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MODIFIED NOOR MULTISTEP ITERATIVE PROCESS WITH ERRORS FOR NONSELF ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS

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Abstract. Many practical problems can be formulated as the fixed point problem $x = Tx$, where T is a nonexpansive mapping. Iterative methods as a powerful tool are often used to approximating the fixed point of such mapping, including Krasnoselskij iteration method, Mann iteration method, Ishikawa iteration method and Noor iteration method etc,. The purpose of this paper is to introduce a modified Noor multistep iterative process with errors for approximating the common fixed point of a finite family of nonself asymptotically quasi-nonexpansive mappings. By using this iterative scheme, we prove several strong convergence theorems for such mappings in uniformly convex Banach spaces. Our results improve and extend some recent results in the literature.

1. Introduction

Let E be a real normed space and K a nonempty subset of E. We use $F(T)$ denotes the set of fixed points of T, i.e., $F(T) = \{x \in K : Tx = x\}$.

Definition 1.1. *A mapping $T : K \to K$ is said to be*

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(i) *asymptotically nonexpansive if there exists a sequence* $\{u_n\} \subset [0, +\infty)$, $\lim_{n\to\infty} u_n = 0$ *such that*

$$\|T^n x - T^n y\| \leq (1 + u_n)\|x - y\|,$$

for all $x, y \in K$ *and* $n \geq 1$.

(ii) *uniformly L-Lipschitzian if there exists a constant* $L > 0$ *such that*

$$\|T^n x - T^n y\| \leq L\|x - y\|,$$

for all $x, y \in K$ *and* $n \geq 1$.

(iii) *asymptotically quasi-nonexpansive if* $F(T) \neq \emptyset$ *and there exists a sequence* $\{u_n\} \in [0, +\infty)$, $\lim_{n\to\infty} u_n = 0$ *such that*

$$\|T^n x - p\| \leq (1 + u_n)\|x - p\|,$$

for all $x \in K, p \in F(T)$ *and* $n \geq 1$.

(iv) *uniformly quasi-Lipschitzian if* $F(T) \neq \emptyset$ *and there exists a constant* $L > 0$ *such that*

$$\|T^n x - p\| \leq L\|x - p\|,$$

for all $x \in K, p \in F(T)$ *and* $n \geq 1$.

Remark 1.2. It is easy to see from Definition 1.1 that asymptotically nonexpansive mapping is asymptotically quasi-nonexpansive mapping as well as uniformly L-Lipschitzian. The asymptotically quasi-nonexpansive mapping is uniformly quasi-Lipschitzian mapping with $L = \sup_{n\geq 1}\{1 + u_n\}$. However, the converse doesn't hold.

The concept of asymptotically nonexpansive mapping was introduced by Goebel and Kirk[9], they proved if K is a nonempty closed and bounded subset of a uniformly convex Banach space, then every asymptotically nonexpansive self-mapping has a fixed point.

Iterative techniques for approximating fixed point of an asymptotically nonexpansive or asymptotically quasi-nonexpansive selfmappings in Hilbert spaces and Banach spaces have been studied by many authors(see, e.g., [2, 10, 31, 3, 36, 29, 12, 24]) and many others.

A subset K of E is said to be a retract if there exists a continuous map $P : E \to K$ such that $Px = x$, for all $x \in K$. Every closed convex subset of a uniformly convex Banach space is a retract. A map $P : E \to E$ is called a retraction if $P^2 = P$. It follows that if a map is a retraction, then $Pz = z$ for all z in the range of P.

In 2003, Chidume et al.[6] generalized the concept of asymptotically nonexpansive self-mappings to the nonself asymptotically nonexpansive mappings as follows.

Definition 1.3. *Let K be a nonempty subset of real normed space E. Let $P : E \to K$ be the nonexpansive retraction of E onto K. A map $T : K \to E$ is said to be a nonself asymptotically nonexpansive mapping if there exists a sequence $\{u_n \subset [0, \infty)\}, \lim_{n\to\infty} u_n = 0$ such that*

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq (1 + u_n)\|x - y\|,$$

for all $x, y \in K$ and $n \geq 1$. T is said to be a nonself uniformly L-Lipschitzian mapping if there exists a constant $L > 0$ such that

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq L\|x - y\|,$$

for all $x, y \in K$ and $n \geq 1$.

If $F(T) \neq \emptyset$, then we can generalize Definition 1.3 to nonself asymptotically quasi-nonexpansive mappings and nonself uniformly quasi-Lipschitzian mappings as follows (see e.g.,[37]).

Definition 1.4. *$T : K \to E$ is said to be a nonself asymptotically quasi-nonexpansive mapping, if there exists a sequence $\{u_n\} \subset [0, \infty), \lim_{n\to\infty} u_n = 0$ such that*

$$\|T(PT)^{n-1}x - p\| \leq (1 + u_n)\|x - p\|,$$

for all $x \in K, p \in F(T)$ and $n \geq 1$. T is said to be a nonself uniformly L-Lipschitzian mapping, if there exists a constant $L > 0$ such that

$$\|T(PT)^{n-1}x - p\| \leq L\|x - p\|,$$

for all $x \in K, p \in F(T)$ and $n \geq 1$.

In [6], they studied the following iteration process(Mann type iteration or one step iteration):

$$\begin{cases} x_1 \in K, \\ x_{n+1} = P((1 - \alpha_n)x_n + \alpha_n T(PT)^{n-1}x_n), \quad n \geq 1, \end{cases} \tag{1.1}$$

where $\{\alpha_n\}$ is a appropriate sequence in $[0, 1]$. They established some convergence theorems for the fixed points of nonself asymptotically nonexpansive mapping T. In 2006, Wang[38] generalized the iteration process (1.1) as follows(Ishikawa type iteration or two step iteration):

$$\begin{cases} x_1 \in K, \\ x_{n+1} = P((1 - \alpha_n)x_n + \alpha_n T_1(PT_1)^{n-1}y_n), \\ y_n = P((1 - \beta_n)x_n + \beta_n T_2(PT_2)^{n-1}x_n), \quad n \geq 1, \end{cases} \tag{1.2}$$

where $T_1, T_2 : K \to E$ are nonself asymptotically nonexpansive mappings and $\{\alpha_n\}, \{\beta_n\}$ are appropriate sequences in $[0, 1]$. He proved several strong and weak convergence theorems of the iterative sequence (1.2) under proper conditions, which generalized the results of [6].

