Nonlinear Functional Analysis and Applications
Vol. 15, No. 1 (2010), pp. 87-162

http://nfaa.kyungnam.ac.kr/jour-nfaa.htm

COINCIDENCE POINT AND HOMOTOPY RESULTS FOR f-HYBRID COMPATIBLE MAPS

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Abstract. Common fixed point, coincidence point, and homotopy results are presented for single-valued as well as multivalued f-hybrid compatible generalized ϕ-contractive maps defined on complete metric spaces and more general spaces called complete gauge spaces (i.e complete uniform spaces). Existence results of coincidence point for single-valued as well as multivalued f-hybrid compatible generalized ϕ-contractive maps are discussed in arbitrary spaces.

1. Introduction

In 1986, Jungck [15] introduced the notion of compatible maps. This notion was extended to multivalued maps independently by Beg and Azam [5], Cho et al. [8], and Kaneko and Sessa [16]. It is worth noting that the class of compatible maps contains the class of commuting maps. It also includes other classes of non-commuting (weakly commuting etc.) maps (see [8, 15]). This paper present new common fixed point theorems, coincidence point theorems, and homotopy results for f-hybrid compatible single-valued and multivalued generalized contractive maps defined on complete gauge spaces.

[0]Received July 10, 2007. Revised February 22, 2009.
[0]2000 Mathematics Subject Classification: 47H10.
[0]Keywords: Common fixed point, homotopy, compatible maps, f-hybrid compatible maps.

The results here extend, improve and complement well known results in the literature (see [1-4, 6, 7, 9, 12-15, 17-21]). In section 2, we present coincidence point and homotopy results for f-hybrid compatible single-valued maps on a complete metric space and, in section 3, we discuss fixed point theory for f-hybrid compatible maps in gauge spaces. The results of section 2 are extended to multivalued compatible maps in section 4 and finally, in section 5, we study the analogue of these results in the setting of gauge spaces.

2. Coincidence points for single valued f-hybrid compatible maps in arbitrary spaces

In this section, we present some local and global coincidence point results for f-hybrid compatible maps. We also establish a homotopy result for a pair of f-hybrid compatible maps. Let Y be an arbitrary space, (X, d) be a complete metric space , $f : Y \to X$ be a bijection map, and $G : Y \to X$ be a mapping. If $x_0 \in X$ and $r > 0$, we let

$$B(Gf^{-1}x_0, r) = \{x \in X : d(x, Gf^{-1}x_0) < r\},$$

$$B(x_0, r) = \{x \in X : d(x, x_0) < r\}$$

and

$$fG^{-1}(B(Gf^{-1}x_0, r)) = \{x \in X : fG^{-1}x \in B(Gf^{-1}x_0, r)\}.$$

Let $F : f^{-1}\Big(\overline{B(Gf^{-1}x_0, r)} \cup G^{-1}(B(Gf^{-1}x_0, r))\Big) \to X$ be a mapping with

$$FG^{-1}(B(Gf^{-1}x_0, r)) \subseteq Gf^{-1}(X).$$

Then F and G are said to be f-hybrid compatible on $\overline{B(Gf^{-1}x_0, r)}$ if

$$\lim_{n\to\infty} d(Ff^{-1}Gf^{-1}x_n, Gf^{-1}Ff^{-1}x_n) = 0,$$

whenever $\{x_n\}$ is a sequence in $fG^{-1}(B(Gf^{-1}x_0, r))$ such that

$$\lim_{n\to\infty} Ff^{-1}x_n = \lim_{n\to\infty} Gf^{-1}x_n = t$$

for some $t \in \overline{B(Gf^{-1}x_0, r)}$.

Remark 2.1. If F and G are f-hybrid compatible and $Ff^{-1}x = Gf^{-1}x$ for some $x \in fG^{-1}(B(Gf^{-1}x_0, r))$, then

$$Ff^{-1}Gf^{-1}x = Gf^{-1}Ff^{-1}x$$

(i.e. Ff^{-1} and Gf^{-1} commute at coincidence point). This is immediate if we set $x_n = x$ for each n. Furthermore, if $Y = X$ and f is the identity map on X then our definition of f-hybrid compatibility of maps F and G reduces to compatibility of maps F and G defined by O'Reagan et al. [18].

Theorem 2.1. *Let Y be an arbitrary space, (X,d) be a complete metric space, $f: Y \to X$ be a bijection map, $x_0 \in X$, $r > 0$ with $F : f^{-1}\Big(\overline{B(Gf^{-1}x_0, r)}\cup G^{-1}(B(Gf^{-1}x_0,r))\Big) \to X$ and $G: Y \to X$ be f-hybrid compatible maps on $\overline{B(Gf^{-1}x_0,r)}$ and*

$$FG^{-1}(B(Gf^{-1}x_0,r)) \subseteq Gf^{-1}(X).$$

Suppose f^{-1} and G are continuous and there exists a continuous, nondecreasing function $\phi : [0,\infty) \to [0,\infty)$ satisfying $\phi(z) < z$ for $z > 0$ such that for $x, y \in \overline{B(Gf^{-1}x_0,r)} \cup fG^{-1}(B(Gf^{-1}x_0,r))$ we have

$$d(Ff^{-1}x, Ff^{-1}y) \le \phi(M(x,y;f^{-1})), \tag{2.1}$$

where

$$M(x,y;f^{-1}) = \max\{d(Gf^{-1}x, Gf^{-1}y), d(Gf^{-1}x, Ff^{-1}x), d(Gf^{-1}y, Ff^{-1}y), \frac{1}{2}[d(Gf^{-1}x, Ff^{-1}y) + d(Gf^{-1}y, Ff^{-1}x)]\}.$$

Also suppose

$$d(Gf^{-1}x_0, Ff^{-1}x_0) < r - \phi(r). \tag{2.2}$$

Then there exists a unique $x \in \overline{B(Gf^{-1}x_0,r)}$ with $x = Ff^{-1}x = Gf^{-1}x$. Moreover, there exists a unique $y \in f^{-1}\Big(\overline{B(Gf^{-1}x_0,r)}\Big)$ with $fy = Fy = Gy$.

