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OPTIMAL RANKS OF NONLINEAR MATRIX EXPRESSIONS WITH APPLICATIONS

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Abstract. This paper presents explicit formulas for the optimal ranks of nonlinear matrix expressions with respect to four variable matrices. As applications, we derive the optimal ranks of the generalized Schur complement and present the optimal ranks of specific nonlinear matrix expressions involving four variable matrices, which represent solutions to four solvable matrix equations. Consequently, we provide equivalent conditions for the existence of solutions to nonlinear matrix equations.

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1. Introduction

Matrix theory and its applications continue to play a central role in modern mathematical analysis, particularly in the study of matrix-valued functions, special functions, and nonlinear structures. Recent developments have highlighted the importance of generalized matrix functions and fractional calculus in extending classical results and providing new analytical tools; see, for instance, [4, 11, 17]. Moreover, investigations into polynomial structures, spectral properties, and numerical approximation methods have further enriched this field and broadened its applicability in both theoretical and computational contexts [3, 1]. In addition, fractional calculus has emerged as a powerful framework for modeling complex systems and enhancing the understanding of matrix-based formulations [2]. These advancements motivate the continued exploration of nonlinear matrix expressions and their associated rank properties.

Throughout this work, all $m \times n$ complex matrices are denoted by $\mathbb{C}^{m\times n}$, and the symbols $r(A)$, A^*, and I_n stand for the rank of A, the conjugate transpose of A, and the identity matrix of order n, respectively. The Moore–Penrose inverse of a matrix $A \in \mathbb{C}^{m\times n}$ is the unique matrix $T \in \mathbb{C}^{n\times m}$ that satisfies the following four matrix equations:

$$(1)\ ATA = A, \quad (2)\ TAT = T, \quad (3)\ (AT)^* = AT, \quad (4)\ (TA)^* = TA.$$

It is frequently denoted by A^+. The Moore–Penrose generalized inverse has been the focus of extensive research efforts; see ([6, 7, 16, 18]). We denote by $E_A = I_m - AA^+$ and $F_A = I_n - A^+A$ the two orthogonal projectors induced by $A \in \mathbb{C}^{m\times n}$.

The rank of a matrix is a fundamental concept in linear algebra. It changes as the matrix's arguments vary. Since the rank is always a finite nonnegative integer, matrix expressions involving variable arguments must have well-defined maximum and minimum ranks. In Lemma 2.1 of the following section, Matsaglia and Styan [15] demonstrate the application of various formulas that simplify intricate matrix expressions or equalities, thereby enhancing the understanding of partitioned matrices with orthogonal projectors. The maximal and minimal ranks of a matrix expression determine the dimensions of the associated row and column spaces. These ranks can be computed efficiently using established elementary block matrix operations.

The maximal and minimal ranks of matrix expressions with variable entries are closely linked to numerous problems in matrix theory and its applications. For instance, all solutions of the matrix expression $A-BXC$ preserve invariant rank if and only if $\max_X r(A-BXC) = \min_X r(A-BXC)$. Furthermore, $BXC =$

A is solvable for X if and only if $\min_X r(A - BXC) = 0$. The matrix equation $BXC = A$ holds for all matrices X if and only if $\max_X r(A - BXC) = 0$.

The maximization and minimization rank problems for matrix expressions have been extensively studied by numerous authors. For instance, in [26], Xiong et al. studied the extremal ranks of the matrix expression $A - B_1X_1C_1 - B_2X_2C_2 - B_3X_3C_3 - B_4X_4C_4$ with respect to X_1, X_2, X_3, and X_4, and then derived applications to the Schur complement and partial matrices. Tian [20] provided closed-form formulas for the extremal ranks of the quadratic matrix function $A - (A_1 - C_1X_1B_1)D(A_2 - C_2X_2B_2)$ with respect to X_1 and X_2, and established equivalent conditions for the solutions of two linear matrix equations to be orthogonal and proportional.

In [22], He investigated rank, inertia, maximization, and minimization problems in the Löwner partial ordering for the quadratic Hermitian matrix function $f(X) = (XC + D)M(XC + D)^* - G$, subject to a consistent linear matrix equation $XA = B$, using straightforward algebraic methods. Guerarra [10] studied optimization problems for a Hermitian quadratic matrix-valued function involving two variable matrices. Furthermore, the author established necessary and sufficient conditions for the existence of solutions satisfying the associated matrix equation and various Hermitian inequalities.

Several papers have addressed related topics on the extremal ranks of matrix expressions; see ([5, 8, 9, 12, 14, 13, 19, 21, 23]). Matrix theory encompasses two primary areas: linear and nonlinear matrix functions and their specific instances. Problems of maximizing or minimizing a quadratic function subject to linear constraints are known as quadratic programming or optimization problems. For example, in [25], Xiong and Qin studied the extremal ranks of nonlinear matrix expressions with respect to two variable matrices and provided solvability conditions for nonlinear matrix functions. In [24], Xie et al. derived sufficient conditions for the existence and uniqueness of a positive semi-definite solution to the nonlinear matrix equation $X^s + A^H F(X)A = Q$, and also considered perturbation analysis for the solution of this class of nonlinear matrix equations.

Consider the nonlinear matrix expression

$$\begin{aligned} \phi(X_1, X_2, X_3, X_4) = Q &- (D_1 - A_1X_1B_1)\, N\, (D_2 - A_2X_2B_2) \\ &- (D_3 - A_3X_3B_3)\, M\, (D_4 - A_4X_4B_4)\,, \end{aligned} \tag{1.1}$$

where $Q \in \mathbb{C}^{m\times n}$, $N \in \mathbb{C}^{l\times k}$, $M \in \mathbb{C}^{t\times h}$, $D_1 \in \mathbb{C}^{m\times l}$, $A_1 \in \mathbb{C}^{m\times p_1}$, $B_1 \in \mathbb{C}^{q_1\times l}$, $D_2 \in \mathbb{C}^{k\times n}$, $A_2 \in \mathbb{C}^{k\times p_2}$, $B_2 \in \mathbb{C}^{q_2\times n}$, $D_3 \in \mathbb{C}^{m\times t}$, $A_3 \in \mathbb{C}^{m\times p_3}$, $B_3 \in \mathbb{C}^{q_3\times t}$, $D_4 \in \mathbb{C}^{h\times n}$, $A_4 \in \mathbb{C}^{h\times p_4}$, and $B_4 \in \mathbb{C}^{q_4\times n}$ are given matrices, while $X_1 \in \mathbb{C}^{p_1\times q_1}$, $X_2 \in \mathbb{C}^{p_2\times q_2}$, $X_3 \in \mathbb{C}^{p_3\times q_3}$, and $X_4 \in \mathbb{C}^{p_4\times q_4}$ are unknown matrices.

Our objective in this article is to derive explicit formulas for the optimal ranks of (1.1) with respect to four variable matrices $X_1 \in \mathbb{C}^{p_1 \times q_1}$, $X_2 \in \mathbb{C}^{p_2 \times q_2}$, $X_3 \in \mathbb{C}^{p_3 \times q_3}$, and $X_4 \in \mathbb{C}^{p_4 \times q_4}$. We then derive consistency conditions for the nonlinear matrix equation

$$Q = (D_1 - A_1 X_1 B_1)\, N\, (D_2 - A_2 X_2 B_2) - (D_3 - A_3 X_3 B_3)\, M\, (D_4 - A_4 X_4 B_4)\,. \tag{1.2}$$

As applications, we derive the extremal ranks of the generalized Schur complement $Q - BH_r^- C - DL_r^- T$ with respect to the reflexive generalized inverses H_r^- and L_r^- of H and L, respectively. Furthermore, we establish the extremal ranks of the nonlinear matrix expression $Q - X_1 N X_2 - X_3 M X_4$ subject to the solutions of four consistent matrix equations, and then provide solvability conditions for the associated nonlinear matrix equations.

This paper is organized as follows. Section 2 presents essential formulas for the ranks of matrices and matrix expressions. Section 3 investigates extremal rank problems for the matrix expression (1.1) and establishes solvability conditions for nonlinear matrix equations. Section 4 presents applications of the results obtained in Section 3. Finally, Section 5 summarizes the main findings.

2. Preliminaries

Given the importance of rank formulas in this paper, this section presents essential results on the ranks of matrix expressions and block matrices.