In 2000, Noor [18] introduced a three-step iterative scheme and studied the approximate solutions of variational inclusion in Hilbert spaces. It is well known that three-step iterative scheme includes Mann(one-step) and Ishikawa(two-step) iterations as special cases. Glowinski and Tallec [8] applied a three-step iterative scheme for finding a approximate solution of the eigenvalue problem and liquid crystal theory. They showed that three-step iterative method perform better than the Mann and Ishikawa iterative methods for solving variational inequalities. Later on, Xu and Noor [40] studied a new three-step scheme to approximate fixed points of asymptotically nonexpansive mappings in Banach spaces. Cho, Zhou and Guo [7] and Plubtieng,Wangkeeree and Runpaeng [27] extended the three-step iterative scheme with errors and obtain many convergence theorems for asymptotically nonexpansive mappings in Banach spaces. The Noor iteration method has been studied extensively by many authors(see for example, [41, 14, 26, 25, 27])

In 2005, Suantain [34] introduced a modified three step iterative sequence which generalized the iterative sequence defined by Xu and Noor [40] and established weak and strong convergence for asymptotically nonexpansive self-mappings. Khan and Hussain [15] extended Suantain‘s results [34] to the nonself asymptotically nonexpansive mappings and introduced a modified three step iterative process for nonself asymptotically nonexpansive mappings as follows(modified three step iteration or modified Noor iteration):

$$\begin{cases} x_1 \in K, \\ z_n = P(a_n T_3 (PT_3)^{n-1} x_n + (1-a_n) x_n), \\ y_n = P(b_n T_2 (PT_2)^{n-1} z_n + c_n T_2 (PT_2)^{n-1} x_n + (1-b_n-c_n) x_n), \\ x_{n+1} = P(\alpha_n T_1 (PT_1)^{n-1} y_n + \beta_n T_1 (PT_1)^{n-1} z_n \\ \qquad + (1-\alpha_n-\beta_n) x_n), \quad n \geq 1, \end{cases} \tag{1.3}$$

where $T_1, T_2, T_3 : K \to E$ are nonself asymptotically nonexpansive mappings and $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}$ and $\{\beta_n\}$ are appropriate sequences in $[0,1]$ satisfy: $b_n + c_n \in [0,1], \alpha_n + \beta_n \in [0,1]$. They proved some strong and weak convergence theorems for nonself asymptotically nonexpansive mappings in a uniformly convex Banach space.

Yang [42] generalized the iteration process (1.3) and introduced a modified multistep iterative process for a finite family of nonself asymptotically nonexpansive mappings as follows(modified multistep iteration):

$$
\begin{cases}
x_1 \in K, \\
y_n = P((1-a_{nr})x_n + a_{nr}T_r(PT_r)^{n-1}x_n), \\
y_{n+1} = P((1-a_{n(r-1)} - b_{n(r-1)})x_n + a_{n(r-1)}T_{r-1}(PT_{r-1})^{n-1}y_n \\
\qquad + b_{n(r-1)}T_{r-1}(PT_{r-1})^{n-1}x_n), \\
y_{n+2} = P((1-a_{n(r-2)} - b_{n(r-2)})x_n + a_{n(r-2)}T_{r-2}(PT_{r-2})^{n-1}y_{n+1} \\
\qquad + b_{n(r-2)}T_{r-2}(PT_{r-2})^{n-1}y_n), \\
\qquad \vdots \\
y_{n+r-1} = P((1-a_{n2}-b_{n2})x_n + a_{n2}T_2(PT_2)^{n-1}y_{n+r-3} \\
\qquad + b_{n2}T_2(PT_2)^{n-1}y_{n+r-4}), \\
x_{n+1} = P((1-a_{n1}-b_{n1})x_n + a_{n1}T_1(PT_1)^{n-1}y_{n+r-2} \\
\qquad + b_{n1}T_1(PT_1)^{n-1}y_{n+r-3}), \quad n \geq 1,
\end{cases} \tag{1.4}
$$

where $T_i : K \to E(i \in \{1, 2, \cdots, r\})$ are nonself asymptotically nonexpansive mappings and $\{a_{ni}\}, \{b_{ni}\}$, $\{1 - a_{ni} - b_{ni}\}$ are appropriate sequences in $[0, 1]$. He proved some strong and weak convergence theorems of the modified multistep iteration for nonself asymptotically nonexpansive mappings in a uniformly convex Banach space.

In 2006, Nammanee et al.[16] introduced the following iterative sequences (modified three step iteration with errors or modified Noor iteration with errors):

$$
\begin{cases}
x_1 \in K, \\
z_n = a_n^{'}x_n + b_n^{'}T^n x_n + r_n u_n, \\
y_n = a_n x_n + b_n T^n x_n + c_n T^n z_n + s_n v_n, \\
x_{n+1} = \alpha_n x_n + \gamma_n T^n y_n + \delta_n T^n z_n + t_n w_n,
\end{cases} \tag{1.5}
$$

where $\{\alpha_n\}, \{\gamma_n\}, \{\delta_n\}, \{a_n\}, \{b_n\}, \{c_n\}, \{a_n^{'}\}, \{b_n^{'}\}, \{r_n\}, \{s_n\}$ and $\{t_n\}$ are appropriate sequences in $[0, 1]$ with $\alpha_n + \gamma_n + \delta_n + t_n = a_n + b_n + c_n + s_n = a_n^{'} + b_n^{'} + r_n = 1$, and $\{u_n\}, \{v_n\}$ and $\{w_n\}$ are bounded sequences in K.

Inspired and motivated by these facts, we introduce a modified Noor multistep iterative sequences with errors for a finite family of nonself asymptotically mappings as follows.

We denote the set $I = \{1, 2, \cdots, r\}$. Let K be a nonempty convex subset of real normed space E. Let $P : E \to K$ be the nonexpansive retraction of E onto K, and let $T_i : K \to E, i \in I$ be a finite family of nonself asymptotically mappings.

$$\begin{cases} x_1 \in K, \\ y_n = P((1-a_{nr}-c_{nr})x_n + a_{nr}T_r(PT_r)^{n-1}x_n + c_{nr}v_{nr}), \\ y_{n+1} = P((1-a_{n(r-1)}-b_{n(r-1)}-c_{n(r-1)})x_n + a_{n(r-1)}T_{r-1}(PT_{r-1})^{n-1}y_n \\ \qquad + b_{n(r-1)}T_{r-1}(PT_{r-1})^{n-1}x_n + c_{n(r-1)}v_{n(r-1)}), \\ y_{n+2} = P((1-a_{n(r-2)}-b_{n(r-2)}-c_{n(r-2)})x_n + a_{n(r-2)}T_{r-2}(PT_{r-1})^{n-1}y_{n+1} \\ \qquad + b_{n(r-2)}T_{r-2}(PT_{r-2})^{n-1}y_n + c_{n(r-2)}v_{n(r-2)}), \\ \quad \vdots \\ y_{n+r-2} = P((1-a_{n2}-b_{n2}-c_{n2})x_n + a_{n2}T_2(PT_2)^{n-1}y_{n+r-3} \\ \qquad + b_{n2}T_2(PT_2)^{n-1}y_{n+r-4} + c_{n2}v_{n2}), \\ x_{n+1} = P((1-a_{n1}-b_{n1}-c_{n1})x_n + a_{n1}T_1(PT_1)^{n-1}y_{n+r-2} \\ \qquad + b_{n1}T_1(PT_1)^{n-1}y_{n+r-3} + c_{n1}v_{n1}), \end{cases} \tag{1.6}$$

where $\{a_{ni}\}, \{b_{ni}\}, \{c_{ni}\}$ and $\{1-a_{ni}-b_{ni}-c_{ni}\}$ are real sequences in $[0,1]$ for any $i \in I$, $\{v_{ni}\}_{n=1}^{\infty}, \forall i \in I$ are bounded sequences in K.

Remark 1.5. Many iteration schemes are special case of (1.6).