Proof. Let $Gf^{-1}x_1 = Ff^{-1}x_0$ for some $x_1 \in X$ (This is possible since $Gf^{-1}x_0 \in B(Gf^{-1}x_0,r)$ and $FG^{-1}(B(Gf^{-1}x_0,r)) \subseteq Gf^{-1}(X)$). Then, by (2.2),

$$d(Gf^{-1}x_1, Gf^{-1}x_0) < r$$

and so

$$Gf^{-1}x_1 \in B(Gf^{-1}x_0, r).$$

Now let $Gf^{-1}x_2 = Ff^{-1}x_1$ (This is possible since $Gf^{-1}x_1 \in B(Gf^{-1}x_0,r)$ and FG^{-1} $(B(Gf^{-1}x_0,r)) \subseteq Gf^{-1}(X)$). For $n \in \{3,4,\cdots\}$, we let

$$Gf^{-1}x_n = Ff^{-1}x_{n-1}.$$

This is possible if we show $Gf^{-1}x_{n-1} \in B(Gf^{-1}x_0,r)$ since $FG^{-1}(B(Gf^{-1}x_0, r)) \subseteq Gf^{-1}(X)$. To show the above we will in fact establish more i.e., we now show

$$\begin{cases} d(Gf^{-1}x_n, Gf^{-1}x_{n+1}) \le \phi(d(Gf^{-1}x_{n-1}, Gf^{-1}x_n)) \\ \qquad \text{for } n \in \{1,2,\cdots\} \\ \text{and } Gf^{-1}x_i \in B(Gf^{-1}x_0,r) \text{ for } i \in \{0,\cdots,n\}. \end{cases} \tag{2.3}$$

Notice

$$\begin{aligned} d(Gf^{-1}x_1, Gf^{-1}x_2) &= d(Ff^{-1}x_0, Ff^{-1}x_1) \\ &\le \phi(M(x_0, x_1; f^{-1})) \\ &= \phi(\max\{d(Gf^{-1}x_0, Gf^{-1}x_1), d(Gf^{-1}x_1, Gf^{-1}x_2), \\ &\quad \frac{1}{2}d(Gf^{-1}x_0, Gf^{-1}x_2)\}) \\ &\le \phi(\max\{d(Gf^{-1}x_0, Gf^{-1}x_1), d(Gf^{-1}x_1, Gf^{-1}x_2), \\ &\quad \frac{1}{2}[d(Gf^{-1}x_0, Gf^{-1}x_1) + d(Gf^{-1}x_1, Gf^{-1}x_2)]\})) \\ &= \phi(\max\{d(Gf^{-1}x_0, Gf^{-1}x_1), d(Gf^{-1}x_1, Gf^{-1}x_2)\}) \\ &\le \phi(d(Gf^{-1}x_0, Gf^{-1}x_1)). \end{aligned}$$

Also from (2.2), we have

$$\begin{aligned} d(Gf^{-1}x_0, Gf^{-1}x_2) &\le d(Gf^{-1}x_0, Gf^{-1}x_1) + d(Gf^{-1}x_1, Gf^{-1}x_2) \\ &\le d(Gf^{-1}x_0, Gf^{-1}x_1) + \phi(d(Gf^{-1}x_0, Gf^{-1}x_1)) \\ &< [r - \phi(r)] + \phi(r) \\ &= r, \end{aligned}$$

so

$$Gf^{-1}x_2 \in B(Gf^{-1}x_0, r).$$

Essentially the same argument as above yields

$$\begin{aligned} d(Gf^{-1}x_2, Gf^{-1}x_3) &= d(Ff^{-1}x_1, Ff^{-1}x_2) \\ &\le \phi(d(Gf^{-1}x_1, Gf^{-1}x_2)). \end{aligned}$$

Now suppose there exists $k \in \{2, 3, \cdots\}$ with

$$d(Gf^{-1}x_m, Gf^{-1}x_{m+1}) \le \phi(d(Gf^{-1}x_{m-1}, Gf^{-1}x_m))$$

and

$$Gf^{-1}x_m \in B(Gf^{-1}x_0, r)$$

for $m \in \{1, 2, \cdots, k\}$. We first show

$$Gf^{-1}x_{k+1} \in B(Gf^{-1}x_0, r).$$

In the proof we will use the inequality

$$d(Gf^{-1}x_k, Gf^{-1}x_1) \le \phi(r).$$

If $k = 2$ this is obvious, from

$$d(Gf^{-1}x_2, Gf^{-1}x_1) \le \phi(d(Gf^{-1}x_1, Gf^{-1}x_0)) \le \phi(r).$$

Next consider $k = 3$. Then

$$\begin{aligned}
d(Gf^{-1}x_3, Gf^{-1}x_1) &= d(Ff^{-1}x_2, Ff^{-1}x_0) \\
&\leq \phi(\max\{d(Gf^{-1}x_2, Gf^{-1}x_0), d(Gf^{-1}x_0, Gf^{-1}x_1), \\
&\quad d(Gf^{-1}x_2, Gf^{-1}x_3), \frac{1}{2}[d(Gf^{-1}x_3, Gf^{-1}x_0) \\
&\quad +d(Gf^{-1}x_2, Gf^{-1}x_1)]\}) \\
&\leq \phi(\max\{d(Gf^{-1}x_2, Gf^{-1}x_0), d(Gf^{-1}x_0, Gf^{-1}x_1), \\
&\quad \phi^2(d(Gf^{-1}x_1, Gf^{-1}x_0)), \frac{1}{2}[d(Gf^{-1}x_3, Gf^{-1}x_0) \\
&\quad +d(Gf^{-1}x_2, Gf^{-1}x_1)]\}) \\
&\leq \phi(\max\{r, r, \phi^2(r), \frac{1}{2}[r + \phi(r)]\})
\end{aligned}$$

since $Gf^{-1}x_2, Gf^{-1}x_3 \in B(Gf^{-1}x_0, r)$ and

$$d(Gf^{-1}x_3, Gf^{-1}x_2) \leq \phi^2(d(Gf^{-1}x_1, Gf^{-1}x_0)).$$

Since $\phi^2(r) \leq r$ and $r + \phi(r) \leq 2r$,
we have

$$d(Gf^{-1}x_3, Gf^{-1}x_1) \leq \phi(r).$$

If $k = 4$, then

$$\begin{aligned}
d(Gf^{-1}x_4, Gf^{-1}x_1) &= d(Ff^{-1}x_3, Ff^{-1}x_0) \\
&\leq \phi(\max\{d(Gf^{-1}x_3, Gf^{-1}x_0), d(Gf^{-1}x_0, Gf^{-1}x_1), \\
&\quad d(Gf^{-1}x_3, Gf^{-1}x_4), \frac{1}{2}[d(Gf^{-1}x_4, Gf^{-1}x_0) \\
&\quad +d(Gf^{-1}x_3, Gf^{-1}x_1)]\}) \\
&\leq \phi(\max\{r, r, \phi^3(r), \frac{1}{2}[r + \phi(r)]\})
\end{aligned}$$

since $Gf^{-1}x_3, Gf^{-1}x_4 \in B(Gf^{-1}x_0, r)$. Thus

$$d(Gf^{-1}x_4, Gf^{-1}x_1) \leq \phi(r).$$

Continuing this process, we obtain for $k \in \{5, 6, \cdots\}$,

$$d(Gf^{-1}x_k, Gf^{-1}x_1) \leq \phi(r).$$

To show $Gf^{-1}x_{k+1} \in B(Gf^{-1}x_0, r)$, notice

$$\begin{aligned} & d(Gf^{-1}x_0, Gf^{-1}x_{k+1}) \\ \leq \; & d(Gf^{-1}x_0, Gf^{-1}x_1) + d(Ff^{-1}x_0, Ff^{-1}x_k) \\ \leq \; & d(Gf^{-1}x_0, Gf^{-1}x_1) + \phi(M(x_0, x_k; f^{-1}) \\ = \; & d(Gf^{-1}x_0, Gf^{-1}x_1) + \phi(\max\{d(Gf^{-1}x_0, Gf^{-1}x_k), \\ & d(Gf^{-1}x_0, Gf^{-1}x_1), d(Gf^{-1}x_k, Gf^{-1}x_{k+1}), \\ & \frac{1}{2}[d(Gf^{-1}x_0, Gf^{-1}x_{k+1}) + d(Gf^{-1}x_k, Gf^{-1}x_1)]\}) \\ \leq \; & d(Gf^{-1}x_0, Gf^{-1}x_1) + \phi(\max\{r, r, \phi^k(d(Gf^{-1}x_0, Gf^{-1}x_1)), \\ & \frac{1}{2}[d(Gf^{-1}x_0, Gf^{-1}x_{k+1}) + d(Gf^{-1}x_k, Gf^{-1}x_1)]\}) \end{aligned}$$