Lemma 2.1. ([15]) *Let $G \in \mathbb{C}^{t \times s}$, $L \in \mathbb{C}^{t \times k}$, and $C \in \mathbb{C}^{l \times s}$. Then,*

$$r\begin{bmatrix} G & L \end{bmatrix} - r(G) = r(E_G L), \quad r\begin{bmatrix} G & L \end{bmatrix} - r(L) = r(E_L G),$$

$$r\begin{bmatrix} G \\ C \end{bmatrix} - r(G) = r(CF_G), \quad r\begin{bmatrix} G \\ C \end{bmatrix} - r(C) = r(GF_C),$$

$$r\begin{bmatrix} G & L \\ C & 0 \end{bmatrix} = r(E_L G F_C) + r(L) + r(C),$$

$$r\begin{bmatrix} G & LF_P \\ E_Q C & 0 \end{bmatrix} = r\begin{bmatrix} G & L & 0 \\ C & 0 & Q \\ 0 & P & 0 \end{bmatrix} - r(P) - r(Q).$$

Lemma 2.2. ([16]) *Let $A \in \mathbb{C}^{k \times s}$ and $D \in \mathbb{C}^{k \times p}$ be given, and let $X \in \mathbb{C}^{s \times p}$ be an unknown matrix. The matrix equation $AX = D$ is consistent if and only if*

$$R(D) \subseteq R(A), \text{ or equivalently } AA^+ D = D. \tag{2.1}$$

In this case, the general solution can be expressed as

$$X = A^+ D + F_A V, \tag{2.2}$$

where $V \in \mathbb{C}^{s\times p}$ *is arbitrary.*

Lemma 2.3. ([6]) *Let* $H \in \mathbb{C}^{n\times l}$*. Then the reflexive generalized inverse* H_r^- *of the matrix* H *can be expressed as*

$$H_r^- = \left(H^+ + F_H U_1\right) H \left(H^+ + U_2 E_H\right), \tag{2.3}$$

where U_1 *and* U_2 *are arbitrary matrices.*

Lemma 2.4. ([19]) *Let*

$$\Psi(X_1, X_2) = G - E_1 X_1 C_1 - E_2 X_2 C_2,$$

where $G \in \mathbb{C}^{k\times s}$, $E_1 \in \mathbb{C}^{k\times p_1}$, $E_2 \in \mathbb{C}^{k\times p_2}$, $C_1 \in \mathbb{C}^{q_1\times s}$, *and* $C_2 \in \mathbb{C}^{q_2\times s}$ *are given, and* $X_1 \in \mathbb{C}^{p_1\times q_1}$ *and* $X_2 \in \mathbb{C}^{p_2\times q_2}$ *are unknown matrices. Then,*

$$\max_{X_1, X_2} r\big(\Psi(X_1, X_2)\big) = \min\left\{\begin{array}{cc} r\begin{bmatrix} G & E_1 & E_2 \end{bmatrix}, & r\begin{bmatrix} G \\ C_1 \\ C_2 \end{bmatrix}, \\ \\ r\begin{bmatrix} G & E_1 \\ C_2 & 0 \end{bmatrix}, & r\begin{bmatrix} G & E_2 \\ C_1 & 0 \end{bmatrix} \end{array}\right\}, \tag{2.4}$$

$$\min_{X_1, X_2} r\big(\Psi(X_1, X_2)\big) = r\begin{bmatrix} G \\ C_1 \\ C_2 \end{bmatrix} + r\begin{bmatrix} G & E_1 & E_2 \end{bmatrix} + \max\{\varepsilon_1, \varepsilon_2\}, \tag{2.5}$$

where

$$\varepsilon_1 = r\begin{bmatrix} G & E_1 \\ C_2 & 0 \end{bmatrix} - r\begin{bmatrix} G & E_1 & E_2 \\ C_2 & 0 & 0 \end{bmatrix} - r\begin{bmatrix} G & E_1 \\ C_1 & 0 \\ C_2 & 0 \end{bmatrix},$$

$$\varepsilon_2 = r\begin{bmatrix} G & E_2 \\ C_1 & 0 \end{bmatrix} - r\begin{bmatrix} G & E_1 & E_2 \\ C_1 & 0 & 0 \end{bmatrix} - r\begin{bmatrix} G & E_2 \\ C_1 & 0 \\ C_2 & 0 \end{bmatrix}.$$

Obviously, we have the next result regarding the definition of the rank of a matrix.

Lemma 2.5. *Let* Υ *be a set of matrices over* $\mathbb{C}^{k\times s}$*. Then*

(a) $0 \in \Upsilon$ *if and only if* $\min\limits_{X\in\Upsilon} r(X) = 0$,

(b) $\Upsilon = \{0\}$ *if and only if* $\max\limits_{X\in\Upsilon} r(X) = 0$.

3. Main results

In this section, we solve the maximal and minimal rank problems for (1.1) and establish solvability conditions for certain nonlinear matrix equations. Let $\phi(X_1, X_2, X_3, X_4)$ be as defined in (1.1). Based on specific block matrix rank transformations, we have

$$
\begin{aligned}
& r\left(\phi\left(X_1, X_2, X_3, X_4\right)\right) \\
& = r\Big(Q-\left(D_1-A_1 X_1 B_1\right) N\left(D_2-A_2 X_2 B_2\right)-\left(D_3-A_3 X_3 B_3\right) M\left(D_4-A_4 X_4 B_4\right)\Big) \\
& = r\begin{bmatrix} Q & \left(D_3-A_3 X_3 B_3\right) M & \left(D_1-A_1 X_1 B_1\right) N \\ N\left(D_2-A_2 X_2 B_2\right) & 0 & N \\ M\left(D_4-A_4 X_4 B_4\right) & M & 0 \end{bmatrix} - r(N) - r(M) \\
& = r\Bigg(\begin{bmatrix} Q & D_3 M & D_1 N \\ N D_2 & 0 & N \\ M D_4 & M & 0 \end{bmatrix} - \begin{bmatrix} A_3 & A_1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} X_3 & 0 \\ 0 & X_1 \end{bmatrix}\begin{bmatrix} 0 & B_3 M & 0 \\ 0 & 0 & B_1 N \end{bmatrix} \\
& \qquad - \begin{bmatrix} 0 & 0 \\ N A_2 & 0 \\ 0 & M A_4 \end{bmatrix}\begin{bmatrix} X_2 & 0 \\ 0 & X_4 \end{bmatrix}\begin{bmatrix} B_2 & 0 & 0 \\ B_4 & 0 & 0 \end{bmatrix}\Bigg) - r(N) - r(M) \\
& = r\left(G_0 - G_1 W_1 L_1 - G_2 W_2 L_2\right) - r(N) - r(M),
\end{aligned} \tag{3.1}
$$

where

$$
G_0 = \begin{bmatrix} Q & D_3 M & D_1 N \\ N D_2 & 0 & N \\ M D_4 & M & 0 \end{bmatrix}, \; G_1 = \begin{bmatrix} A_3 & A_1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \; G_2 = \begin{bmatrix} 0 & 0 \\ N A_2 & 0 \\ 0 & M A_4 \end{bmatrix},
$$

$$
L_1 = \begin{bmatrix} 0 & B_3 M & 0 \\ 0 & 0 & B_1 N \end{bmatrix}, \; L_2 = \begin{bmatrix} B_2 & 0 & 0 \\ B_4 & 0 & 0 \end{bmatrix}
$$

and

$$
W_1 = \begin{bmatrix} X_3 & 0 \\ 0 & X_1 \end{bmatrix}, \quad W_2 = \begin{bmatrix} X_2 & 0 \\ 0 & X_4 \end{bmatrix}
$$

are arbitrary matrices.

Now, applying Lemma 2.4 to the linear matrix expression in (3.1) leads to the main result of this paper.