(i) If $c_{ni} = 0$ for all $i \in I$, then (1.6) reduces to (1.4);

(ii) If $c_{ni} = 0$ for all $i \in I$ and $b_{ni} = 0, i = 1, 2, \cdots, r-1$, then (1.6) reduces to the iteration defined by Chidume and Basir[5]. Furthermore, if all the mappings are selfmappings then it reduces to iteration defined by Khan et al.[14].

(iii) If $r = 3, y_n = z_n, y_{n+1} = y_n$, then (1.6) reduces to the modified three step iteration with errors:

$$\begin{cases} x_1 \in K, \\ z_n = P((1-a_{n3}-c_{n3})x_n + a_{n3}T_3(PT_3)^{n-1}x_n + c_{n3}v_{n3}), \\ y_n = P((1-a_{n2}-b_{n2}-c_{n2})x_n + a_{n2}T_2(PT_2)^{n-1}z_n \\ \qquad + b_{n2}T_2(PT_2)^{n-1}x_n + c_{n2}v_{n2}), \\ x_{n+1} = P((1-a_{n1}-b_{n1}-c_{n1})x_n + a_{n1}T_1(PT_1)^{n-1}y_n \\ \qquad + b_{n1}T_1(PT_1)^{n-1}z_n + c_{n1}v_{n1}), \end{cases} \tag{1.7}$$

where $\{a_{ni}\}, \{b_{ni}\}, \{c_{ni}\}$ and $\{1-a_{ni}-b_{ni}-c_{ni}\}$ are real sequences in $[0,1]$ for any $i \in \{1,2,3\}$, $\{v_{ni}\}_{n=1}^{\infty}, i \in \{1,2,3\}$ are bounded sequences in K. If $T_1 = T_2 = T_3 = T$ are selfmappings, then iteration scheme (1.7) reduces to (1.5).

(iv) If $c_{n1} = c_{n2} = c_{n3} = 0, \forall n \geq 1$, then (1.7) reduces to (1.3).

(v) If $b_{n1} = b_{n2} = 0, \forall n \geq 1$, then (1.7) reduces to the three step iteration with errors studied in Wang and Zhu[37]. If $T_1 = T_2 = T_3 = T$, then it reduces to iteration scheme studied in Su et al.[35].

In this paper, we construct a modified Noor multistep iterative process with errors and prove some strong convergence theorems by using this iteration scheme for approximating the common fixed point of a finite family of nonself asymptotically quasi-nonexpansive mappings in a uniformly convex Banach space. Our results improve and generalize the corresponding results of [6, 37, 38, 15, 42, 16, 5] and many others.

2. Preliminaries

In the sequel, we shall need the following definitions and results.

The modulus of convexity of a real normed space E is the function $\delta_E : (0,2] \to [0,1]$ defined by

$$\delta_E(\varepsilon) = \inf\{1 - \|\frac{x+y}{2}\| : \|x\| = \|y\| = 1, \varepsilon = \|x-y\|\}.$$

E is said to be uniformly convex if and only if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$.

A mapping $T : K \to E$ is said to be semicompact if for any bounded sequence $\{x_n\}$ in K such that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to $x^* \in K$. T is said to be completely continuous if for every bounded sequence $\{x_n\}$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that the sequence $\{Tx_{n_j}\}$ converges to some element of the range of T.

A mapping $T : K \to E$ with $F(T) \neq \emptyset$ is said to satisfy condition(A) [32] if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0, f(t) > 0$ for all $t > 0$ such that

$$\|x - Tx\| \geq f(d(x, F(T))),$$

for all $x \in K$, where $d(x, F(T)) = \inf_{p \in F(T)} \|x - p\|$.

Yang [42] modified this condition for a finite family of nonself asymptotically nonexpansive mappings as follows: The mapping $T_i : K \to E, i \in I$ is said to satisfy condition(A') if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0, f(t) > 0$ for all $t > 0$ such that

$$\frac{1}{r}(\|x - T_1x\| + \|x - T_2x\| + \cdots + \|x - T_rx\|) \geq f(d(x,F)),$$

for all $x \in K$, where $F = \bigcap_{i=1}^r F(T_i)$. If $r = 3$, then condition(A') reduces to

$$\frac{1}{3}(\|x - T_1x\| + \|x - T_2x\| + \|x - T_3x\|) \geq f(d(x,F)),$$

for all $x \in K$, where $F = \bigcap_{i=1}^{3} F(T_i)$. Note that condition(A) is a special case of (A') for $T_i = T, i \in I$.

Lemma 2.1. *([7]) Let E be a uniformly convex Banach space and $B_D = \{x \in E : \|x\| \le D\}$, $D > 0$. Then there exists a continuous strictly increasing and convex function $g_1 : [0,\infty) \to [0,\infty), g_1(0) = 0$ such that*

$$\|\lambda x + \beta y + \gamma z\|^2 \le \lambda\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \lambda\beta g_1(\|x - y\|),$$

for all $x, y, z \in B_D$ and $\lambda, \beta, \gamma \in [0,1]$ with $\lambda + \beta + \gamma = 1$.

Lemma 2.2. *([16]) Let E be a uniformly convex Banach space and $B_D = \{x \in E : \|x\| \le D\}$, $D > 0$. Then there exists a continuous strictly increasing and convex function $g_2 : [0,\infty) \to [0,\infty), g_2(0) = 0$ such that*

$$\|\alpha x + \beta y + \mu z + \lambda w\|^2 \le \alpha\|x\|^2 + \beta\|y\|^2 + \mu\|z\|^2 + \lambda\|w\|^2 - \alpha\beta g_2(\|x - y\|),$$

for all $x, y, z, w \in B_D$ and $\alpha, \beta, \mu, \lambda \in [0,1]$ with $\alpha + \beta + \mu + \lambda = 1$.

Lemma 2.3. *([24]) Let $\{a_n\}, \{b_n\}, \{\lambda_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \le (1 + \lambda_n)a_n + b_n, \qquad n \ge 1.$$

Suppose that $\sum_{n=1}^{\infty} \lambda_n < +\infty$, $\sum_{n=1}^{\infty} b_n < +\infty$. Then we have the folloeings.

(i) $\lim_{n\to\infty} a_n$ *exists.*
(ii) *If* $\liminf_{n\to\infty} a_n = 0$, *then* $\lim_{n\to\infty} a_n = 0$.

3. Main results

In order to prove our main results, we shall need the following lemmas. For convenience, we assume $r = 3$ in (1.6). The method used in our proofs can easily be extended to the case of $r > 3$.

