsf{s} \rfloor_{\mathfrak{q}} |\alpha_{\mathsf{s}}| \leq |\vartheta_{\mathfrak{q}}|^{\mathsf{s}-1}\varphi_{\mathsf{s}}(\mathfrak{q}),$$

which establishes (3.7) is such that $\mathsf{z}\,\eth_{\mathfrak{q}}\langle \widetilde{f}(\mathsf{z})\rangle = \Upsilon_0(\mathsf{z};\mathfrak{q})$, where the function $\widetilde{\Upsilon}(\mathsf{z};\mathfrak{q})$ is given in (3.8), and hence by (4.3), it follows $\widetilde{f} \in \mathsf{KSL}_{\mathfrak{q}}$. Moreover, by (3.8), we get

$$\widetilde{f}(\mathsf{z}) = \mathsf{z} + \sum_{\mathsf{s}=2}^{\infty} \frac{|\vartheta_{\mathfrak{q}}|^{\mathsf{s}-1}\varphi_{\mathsf{s}}(\mathfrak{q})}{\lceil \mathsf{s} \rfloor_{\mathfrak{q}}}\mathsf{z}^s \qquad (\mathsf{z} \in \mathscr{O}). \tag{4.6}$$

Thus, the result (4.1) is sharp. $\square$

Corollary 4.2. *If a function f of the form* (1.1) *is in the class* $\mathsf{KSL} = \lim\limits_{\mathsf{q}\to 1^-} \mathsf{KSL}_\mathsf{q}$, *Then, the coefficients satisfy the bound*

$$|\alpha_\mathsf{s}| \leq \frac{|\vartheta|^{\mathsf{s}-1}\varphi_\mathsf{s}}{\mathsf{s}}, \qquad \mathsf{s} = 2, 3, 4, \cdots,$$

where $\vartheta = \frac{1-\sqrt{5}}{2}$. *Moreover, this bound is sharp and is attained by the function*

$$\widetilde{\mathsf{f}}(\mathsf{z}) = \frac{1}{\left(1+\vartheta^2\right)} \log\left(\frac{1+\mathsf{z}}{1-\vartheta^2\mathsf{z}}\right).$$

Corollary 4.3. ([27]) *If the function* f *belongs to the class* SL, *then*

$$\lim_{n\to\infty} |\alpha_\mathsf{s}| \leq |\vartheta|^{\mathsf{s}-1}\varphi_\mathsf{s}.$$

This result is sharp for the function

$$\widetilde{\Upsilon}(\mathsf{z}; 1) = \frac{\mathsf{z}}{1-\vartheta\,\mathsf{z}-\vartheta^2\,\mathsf{z}^2}, \qquad \vartheta = \frac{1-\sqrt{5}}{2}.$$

Theorem 4.4. *A function f of the form* (1.1) *is in the class* KSL_q *if and only if there exists a function μ where $\mu \prec \Upsilon$ such that*

$$f(\mathsf{z}) = \left(\frac{\log\mathsf{q}}{\mathsf{q}-1}\right)^2 \int_0^{\mathsf{z}} \exp_\mathsf{q}\left[\int_0^{\zeta} \frac{\mu(\xi)-1}{\xi}\eth_\mathsf{q}\xi\right]\eth_\mathsf{q}\zeta, \quad \mathsf{z} \in \mathscr{O}. \tag{4.7}$$

Proof. It is known that a function $g \in \mathsf{SL}_\mathsf{q}$ can be expressed by

$$g(\mathsf{z}) = z\cdot\left(\frac{\log\mathsf{q}}{\mathsf{q}-1}\right)\exp_\mathsf{q}\int_0^{\mathsf{z}} \frac{\mu(\xi)-1}{\xi}\eth_\mathsf{q}\xi, \tag{4.8}$$

where $\mu \prec \Upsilon$. In view of (4.3), we infer that $f \in \mathsf{KSL}_\mathsf{q}$ if and only if $g(\mathsf{z}) = \mathsf{z}\,\eth_\mathsf{q}\langle f(\mathsf{z})\rangle$ with $g \in \mathsf{SL}_\mathsf{q}$. Applying q-Jackson integral on (4.8), we obtain (4.7). □

Corollary 4.5. *A function f of the form* (1.1) *is in the class* $\mathsf{KSL} = \lim\limits_{\mathsf{q}\to 1^-} \mathsf{KSL}_\mathsf{q}$ *if and only if there exists a function μ where $\mu \prec \Upsilon$ such that*

$$f(\mathsf{z}) = \int_0^{\mathsf{z}} \exp\left[\int_0^{\zeta} \frac{\mu(\xi)-1}{\xi}d\xi\right]d\zeta, \quad \mathsf{z} \in \mathscr{O}. \tag{4.9}$$

Theorem 4.4 provides us a method of finding the members of the class KSL_q.