since $Gf^{-1}x_m \in B(Gf^{-1}x_0, r)$ for $m \in \{1, \cdots, k\}$. Since

$$\phi^k(d(Gf^{-1}x_0, Gf^{-1}x_1)) \leq \phi^k(r) \leq r,$$

it follows that

$$\begin{aligned} d(Gf^{-1}x_0, Gf^{-1}x_{k+1}) \leq & \; d(Gf^{-1}x_0, Gf^{-1}x_1) \\ & + \phi(\max\{r, \frac{1}{2}[d(Gf^{-1}x_0, Gf^{-1}x_{k+1}) + \phi(r))]\}). \end{aligned}$$

Let $\tau_k = \max\{r, \frac{1}{2}[d(Gf^{-1}x_0, Gf^{-1}x_{k+1}) + \phi(r)]\}$. If $\tau_k = r$, then the preceeding inequality gives

$$\begin{aligned} d(Gf^{-1}x_0, Gf^{-1}x_{k+1}) & \leq d(Gf^{-1}x_0, Gf^{-1}x_1) + \phi(r) \\ & < [r - \phi(r)] + \phi(r) = r. \end{aligned}$$

Thus, we have

$$Gf^{-1}x_{k+1} \in B(Gf^{-1}x_0, r).$$

On the other hand, if $\tau_k = \frac{1}{2}[d(Gf^{-1}x_0,\, Gf^{-1}x_{k+1}) + \phi(r)]$, then

$$\begin{aligned} d(Gf^{-1}x_0, Gf^{-1}x_{k+1}) \leq & \; d(Gf^{-1}x_0, Gf^{-1}x_1) \\ & + \phi(\frac{1}{2}[d(Gf^{-1}x_0, Gf^{-1}x_{k+1}) + \phi(r)]) \end{aligned}$$

and so

$$\begin{aligned} d(Gf^{-1}x_0, Gf^{-1}x_{k+1}) < & \; d(Gf^{-1}x_0, Gf^{-1}x_1) \\ & + \frac{1}{2}[d(Gf^{-1}x_0, Gf^{-1}x_{k+1}) + \phi(r)]. \end{aligned}$$

This implies that

$$\frac{1}{2}d(Gf^{-1}x_0, Gf^{-1}x_{k+1}) < d(Gf^{-1}x_0, Gf^{-1}x_1) + \frac{1}{2}\phi(r).$$

As a result

$$\begin{aligned}\tau_k &= \frac{1}{2}[d(Gf^{-1}x_0, Gf^{-1}x_{k+1}) + \phi(r)]\\ &< d(Gf^{-1}x_0, Gf^{-1}x_1) + \phi(r)\\ &< [r - \phi(r)] + \phi(r)\\ &= r,\end{aligned}$$

which contradicts the definition of τ_k. Consequently, we have

$$Gf^{-1}x_{k+1} \in B(Gf^{-1}x_0, r).$$

Also

$$\begin{aligned}&d(Gf^{-1}x_{k+1}, Gf^{-1}x_{k+2})\\ &= d(Ff^{-1}x_k, Ff^{-1}x_{k+1})\\ &\leq \phi(M(x_k, x_{k+1}; f^{-1})\\ &= \phi(\max\{d(Gf^{-1}x_k, Gf^{-1}x_{k+1}), d(Gf^{-1}x_{k+1}, Gf^{-1}x_{k+2}),\\ &\quad \frac{1}{2}d(Gf^{-1}x_k, Gf^{-1}x_{k+2})\})\\ &\leq \phi(\max\{d(Gf^{-1}x_k, Gf^{-1}x_{k+1}), d(Gf^{-1}x_{k+1}, Gf^{-1}x_{k+2}),\\ &\quad \frac{1}{2}[d(Gf^{-1}x_k, Gf^{-1}x_{k+1}) + d(Gf^{-1}x_{k+1}, Gf^{-1}x_{k+2})]\}))\\ &= \phi(\max\{d(Gf^{-1}x_k, Gf^{-1}x_{k+1}), d(Gf^{-1}x_{k+1}, Gf^{-1}x_{k+2})\})\\ &\leq \phi(d(Gf^{-1}x_k, Gf^{-1}x_{k+1})).\end{aligned}$$

Thus, by induction $Gf^{-1}x_n \in B(Gf^{-1}x_0, r)$ for $n \in \{0, 1, 2, \cdots\}$ and

$$d(Gf^{-1}x_n, Gf^{-1}x_{n+1}) \leq \phi(d(Gf^{-1}x_{n-1}, Gf^{-1}x_n)) \tag{2.4}$$

for $n \in \{1, 2, \cdots\}$. This implies that

$$d(Gf^{-1}x_n, Gf^{-1}x_{n+1}) \leq \phi^n(d(Gf^{-1}x_0, Gf^{-1}x_1))$$

for $n \in \{1, 2, \cdots\}$. We now claim

$$\{Gf^{-1}x_n\} \text{ is a Cauchy sequence.} \tag{2.5}$$

Suppose not. Then we can find a $\delta > 0$ and two sequences of integers $\{m(k)\}$, $\{n(k)\}$, $m(k) > n(k) \geq k$ with

$$r_k = d(Gf^{-1}x_{n(k)}, Gf^{-1}x_{m(k)}) \geq \delta \tag{2.6}$$

for $k \in \{1, 2, \cdots\}$. Choose $m(k)$ to be the smallest number exceeding $n(k)$ for which (2.6) holds. Then we may assume

$$d(Gf^{-1}x_{m(k)-1}, Gf^{-1}x_{n(k)}) < \delta. \tag{2.7}$$

In view of (2.4), (2.6) and (2.7), we have

$$\begin{aligned}\delta &\le r_k\\ &\le d(Gf^{-1}x_{m(k)}, Gf^{-1}x_{m(k)-1}) + d(Gf^{-1}x_{m(k)-1}, Gf^{-1}x_{n(k)})\\ &\le \phi^{m(k)-1}(d(Gf^{-1}x_1, Gf^{-1}x_0)) + \delta.\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} r_k = \delta$$