Lemma 3.1. *Let K be a nonempty closed convex subset of a Banach space E which is also a nonexpansive retract of E. Let $T_1, T_2, T_3 : K \to E$ be nonself asymptotically quasi-nonexpansive mappings with sequences $\{u_{ni}\}$ such that $\sum_{n=1}^{\infty} u_{ni} < +\infty$, for $i = 1, 2, 3$ and*

$$F := \bigcap_{i=1}^{3} F(T_i) \ne \emptyset.$$

Suppose the iterative sequences $\{x_n\}$ defined by (1.7) satisfying $\sum_{n=1}^{\infty} c_{ni} < +\infty, i = 1, 2, 3$. Then

(i) *there exists two sequences* $\{r_n\}, \{s_n\} \subset (0, \infty)$ *such that* $\sum_{n=1}^{\infty} r_n < +\infty, \sum_{n=1}^{\infty} s_n < +\infty$ *and*

$$\|x_{n+1} - p\| \leq (1 + r_n)\|x_n - p\| + s_n, \quad \forall p \in F, n \geq 1.$$

(ii) *there exists a constant* $M_1 > 0$*, for any integer* $m \geq 1$ *such that*

$$\|x_{n+m} - p\| \leq M_1\|x_n - p\| + M_1 \sum_{k=n}^{n+m-1} s_k, \quad \forall p \in F.$$

Proof. Let $p \in F$. Since $\{v_{n1}\}, \{v_{n2}\}$ and $\{v_{n3}\}$ are bounded sequences in K, so there exists a constant $M > 0$ such that

$$M = \max\left\{\sup_{n\geq 1}\|v_{n1} - p\|, \sup_{n\geq 1}\|v_{n2} - p\|, \sup_{n\geq 1}\|v_{n3} - p\|\right\}.$$

Let $u_n = \max\{u_{n1}, u_{n2}, u_{n3}\}$. Then $\sum_{n=1}^{\infty} u_n < +\infty$. It follows from (1.7) that

$$\begin{aligned} \|z_n - p\| &= \|P((1 - a_{n3} - c_{n3})x_n + a_{n3}T_3(PT_3)^{n-1}x_n + c_{n3}v_{n3}) - P(p)\| \\ &\leq (1 - a_{n3} - c_{n3})\|x_n - p\| + a_{n3}(1 + u_n)\|x_n - p\| + c_{n3}M \\ &\leq (1 + u_n)\|x_n - p\| + c_{n3}M, \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} &\|y_n - p\| \\ &= \|P((1 - a_{n2} - b_{n2} - c_{n2})x_n + a_{n2}T_2(PT_2)^{n-1}z_n + b_{n2}T_2(PT_2)^{n-1}x_n \\ &\qquad + c_{n2}v_{n2}) - P(p)\| \\ &\leq (1 - a_{n2} - b_{n2} - c_{n2})\|x_n - p\| + a_{n2}(1 + u_n)\|z_n - p\| \\ &\quad + b_{n2}(1 + u_n)\|x_n - p\| + c_{n2}M \\ &\leq (1 - a_{n2} - b_{n2})\|x_n - p\| + a_{n2}(1 + u_n)^2\|x_n - p\| \\ &\quad + b_{n2}(1 + u_n)\|x_n - p\| + a_{n2}c_{n3}(1 + u_n)M + c_{n2}M \\ &\leq (1 - a_{n2} - b_{n2})\|x_n - p\| + (a_{n2} + b_{n2})(1 + u_n)^2\|x_n - p\| \\ &\quad + c_{n3}(1 + u_n)M + c_{n2}M \\ &\leq (1 + u_n)^2\|x_n - p\| + c_{n3}(1 + u_n)M + c_{n2}M. \end{aligned} \tag{3.2}$$

By (1.7), (3.1) and (3.2), we have

$$\begin{aligned}
&\|x_{n+1}-p\| \\
&= \|P((1-a_{n1}-b_{n1}-c_{n1})x_n + a_{n1}T_1(PT_1)^{n-1}y_n \\
&\quad + b_{n1}T_1(PT_1)^{n-1}z_n + c_{n1}v_{n1}) - P(p)\| \\
&\le (1-a_{n1}-b_{n1})\|x_n-p\| + a_{n1}(1+u_n)\|y_n-p\| \\
&\quad + b_{n1}(1+u_n)\|z_n-p\| + c_{n1}M \\
&\le (1-a_{n1}-b_{n1})\|x_n-p\| + a_{n1}(1+u_n)^3\|x_n-p\| \\
&\quad + b_{n1}(1+u_n)^2\|x_n-p\| + a_{n1}c_{n3}(1+u_n)^2M \\
&\quad + a_{n1}c_{n2}(1+u_n)M + b_{n1}c_{n3}(1+u_n)M + c_{n1}M \\
&\le (1-a_{n1}-b_{n1})\|x_n-p\| + (a_{n1}+b_{n1})(1+u_n)^3\|x_n-p\| \\
&\quad + c_{n3}(1+u_n)^2M + c_{n2}(1+u_n)M + c_{n3}(1+u_n)M + c_{n1}M \\
&\le (1+u_n)^3\|x_n-p\| + c_{n3}(1+u_n)^2M + c_{n2}(1+u_n)M \\
&\quad + c_{n3}(1+u_n)M + c_{n1}M \\
&= (1+\alpha_n)\|x_n-p\| + \beta_n, \qquad (3.3)
\end{aligned}$$

where $\alpha_n = (1+u_n)^3-1, \beta_n = c_{n3}(1+u_n)^2M + c_{n2}(1+u_n)M + c_{n3}(1+u_n)M + c_{n1}M$. Since $\sum_{n=1}^{\infty} u_n < +\infty, \sum_{n=1}^{\infty} c_{ni} < +\infty, i = 1,2,3$, so $\sum_{n=1}^{\infty}\alpha_n < +\infty, \sum_{n=1}^{\infty}\beta_n < +\infty$. By Lemma 2.3(i), $\lim_{n\to\infty}\|x_n-p\|$ exists. Hence, $\{x_n\}$ is bounded.

From (3.1), we have

$$\begin{aligned}
\|z_n-p\| &= \|P((1-a_{n3}-c_{n3})x_n + a_{n3}T_3(PT_3)^{n-1}x_n + c_{n3}v_{n3}) - P(p)\| \\
&\le (1-a_{n3}-c_{n3})\|x_n-p\| + a_{n3}(1+u_n)\|x_n-p\| \\
&\quad + c_{n3}(\|v_{n3}-x_n\| + \|x_n-p\|) \\
&\le (1+u_n)\|x_n-p\| + c_{n3}\|v_{n3}-x_n\|.
\end{aligned}$$

Similarly, we get from (3.2) and (3.3)

$$\begin{aligned}
\|y_n-p\| &\le (1+u_n)^2\|x_n-p\| + c_{n3}(1+u_n)\|v_{n3}-x_n\| + c_{n2}\|v_{n2}-x_n\|, \\
\|x_{n+1}-p\| &\le (1+u_n)^3\|x_n-p\| + c_{n3}(1+u_n)^2\|v_{n3}-x_n\| \\
&\quad + c_{n2}(1+u_n)\|v_{n2}-x_n\| + c_{n3}(1+u_n)\|v_{n3}-x_n\| \\
&\quad + c_{n1}\|v_{n1}-x_n\| \\
&= (1+r_n)\|x_n-p\| + s_n,
\end{aligned}$$

where $r_n = (1+u_n)^3-1, s_n = c_{n3}(1+u_n)^2\|v_{n3}-x_n\| + c_{n2}(1+u_n)\|v_{n2}-x_n\| + c_{n3}(1+u_n)\|v_{n3}-x_n\| + c_{n1}\|v_{n1}-x_n\|$. Since $\sum_{n=1}^{\infty}u_n < +\infty, \sum_{n=1}^{\infty}c_{ni} <$

$+\infty, i = 1,2,3$, $\{v_{ni}\}_{n\geq 1}, i = 1,2,3$ and $\{x_n\}$ are bounded, then $\sum_{n=1}^{\infty} r_n < +\infty, \sum_{n=1}^{\infty} s_n < +\infty$. This completes the proof of part(i).