Theorem 3.1. *Let* $\phi(X_1, X_2, X_3, X_4)$ *be as given in* (1.1). *Then,*

$$\max_{X_i \in \mathbb{C}^{p_i \times q_i},\, i=\overline{1,4}} r\big(\phi(X_1, X_2, X_3, X_4)\big)$$

$$= \min \left\{ r\begin{bmatrix} Q - D_1ND_2 - D_3MD_4 & A_3 & A_1 & D_1NA_2 & D_3MA_4 \end{bmatrix}, \right.$$

$$r\begin{bmatrix} Q - D_1ND_2 - D_3MD_4 \\ B_3MD_4 \\ B_1ND_2 \\ B_2 \\ B_4 \end{bmatrix}, \quad r\begin{bmatrix} Q - D_1ND_2 - D_3MD_4 & A_3 & A_1 \\ B_2 & 0 & 0 \\ B_4 & 0 & 0 \end{bmatrix},$$

$$\left. r\begin{bmatrix} D_1ND_2 + D_3MD_4 - Q & D_1NA_2 & D_3MA_4 \\ B_3MD_4 & 0 & B_3MA_4 \\ B_1ND_2 & B_1NA_2 & 0 \end{bmatrix} \right\}$$

and

$$\min_{X_i \in \mathbb{C}^{p_i \times q_i},\, i=\overline{1,4}} r\big(\phi(X_1, X_2, X_3, X_4)\big)$$

$$= r\begin{bmatrix} Q - D_1ND_2 - D_3MD_4 \\ B_3MD_4 \\ B_1ND_2 \\ B_2 \\ B_4 \end{bmatrix}$$

$$+ r\begin{bmatrix} Q - D_1ND_2 - D_3MD_4 & A_3 & A_1 & D_1NA_2 & D_3MA_4 \end{bmatrix}$$

$$+ \max\{\omega_1, \omega_2\},$$

$$\omega_1 = r\begin{bmatrix} Q - D_1ND_2 - D_3MD_4 & A_3 & A_1 \\ B_2 & 0 & 0 \\ B_4 & 0 & 0 \end{bmatrix}$$

$$- r\begin{bmatrix} Q - D_1ND_2 - D_3MD_4 & A_3 & A_1 \\ B_3MD_4 & 0 & 0 \\ B_1ND_2 & 0 & 0 \\ B_2 & 0 & 0 \\ B_4 & 0 & 0 \end{bmatrix}$$

$$- r\begin{bmatrix} Q - D_1ND_2 - D_3MD_4 & A_3 & A_1 & D_1NA_2 & D_3MA_4 \\ B_2 & 0 & 0 & 0 & 0 \\ B_4 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\begin{aligned}\omega_2 = r&\begin{bmatrix} D_1ND_2 + D_3MD_4 - Q & D_1NA_2 & D_3MA_4 \\ B_3MD_4 & 0 & B_3MA_4 \\ B_1ND_2 & B_1NA_2 & 0 \end{bmatrix} \\ &- r\begin{bmatrix} D_1ND_2 + D_3MD_4 - Q & D_1NA_2 & D_3MA_4 \\ B_3MD_4 & 0 & B_3MA_4 \\ B_1ND_2 & B_1NA_2 & 0 \\ B_2 & 0 & 0 \\ B_4 & 0 & 0 \end{bmatrix} \\ &- r\begin{bmatrix} D_1ND_2 + D_3MD_4 - Q & A_3 & A_1 & D_1NA_2 & D_3MA_4 \\ B_3MD_4 & 0 & 0 & 0 & B_3MA_4 \\ B_1ND_2 & 0 & 0 & B_1NA_2 & 0 \end{bmatrix}.\end{aligned}$$

Regarding the consistency of some nonlinear matrix equations, we derive immediate consequences of Theorem 3.1 in the following part of this section.

Corollary 3.2. *Let $\phi(X_1, X_2, X_3, X_4)$ be as defined in* (1.1)*, and suppose that the matrix equations $D_i = A_iX_iB_i$ are consistent; that is, $R(D_i) \subseteq R(A_i)$ and $R(D_i^*) \subseteq R(B_i^*)$ for $i = 1, \ldots, 4$. Then,*

$$\begin{aligned}&\max_{X_i \in \mathbb{C}^{p_i \times q_i},\, i=\overline{1,4}} r(\phi(X_1, X_2, X_3, X_4)) \\ &= \min \left\{ \begin{array}{c} r\begin{bmatrix} Q & A_3 & A_1 \end{bmatrix}, \quad r\begin{bmatrix} Q \\ B_2 \\ B_4 \end{bmatrix}, \\ r\begin{bmatrix} Q & A_3 & A_1 \\ B_2 & 0 & 0 \\ B_4 & 0 & 0 \end{bmatrix}, \quad r(B_1NA_2) + r(B_3MA_4) + r(Q) \end{array} \right\}\end{aligned}$$

and

$$\begin{aligned}&\min_{X_i \in \mathbb{C}^{p_i \times q_i},\, i=\overline{1,4}} r(\phi(X_1, X_2, X_3, X_4)) \\ &= \max \left\{ \begin{array}{c} r\begin{bmatrix} Q \\ B_2 \\ B_4 \end{bmatrix} + r\begin{bmatrix} Q & A_3 & A_1 \end{bmatrix} - r\begin{bmatrix} Q & A_3 & A_1 \\ B_2 & 0 & 0 \\ B_4 & 0 & 0 \end{bmatrix}, \\ r(Q) - r(B_1NA_2) - r(B_3MA_4) \end{array} \right\}.\end{aligned}$$

Hence, the equation

$$(D_1 - A_1X_1B_1)\, N\, (D_2 - A_2X_2B_2) + (D_3 - A_3X_3B_3)\, M\, (D_4 - A_4X_4B_4) = Q$$

is consistent if and only if

$$
\begin{aligned}
&R(Q) \subseteq R\begin{bmatrix} A_3 & A_1 \end{bmatrix}, \quad R(Q^*) \subseteq R\begin{bmatrix} B_2^* & B_4^* \end{bmatrix},\\
&R(Q) \leq r(B_1NA_2) + r(B_3MA_4).
\end{aligned}
$$

Setting $B_1 = I_{p_1}$, $B_3 = I_t$, $A_2 = I_k$, and $A_4 = I_h$ in Theorem 3.1, we obtain the subsequent result.

Corollary 3.3. *Let*

$$
\begin{aligned}
\phi(X_1, X_2, X_3, X_4) = {} & Q - (D_1 - A_1X_1)\, N\, (D_2 - X_2B_2)\\
& - (D_3 - A_3X_3)\, M\, (D_4 - X_4B_4)\,,
\end{aligned}
\tag{3.2}
$$

where $Q \in \mathbb{C}^{m\times n}$, $N \in \mathbb{C}^{l\times k}$, $M \in \mathbb{C}^{t\times h}$, $D_1 \in \mathbb{C}^{m\times l}$, $A_1 \in \mathbb{C}^{m\times p_1}$, $D_2 \in \mathbb{C}^{k\times n}$, $B_2 \in \mathbb{C}^{q_2\times n}$, $D_3 \in \mathbb{C}^{m\times t}$, $A_3 \in \mathbb{C}^{m\times p_3}$, $D_4 \in \mathbb{C}^{h\times n}$, and $B_4 \in \mathbb{C}^{q_4\times n}$ are given matrices, while $X_1 \in \mathbb{C}^{p_1\times l}$, $X_2 \in \mathbb{C}^{k\times q_2}$, $X_3 \in \mathbb{C}^{p_3\times t}$, and $X_4 \in \mathbb{C}^{h\times q_4}$ are unknown matrices. Then,

$$
\begin{aligned}
&\max_{\substack{X_1 \in \mathbb{C}^{p_1\times l},\ X_2 \in \mathbb{C}^{k\times q_2},\\ X_3 \in \mathbb{C}^{p_3\times t},\ X_4 \in \mathbb{C}^{h\times q_4}}} r(\phi(X_1, X_2, X_3, X_4))\\
&= \min\left\{
\begin{matrix}
r\begin{bmatrix} Q & A_3 & A_1 & D_1N & D_3M \end{bmatrix}, \quad r\begin{bmatrix} Q \\ MD_4 \\ ND_2 \\ B_2 \\ B_4 \end{bmatrix}, \\[2ex]
r\begin{bmatrix} Q - D_1ND_2 - D_3MD_4 & A_3 & A_1 \\ B_2 & 0 & 0 \\ B_4 & 0 & 0 \end{bmatrix}, \quad r(Q) + r(N) + r(M)
\end{matrix}
\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
&\min_{\substack{X_1 \in \mathbb{C}^{p_1\times l},\ X_2 \in \mathbb{C}^{k\times q_2},\\ X_3 \in \mathbb{C}^{p_3\times t},\ X_4 \in \mathbb{C}^{h\times q_4}}} r(\phi(X_1, X_2, X_3, X_4))\\
&= r\begin{bmatrix} Q \\ MD_4 \\ ND_2 \\ B_2 \\ B_4 \end{bmatrix} + r\begin{bmatrix} Q & A_3 & A_1 & D_1N & D_3M \end{bmatrix} + \max\{\tau_1, \tau_2\},
\end{aligned}
$$

where

$$\tau_1 = r\begin{bmatrix} Q - D_1ND_2 - D_3MD_4 & A_3 & A_1 \\ B_2 & 0 & 0 \\ B_4 & 0 & 0 \end{bmatrix}$$
$$- r\begin{bmatrix} Q & A_3 & A_1 & D_1N & D_3M \\ B_2 & 0 & 0 & 0 & 0 \\ B_4 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$- r\begin{bmatrix} Q & A_3 & A_1 \\ MD_4 & 0 & 0 \\ ND_2 & 0 & 0 \\ B_2 & 0 & 0 \\ B_4 & 0 & 0 \end{bmatrix},$$

$$\tau_2 = r(Q) - r(N) - r(M)$$
$$- r\begin{bmatrix} Q & A_3 & A_1 \end{bmatrix} - r\begin{bmatrix} Q \\ B_2 \\ B_4 \end{bmatrix}.$$