(Note $\lim_{n\to\infty}\phi^n(a) = 0$ for any $a > 0$ since if we take $a > 0$ and $a_n = \phi^n(a)$, then $a_n = \phi(a_{n-1}) \le a_{n-1}$. Thus $a_n \downarrow c$ (say). Since $c = \phi(c)$, we have $c = 0$). From (2.4) we have

$$\begin{aligned}\delta &\le r_k\\ &\le d(Gf^{-1}x_{n(k)}, Gf^{-1}x_{n(k)+1}) + d(Gf^{-1}x_{m(k)+1}, Gf^{-1}x_{m(k)})\\ &\quad + d(Gf^{-1}x_{n(k)+1}, Gf^{-1}x_{m(k)+1})\\ &\le \phi^{n(k)}(d(Gf^{-1}x_0, Gf^{-1}x_1))\\ &\quad + \phi^{m(k)}(d(Gf^{-1}x_0, Gf^{-1}x_1)) + d(Ff^{-1}x_{n(k)}, Ff^{-1}x_{m(k)}).\end{aligned}$$

Notice

$$\begin{aligned}&d(Ff^{-1}x_{n(k)}, Ff^{-1}x_{m(k)})\\ &\le \phi(\max\{d(Gf^{-1}x_{n(k)}, Gf^{-1}x_{m(k)}), d(Gf^{-1}x_{n(k)}, Gf^{-1}x_{n(k)+1}),\\ &\quad d(Gf^{-1}x_{m(k)}, Gf^{-1}x_{m(k)+1}), \frac{1}{2}[d(Gf^{-1}x_{n(k)}, Gf^{-1}x_{m(k)+1})\\ &\quad + d(Gf^{-1}x_{m(k)}, Gf^{-1}x_{n(k)+1})]\})\\ &\le \phi(\max\{r_k, \phi^{n(k)}(d(Gf^{-1}x_0, Gf^{-1}x_1)), \phi^{m(k)}(d(Gf^{-1}x_0, Gf^{-1}x_1)),\\ &\quad \frac{1}{2}[2r_k + d(Gf^{-1}x_{n(k)}, Gf^{-1}x_{n(k)+1}) + d(Gf^{-1}x_{m(k)}, Gf^{-1}x_{m(k)+1})]\})\\ &\le \phi(\max\{r_k, \phi^{n(k)}(d(Gf^{-1}x_0, Gf^{-1}x_1)), \phi^{m(k)}(d(Gf^{-1}x_0, Gf^{-1}x_1)),\\ &\quad r_k + \frac{1}{2}[\phi^{n(k)}(d(Gf^{-1}x_0, Gf^{-1}x_1)) + \phi^{m(k)}d(Gf^{-1}x_0, Gf^{-1}x_1)]\})\\ &\le \phi(r_k + \phi^{n(k)}(d(Gf^{-1}x_0, Gf^{-1}x_1)) + \phi^{m(k)}(d(Gf^{-1}x_0, Gf^{-1}x_1))).\end{aligned}$$

Therefore,

$$\begin{aligned}\delta &\le r_k\\ &\le \phi^{n(k)}(d(Gf^{-1}x_0, Gf^{-1}x_1)) + \phi^{m(k)}(d(Gf^{-1}x_0, Gf^{-1}x_1))\\ &\quad \phi(r_k + \phi^{n(k)}(d(Gf^{-1}x_0, Gf^{-1}x_1)) + \phi^{m(k)}(d(Gf^{-1}x_0, Gf^{-1}x_1))).\end{aligned}$$

Taking the limit as $k \to \infty$ yields $\delta \le \phi(\delta)$ since $\lim_{k\to\infty} r_k = \delta$ and $\lim_{n\to\infty} \phi^n(a) = 0$ for any $a > 0$. This is a contradiction. Hence $\{Gf^{-1}x_n\}$ is a

Cauchy sequence and (2.5) holds. Since (X, d) is complete, there exists $x \in \overline{B(Gf^{-1}x_0, r)}$ with $Gf^{-1}x_n \longrightarrow x$ as $n \to \infty$. Also $Ff^{-1}x_n = Gf^{-1}x_{n+1} \longrightarrow x$ as $n \to \infty$. Since $\lim_n Ff^{-1}x_n = x = \lim_n Gf^{-1}x_n$ and $Ff^{-1}x_n = Gf^{-1}x_{n+1} \in B(Gf^{-1}x_0, r)$ for $n \in \{1, 2, \cdots\}$, the continuity of f^{-1} and G and f-hybrid compatibility of F and G imply that

$$\lim_{n\to\infty} d(Ff^{-1}Gf^{-1}x_n, Gf^{-1}x) = \lim_{n\to\infty} d(Ff^{-1}Gf^{-1}x_n, Gf^{-1}Ff^{-1}x_n) = 0$$

since

$$\begin{aligned} d(Ff^{-1}Gf^{-1}x_n, Gf^{-1}x) &\leq d(Ff^{-1}Gf^{-1}x_n, Gf^{-1}Ff^{-1}x_n) \\ &\quad + d(Gf^{-1}Ff^{-1}x_n, Gf^{-1}x). \end{aligned}$$

Thus $Ff^{-1}Gf^{-1}x_n \to Gf^{-1}x$ as $n \to \infty$. Now $Ff^{-1}x = Gf^{-1}x$ since

$$\begin{aligned} &d(Ff^{-1}x, Gf^{-1}x) \\ &\leq d(Ff^{-1}x, Ff^{-1}Gf^{-1}x_n) + d(Ff^{-1}Gf^{-1}x_n, Gf^{-1}x) \\ &\leq \phi(\max\{d(Gf^{-1}x, Gf^{-1}Gf^{-1}x_n), d(Gf^{-1}x, Ff^{-1}x), \\ &\quad d(Gf^{-1}Gf^{-1}x_n, Ff^{-1}Gf^{-1}x_n), \frac{1}{2}[d(Gf^{-1}x, Ff^{-1}Gf^{-1}x_n) \\ &\quad + d(Gf^{-1}Gf^{-1}x_n, Ff^{-1}x)]\}) + d(Ff^{-1}Gf^{-1}x_n, Gf^{-1}x). \end{aligned}$$

Taking the limit as $n \to \infty$, we have (from above we know $Ff^{-1}Gf^{-1}x_n \to Gf^{-1}x$ and $Gf^{-1}x_n \to x$)

$$\begin{aligned} d(Ff^{-1}x, Gf^{-1}x) &\leq \phi(\max\{0, d(Gf^{-1}x, Ff^{-1}x), 0, \frac{1}{2}d(Gf^{-1}x, Ff^{-1}x)\}) \\ &= \phi(d(Gf^{-1}x, Ff^{-1}x)). \end{aligned}$$