Theorem 4.6. *If a function f belongs to the class* KSL_q, *then there exists a function* $g \in \mathsf{S}^*(0, -\vartheta_\mathsf{q}^2; \mathsf{q})$ *and a function* $\hbar \in \mathsf{S}_\mathsf{q}^*(\frac{1}{2})$ *such that*

$$\mathsf{z}^2\,\eth_\mathsf{q}\langle f(\mathsf{z})\rangle = g(\mathsf{z})\cdot\hbar(\mathsf{z}) \qquad (\mathsf{z} \in \mathscr{O}). \tag{4.10}$$

Proof. $f \in \mathsf{KSL}_{\mathfrak{q}}$, then by Theorem 4.4, there exists analytic function $\omega(\mathsf{z})$ with $\omega(0) = 0$ and $|\omega(\mathsf{z})| < 1$ for $\mathsf{z} \in \mathscr{O}$ such that

$$\mathsf{z}^2\, \eth_{\mathfrak{q}} \langle f(\mathsf{z})\rangle = \frac{\log \mathfrak{q}}{\mathfrak{q}-1}\, \mathsf{z}^2 \exp_{\mathfrak{q}} \left[\int_0^{\mathsf{z}} \frac{\Upsilon(\omega(\xi);\mathfrak{q}) - 1}{\xi}\, \eth_{\mathfrak{q}}\xi\right]. \tag{4.11}$$

Observe that

$$\Upsilon(\omega(\xi);\mathfrak{q}) = \frac{1}{1-\mathfrak{q}^2\vartheta_{\mathfrak{q}}\omega(\mathsf{z})} + \frac{1}{1+\omega(\mathsf{z})} - 1, \tag{4.12}$$

so we can rewrite (4.12) in the form

$$\begin{aligned}
\mathsf{z}^2\, \eth_{\mathfrak{q}} \langle f(\mathsf{z})\rangle &= \left(\frac{\log \mathfrak{q}}{\mathfrak{q}-1}\right)^2 \mathsf{z}^2 \exp_{\mathfrak{q}} \left[\int_0^{\mathsf{z}} \frac{\frac{1}{1-\mathfrak{q}^2\vartheta_{\mathfrak{q}}\omega(\mathsf{z})} - 1}{\xi}\, \eth_{\mathfrak{q}}\xi + \int_0^{\mathsf{z}} \frac{\frac{1}{1+\omega(\mathsf{z})} - 1}{\xi}\, \eth_{\mathfrak{q}}\xi\right]\\
&= \frac{\log \mathfrak{q}}{\mathfrak{q}-1}\, \mathsf{z} \exp_{\mathfrak{q}} \left[\int_0^{\mathsf{z}} \frac{\frac{1}{1-\mathfrak{q}^2\vartheta_{\mathfrak{q}}\omega(\mathsf{z})} - 1}{\xi}\, \eth_{\mathfrak{q}}\xi\right]\\
&\quad \times \frac{\log \mathfrak{q}}{\mathfrak{q}-1}\, \mathsf{z} \exp_{\mathfrak{q}} \left[\int_0^{\mathsf{z}} \frac{\frac{1}{1+\omega(\mathsf{z})} - 1}{\xi}\, \eth_{\mathfrak{q}}\xi\right]\\
&= g(\mathsf{z}) \cdot \hbar(\mathsf{z}).
\end{aligned} \tag{4.13}$$

Using the structural formulas for the class $\mathsf{S}^*(A, B;\mathfrak{q})$ and the class $\mathsf{S}^*_{\mathfrak{q}}(\alpha)$ we can find the functions g and $\hbar$ defined by (4.13) satisfies $g \in \mathsf{S}^*(0, -\vartheta^2_{\mathfrak{q}};\mathfrak{q})$ and a function $\hbar \in \mathsf{S}^*_{\mathfrak{q}}(\frac{1}{2})$, respectively. □