Hence, the equation

$$(D_1 - A_1X_1)\, N\, (D_2 - X_2B_2) + (D_3 - A_3X_3)\, M\, (D_4 - X_4B_4) = Q$$

is consistent if and only if

$$R(Q) \subseteq R\begin{bmatrix} A_3 & A_1 & D_1N & D_3M \end{bmatrix},$$

$$R(Q^*) \subseteq R\begin{bmatrix} D_4^*M^* & D_2^*N^* & B_2^* & B_4^* \end{bmatrix},$$

$$r\begin{bmatrix} Q - D_1ND_2 - D_3MD_4 & A_3 & A_1 \\ B_2 & 0 & 0 \\ B_4 & 0 & 0 \end{bmatrix} = r\begin{bmatrix} A_3 & A_1 \end{bmatrix} + r\begin{bmatrix} B_2 \\ B_4 \end{bmatrix},$$

$$r\begin{bmatrix} MD_4 \\ ND_2 \\ B_2 \\ B_4 \end{bmatrix} + r\begin{bmatrix} A_3 & A_1 & D_1N & D_3M \end{bmatrix} \leqslant r\,(N) + r\,(M)$$
$$+ r\begin{bmatrix} Q & A_3 & A_1 \end{bmatrix} + r\begin{bmatrix} Q \\ B_2 \\ B_4 \end{bmatrix} - r\,(Q)\,.$$

Setting $B_2 = B_4 = I_n$, $A_1 = A_3 = I_m$ in Theorem 3.1 yields the following result.

Corollary 3.4. *Let*

$$\begin{aligned}\phi(X_1, X_2, X_3, X_4) = Q &- (D_1 - X_1B_1)\, N\, (D_2 - A_2X_2)\\ &- (D_3 - X_3B_3)\, M\, (D_4 - A_4X_4),\end{aligned}$$

where $Q \in \mathbb{C}^{m\times n}$, $N \in \mathbb{C}^{l\times k}$, $M \in \mathbb{C}^{t\times h}$, $D_1 \in \mathbb{C}^{m\times l}$, $B_1 \in \mathbb{C}^{q_1\times l}$, $D_2 \in \mathbb{C}^{k\times n}$, $A_2 \in \mathbb{C}^{k\times p_2}$, $D_3 \in \mathbb{C}^{m\times t}$, $B_3 \in \mathbb{C}^{q_3\times t}$, $D_4 \in \mathbb{C}^{h\times n}$, *and* $A_4 \in \mathbb{C}^{h\times p_4}$ *are given matrices, while* $X_1 \in \mathbb{C}^{m\times q_1}$, $X_2 \in \mathbb{C}^{p_2\times n}$, $X_3 \in \mathbb{C}^{m\times q_3}$, *and* $X_4 \in \mathbb{C}^{p_4\times n}$ *are variable matrices. Then,*

$$\begin{aligned}&\max_{\substack{X_1 \in \mathbb{C}^{m\times q_1},\ X_2 \in \mathbb{C}^{p_2\times n},\\ X_3 \in \mathbb{C}^{m\times q_3},\ X_4 \in \mathbb{C}^{p_4\times n}}} r(\phi(X_1, X_2, X_3, X_4))\\ &= \min\left\{ m,\ n,\ r\begin{bmatrix} D_1ND_2 + D_3MD_4 - Q & D_1NA_2 & D_3MA_4\\ B_3MD_4 & 0 & B_3MA_4\\ B_1ND_2 & B_1NA_2 & 0\end{bmatrix}\right\}\end{aligned}$$

and

$$\min_{\substack{X_1 \in \mathbb{C}^{m\times q_1},\ X_2 \in \mathbb{C}^{p_2\times n},\\ X_3 \in \mathbb{C}^{m\times q_3},\ X_4 \in \mathbb{C}^{p_4\times n}}} r(\phi(X_1, X_2, X_3, X_4)) = \max\{0, \sigma\},$$

where

$$\begin{aligned}\sigma = r&\begin{bmatrix} D_1ND_2 + D_3MD_4 - Q & D_1NA_2 & D_3MA_4\\ B_3MD_4 & 0 & B_3MA_4\\ B_1ND_2 & B_1NA_2 & 0\end{bmatrix}\\ - r&\begin{bmatrix} D_1NA_2 & D_3MA_4\\ 0 & B_3MA_4\\ B_1NA_2 & 0\end{bmatrix}\\ - r&\begin{bmatrix} B_3MD_4 & 0 & B_3MA_4\\ B_1ND_2 & B_1NA_2 & 0\end{bmatrix}.\end{aligned}$$

Hence, the equation

$$(D_1 - X_1B_1)\, N\, (D_2 - A_2X_2) + (D_3 - X_3B_3)\, M\, (D_4 - A_4X_4) = Q$$

is consistent if and only if

$$\begin{aligned}r&\begin{bmatrix} D_1ND_2 + D_3MD_4 - Q & D_1NA_2 & D_3MA_4\\ B_3MD_4 & 0 & B_3MA_4\\ B_1ND_2 & B_1NA_2 & 0\end{bmatrix}\\ &\leq\ r\begin{bmatrix} B_3MD_4 & 0 & B_3MA_4\\ B_1ND_2 & B_1NA_2 & 0\end{bmatrix} + r\begin{bmatrix} D_1NA_2 & D_3MA_4\\ 0 & B_3MA_4\\ B_1NA_2 & 0\end{bmatrix}.\end{aligned}$$

Next, let D_3, A_3, B_3, D_4, A_4, and B_4 vanish. We then obtain the following result, as derived by Tian in [20].

Corollary 3.5. *Let*

$$g(X_1, X_2) = Q - (D_1 - A_1X_1B_1)\, N\, (D_2 - A_2X_2B_2)\,, \tag{3.3}$$

where $Q \in \mathbb{C}^{m\times n}$, $N \in \mathbb{C}^{l\times k}$, $D_1 \in \mathbb{C}^{m\times l}$, $A_1 \in \mathbb{C}^{m\times p_1}$, $B_1 \in \mathbb{C}^{q_1\times l}$, $D_2 \in \mathbb{C}^{k\times n}$, $A_2 \in \mathbb{C}^{k\times p_2}$, *and* $B_2 \in \mathbb{C}^{q_2\times n}$ *are given matrices, while* $X_1 \in \mathbb{C}^{p_1\times q_1}$ *and* $X_2 \in \mathbb{C}^{p_2\times q_2}$ *are unknown matrices. Then,*

$$\begin{aligned}
&\max_{X_1\in\mathbb{C}^{p_1\times q_1},\, X_2\in\mathbb{C}^{p_2\times q_2}} r(g(X_1, X_2)) \\
&= \min\left\{\begin{array}{cc}
r\begin{bmatrix} Q - D_1ND_2 & A_1 & D_1NA_2 \end{bmatrix}, & r\begin{bmatrix} Q - D_1ND_2 \\ B_1ND_2 \\ B_2 \end{bmatrix}, \\
r\begin{bmatrix} Q - D_1ND_2 & A_1 \\ B_2 & 0 \end{bmatrix}, & r\begin{bmatrix} D_1ND_2 - Q & D_1NA_2 \\ B_1ND_2 & B_1NA_2 \end{bmatrix}
\end{array}\right\}
\end{aligned}$$

and

$$\begin{aligned}
&\min_{X_1\in\mathbb{C}^{p_1\times q_1},\, X_2\in\mathbb{C}^{p_2\times q_2}} r(g(X_1, X_2)) \\
&= r\begin{bmatrix} Q - D_1ND_2 \\ B_1ND_2 \\ B_2 \end{bmatrix} + r\begin{bmatrix} Q - D_1ND_2 & A_1 & D_1NA_2 \end{bmatrix} + \max\{\theta_1,\, \theta_2\},
\end{aligned}$$

where

$$\begin{aligned}
\theta_1 = r\begin{bmatrix} Q - D_1ND_2 & A_1 \\ B_2 & 0 \end{bmatrix} - r\begin{bmatrix} Q - D_1ND_2 & A_1 \\ B_1ND_2 & 0 \\ B_2 & 0 \end{bmatrix} \\
- r\begin{bmatrix} Q - D_1ND_2 & A_1 & D_1NA_2 \\ B_2 & 0 & 0 \end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
\theta_2 = r\begin{bmatrix} D_1ND_2 - Q & D_1NA_2 \\ B_1ND_2 & B_1NA_2 \end{bmatrix} - r\begin{bmatrix} D_1ND_2 - Q & D_1NA_2 \\ B_1ND_2 & B_1NA_2 \\ B_2 & 0 \end{bmatrix} \\
- r\begin{bmatrix} D_1ND_2 - Q & A_1 & D_1NA_2 \\ B_1ND_2 & 0 & B_1NA_2 \end{bmatrix}.
\end{aligned}$$

4. Some applications

This section presents direct applications of the results from Section 3, beginning with the maximal and minimal ranks of the generalized Schur complement $Q - BH_r^- C - DL_r^- T$. In addition, we provide solvability conditions for the matrix equation $Q = BH_r^- C + DL_r^- T$. Further applications are presented at the end of this section.