(ii) For any $x > 0, 1 + x \leq e^x$. From part(i) and for each $m \geq 1$, we have

$$\begin{aligned}
\|x_{n+m} - p\| &\leq (1 + r_{n+m-1})\|x_{n+m-1} - p\| + s_{n+m-1} \\
&\leq e^{r_{n+m-1}}\|x_{n+m-1} - p\| + s_{n+m-1} \\
&\leq e^{r_{n+m-1}} e^{r_{n+m-2}}\|x_{n+m-2} - p\| + e^{r_{n+m-1}} s_{n+m-2} + s_{n+m-1} \\
&\quad \cdots \\
&\leq e^{\sum_{k=n}^{n+m-1} r_k}\|x_n - p\| + e^{\sum_{k=n}^{n+m-1} r_k} \sum_{k=n}^{n+m-1} s_k \\
&\leq e^{\sum_{n=1}^{\infty} r_n}\|x_n - p\| + e^{\sum_{n=1}^{\infty} r_n} \sum_{k=n}^{n+m-1} s_k \\
&= M_1\|x_n - p\| + M_1 \sum_{k=n}^{n+m-1} s_k,
\end{aligned}$$

where $M_1 = e^{\sum_{n=1}^{\infty} r_n}$. This completes the proof of part(ii). □

Remark 3.2. Since the sequences $\{r_n\}$ and $\{s_n\}$ are independent of $p \in F$, so $\{x_n\}$ is said to be monotone type (1)(see Definition 2.1 of [30]).

Lemma 3.3. *Let K be a nonempty closed convex subset of a real uniformly convex Banach space E which is also a nonexpansive retract of E. Let $T_1, T_2, T_3 : K \to E$ be uniformly L_i-Lipschitzian nonself asymptotically quasi-nonexpansive mappings with sequences $\{u_{ni}\}$ such that $\sum_{n=1}^{\infty} u_{ni} < +\infty$, for $i = 1,2,3$ and*

$$F := \bigcap_{i=1}^{3} F(T_i) \neq \emptyset.$$

Suppose the iterative sequences $\{x_n\}$ defined by (1.7) and satisfying the following conditions:

(i) $\sum_{n=1}^{\infty} c_{ni} < +\infty, i = 1,2,3,$

(ii) $0 < \liminf_{n\to\infty} a_{ni} \leq \limsup_{n\to\infty}(a_{ni} + b_{ni} + c_{ni}) < 1, i = 1,2,3.$

Then $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0, \quad i = 1,2,3.$

Proof. Let $L = \max\{L_i, i = 1,2,3\}$. For any $p \in F$, by Lemma 3.1, we know $\lim_{n\to\infty} \|x_n - p\|$ exists. Hence $\{\|x_n - p\|\}$ is bounded. By virtue of Lemma

2.1 and (1.7), we have

$$
\begin{aligned}
&\|z_n - p\|^2 \\
&= \|P((1-a_{n3}-c_{n3})x_n + a_{n3}T_3(PT_3)^{n-1}x_n + c_{n3}v_{n3}) - P(p)\|^2 \\
&\leq (1-a_{n3}-c_{n3})\|x_n - p\|^2 + a_{n3}(1+u_n)^2\|x_n - p\|^2 + c_{n3}M^2 \\
&\quad - (1-a_{n3}-c_{n3})a_{n3}g_1(\|x_n - T_3(PT_3)^{n-1}x_n\|) \\
&\leq (1+u_n)^2\|x_n - p\|^2 + c_{n3}M^2 \\
&\quad - (1-a_{n3}-c_{n3})a_{n3}g_1(\|x_n - T_3(PT_3)^{n-1}x_n\|).
\end{aligned} \tag{3.4}
$$

Furthermore, by Lemma 2.2 and (3.4), we obtain

$$
\begin{aligned}
&\|y_n - p\|^2 \\
&= \|P((1-a_{n2}-b_{n2}-c_{n2})x_n + a_{n2}T_2(PT_2)^{n-1}z_n \\
&\quad + b_{n2}T_2(PT_2)^{n-1}x_n + c_{n2}v_{n2}) - P(p)\|^2 \\
&\leq (1-a_{n2}-b_{n2}-c_{n2})\|x_n - p\|^2 + a_{n2}(1+u_n)^2\|y_n - p\|^2 \\
&\quad + b_{n2}(1+u_n)^2\|x_n - p\|^2 + c_{n2}M^2 \\
&\quad - (1-a_{n2}-b_{n2}-c_{n2})a_{n2}g_2(\|x_n - T_2(PT_2)^{n-1}z_n\|) \\
&\leq (1-a_{n2}-b_{n2})\|x_n - p\|^2 + a_{n2}(1+u_n)^4\|x_n - p\|^2 \\
&\quad + b_{n2}(1+u_n)^2\|x_n - p\|^2 + c_{n3}(1+u_n)^2M^2 + c_{n2}M^2 \\
&\quad - (1-a_{n3}-c_{n3})a_{n2}a_{n3}g_1(\|x_n - T_3(PT_3)^{n-1}x_n\|) \\
&\quad - (1-a_{n2}-b_{n2}-c_{n2})a_{n2}g_2(\|x_n - T_2(PT_2)^{n-1}z_n\|) \\
&\leq (1-a_{n2}-b_{n2})\|x_n - p\|^2 + (a_{n2}+b_{n2})(1+u_n)^4\|x_n - p\|^2 \\
&\quad + c_{n3}(1+u_n)^2M^2 \\
&\quad + c_{n2}M^2 - (1-a_{n3}-c_{n3})a_{n2}a_{n3}g_1(\|x_n - T_3(PT_3)^{n-1}x_n\|) \\
&\quad - (1-a_{n2}-b_{n2}-c_{n2})a_{n2}g_2(\|x_n - T_2(PT_2)^{n-1}z_n\|) \\
&\leq (1+u_n)^4\|x_n - p\|^2 + c_{n3}(1+u_n)^2M^2 + c_{n2}M^2 \\
&\quad - (1-a_{n3}-c_{n3})a_{n2}a_{n3}g_1(\|x_n - T_3(PT_3)^{n-1}x_n\|) \\
&\quad - (1-a_{n2}-b_{n2}-c_{n2})a_{n2}g_2(\|x_n - T_2(PT_2)^{n-1}z_n\|),
\end{aligned} \tag{3.5}
$$