Theorem 4.7. *If $f \in \mathsf{KSL}_{\mathfrak{q}}$, then*

$$\frac{(1+\mathsf{r})^{2\sigma-1} - 1}{2\sigma - 1} \le |f(\mathsf{z})| \le \left|\widetilde{f}(-\mathsf{r})\right| = \frac{1}{1+\mathfrak{q}\vartheta^2_{\mathfrak{q}}} \log_{\mathfrak{q}}\left(\frac{1+\mathfrak{q}\vartheta^2_{\mathfrak{q}}\mathsf{r}}{1-\mathsf{r}}\right) \tag{4.14}$$

and

$$\frac{1}{1+(1+\mathfrak{q}\vartheta^2_{\mathfrak{q}})\mathsf{r} + \mathfrak{q}\vartheta^2_{\mathfrak{q}}\mathsf{r}} \le |\eth_{\mathfrak{q}}\langle f(\mathsf{z})\rangle| \le \eth_{\mathfrak{q}}\langle \widetilde{f}(-\mathsf{r})\rangle = \frac{\frac{\log \mathfrak{q}}{\mathfrak{q}-1}}{1+\vartheta_{\mathfrak{q}}\mathsf{r} - \mathfrak{q}\vartheta^2_{\mathfrak{q}}\mathsf{r}^2}. \tag{4.15}$$

Proof. Let $f \in \mathsf{KSL}_{\mathfrak{q}}$ be of the form (1.1) , and let us denote by $\widetilde{\mathsf{p}}$ the coefficients of the function $\widetilde{f}$ given by (4.6). Then by (4.1), we have $|\alpha_{\mathsf{s}}| \le |\widetilde{\mathsf{p}}_{\mathsf{s}}|$ for each integer $\mathsf{s} \ge 0$. Notice that the even coefficients $\widetilde{\mathsf{p}}_{\mathsf{s}} = \frac{\vartheta^{\mathsf{s}-1}_{\mathfrak{q}}\varphi_{\mathsf{s}}(\mathfrak{q})}{\lceil \mathsf{s} \rfloor_{\mathfrak{q}}}$ are negative, while the odd coefficients are positive. If $\mathsf{z} = \mathsf{r}e^{i\theta}$, then from the coefficient inequality $|\alpha_{\mathsf{s}}| \le |\widetilde{\mathsf{p}}_{\mathsf{s}}|$, we obtain

$$|f(\mathsf{z})| = \mathsf{r} + \sum_{\mathsf{s}=2}^{\infty} |\alpha_{\mathsf{s}}|\mathsf{r}^s \le \mathsf{r} + \sum_{\mathsf{s}=2}^{\infty} |\widetilde{\mathsf{p}}_{\mathsf{s}}| = \mathsf{r} + \sum_{\mathsf{s}=2}^{\infty} \widetilde{\mathsf{p}}_{\mathsf{s}}(-1)^{\mathsf{s}-1} = -\widetilde{f}(-\mathsf{r}),$$

which yields the upper bound of (4.14). Analogously, we obtain

$$\begin{aligned}
|\eth_{\mathsf{q}}\langle f(\mathsf{z})\rangle| &= \left|1+\sum_{\mathsf{s}=2}^{\infty}\lceil \mathsf{s}\rfloor_{\mathsf{q}}\alpha_{\mathsf{s}}\mathsf{z}^{\mathsf{s}-1}\right| \\
&\le 1+\sum_{\mathsf{s}=2}^{\infty}\lceil \mathsf{s}\rfloor_{\mathsf{q}}|\alpha_{\mathsf{s}}||\mathsf{z}|^{\mathsf{s}-1} \\
&\le 1+\sum_{\mathsf{s}=2}^{\infty}\lceil \mathsf{s}\rfloor_{\mathsf{q}}|\widetilde{\mathsf{p}}_{\mathsf{s}}||\mathsf{z}|^{\mathsf{s}-1} \\
&\le 1+\sum_{\mathsf{s}=2}^{\infty}\lceil \mathsf{s}\rfloor_{\mathsf{q}}|\widetilde{\mathsf{p}}_{\mathsf{s}}|\mathsf{r}^{\mathsf{s}-1} \\
&= 1-\lceil 2\rfloor_{\mathsf{q}}\widetilde{\mathsf{p}}_2\mathsf{r}+\lceil 3\rfloor_{\mathsf{q}}\widetilde{\mathsf{p}}_3\mathsf{r}^2-\lceil 4\rfloor_{\mathsf{q}}\widetilde{\mathsf{p}}_4\mathsf{r}^3+\cdots \\
&= \eth_{\mathsf{q}}\langle \widetilde{f}(-\mathsf{r})\rangle,
\end{aligned}$$