Theorem 4.1. *Let $Q \in \mathbb{C}^{k\times n}$, $B \in \mathbb{C}^{k\times s}$, $C \in \mathbb{C}^{l\times n}$, $D \in \mathbb{C}^{k\times p}$, $T \in \mathbb{C}^{q\times n}$, $H \in \mathbb{C}^{l\times s}$, and $L \in \mathbb{C}^{q\times p}$ be given. Then*

$$\begin{aligned}
&\max_{H_r^-,\, L_r^-} r\big(Q - BH_r^- C - DL_r^- T\big) \\
&= \min \left\{ \begin{array}{c} r\begin{bmatrix} Q & D & B \end{bmatrix}, \quad r\begin{bmatrix} Q \\ C \\ T \end{bmatrix}, \quad r(Q) + r(H) + r(L), \\[2ex] r\begin{bmatrix} Q & D & -B \\ C & 0 & H \\ -T & L & 0 \end{bmatrix} - r(H) - r(L) \end{array} \right\}
\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}
&\min_{H_r^-,\, L_r^-} r\big(Q - BH_r^- C - DL_r^- T\big) \\
&= r\begin{bmatrix} Q \\ C \\ T \end{bmatrix} + r\begin{bmatrix} Q & D & B \end{bmatrix} + \max\{s_1,\, s_2\},
\end{aligned} \tag{4.2}$$

where

$$\begin{aligned}
s_1 = r\begin{bmatrix} Q & D & -B \\ C & 0 & H \\ -T & L & 0 \end{bmatrix} - r\begin{bmatrix} Q & D & B & 0 & 0 \\ C & 0 & 0 & H & 0 \\ T & 0 & 0 & 0 & L \end{bmatrix} \\
- r\begin{bmatrix} Q & D & B \\ C & 0 & 0 \\ T & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & H \end{bmatrix} + r(H) + r(L),
\end{aligned}$$

$$s_2 = r(Q) + r(H) + r(L) - r\begin{bmatrix} Q & D & B \\ 0 & L & 0 \\ 0 & 0 & H \end{bmatrix} - r\begin{bmatrix} Q & 0 & 0 \\ C & H & 0 \\ T & 0 & L \end{bmatrix}.$$

Proof. From Lemma 2.3, we have

$$H_r^- = \left(H^+ - F_H U_1\right) H \left(H^+ - U_2 E_H\right), \tag{4.3}$$

$$L_r^- = \left(L^+ - F_L U_3\right) L \left(L^+ - U_4 E_L\right), \tag{4.4}$$

where U_i, for $i = \overline{1,4}$, are arbitrary matrices. Substituting (4.3) and (4.4) into $Q - BH_r^- C - DL_r^- T$ yields

$$Q - BH_r^- C - DL_r^- T = Q - \left(BH^+ - BF_H U_1\right) H \left(H^+ C - U_2 E_H C\right) - \left(DL^+ - DF_L U_3\right) L \left(L^+ T - U_4 E_L T\right). \tag{4.5}$$

Applying Corollary 3.3 to (4.5) yields

$$\max_{H_r^-,\, L_r^-} r\left(Q - BH_r^- C - DL_r^- T\right) = \min\left\{ r\begin{bmatrix} Q & DF_L & BF_H & BH^+H & DL^+L \end{bmatrix}, r\begin{bmatrix} Q \\ LL^+T \\ HH^+C \\ E_H C \\ E_L T \end{bmatrix}, r\begin{bmatrix} Q - BH^+C - DL^+T & DF_L & BF_H \\ E_H C & 0 & 0 \\ E_L T & 0 & 0 \end{bmatrix}, r(Q) + r(H) + r(L) \right\}, \tag{4.6}$$

$$\min_{H_r^-,\, L_r^-} r\big(Q - BH_r^- C - DL_r^- T\big) \tag{4.7}$$
$$= r\begin{bmatrix} Q \\ LL^+T \\ HH^+C \\ E_HC \\ E_LT \end{bmatrix} + r\begin{bmatrix} Q & DF_L & BF_H & BH^+H & DL^+L \end{bmatrix} + \max\{s_1,\, s_2\},$$

where

$$s_1 = r\begin{bmatrix} Q - BH^+C - DL^+T & DF_L & BF_H \\ E_HC & 0 & 0 \\ E_LT & 0 & 0 \end{bmatrix}$$
$$- r\begin{bmatrix} Q & DF_L & BF_H & BH^+H & DL^+L \\ E_HC & 0 & 0 & 0 & 0 \\ E_LT & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$- r\begin{bmatrix} Q & DF_L & BF_H \\ LL^+T & 0 & 0 \\ HH^+C & 0 & 0 \\ E_HC & 0 & 0 \\ E_LT & 0 & 0 \end{bmatrix},$$

$$s_2 = r(Q) - r(H) - r(L) - r\begin{bmatrix} Q & DF_L & BF_H \end{bmatrix} - r\begin{bmatrix} Q \\ E_HC \\ E_LT \end{bmatrix}.$$

Applying Lemma 2.1 and elementary block matrix operations, we obtain

$$r\begin{bmatrix} Q & DF_L & BF_H & BH^+H & DL^+L \end{bmatrix}$$
$$= r\begin{bmatrix} Q & D & B & BH^+H & DL^+L \\ 0 & L & 0 & 0 & 0 \\ 0 & 0 & H & 0 & 0 \end{bmatrix} - r(H) - r(N)$$
$$= r\begin{bmatrix} Q & D & B & 0 & 0 \\ 0 & 0 & 0 & 0 & L \\ 0 & 0 & 0 & H & 0 \end{bmatrix} - r(H) - r(N) = r\begin{bmatrix} Q & D & B \end{bmatrix}. \tag{4.8}$$

$$r\begin{bmatrix} Q \\ LL^{+}T \\ HH^{+}C \\ E_HC \\ E_LT \end{bmatrix} = r\begin{bmatrix} Q & 0 & 0 \\ LL^{+}T & 0 & 0 \\ HH^{+}C & 0 & 0 \\ C & H & 0 \\ T & 0 & L \end{bmatrix} - r\left(H\right) - r\left(N\right) = r\begin{bmatrix} Q \\ C \\ T \end{bmatrix}, \quad (4.9)$$

$$r\begin{bmatrix} Q - BH^{+}C - DL^{+}T & DF_L & BF_H \\ E_HC & 0 & 0 \\ E_LT & 0 & 0 \end{bmatrix}$$

$$= r\begin{bmatrix} Q - BH^{+}C - DL^{+}T & D & B & 0 & 0 \\ C & 0 & 0 & 0 & H \\ T & 0 & 0 & L & 0 \\ 0 & L & 0 & 0 & 0 \\ 0 & 0 & H & 0 & 0 \end{bmatrix} - 2r(H) - 2r(L)$$

or

$$r\begin{bmatrix} Q - BH^{+}C - DL^{+}T & DF_L & BF_H \\ E_HC & 0 & 0 \\ E_LT & 0 & 0 \end{bmatrix}$$

$$= r\begin{bmatrix} Q & D & B & DL^{+}L & BH^{+}H \\ C & 0 & 0 & 0 & H \\ T & 0 & 0 & L & 0 \\ 0 & L & 0 & 0 & 0 \\ 0 & 0 & H & 0 & 0 \end{bmatrix} - 2r(H) - 2r(L)$$