From Lemma 2.2, (3.4) and (3.5), we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|P((1-a_{n1}-b_{n1}-c_{n1})x_n + a_{n1}T_1(PT_1)^{n1}y_n \\
&\quad + b_{n1}T_1(PT_1)^{n-1}z_n + c_{n1}v_{n1}) - P(p)\|^2 \\
&\le (1-a_{n1}-b_{n1}-c_{n1})\|x_n-p\|^2 + a_{n1}(1+u_n)^2\|y_n-p\|^2 \\
&\quad + b_{n1}(1+u_n)^2\|z_n-p\|^2 + c_{n1}M^2 \\
&\quad - (1-a_{n1}-b_{n1}-c_{n1})a_{n1}g_2(\|x_n - T_1(PT_1)^{n-1}y_n\|) \\
&\le (1-a_{n1}-b_{n1})\|x_n-p\|^2 + a_{n1}(1+u_n)^6\|x_n-p\|^2 \\
&\quad + b_{n1}(1+u_n)^4\|x_n-p\|^2 + c_{n1}M^2 + c_{n3}(1+u_n)^4 \\
&\quad + c_{n2}(1+u_n)^2M^2 + c_{n3}(1+u_n)^2M^2 \\
&\quad - (1-a_{n1}-b_{n1}-c_{n1})a_{n1}g_2(\|x_n - T_1(PT_1)^{n-1}y_n\|) \\
&\quad - (1-a_{n3}-c_{n3})a_{n1}a_{n2}a_{n3}g_1(\|x_n - T_3(PT_3)^{n-1}x_n\|) \\
&\quad - (1-a_{n2}-b_{n2}-c_{n2})a_{n2}g_2(\|x_n - T_2(PT_2)^{n-1}z_n\|) \\
&\quad - (1-a_{n3}-c_{n3})a_{n3}b_{n1}g_1(\|x_n - T_3(PT_3)^{n-1}x_n\|) \\
&\le (1+u_n)^6\|x_n-p\|^2 + l_n n \qquad (3.6) \\
&\quad - (1-a_{n1}-b_{n1}-c_{n1})a_{n1}g_2(\|x_n - T_1(PT_1)^{n-1}y_n\|) \\
&\quad - (1-a_{n3}-c_{n3})a_{n1}a_{n2}a_{n3}g_1(\|x_n - T_3(PT_3)^{n-1}x_n\|) \\
&\quad - (1-a_{n2}-b_{n2}-c_{n2})a_{n2}g_2(\|x_n - T_2(PT_2)^{n-1}z_n\|) \\
&\quad - (1-a_{n3}-c_{n3})a_{n3}b_{n1}g_1(\|x_n - T_3(PT_3)^{n-1}x_n\|),
\end{aligned}$$

where $l_n = c_{n1}M^2 + c_{n3}(1+u_n)^4 + c_{n2}(1+u_n)^2M^2 + c_{n3}(1+u_n)^2M^2$. Since $\sum_{n=1}^{\infty} c_{ni} < +\infty, i = 1,2,3$, so $\sum_{n=1}^{\infty} l_n < +\infty$. It follows from (3.6) that

$$\begin{aligned}
&(1-a_{n1}-b_{n1}-c_{n1})a_{n1}g_2(\|x_n - T_1(PT_1)^{n-1}y_n\|) \\
&\le \|x_n-p\|^2 - \|x_{n+1}-p\|^2 \\
&\quad + l_n + ((1+u_n)^6-1)\|x_n-p\|^2, \qquad (3.7)
\end{aligned}$$

Notice the conditions $0 < \liminf_{n\to\infty} a_{n1} \le \limsup_{n\to\infty}(a_{n1}+b_{n1}+c_{n1}) < 1$, then there exists a natural number n_1 and $\eta, \eta^{'} \in (0,1)$ such that $0 < \eta < a_{n1}$ and $a_{n1}+b_{n1}+c_{n1} < \eta^{'} < 1$ for all $n \ge n_1$. Therefore from (3.7), we have

$$\begin{aligned}
\eta(1-\eta^{'})g_2(\|x_n - T_1(PT_1)^{n-1}y_n\|) &\le \|x_n-p\|^2 - \|x_{n+1}-p\|^2 \\
&\quad + l_n + ((1+u_n)^6-1)\|x_n-p\|^2,
\end{aligned}$$

for all $n \ge n_1$. Thus for $m \ge n_1$, we obtain

$$
\begin{aligned}
\sum_{n=n_1}^{m} g_2(\|x_n - T_1(PT_1)^{n-1}y_n\|) &\le \frac{1}{\eta(1-\eta')}\Big(\sum_{n=n_1}^{m}(\|x_n-p\|^2 - \|x_{n+1}-p\|^2) \\
&\quad + \sum_{n=n_1}^{m} l_n + \sum_{n=n_1}^{m}((1+u_n)^6-1)\|x_n-p\|^2\Big) \\
&\le \frac{1}{\eta(1-\eta')}\Big(\|x_{n_1}-p\|^2 + \sum_{n=n_1}^{m} l_n \\
&\quad + \sum_{n=n_1}^{m}((1+u_n)^6-1)\|x_n-p\|^2\Big).
\end{aligned}
$$

Since $\sum_{n=1}^{\infty} l_n < +\infty$ and $\{\|x_n - p\|\}$ is bounded, so letting $m \to \infty$, we have $\sum_{n=n_1}^{\infty} g_2(\|x_n - T_1(PT_a)^{n-1}y_n) < +\infty$, therefore , we have

$$\lim_{n\to\infty} g_2(\|x_n - T_1(PT_1)^{n-1}y_n) = 0.$$

Since g_2 is continuous strictly increasing function with $g_2(0) = 0$,

$$\lim_{n\to\infty} \|x_n - T_1(PT_1)^{n-1}y_n = 0.$$

By the same method and together with (3.6), we can obtain

$$\lim_{n\to\infty} \|x_n - T_2(PT_2)^{n-1}z_n\| = 0, \lim_{n\to\infty} \|x_n - T_3(PT_3)^{n-1}x_n\| = 0.$$

Finally, we prove $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0, i = 1, 2, 3$. Since $\{\|x_n - p\|\}$ and $\{\|v_{ni}\|, i = 1, 2, 3\}$ are bounded, so $\{\|v_{ni} - x_n\|, i = 1, 2, 3\}$ are bounded too.

First, by virtue of the above results and (1.7), we obtain

$$
\begin{aligned}
\|z_n - x_n\| &= \|P((1-a_{n3}-c_{n3})x_n + a_{n3}T_3(PT_3)^{n-1}x_n + c_{n3}v_{n3}) - x_n\| \\
&\le a_{n3}\|x_n - T_3(PT_3)^{n-1}x_n\| + c_{n3}\|v_{n3} - x_n\| \to 0 \quad \text{as } n \to \infty.
\end{aligned} \tag{3.8}
$$

Since T_2 is uniformly L-Lipschitzian and it follows from (3.8) that

$$
\begin{aligned}
&\|T_2(PT_2)^{n-1}x_n - x_n\| \\
&\le \|T_2(PT_2)^{n-1}x_n - T_2(PT_2)^{n-1}z_n\| + \|T_2(PT_2)^{n-1}z_n - x_n\| \\
&\le L\|x_n - z_n\| + \|T_2(PT_2)^{n-1}z_n - x_n\| \to 0 \quad \text{as } n \to \infty,
\end{aligned} \tag{3.9}
$$