We claim that $x = Gf^{-1}x$. Suppose that $d(x, Gf^{-1}x) = s$ for some $s > 0$. Since

$$\begin{aligned} &d(Ff^{-1}x_n, Ff^{-1}Gf^{-1}x_n) \\ &\leq \phi(\max\{d(Gf^{-1}x_n, Gf^{-1}Gf^{-1}x_n), \\ &\quad d(Gf^{-1}x_n, Ff^{-1}x_n), d(Gf^{-1}Gf^{-1}x_n, Ff^{-1}Gf^{-1}x_n), \\ &\quad \frac{1}{2}[d(Gf^{-1}x_n, Ff^{-1}Gf^{-1}x_n) + d(Gf^{-1}Gf^{-1}x_n, Ff^{-1}x_n)]\}), \end{aligned}$$

which on letting $n \to \infty$ gives (recall from above that $Ff^{-1}Gf^{-1}x_n \to Gf^{-1}x$ and $Ff^{-1}x_n \to x$)

$$\begin{aligned} s = d(x, Gf^{-1}x) &\leq \phi(\max\{d(x, Gf^{-1}x), 0, 0, \frac{1}{2}[d(x, Gf^{-1}x) + d(Gf^{-1}x, x)]\}) \\ &= \phi(d(x, Gf^{-1}x)) = \phi(s) < s, \end{aligned}$$

a contradiction. Hence $x = Gf^{-1}x = Ff^{-1}x$. Uniqueness of common fixed point of Gf^{-1} and Ff^{-1} follows easily from (2.1). Indeed, if $x' = Gx' = Fx'$ with $x \neq x'$, then

$$\begin{aligned} d(x,x') &= d(Ff^{-1}x, Ff^{-1}y) \leq \phi(M(x,x';f^{-1})) \\ &= \phi(\max\{d(x,x'),0,0,\frac{1}{2}[d(x,x')+d(x',x)]\}) = \phi(d(x,x')), \end{aligned}$$

which gives a contradiction. Further, let $f^{-1}x = y$ then since f is a surjective map we have a unique $y \in f^{-1}\Big(\overline{B(Gf^{-1}x_0, r)}\Big)$ with $fy = Fy = Gy$. □

If $Y = X$ and f is the identity map on X then our theorem 2.1 reduces to the following result of O'Regan et al. [18, Theorem 2.1].

Corollary 2.2. *Let* (X,d) *be a complete metric space,* $x_0 \in X$ *,* $r > 0$ *with* $F : \overline{B(Gx_0,r)}) \cup G^{-1}(B(Gx_0,r))\Big) \to X$ *and* $G : X \to X$ *be compatible maps on* $\overline{B(Gx_0,r)}$ *and* $FG^{-1}(B(Gx_0,r)) \subseteq G(X)$*. Suppose* G *is continuous and there exists a continuous, nondecreasing function* $\phi : [0,\infty) \to [0,\infty)$ *satisfying* $\phi(z) < z$ *for* $z > 0$ *such that for* $x, y \in \overline{B(Gx_0,r)} \cup G^{-1}(B(Gx_0,r))$ *we have*

$$d(Fx,Fy) \leq \phi(M(x,y)),$$

where

$$M(x,y) = \max\{d(Gx,Gy), d(Gx,Fx), d(Gy,Fy), \frac{1}{2}[d(Gx,Fy)+d(Gy,Fx)]\}.$$

Also suppose

$$d(Gx_0,Fx_0) < r - \phi(r).$$

Then there exists a unique $x \in \overline{B(Gx_0,r)}\Big)$ *with* $x = Fx = Gx$ *.*

Corollary 2.3. *Let* Y *be an arbitrary space,* (X,d) *be a complete metric space,* $f : Y \to X$ *be a bijection map,* $x_0 \in X$ *,* $r > 0$ *and* $F : f^{-1}\Big(\overline{B(x_0,r)}\Big) \to X$ *. Suppose* f^{-1} *is continuous and there exists a continuous, nondecreasing function* $\phi : [0,\infty) \to [0,\infty)$ *satisfying* $\phi(z) < z$ *for* $z > 0$ *such that for* $x, y \in \overline{B(x_0,r)}$ *we have*

$$d(Ff^{-1}x, Ff^{-1}y) \leq \phi(M(x,y;f^{-1})),$$

where

$$M(x,y;f^{-1}) = \max\{d(f^{-1}x,f^{-1}y), d(f^{-1}x,Ff^{-1}x), d(f^{-1}y,Ff^{-1}y), \frac{1}{2}[d(f^{-1}x,Ff^{-1}y)+d(f^{-1}y,Ff^{-1}x)]\}.$$

Also suppose

$$d(f^{-1}x_0, Ff^{-1}x_0) < r - \phi(r).$$

Then there exists a unique $y \in f^{-1}\left(\overline{B(x_0, r)}\right)$ *with* $fy = Fy$ *.*

If $Y = X$ and f is the identity map on X then our corollary 2.3 reduces to the following result of O'Regan et al. [18, Corollary 2.2].

Corollary 2.4. *Let* (X, d) *be a complete metric space,* $x_0 \in X$ *,* $r > 0$ *and* $F : \overline{B(x_0, r)} \to X$ *. Suppose there exists a continuous, nondecreasing function* $\phi : [0, \infty) \to [0, \infty)$ *satisfying* $\phi(z) < z$ *for* $z > 0$ *such that for* $x, y \in \overline{B(x_0, r)}$ *we have*

$$d(Fx, Fy) \leq \phi(M(x, y)),$$

where

$$M(x, y) = \max\{d(x, y), d(x, Fx), d(y, Fy), \frac{1}{2}[d(x, Fy) + d(y, Fx)]\}.$$

Also suppose

$$d(x_0, Fx_0) < r - \phi(r).$$

Then there exists a unique $x \in \overline{B(x_0, r)}$ *with* $x = Fx$ *.*

We now state the global result corresponding to Theorem 2.1.

Theorem 2.5. *Let* Y *be an arbitrary space,* (X, d) *be a complete metric space,* $f : Y \to X$ *be a bijection map,* $F : Y \to X$ *and* $G : Y \to X$ *be* f*-hybrid compatible maps and* $Ff^{-1}(X) \subseteq Gf^{-1}(X)$*. Suppose* f^{-1} *and* G *are continuous and there exists a continuous, nondecreasing function* $\phi : [0, \infty) \to [0, \infty)$ *satisfying* $\phi(z) < z$ *for* $z > 0$ *such that for* $x, y \in X$ *we have*

$$d(Ff^{-1}x, Ff^{-1}y) \leq \phi(M(x, y; f^{-1})),$$

where

$$M(x, y; f^{-1}) = \max\{d(Gf^{-1}x, Gf^{-1}y), d(Gf^{-1}x, Ff^{-1}x), d(Gf^{-1}y, Ff^{-1}y), \frac{1}{2}[d(Gf^{-1}x, Ff^{-1}y) + d(Gf^{-1}y, Ff^{-1}x)]\}.$$

Then there exists a unique $y \in f^{-1}\left(\overline{B(Gf^{-1}x_0, r)}\right)$ *with* $fy = Fy = Gy$ *.*

If $Y = X$ and f is the identity map on X then our Theorem 2.5 reduces to the following result of O'Regan et al. [18, Theorem 2.3].

Corollary 2.6. *Let* (X, d) *be a complete metric space with* $F : X \to X$ *and* $G : X \to X$ *compatible maps and* $F(X) \subseteq G(X)$*. Suppose* G *is continuous* *and there exists a continuous, nondecreasing function $\phi : [0,\infty) \to [0,\infty)$ satisfying $\phi(z) < z$ for $z > 0$ such that for $x, y \in X$ we have*

$$d(Fx, Fy) \le \phi(M(x, y)),$$

where

$$M(x,y) = \max\{d(Gx, Gy), d(Gx, Fx), d(Gy, Fy), \frac{1}{2}[d(Gx, Fy) + d(Gy, Fx)]\}.$$

Then there exists a unique $x \in X$ with $x = Fx = Gx$.