$$= r\begin{bmatrix} Q & D & B & 0 & 0 \\ C & 0 & 0 & 0 & H \\ T & 0 & 0 & L & 0 \\ 0 & L & 0 & L & 0 \\ 0 & 0 & H & 0 & H \end{bmatrix} - 2r(H) - 2r(L)$$

$$= r\begin{bmatrix} Q & D & -B \\ C & 0 & H \\ -T & L & 0 \end{bmatrix} - r(H) - r(L). \quad (4.10)$$

$$
\begin{aligned}
& r\begin{bmatrix} Q & DF_L & BF_H & BH^+H & DL^+L \\ E_HC & 0 & 0 & 0 & 0 \\ E_LT & 0 & 0 & 0 & 0 \end{bmatrix} \\
& = r\begin{bmatrix} Q & D & B & BH^+H & DL^+L & 0 & 0 \\ C & 0 & 0 & 0 & 0 & H & 0 \\ T & 0 & 0 & 0 & 0 & 0 & L \\ 0 & L & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & H & 0 & 0 & 0 & 0 \end{bmatrix} - 2r(H) - 2r(L) \\
& = r\begin{bmatrix} Q & D & B & 0 & 0 & 0 & 0 \\ C & 0 & 0 & 0 & 0 & H & 0 \\ T & 0 & 0 & 0 & 0 & 0 & L \\ 0 & 0 & 0 & 0 & L & 0 & 0 \\ 0 & 0 & 0 & H & 0 & 0 & 0 \end{bmatrix} - 2r(H) - 2r(L) \\
& = r\begin{bmatrix} Q & D & B & 0 & 0 \\ C & 0 & 0 & H & 0 \\ T & 0 & 0 & 0 & L \end{bmatrix} - r(H) - r(L). \qquad (4.11)
\end{aligned}
$$

$$
r\begin{bmatrix} Q & DF_L & BF_H \\ LL^+T & 0 & 0 \\ HH^+C & 0 & 0 \\ E_HC & 0 & 0 \\ E_LT & 0 & 0 \end{bmatrix} = r\begin{bmatrix} Q & D & B & 0 & 0 \\ 0 & 0 & 0 & 0 & L \\ 0 & 0 & 0 & H & 0 \\ C & 0 & 0 & 0 & 0 \\ T & 0 & 0 & 0 & 0 \\ 0 & L & 0 & 0 & 0 \\ 0 & 0 & H & 0 & 0 \end{bmatrix} - 2r(H) - 2r(L)
$$

or

$$
\begin{aligned}
r\begin{bmatrix} Q & DF_L & BF_H \\ LL^+T & 0 & 0 \\ HH^+C & 0 & 0 \\ E_HC & 0 & 0 \\ E_LT & 0 & 0 \end{bmatrix} &= r\begin{bmatrix} Q & D & B & 0 & 0 \\ C & 0 & 0 & 0 & 0 \\ T & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & L \\ 0 & 0 & 0 & H & 0 \\ 0 & L & 0 & 0 & 0 \\ 0 & 0 & H & 0 & 0 \end{bmatrix} - 2r(H) - 2r(L) \\
&= r\begin{bmatrix} Q & D & B \\ C & 0 & 0 \\ T & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & H \end{bmatrix} - r(H) - r(L), \qquad (4.12)
\end{aligned}
$$

$$r\begin{bmatrix} Q & DF_L & BF_H \end{bmatrix} = r\begin{bmatrix} Q & D & B \\ 0 & L & 0 \\ 0 & 0 & H \end{bmatrix} - r(H) - r(L), \tag{4.13}$$

$$r\begin{bmatrix} Q \\ E_H C \\ E_L T \end{bmatrix} = r\begin{bmatrix} Q & 0 & 0 \\ C & H & 0 \\ T & 0 & L \end{bmatrix} - r(H) - r(L). \tag{4.14}$$

Substituting (4.8)-(4.10) into (4.6) and (4.8)-(4.14) into (4.7) yields respectively (4.1) and (4.2). □

We next derive the consistency conditions for Equation $Q = BH_r^- C + DL_r^- T$ in the following corollary.

Corollary 4.2. *Let $Q \in \mathbb{C}^{k\times n}$, $B \in \mathbb{C}^{k\times s}$, $C \in \mathbb{C}^{l\times n}$, $D \in \mathbb{C}^{k\times p}$, $T \in \mathbb{C}^{q\times n}$, $H \in \mathbb{C}^{l\times s}$, and $L \in \mathbb{C}^{q\times p}$ be given. Then*

(1) *The equality $Q = BH_r^- C + DL_r^- T$ holds for all $H_r^- \in \mathbb{C}^{s\times l}$ and $L_r^- \in \mathbb{C}^{p\times q}$ if and only if any one of the following equalities holds:*

$$\begin{bmatrix} Q & D & B \end{bmatrix} = 0, \quad \begin{bmatrix} Q \\ C \\ T \end{bmatrix} = 0, \quad r\begin{bmatrix} Q & D & -B \\ C & 0 & H \\ -T & L & 0 \end{bmatrix} = r(H) + r(L),$$
$$Q = 0 \quad \text{and} \quad H = 0 \quad \text{and} \quad L = 0.$$

(2) *The equation $Q = BH_r^- C + DL_r^- T$ is consistent if and only if*

$$R(Q) \subseteq R\begin{bmatrix} D & B \end{bmatrix}, \quad R(Q^*) \subseteq R\begin{bmatrix} C^* & T^* \end{bmatrix},$$

$$r\begin{bmatrix} Q & D & -B \\ C & 0 & H \\ -T & L & 0 \end{bmatrix} = r\begin{bmatrix} C & 0 & H \\ T & L & 0 \end{bmatrix} + r\begin{bmatrix} D & B \\ L & 0 \\ 0 & H \end{bmatrix} - r(H) - r(L),$$

$$r\begin{bmatrix} C \\ T \end{bmatrix} + r\begin{bmatrix} D & B \end{bmatrix} + r(Q) + r(H) + r(L) \leq r\begin{bmatrix} Q & D & B \\ 0 & L & 0 \\ 0 & 0 & H \end{bmatrix} + r\begin{bmatrix} Q & 0 & 0 \\ C & H & 0 \\ T & 0 & L \end{bmatrix}.$$

In the remainder of this section, as a direct application of the results in Section 2, we assume that $Q \in \mathbb{C}^{m\times n}$, $N \in \mathbb{C}^{l\times k}$, $M \in \mathbb{C}^{t\times s}$, $T_1 \in \mathbb{C}^{l\times p_1}$, $T_2 \in \mathbb{C}^{q_2\times k}$, $T_3 \in \mathbb{C}^{t\times p_3}$, $T_4 \in \mathbb{C}^{q_4\times s}$, $S_1 \in \mathbb{C}^{m\times p_1}$, $S_2 \in \mathbb{C}^{q_2\times n}$, $S_3 \in \mathbb{C}^{m\times p_3}$, and $S_4 \in \mathbb{C}^{q_4\times n}$ are given matrices, while $X_1 \in \mathbb{C}^{m\times l}$, $X_2 \in \mathbb{C}^{k\times n}$, $X_3 \in \mathbb{C}^{m\times t}$, and $X_4 \in \mathbb{C}^{s\times n}$ are unknown matrices. We determine the extremal ranks of the following nonlinear matrix equation:

$$\varphi(X_1, X_2, X_3, X_4) = Q - X_1NX_2 - X_3MX_4 \tag{4.15}$$

subject to X_1, X_2, X_3, and X_4 being solutions of the consistent linear matrix equations $X_1T_1 = S_1$, $T_2X_2 = S_2$, $X_3T_3 = S_3$, and $T_4X_4 = S_4$, respectively. As a consequence, we establish equivalent conditions for the consistency of the nonlinear equation $Q = X_1NX_2 - X_3MX_4$. From Lemma 2.2, if we adopt the following representations of the sets Ω_i for $i = \overline{1,4}$, then

$$\begin{aligned}
\Omega_1 &= \left\{X_1 \in \mathbb{C}^{m\times l} \,\middle|\, X_1T_1 = S_1,\ S_1T_1^{+}T_1 = S_1\right\},\\
\Omega_2 &= \left\{X_2 \in \mathbb{C}^{k\times n} \,\middle|\, T_2X_2 = S_2,\ T_2T_2^{+}S_2 = S_2\right\},\\
\Omega_3 &= \left\{X_3 \in \mathbb{C}^{m\times t} \,\middle|\, X_3T_3 = S_3,\ S_3T_3^{+}T_3 = S_3\right\},\\
\Omega_4 &= \left\{X_4 \in \mathbb{C}^{s\times n} \,\middle|\, T_4X_4 = S_4,\ T_4T_4^{+}S_4 = S_4\right\},
\end{aligned} \tag{4.16}$$

then the sets Ω_i, for $i = \overline{1,4}$, are nonempty.