and we obtain the upper bound of inequality (4.15). It is easy to see that the upper bounds are sharp, being attained by the function $\widetilde{f} \in \mathsf{KSL}_{\mathsf{q}}$ at the point $\mathsf{z} = -\mathsf{r}$. To find the left-hand side of the inequality (4.14), since $g \in \mathsf{S}^*(0, -\vartheta_{\mathsf{q}}^2; \mathsf{q})$, then for $\mathsf{z} = \mathsf{r}$ for $0 \le \mathsf{r} < 1$, we have

$$\frac{\mathsf{r}}{1+\mathsf{q}\vartheta_{\mathsf{q}}^2\mathsf{r}} \le |g(\mathsf{z})|. \tag{4.16}$$

Moreover, if $\hbar \in \mathsf{S}_{\mathsf{q}}^*(\frac{1}{2})$, $\mathsf{z} = \mathsf{r}$ for $0 \le \mathsf{r} < 1$, we obtain

$$\frac{\mathsf{r}}{1+\mathsf{r}} \le |\hbar(\mathsf{z})|. \tag{4.17}$$

By Theorem 4.6, we have $|\mathsf{z}^2\, \eth_q\langle f(\mathsf{z})\rangle| = |g(\mathsf{z})| \cdot |\hbar(\mathsf{z})|$ with $g \in \mathsf{S}^*(0, -\vartheta_{\mathsf{q}}^2; \mathsf{q})$ and $\hbar \in \mathsf{S}_{\mathsf{q}}^*(\frac{1}{2})$, and multiplying the respective sides of (4.16) with those of (4.17), we thus obtain the left-hand side of (4.15).

To prove the left-hand side of the inequality (4.14), we note that by (2.2) if $f \in \mathsf{KSL}_{\mathsf{q}}$, then $f \in \mathsf{K}_{\mathsf{q}}(\sigma)$ is convex of order σ. The desired inequality will now follow from the well-known inequality for $|f(\mathsf{z})|$ with $f \in \mathsf{K}_{\mathsf{q}}(\sigma)$. $\square$

Corollary 4.8. *If a function f of the form* (1.1) *is in the class* $\mathsf{KSL} = \lim\limits_{\mathsf{q}\to 1^-} \mathsf{KSL}_{\mathsf{q}}$, *then*

$$\frac{(1+\mathsf{r})^{2\sigma-1}-1}{2\sigma-1} \le |f(\mathsf{z})| \le \left|\widetilde{f}(-\mathsf{r})\right| = \frac{1}{1+\vartheta^2}\log\left(\frac{1+\vartheta^2\mathsf{r}}{1-\mathsf{r}}\right)$$

and

$$\frac{1}{1+(1+\vartheta^2)\mathsf{r}+\vartheta^2\mathsf{r}} \le |f'(\mathsf{z})| \le \widetilde{f}'(-\mathsf{r}) = \frac{1}{1+\vartheta\mathsf{r}-\vartheta^2\mathsf{r}^2},$$

where $\vartheta = \frac{1-\sqrt{5}}{2}$.

5. Conclusion

This study introduces and rigorously investigates a novel subclass of convex functions associated with $\mathfrak{q}$-Fibonacci numbers and $\mathfrak{q}$-polynomials, intricately linked to shell-like starlike curves. Employing subordination principles and using the structural properties of $\mathfrak{q}$-Fibonacci sequences, we perform a comprehensive analysis of the geometric and analytic characteristics of the subclass. Specifically, we delineate the intervals of univalence and nonunivalence for certain functions within this class and establish its connections with existing convex function families. Furthermore, we derive sharp upper bounds for growth and distortion of the coefficients associated with functions in this convex class, thereby providing significant insights into the interplay between Fibonacci-type sequences and the geometric behavior of analytic functions.

The results given are expected to inspire curiosity among the scholarly community and encourage more research on the uses of $\mathfrak{q}$-calculus in geometric function theory and related fields. This work underlines the prospect for new interactions between Fibonacci-type sequences and the geometric features of analytic functions, therefore laying a basis for next investigations and advancing both theoretical and practical realms.

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