Next we present a homotopy result for f-hybrid compatible maps.

Theorem 2.7. *Let Y be an arbitrary space, (X, d) be a complete metric space, $f : Y \to X$ be a bijection map, and U an open subset of X with $H : (f^{-1}(\overline{U}) \cup G^{-1}(U)) \times [0,1] \to X$ and $G : Y \to X$ and for each $\lambda \in [0,1]$, H_λ and G are f-hybrid compatible on $\overline{U}$, and $H_\lambda(G^{-1}(U)) \subseteq Gf^{-1}(X)$. Assume the following conditions hold:*

(i) *for $\lambda \in [0,1]$, $f(y) = G(y) = H(y,\lambda)$ cannot occur for $y \in f^{-1}(\partial(U))$ (where $\partial(U)$ denotes the boundary of U in X);*

(ii) *f^{-1} and G are continuous;*

(iii) *there exists a continuous, nondecreasing function $\phi : [0,\infty) \to [0,\infty)$ satisfying $\phi(z) < z$ for $z > 0$ such that for all $\lambda \in [0,1]$ and $x, y \in \overline{U} \cup fG^{-1}(U)$ we have*

$$d(Hf^{-1}(x,\lambda), Hf^{-1}(y,\lambda)) \le \phi(M(x, y, \lambda; f^{-1}),$$

where

$$\begin{aligned} M(x,y,\lambda;f^{-1}) = \max\{&d(Gf^{-1}(x), Gf^{-1}(y)), d(Gf^{-1}(x), H(f^{-1}x,\lambda)),\\ &d(Gf^{-1}(y), H(f^{-1}y,\lambda)), \frac{1}{2}[d(Gf^{-1}(x), H(f^{-1}y,\lambda))\\ &+ d(Gf^{-1}(y), H(f^{-1}x,\lambda))]\};\end{aligned}$$

(iv) *$H(f^{-1}x,\lambda)$ is continuous in λ uniformly for $x \in \overline{U}$;*

(v) *$\phi(a + b) \le \phi(a) + \phi(b)$ for $a \ge 0$, $b \ge 0$;*

(vi) *$H(f^{-1}(U) \times [0,1])$ is bounded.*

In addition assume H_0f^{-1} and Gf^{-1} have a coincident point (i.e. there exists $x \in \overline{U} \cup fG^{-1}(U)$ with $H_0f^{-1}(x) = Gf^{-1}(x)$)). Then for each $\lambda \in [0,1]$, we have that f, H_λ and G have a unique coincidence point $y_\lambda \in G^{-1}(U)$ (here $H_\lambda f^{-1}(\cdot) = Hf^{-1}(\cdot,\lambda)$). Moreover, for each $\lambda \in [0,1]$, $H_\lambda f^{-1}$ and Gf^{-1} have a unique common fixed point $x_\lambda \in fG^{-1}(U)$ (i.e. $x_\lambda = Hf^{-1}(x_\lambda,\lambda) = Gf^{-1}(x_\lambda)$).

Remark 2.2. In Theorem 2.7, we assume there exists $x \in \overline{U} \cup fG^{-1}(U)$ with

$$H_0f^{-1}(x) = Gf^{-1}(x).$$

In fact H_0f^{-1} and Gf^{-1} have a common fixed point $Gf^{-1}(x)$. To see this, notice

$$Gf^{-1}(Gf^{-1}(x)) = Gf^{-1}(H_0f^{-1}(x)) = H_0f^{-1}(Gf^{-1}(x)) = Hf^{-1}(Gf^{-1}(x), 0)$$

(since H_0f^{-1} and Gf^{-1} commute at the coincidence point x). Now

$$\begin{aligned}
&d(Gf^{-1}(x), Gf^{-1}(Gf^{-1}(x)))\\
&= d(Hf^{-1}(x,0), Hf^{-1}(Gf^{-1}(x),0))\\
&\leq \phi(\max\{d(Gf^{-1}(x), Gf^{-1}(Gf^{-1}(x))), d(Gf^{-1}(x), Hf^{-1}(x,0)),\\
&\quad d(Gf^{-1}(Gf^{-1}(x)), Hf^{-1}(Gf^{-1}(x),0)), \frac{1}{2}[d(Gf^{-1}(x), Hf^{-1}(Gf^{-1}(x),0))\\
&\quad + d(Gf^{-1}(Gf^{-1}(x)), Hf^{-1}(x,0))]\})\\
&\leq \phi(\max\{d(Gf^{-1}(x), Gf^{-1}(Gf^{-1}(x))), 0, 0,\\
&\quad \frac{1}{2}[d(Gf^{-1}(x), Gf^{-1}(Gf^{-1}(x))) + d(Gf^{-1}(Gf^{-1}(x)), Gf^{-1}(x))]\})\\
&= \phi(d(Gf^{-1}(x), Gf^{-1}(Gf^{-1}(x))),
\end{aligned}$$

which gives $Gf^{-1}(x) = Gf^{-1}(Gf^{-1}(x)))$. Hence, we have

$$Gf^{-1}(x) = Gf^{-1}(Gf^{-1}(x)) = H_0f^{-1}(Gf^{-1}(x)).$$

We also use the fact in the proof of Theorem 2.7 that a topological space X is connected iff the only open and subsets of X are X and $\emptyset$.

Proof. First, we shall prove that for each $\lambda \in [0,1]$, $H_\lambda f^{-1}$ and Gf^{-1} have a common fixed point x_λ. To see this, let

$$A = \{\lambda \in [0,1] : Hf^{-1}(x,\lambda) = Gf^{-1}(x) \text{ for some } x \in \overline{U} \cup fG^{-1}(U)\}.$$

Since H_0f^{-1} and Gf^{-1} have a coincidence point, $0 \in A$ and so A is nonempty. We now show that A is both open and closed in $[0,1]$ and so by the connectedness of $[0,1]$, we have $A = [0,1]$.

First we show A is open in $[0,1]$. Let $\lambda_0 \in A$. Then there exists $x_0 \in fG^{-1}(U)$ with

$$Hf^{-1}(x_0,\lambda_0) = Gf^{-1}(x_0).$$

Then $Gf^{-1}x_0 \in U$ and since U is open, there exists a ball $B(Gf^{-1}(x_0),\delta)$, $\delta > 0$, with

$$\overline{B(Gf^{-1}(x_0),\delta)} \subseteq U.$$

Now, by (iv), there exists $\eta(\delta) > 0$ with

$$\begin{aligned}
d(Gf^{-1}(x_0), Hf^{-1}(x_0,\lambda) &= d(Hf^{-1}(x_0,\lambda_0), Hf^{-1}(x_0,\lambda))\\
&< \delta - \phi(\delta)
\end{aligned}$$

for $\lambda \in [0,1]$ and $|\lambda-\lambda_0| < \eta$. Now (iii) with Theorem 2.1 (here $r = \delta$, $F = H_\lambda$ and $G = G$) guarantees that there exists $x_\lambda \in \overline{B(Gf^{-1}(x_0), \delta)} \subseteq U$ with

$$x_\lambda = H_\lambda f^{-1}(x_\lambda) = Gf^{-1}(x_\lambda)$$

for $\lambda \in [0,1]$ and $|\lambda - \lambda_0| < \eta$ (note if $y_\lambda = f^{-1}(x_\lambda) \in f^{-1}(U)$ then $f(y_\lambda) = H_\lambda(y_\lambda) = G(y_\lambda) \in U$ and $f(y_\lambda) = G(y_\lambda)$ implies $y_\lambda = f^{-1}(x_\lambda) \in G^{-1}(U)$, i.e., $x_\lambda \in fG^{-1}(U)$). As a result A is open in $[0,1]$.