Theorem 4.3. *Let $\varphi(X_i)$ and Ω_i for $i = \overline{1,4}$ be as given in* (4.15) *and* (4.16) *respectively. Then,*

$$\max_{X_i\in\Omega_i,\, i=\overline{1,4}} r\big(\varphi(X_i)\big) = \min\left\{m,\ n,\ r\begin{bmatrix} -Q & S_3 & S_1 & 0 & 0\\ S_2 & 0 & 0 & T_2 & 0\\ S_4 & 0 & 0 & 0 & T_4\\ 0 & T_3 & 0 & 0 & M\\ 0 & 0 & T_1 & N & 0 \end{bmatrix} - \sum_{i=1}^{4} r(T_i)\right\}, \tag{4.17}$$

$$\min_{X_i\in\Omega_i,\, i=\overline{1,4}} r\big(\varphi(X_i)\big) = \max\{0, \delta\}, \tag{4.18}$$

where

$$\begin{aligned}
\delta = r\begin{bmatrix} -Q & S_3 & S_1 & 0 & 0\\ S_2 & 0 & 0 & T_2 & 0\\ S_4 & 0 & 0 & 0 & T_4\\ 0 & T_3 & 0 & 0 & M\\ 0 & 0 & T_1 & N & 0 \end{bmatrix} &- r\begin{bmatrix} 0 & 0 & M & T_3 & 0\\ 0 & N & 0 & 0 & T_1\\ S_2 & T_2 & 0 & 0 & 0\\ S_4 & 0 & T_4 & 0 & 0 \end{bmatrix}\\
- r\begin{bmatrix} 0 & 0 & S_3 & S_1\\ 0 & M & T_3 & 0\\ N & 0 & 0 & T_1\\ T_2 & 0 & 0 & 0\\ 0 & T_4 & 0 & 0 \end{bmatrix} &+ \sum_{i=1}^{4} r(T_i).
\end{aligned}$$

Proof. From formula (2.2) in Lemma 2.2, the general expressions of X_i for $i = \overline{1,4}$ in (4.16) are given by

$$X_1 = S_1T_1^+ + V_1E_{T_1}, \tag{4.19}$$

$$X_2 = T_2^+S_2 + F_{T_2}V_2, \tag{4.20}$$

$$X_3 = S_3T_3^+ + V_3E_{T_3}, \tag{4.21}$$

$$X_4 = T_4^+S_4 + F_{T_4}V_4, \tag{4.22}$$

where V_i, for $i = \overline{1,4}$, are arbitrary matrices of appropriate sizes. Substituting (4.19)–(4.22) into $\varphi(X_i)$ in (4.15) yields

$$\begin{aligned}\varphi(X_i) &= Q - X_1NX_2 - X_3MX_4 \\ &= Q - \left(S_1T_1^+ + V_1E_{T_1}\right)N\left(T_2^+S_2 + F_{T_2}V_2\right) \\ &\quad - \left(S_3T_3^+ + V_3E_{T_3}\right)M\left(T_4^+S_4 + F_{T_4}V_4\right).\end{aligned} \tag{4.23}$$

Applying Corollary 3.4 to $\varphi(X_i)$ in (4.23), we obtain Set

$$\Lambda = S_1T_1^+NT_2^+S_2 + S_3T_3^+MT_4^+S_4 - Q.$$

Then

$$\begin{aligned}&\max_{X_i\in\Omega_i,\, i=\overline{1,4}} r(\varphi(X_i)) \\ &= \max_{V_i} r\Big[Q - (S_1T_1^+ + V_1E_{T_1})N(T_2^+S_2 + F_{T_2}V_2) \\ q &\quad - (S_3T_3^+ + V_3E_{T_3})M(T_4^+S_4 + F_{T_4}V_4)\Big] \\ &= \min\left\{m,\ n,\ r\begin{bmatrix} \Lambda & S_1T_1^+NF_{T_2} & S_3T_3^+MF_{T_4} \\ E_{T_3}MT_4^+S_4 & 0 & E_{T_3}MF_{T_4} \\ E_{T_1}NT_2^+S_2 & E_{T_1}NF_{T_2} & 0 \end{bmatrix}\right\}.\end{aligned} \tag{4.24}$$

$$\min_{X_i\in\Omega_i,\, i=\overline{1,4}} r(\varphi(X_i)) = \max\{0,\, \delta_1\},$$

where

$$\begin{aligned}\delta_1 = r&\begin{bmatrix} S_1T_1^+NT_2^+S_2 + S_3T_3^+MT_4^+S_4 - Q & S_1T_1^+NF_{T_2} & S_3T_3^+MF_{T_4} \\ E_{T_3}MT_4^+S_4 & 0 & E_{T_3}MF_{T_4} \\ E_{T_1}NT_2^+S_2 & E_{T_1}NF_{T_2} & 0 \end{bmatrix} \\ &- r\begin{bmatrix} E_{T_3}MT_4^+S_4 & 0 & E_{T_3}MF_{T_4} \\ E_{T_1}NT_2^+S_2 & E_{T_1}NF_{T_2} & 0 \end{bmatrix} \\ &- r\begin{bmatrix} S_1T_1^+NF_{T_2} & S_3T_3^+MF_{T_4} \\ 0 & E_{T_3}MF_{T_4} \\ E_{T_1}NF_{T_2} & 0 \end{bmatrix}.\end{aligned} \tag{4.25}$$

Applying Lemma 2.1 and three types of elementary block matrix operations, and simplify by $S_1T_1^+T_1 = S_1$, $T_2T_2^+S_2 = S_2$, $S_3T_3^+T_3 = S_3$ and $T_4T_4^+S_4 = S_4$, we get

$$r\begin{bmatrix} S_1T_1^+NT_2^+S_2 + S_3T_3^+MT_4^+S_4 - Q & S_1T_1^+NF_{T_2} & S_3T_3^+MF_{T_4} \\ E_{T_3}MT_4^+S_4 & 0 & E_{T_3}MF_{T_4} \\ E_{T_1}NT_2^+S_2 & E_{T_1}NF_{T_2} & 0 \end{bmatrix}$$

$$= r\begin{bmatrix} S_1T_1^+NT_2^+S_2 + S_3T_3^+MT_4^+S_4 - Q & S_1T_1^+N & S_3T_3^+M & 0 & 0 \\ MT_4^+S_4 & 0 & M & T_3 & 0 \\ NT_2^+S_2 & N & 0 & 0 & T_1 \\ 0 & T_2 & 0 & 0 & 0 \\ 0 & 0 & T_4 & 0 & 0 \end{bmatrix}$$

$$- \sum_{i=1}^{4} r(T_i)$$

or

$$r\begin{bmatrix} S_1T_1^+NT_2^+S_2 + S_3T_3^+MT_4^+S_4 - Q & S_1T_1^+NF_{T_2} & S_3T_3^+MF_{T_4} \\ E_{T_3}MT_4^+S_4 & 0 & E_{T_3}MF_{T_4} \\ E_{T_1}NT_2^+S_2 & E_{T_1}NF_{T_2} & 0 \end{bmatrix}$$

$$= r\begin{bmatrix} -Q & 0 & 0 & -S_3T_3^+T_3 & -S_1T_1^+T_1 \\ 0 & 0 & M & T_3 & 0 \\ 0 & N & 0 & 0 & T_1 \\ -T_2T_2^+S_2 & T_2 & 0 & 0 & 0 \\ -T_4T_4^+S_4 & 0 & T_4 & 0 & 0 \end{bmatrix} - \sum_{i=1}^{4} r(T_i)$$

$$= r\begin{bmatrix} -Q & S_3 & S_1 & 0 & 0 \\ S_2 & 0 & 0 & T_2 & 0 \\ S_4 & 0 & 0 & 0 & T_4 \\ 0 & T_3 & 0 & 0 & M \\ 0 & 0 & T_1 & N & 0 \end{bmatrix} - \sum_{i=1}^{4} r(T_i), \tag{4.26}$$

$$r\begin{bmatrix} E_{T_3}MT_4^{+}S_4 & 0 & E_{T_3}MF_{T_4} \\ E_{T_1}NT_2^{+}S_2 & E_{T_1}NF_{T_2} & 0 \end{bmatrix}$$