Hence

$$\begin{aligned}
&\|y_n - x_n\| \\
&= \|P((1 - a_{n2} - b_{n2} - c_{n2})x_n + a_{n2}T_2(PT_2)^{n-1}z_n \\
&\quad + b_{n2}T_2(PT_2)^{n-1}x_n + c_{n2}v_{n2}) - x_n\| \\
&\le a_{n2}\|x_n - T_2(PT_2)^{n-1}z_n\| \\
&\quad + b_{n2}\|x_n - T_2(PT_2)^{n-1}x_n\| + c_{n2}\|v_{n2} - x_n\| \to 0, \quad \text{as } n \to \infty. \qquad (3.10)
\end{aligned}$$

Thus,

$$\begin{aligned}
&\|T_1(PT_1)^{n-1}x_n - x_n\| \\
&\le \|T_1(PT_1)^{n-1}x_n - T_1(PT_1)^{n-1}y_n\| + \|T_1(PT_1)^{n-1}y_n - x_n\| \\
&\le L\|x_n - y_n\| + \|T_1(PT_1)^{n-1}y_n - x_n\| \to 0, \quad \text{as } n \to \infty \qquad (3.11)
\end{aligned}$$

Now, with the help of (3.8) and (3.11), we have

$$\begin{aligned}
&\|T_1(PT_1)^{n-1}z_n - x_n\| \\
&\le \|T_1(PT_1)^{n-1}z_n - T_1(PT_1)^{n-1}x_n\| + \|T_1(PT_1)^{n-1}x_n - x_n\| \\
&\le L\|z_n - x_n\| + \|T_1(PT_1)^{n-1}x_n - x_n\| \to 0, \quad \text{as } n \to \infty. \qquad (3.12)
\end{aligned}$$

Observe that

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|P((1 - a_{n1} - b_{n1} - c_{n1})x_n + a_{n1}T_1(PT_1)^{n-1}y_n \\
&\quad + b_{n1}T_1(PT_1)^{n-1}z_n + c_{n1}v_{n1}) - x_n\| \\
&\le a_{n1}\|T_1(PT_1)^{n-1}y_n - x_n\| + b_{n1}\|T_1(PT_1)^{n-1}z_n - x_n\| \\
&\quad + c_{n1}\|v_n - x_n\| \to 0, \quad \text{as } n \to \infty \qquad (3.13)
\end{aligned}$$

Together with (3.11)and (3.13), we have

$$\begin{aligned}
\|x_n - T_1x_n\| &\le \|x_n - T_1(PT_1)^{n-1}x_n\| + \|T_1(PT_1)^{n-1}x_n - T_1x_n\| \\
&\le \|x_n - T_1(PT_1)^{n-1}x_n\| + L\|T_1(PT_1)^{n-2}x_n - x_n\| \\
&\le \|x_n - T_1(PT_1)^{n-1}x_n\| + L\|T_1(PT_1)^{n-2}x_n - T_1(PT_1)^{n-2}x_{n-1}\| \\
&\quad + L\|T_1(PT_1)^{n-2}x_{n-1} - x_{n-1}\| + L\|x_{n-1} - x_n\| \\
&\le \|x_n - T_1(PT_1)^{n-1}x_n\| + L(L+1)\|x_{n-1} - x_n\| \\
&\quad + L\|T_1(PT_1)^{n-2}x_{n-1} - x_{n-1}\| \to 0, \text{ as } n \to \infty. \qquad (3.14)
\end{aligned}$$

It follows from (3.9) and (3.13) that

$$\begin{aligned}\|x_n - T_2x_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T_2(PT_2)^n x_{n+1}\| \\ &\quad + \|T_2(PT_2)^n x_{n+1} - T_2(PT_2)^n x_n\| + \|T_2(PT_2)^n x_n - T_2x_n\| \\ &\leq (L+1)\|x_n - x_{n+1}\| + \|x_{n+1} - T_2(PT_2)^n x_{n+1}\| \\ &\quad + L\|T_2(PT_2)^{n-1}x_n - x_n\| \to 0, \text{ as } n \to \infty. \qquad (3.15)\end{aligned}$$

and

$$\begin{aligned}\|x_n - T_3x_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T_3(PT_3)^n x_{n+1}\| \\ &\quad + \|T_3(PT_3)^n x_{n+1} - T_3(PT_3)^n x_n\| + \|T_3(PT_3)^n x_n - T_2x_n\| \\ &\leq (L+1)\|x_n - x_{n+1}\| + \|x_{n+1} - T_3(PT_3)^n x_{n+1}\| \\ &\quad + L\|T_3(PT_3)^{n-1}x_n - x_n\| \to 0, \text{ as } n \to \infty. \qquad (3.16)\end{aligned}$$

Therefore, $\lim_{n\to\infty} \|x_n - T_ix_n\| = 0, \quad i = 1, 2, 3$. This completes the proof. □

Theorem 3.4. *Let $E, K, T_i (i = 1, 2, 3)$ and $\{x_n\}$ be as in Lemma 3.3. If $T_1, T_2, T_3 : K \to E$ satisfy condition(A') and $F := \bigcap_{i=1}^3 F(T_i) \neq \emptyset$, then $\{x_n\}$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^3$.*

Proof. By Lemma 3.3, we know that $\lim_{n\to\infty} \|x_n - T_ix_n\| = 0, i = 1, 2, 3$. Since T_1, T_2, T_3 satisfy condition(A'), we obtain that $\lim_{n\to\infty} f(d(x_n, F)) = 0$. While the function f is nondecreasing with $f(0) = 0$, so $\lim_{n\to\infty} d(x_n, F) = 0$.

Next, we prove $\{x_n\}$ is a cauchy sequence. By Lemma 3.1(i), we have

$$d(x_{n+1}, F) \leq (1 + r_n)d(x_n, F) + s_n,$$

where $\sum_{n=1}^{\infty} r_n < +\infty$ and $\sum_{n=1}^{\infty} s_n < +\infty$.

For an arbitrary $\varepsilon > 0$, since $\lim_{n\to\infty} d(x_n, F) = 0$ and $\sum_{n=1}^{\infty} s_n < +\infty$, there exists a positive integer N_0 such that

$$d(x_n, F) < \frac{\varepsilon}{12M_1}, \quad \sum_{n=N_0}^{\infty} s_n < \frac{\varepsilon}{4M_1}, \quad \forall n \geq N_0.$$

It means that there exists a $p_1 \in F$ such that $\|x_{N_0} - p_1\| < \frac{\varepsilon}{4M_1}$. By Lemma 3.1(ii) and for any $n \geq N_0$, we have

$$\begin{aligned}\|x_{n+m} - x_n\| &\leq \|x_{n+m} - p_1\| + \|p_1 - x_n\| \\ &\leq 2M_1\|x_{N_0} - p_1\| + 2M_1 \sum_{k=N_0}^{\infty} s_k \\ &\leq 2M_1 \frac{\varepsilon}{4M_1} + 2M_1 \frac{\varepsilon}{4M_1} = \varepsilon,\end{aligned}$$

Thus, $\{x_n\}$ is a cauchy sequence. Since K is closed and E is a Banach space, so $\lim_{n\to\infty} x_n$ exists, i.e., there exists a $q \in K$ such that $\lim_{n\to\infty} x_n = q$. Finally, we prove $q \in F$. Notice that $\lim_{n\to\infty} d(x_n, F) = 0$, therefore $d(q, F) = 0$. For an arbitrary $\varepsilon_1 > 0$, there exists a $p_2 \in F$ such that $\|q - p_2\| \le \varepsilon_1$. Thus,

$$\begin{aligned}\|q - T_i q\| &\le \|q - p_2\| + \|p_2 - T_i q\| \\ &\le (1 + L)\|q - p_2\| \le (1 + L)\varepsilon_1,\end{aligned}$$