Next we show A is closed in $[0,1]$. Let $\{\lambda_k\} \subseteq A$ be such that $\lambda_k \to \lambda \in [0,1]$ as $k \to \infty$. Then for each k, there exists $x_k \in fG^{-1}(U)$ with

$$Hf^{-1}(x_k, \lambda_k) = Gf^{-1}(x_k).$$

We claim $\{Gf^{-1}(x_k)\} \subset U$ is a Cauchy sequence. Suppose not. Then we can find a $\delta > 0$ and two subsequences of integers $\{m(k)\}$, $\{n(k)\}$ such that $m(k) > n(k) \geq k$ with

$$r_k = d(Gf^{-1}(x_{n(k)}), Gf^{-1}(x_{m(k)})) \geq \delta \tag{2.8}$$

for $k \in \{1, 2, \cdots\}$. Notice

$$\begin{aligned}
\delta \leq r_k \quad \leq \quad & d(Gf^{-1}(x_{n(k)}), Hf^{-1}(x_{n(k)}, \lambda)) \\
& + d(Hf^{-1}(x_{n(k)}, \lambda), Hf^{-1}(x_{m(k)}), \lambda)) \\
& + d(Hf^{-1}(x_{m(k)}, \lambda), Gf^{-1}(x_{m(k)})) \\
\leq \quad & d(Hf^{-1}(x_{n(k)}, \lambda_{n(k)}), Hf^{-1}(x_{n(k)}, \lambda)) \\
& + d(Hf^{-1}(x_{n(k)}, \lambda), Hf^{-1}(x_{m(k)}, \lambda)) \\
& + d(Hf^{-1}(x_{m(k)}, \lambda), Hf^{-1}(x_{m(k)}, \lambda_{m(k)})).
\end{aligned}$$

Also

$$\begin{aligned}
& d(Hf^{-1}(x_{n(k)}, \lambda), Hf^{-1}(x_{m(k)}, \lambda)) \\
& \leq \phi(\max\{d(Gf^{-1}(x_{n(k)}), Gf^{-1}(x_{m(k)})), \\
& \quad d(Gf^{-1}(x_{n(k)}), Hf^{-1}(x_{n(k)}, \lambda)), d(Gf^{-1}(x_{m(k)}), Hf^{-1}(x_{m(k)}, \lambda)), \\
& \quad \frac{1}{2}[d(Gf^{-1}(x_{n(k)}), Hf^{-1}(x_{m(k)}, \lambda)) + d(Gf^{-1}(x_{m(k)}), Hf^{-1}(x_{n(k)}, \lambda))]\}) \\
& \leq \phi(\max\{d(Gf^{-1}(x_{n(k)}), Gf^{-1}(x_{m(k)})), d(Gf^{-1}(x_{n(k)}), Hf^{-1}(x_{n(k)}, \lambda)), \\
& \quad d(Gf^{-1}(x_{m(k)}), Hf^{-1}(x_{m(k)}, \lambda)), d(Gf^{-1}(x_{n(k)}), Gf^{-1}(x_{m(k)})) \\
& \quad + \frac{1}{2}[d(Gf^{-1}(x_{m(k)}), Hf^{-1}(x_{m(k)}, \lambda)) + d(Gf^{-1}(x_{n(k)}), Hf^{-1}(x_{n(k)}, \lambda))]\}) \\
& \leq \phi(\max\{r_k, d(Gf^{-1}(x_{n(k)}), Hf^{-1}(x_{n(k)}, \lambda)), d(Gf^{-1}(x_{m(k)}, \lambda), \\
& \quad Hf^{-1}(x_{m(k)}, \lambda)), r_k + \frac{1}{2}[d(Gf^{-1}(x_{m(k)}), Hf^{-1}(x_{m(k)}, \lambda)) + d(Gf^{-1}(x_{n(k)}), \\
& \quad Hf^{-1}(x_{n(k)}, \lambda))]\}).
\end{aligned}$$

From (iv) we may find k large enough (i.e., $k \geq k_0$) such that

$$d(Hf^{-1}(x_{n(k)}, \lambda_{n(k)}), Hf^{-1}(x_{n(k)}, \lambda)) \leq \delta$$

and

$$d(Hf^{-1}(x_{m(k)}, \lambda_{m(k)}), Hf^{-1}(x_{m(k)}, \lambda)) \leq \delta.$$

Now for $k \geq k_0$, we have

$$\begin{aligned} \delta \leq r_k \quad \leq \quad & d(Hf^{-1}(x_{n(k)}, \lambda_{n(k)}), Hf^{-1}(x_{n(k)}, \lambda)) \\ & + d(Hf^{-1}(x_{m(k)}, \lambda), Hf^{-1}(x_{m(k)}, \lambda_{m(k)})) \\ & + \phi(r_k + \frac{1}{2}[d(Gf^{-1}(x_{m(k)}), Hf^{-1}(x_{m(k)}, \lambda)) \\ & + d(Gf^{-1}(x_{n(k)}), Hf^{-1}(x_{n(k)}, \lambda))]). \end{aligned}$$

Therefore, for $k \geq k_0$, we have using (v) that

$$\begin{aligned} 0 < \Phi(r_k) = r_k - \phi(r_k) \quad \leq \quad & d(Hf^{-1}(x_{n(k)}, \lambda_{n(k)}), Hf^{-1}(x_{n(k)}, \lambda)) \\ & + d(Hf^{-1}(x_{m(k)}, \lambda), Hf^{-1}(x_{m(k)}, \lambda_{m(k)})) \\ & + \phi(\frac{1}{2}[d(Gf^{-1}(x_{m(k)}), Hf^{-1}(x_{m(k)}, \lambda)) \\ & + d(Gf^{-1}(x_{n(k)}), Hf^{-1}(x_{n(k)}, \lambda))]). \end{aligned}$$

In view of (iv), we obtain

$$\begin{aligned} & \lim_{k\to\infty} d(Gf^{-1}x_{n(k)}), Hf^{-1}(x_{n(k)}, \lambda)) \\ = & \lim_{k\to\infty} d(Hf^{-1}(x_{n(k)}, \lambda), Hf^{-1}(x_{n(k)}, \lambda_{n(k)})) \\ = & \ 0. \end{aligned}$$

Similarly, we have

$$\begin{aligned} & \lim_{k\to\infty} d(Gf^{-1}(x_{m(k)}), Hf^{-1}(x_{m(k)}, \lambda)) \\ = & \lim_{k\to\infty} d(Hf^{-1}(x_{m(k)}, \lambda), Hf^{-1}(x_{m(k)}, \lambda_{m(k)})) \\ = & \ 0. \end{aligned}$$