$$= r\begin{bmatrix} MT_4^{+}S_4 & 0 & M & T_3 & 0 \\ NT_2^{+}S_2 & N & 0 & 0 & T_1 \\ 0 & T_2 & 0 & 0 & 0 \\ 0 & 0 & T_4 & 0 & 0 \end{bmatrix} - \sum_{i=1}^{4} r(T_i)$$

$$= r\begin{bmatrix} 0 & 0 & M & T_3 & 0 \\ 0 & N & 0 & 0 & T_1 \\ -T_2T_2^{+}S_2 & T_2 & 0 & 0 & 0 \\ -T_4T_4^{+}S_4 & 0 & T_4 & 0 & 0 \end{bmatrix} - \sum_{i=1}^{4} r(T_i)$$

$$= r\begin{bmatrix} 0 & 0 & M & T_3 & 0 \\ 0 & N & 0 & 0 & T_1 \\ S_2 & T_2 & 0 & 0 & 0 \\ S_4 & 0 & T_4 & 0 & 0 \end{bmatrix} - \sum_{i=1}^{4} r(T_i), \tag{4.27}$$

$$r\begin{bmatrix} S_1T_1^{+}NF_{T_2} & S_3T_3^{+}MF_{T_4} \\ 0 & E_{T_3}MF_{T_4} \\ E_{T_1}NF_{T_2} & 0 \end{bmatrix}$$

$$= r\begin{bmatrix} S_1T_1^{+}N & S_3T_3^{+}M & 0 & 0 \\ 0 & M & T_3 & 0 \\ N & 0 & 0 & T_1 \\ T_2 & 0 & 0 & 0 \\ 0 & T_4 & 0 & 0 \end{bmatrix} - \sum_{i=1}^{4} r(T_i)$$

or

$$r\begin{bmatrix} S_1T_1^{+}NF_{T_2} & S_3T_3^{+}MF_{T_4} \\ 0 & E_{T_3}MF_{T_4} \\ E_{T_1}NF_{T_2} & 0 \end{bmatrix}$$

$$= r\begin{bmatrix} 0 & 0 & -S_3T_3^{+}T_3 & -S_1T_1^{+}T_1 \\ 0 & M & T_3 & 0 \\ N & 0 & 0 & T_1 \\ T_2 & 0 & 0 & 0 \\ 0 & T_4 & 0 & 0 \end{bmatrix} - \sum_{i=1}^{4} r(T_i)$$

$$= r\begin{bmatrix} 0 & 0 & S_3 & S_1 \\ 0 & M & T_3 & 0 \\ N & 0 & 0 & T_1 \\ T_2 & 0 & 0 & 0 \\ 0 & T_4 & 0 & 0 \end{bmatrix} - \sum_{i=1}^{4} r(T_i). \tag{4.28}$$

Substituting (4.26) into (4.24) and (4.26)-(4.28) into (4.25) leads respectively to (4.17) and (4.18). □

Corollary 4.4. *Let $\varphi(X_i)$ and Ω_i, for $i = \overline{1,4}$, be as given in* (4.15) *and* (4.16)*, respectively. Then*

(1) *The nonlinear matrix equation $Q = X_1NX_2 + X_3MX_4$ holds for all $X_i \in \Omega_i$, for $i = \overline{1,4}$, if and only if*

$$r\begin{bmatrix} -Q & S_3 & S_1 & 0 & 0 \\ S_2 & 0 & 0 & T_2 & 0 \\ S_4 & 0 & 0 & 0 & T_4 \\ 0 & T_3 & 0 & 0 & M \\ 0 & 0 & T_1 & N & 0 \end{bmatrix} = \sum_{i=1}^{4} r\left(T_i\right).$$

(2) *There exist $X_i \in \Omega_i$ for $i = \overline{1,4}$ such that $Q = X_1NX_2 + X_3MX_4$ if and only if*

$$r\begin{bmatrix} -Q & S_3 & S_1 & 0 & 0 \\ S_2 & 0 & 0 & T_2 & 0 \\ S_4 & 0 & 0 & 0 & T_4 \\ 0 & T_3 & 0 & 0 & M \\ 0 & 0 & T_1 & N & 0 \end{bmatrix} \le r\begin{bmatrix} 0 & 0 & M & T_3 & 0 \\ 0 & N & 0 & 0 & T_1 \\ S_2 & T_2 & 0 & 0 & 0 \\ S_4 & 0 & T_4 & 0 & 0 \end{bmatrix}$$

$$+ r\begin{bmatrix} 0 & 0 & S_3 & S_1 \\ 0 & M & T_3 & 0 \\ N & 0 & 0 & T_1 \\ T_2 & 0 & 0 & 0 \\ 0 & T_4 & 0 & 0 \end{bmatrix} - \sum_{i=1}^{4} r(T_i).$$

As a special case of the nonlinear function φ in (4.15), the following corollary determines the extremal ranks of the nonlinear expression $Q - X_1NX_2$, subject to the two consistent matrix equations $X_1T_1 = S_1$ and $T_2X_2 = S_2$, where $T_1 \in$

$\mathbb{C}^{l\times p_1}$, $T_2 \in \mathbb{C}^{q_2\times k}$, $S_1 \in \mathbb{C}^{m\times p_1}$, and $S_2 \in \mathbb{C}^{q_2\times n}$. We also establish equivalent conditions for the consistency of the nonlinear equation $Q = X_1NX_2$.

Corollary 4.5. *Let $Q \in \mathbb{C}^{m\times n}$, $N \in \mathbb{C}^{l\times k}$, $T_1 \in \mathbb{C}^{l\times p_1}$, $T_2 \in \mathbb{C}^{q_2\times k}$, $S_1 \in \mathbb{C}^{m\times p_1}$, $S_2 \in \mathbb{C}^{q_2\times n}$ be given, and let Ω_1, Ω_2 be as defined in* (4.16). *Then*

$$\max_{X_1\in\Omega_1,\, X_2\in\Omega_2} r(Q-X_1NX_2) = \min\left\{m,\ n,\ r\begin{bmatrix} -Q & S_1 & 0 \\ S_2 & 0 & T_2 \\ 0 & T_1 & N \end{bmatrix} - r(T_1) - r(T_2)\right\},$$

$$\min_{X_1\in\Omega_1,\, X_2\in\Omega_2} r(Q - X_1NX_2) = \max\{0, \beta\}, \tag{4.29}$$

where

$$\beta = r\begin{bmatrix} -Q & S_1 & 0 & 0 \\ S_2 & 0 & T_2 & 0 \\ 0 & T_1 & N & 0 \end{bmatrix} - r\begin{bmatrix} 0 & N & T_1 \\ S_2 & T_2 & 0 \end{bmatrix} - r\begin{bmatrix} 0 & S_1 \\ N & T_1 \\ T_2 & 0 \end{bmatrix} + r(T_1) + r(T_2).$$

Hence

(1) *All solutions of $X_1T_1 = S_1$ and $T_2X_2 = S_2$ satisfy $Q = X_1NX_2$ if and only if*
$$r\begin{bmatrix} -Q & S_1 & 0 \\ S_2 & 0 & T_2 \\ 0 & T_1 & N \end{bmatrix} = r(T_1) + r(T_2).$$

(2) *There exist $X_1 \in \Omega_1$ and $X_2 \in \Omega_2$ such that $Q = X_1NX_2$ if and only if*
$$r\begin{bmatrix} -Q & S_1 & 0 \\ S_2 & 0 & T_2 \\ 0 & T_1 & N \end{bmatrix} \leqslant r\begin{bmatrix} 0 & N & T_1 \\ S_2 & T_2 & 0 \end{bmatrix} + r\begin{bmatrix} 0 & S_1 \\ N & T_1 \\ T_2 & 0 \end{bmatrix} - r\left(T_1\right) - r\left(T_2\right).$$

5. Concluding remarks

In this paper, we use generalized matrix inverses and elementary block matrix operations to derive explicit formulas for the optimal ranks of the nonlinear matrix equation (1.1). We also establish consistency conditions for the nonlinear matrix equation (1.2). As applications, we determine properties of the generalized Schur complement with respect to reflexive generalized inverses of given matrices. Furthermore, we obtain the extremal ranks of the nonlinear matrix expression (4.15), subject to the solutions of four consistent matrix equations, and provide solvability conditions for the corresponding nonlinear matrix equations. These results contribute to matrix theory and may benefit researchers in optimization theory.

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