By the arbitrary of ε_1, we know $T_i q = q, i = 1, 2, 3$, i.e., $q \in F$. This completes the proof. □

Remark 3.5. Theorem 3.4 improves and extends the corresponding results of [39, 6, 37, 38, 15, 42, 16]in the following aspects:

(i) The mappings in Theorem 2.5 of Yang[42] is replaced by asymptotically quasi-nonexpansive mappings, and the iteration scheme is extended to the modified multistep iteration with errors.
(ii) The iteration scheme in Theorem 3.4 and Theorem 3.6 of Xiao et al.[39], Theorem 3.3 of Wang and Zhu[37], Theorem 7 of Khan and Hussian[15] are extended to (1.7).
(iii) The mappings in [39, 16] is extended to a more general class of nonself mappings.
(iv) Our method in Theorem 3.4 is independent from [6, 38].

Theorem 3.6. *Let $E, K, T_i (i = 1, 2, 3)$ and $\{x_n\}$ be as in Lemma 3.3. If $F := \bigcap_{i=1}^3 F(T_i) \neq \emptyset$, then $\{x_n\}$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^3$ if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$, where $d(x, F) = inf_{p\in F}\|x - p\|$.*

Proof. The necessity is obvious. We will only prove the sufficiency. From Lemma 3.1(i), we have

$$d(x_{n+1}, F) \le (1 + r_n)d(x_n, F) + s_n,$$

where $\sum_{n=1}^\infty r_n < +\infty$ and $\sum_{n=1}^\infty s_n < +\infty$. By Lemma 2.3(ii) and

$$\liminf_{n\to\infty} d(x_n, F) = 0,$$

we get that $\lim_{n\to\infty} d(x_n, F) = 0$. The rest of proof is the same to Theorem 3.4. So is omitted. □

Theorem 3.7. *Let $E, K, T_i(i = 1, 2, 3)$ and $\{x_n\}$ be as in Lemma 3.3. If $F := \bigcap_{i=1}^3 F(T_i) \neq \emptyset$ and at least one of $T_i(i = 1, 2, 3)$ is completely continuous or semicompact, then $\{x_n\}$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^3$.*

Proof. By the proof of Lemma 3.1(i), $\{x_n\}$ is bounded. Without loss of generality, we assume T_1 is semicompact. By Lemma 3.3, $\lim_{n\to\infty} \|x_n - T_1 x_n\| = 0$, therefore there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to $x^* \in K$. Hence, we have

$$\begin{aligned}\|x^* - T_ix^*\| &\leq \|x^* - x_{n_j}\| + \|x_{n_j} - T_ix_{n_j}\| + \|T_ix_{n_j} - T_ix^*\| \\ &\leq (1+L)\|x^* - x_{n_j}\| + \|x_{n_j} - T_ix_{n_j}\| \to 0 \text{ as } n \to \infty, i = 1,2,3.\end{aligned}$$

This implies that $x^* \in F$. Moreover, as $\lim_{n\to\infty} \|x_n - p\|$ exists for any $p \in F$, therefore $\lim_{n\to\infty} \|x_n - x^*\|$ exists. Since $\lim_{j\to\infty} \|x_{n_j} - x^*\| = 0$, that is $\lim_{n\to\infty} \|x_n - x^*\| = 0$, i.e., $\{x_n\}$ converges strongly to x^*.

Next, we assume T_1 is completely continuous, there exists a subsequence $\{T_1x_{n_j}\}$ of $\{T_1x_n\}$ such that $T_1x_{n_j} \to x^*$ as $j \to \infty$. By Lemma 3.3,

$$\lim_{n\to\infty} \|x_{n_j} - T_1x_{n_j}\| = 0.$$

Hence,

$$\|x_{n_j} - x^*\| \leq \|x_{n_j} - T_1x_{n_j}\| + \|T_1x_{n_j} - x^*\| \to 0, \text{ as } j \to \infty.$$

This implies that $\lim_{j\to\infty} x_{n_j} = x^*$, by Lemma 3.3, $\lim_{n\to\infty} \|x_n - T_ix_n\| = 0, i = 1,2,3$, we obtain $T_ix^* = x^*, i = 1,2,3$. Thus $x^* \in F$. By Lemma 2.3(ii), we have $\lim_{n\to\infty} \|x_n - x^*\| = 0$. This completes the proof. □

Remark 3.8. Theorem 3.7 improves and generalizes Theorem 3.3 of Xiao et al.[39], Theorem 7 of Khan and Hussain [15], Theorem 3.4 of Chidume and Ali [5], Theorem 3.5 of Wang and Zhu [37], and others. If T_i are selfmappings, Theorem 3.3 of Khan et al.[14] are special case of Theorem 3.7.

Since every nonself asymptotically nonexpansive mapping is uniformly L-Lipschitizian and nonself asymptotically quasi-nonexpansive mapping, therefore, by Theorem 3.4 and Theorem 3.7, we can get the following corollaries immediately.

Corollary 3.9. *Let K be a nonempty closed convex subset of a real uniformly convex Banach space E which is also a nonexpansive retract of E. Let $T_1, T_2, T_3 : K \to E$ be nonself asymptotically nonexpansive mappings with sequences $\{u_{ni}\}$ such that $\sum_{n=1}^{\infty} u_{ni} < +\infty$, for $i = 1,2,3$ and satisfying condition(A'). Let $\{x_n\}$ be defined as in Lemma 3.3. If $F := \bigcap_{i=1}^{3} F(T_i) \neq \emptyset$, then $\{x_n\}$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^{3}$.*

Corollary 3.10. *Let K be a nonempty closed convex subset of a real uniformly convex Banach space E which is also a nonexpansive retract of E. Let $T_1, T_2, T_3 : K \to E$ be nonself asymptotically nonexpansive mappings with sequences $\{u_{ni}\}$ such that $\sum_{n=1}^{\infty} u_{ni} < +\infty$, for $i = 1,2,3$. Let $\{x_n\}$ be defined as in Lemma 3.3. If $F := \bigcap_{i=1}^{3} F(T_i) \neq \emptyset$ and at least one of $T_i (i = 1,2,3)$ is completely continuous or semicompact, then $\{x_n\}$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^{3}$.*

4. Applications

Variational inequalities theory, which was introduced in 1960's, has emerged as an interesting and fascinating field of mathematical and engineering sciences(see Noor's book [20] and Noor [19, 21, 22], and also Chang's book [4] for details). Haubruge et al.[11] studied the convergence analysis of three-step schemes of [8] and applied this scheme to obtain new splitting-type algorithms for solving variation inequalities, separable convex programming and minimization of a sum of convex functions. They proved that the three-step iterations lead to highly parallelized algorithms under certain conditions.

Our results enrich and develop the theory of three-step iterative scheme introduced by Noor(see e.g., [1, 23, 13, 17]).

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