Letting $k \to \infty$, we have

$$\lim_{n\to\infty} \Phi(r_k) = 0. \tag{2.9}$$

Now (vi) implies that there exists $M > 0$ with $r_k \leq M$ for $k \in \{1, 2, \cdots\}$. Consequently, for $k \in \{1, 2, \cdots\}$ we obtain

$$\Phi(r_k) \geq \min_{x\in[\delta,M]} \Phi(x) = \Phi(r_0) \text{ for some } r_0 \in [\delta, M],$$

which contradicts (2.9). Hence (2.8) holds. Since (X, d) is complete there exists $x \in \overline{U}$ with $d(Gf^{-1}(x_k), x) \to 0$ as $k \to \infty$. Now

$$\begin{aligned} d(x, H_\lambda f^{-1}(x_k)) &\leq d(x, Gf^{-1}(x_k)) + d(Gf^{-1}(x_k), H_\lambda f^{-1}(x_k)) \\ &= d(x, Gf^{-1}(x_k)) + d(Hf^{-1}(x_k, \lambda_k), Hf^{-1}(x_k, \lambda)) \end{aligned}$$

which on letting $k \to \infty$ yields $\lim_{k\to\infty} H_\lambda f^{-1}(x_k) = x$. Using the continuity of G, f^{-1} and f-hybrid compatibility of $H_\lambda f^{-1}$ and Gf^{-1}, we obtain

$$\begin{aligned} &\lim_{k\to\infty} d(H_\lambda f^{-1}(Gf^{-1}(x_k)), Gf^{-1}(x)) \\ &= \lim_{k\to\infty} d(H_\lambda f^{-1}(G(x_k)), Gf^{-1}(H_\lambda f^{-1}(x_k))) \\ &= 0 \end{aligned}$$

since

$$\begin{aligned} d(H_\lambda f^{-1}(Gf^{-1}(x_k)), Gf^{-1}(x)) \leq\ & d(H_\lambda f^{-1}(Gf^{-1}(x_k)), Gf^{-1}(H_\lambda f^{-1}(x_k))) \\ &+ d(Gf^{-1}(H_\lambda f^{-1}(x_k)), Gf^{-1}(x)). \end{aligned}$$

Thus $H_\lambda f^{-1}(Gf^{-1}(x_k)) \to Gf^{-1}x$ as $k \to \infty$. Next we show

$$Hf^{-1}(x, \lambda) = Gf^{-1}(x).$$

Notice

$$\begin{aligned} d(Hf^{-1}(x,\lambda), Gf^{-1}(x)) &\leq d(Hf^{-1}(x,\lambda), Hf^{-1}(Gf^{-1}(x_k),\lambda)) \\ &\quad +d(Hf^{-1}(Gf^{-1}(x_k),\lambda), Gf^{-1}(x)) \\ &\leq d(Hf^{-1}(Gf^{-1}(x_k),\lambda), Gf^{-1}(x)) \\ &\quad +\phi(\max\{d(Gf^{-1}(x), Gf^{-1}(Gf^{-1}(x_k))), \\ &\quad d(Gf^{-1}(x), Hf^{-1}(x,\lambda)), \\ &\quad d(Gf^{-1}(Gf^{-1}(x_k)), Hf^{-1}(Gf^{-1}(x_k),\lambda)), \\ &\quad \frac{1}{2}[d(Gf^{-1}(x), Hf^{-1}(Gf^{-1}(x_k),\lambda)) \\ &\quad +d(Gf^{-1}(Gf^{-1}(x_k)), Hf^{-1}(x,\lambda))]\}). \end{aligned}$$

Taking the limit as $k \to \infty$ yields (here we use $H_\lambda f^{-1}(Gf^{-1}(x_k)) \to Gf^{-1}x$ and $Gf^{-1}x_k \to x$)

$$\begin{aligned} d(Hf^{-1}(x,\lambda), Gf^{-1}(x)) &\leq \phi(\max\{0, d(Gf^{-1}(x), Hf^{-1}(x,\lambda)), 0, \\ &\quad \frac{1}{2}[d(Gf^{-1}(x), Hf^{-1}(x,\lambda))]\}) \\ &= \phi(d(Hf^{-1}(x,\lambda), Gf^{-1}(x))). \end{aligned}$$

This implies that $Hf^{-1}(x,\lambda) = Gf^{-1}(x)$ and so

$$Hf^{-1}(x, \lambda) = Gf^{-1}(x).$$

We claim that $x = Gf^{-1}x$. Suppose that $d(x, Gf^{-1}x) = s$ for some $s > 0$. Since

$$\begin{aligned} d(Hf^{-1}(x_k,\lambda), Hf^{-1}(Gf^{-1}(x_k),\lambda)) \le \phi(\max\{&d(Gf^{-1}(x_k), Gf^{-1}(Gf^{-1}(x_k))),\\ &d(Gf^{-1}(x_k), H_\lambda f^{-1}(x_k)),\\ &d(Gf^{-1}(Gf^{-1}(x_k)), H_\lambda f^{-1}(Gf^{-1}(x_k))),\\ &\frac{1}{2}[d(Gf^{-1}(x_k), H_\lambda f^{-1}(Gf^{-1}(x_k)))\\ &+ d(Gf^{-1}(Gf^{-1}(x_k)), H_\lambda f^{-1}(x_k))]\}),\end{aligned}$$

which on letting $k \to \infty$ gives (recall from above that $H_\lambda f^{-1}(Gf^{-1}(x_k)) \to Gf^{-1}(x)$ and $H_\lambda f^{-1}(x_k) \to x$)

$$\begin{aligned} s = d(x, Gf^{-1}x) &\le \phi(\max\{d(x, Gf^{-1}x), 0, 0,\\ &\qquad \frac{1}{2}[d(x, Gf^{-1}x) + d(Gf^{-1}x, x)]\})\\ &= \phi(d(x, Gf^{-1}x)) = \phi(s) < s,\end{aligned}$$

a contradiction. Hence

$$x = Gf^{-1}(x) = Hf^{-1}(x,\lambda).$$

From (i), $x \in U$ and so $x = Gf^{-1}(x)$ implies $x \in fG^{-1}(U)$. Consequently, $\lambda \in A$. Hence A is closed in $[0,1]$. Thus we can deduce that $A = [0,1]$ and so for each $\lambda \in [0,1]$, $H_\lambda f^{-1}$ and Gf^{-1} have a common fixed point $x_\lambda \in fG^{-1}(U)$ (i.e. $x_\lambda = Hf^{-1}(x_\lambda,\lambda) = Gf^{-1}(x_\lambda)$). Now fix $\lambda \in [0,1]$. It remains to show the uniqueness. If $y_\lambda = Gf^{-1}(y_\lambda) = H_\lambda f^{-1}(y_\lambda)$ with $x_\lambda \ne x'_\lambda$, then

$$\begin{aligned} d(x_\lambda, x'_\lambda) &= d(Hf^{-1}(x_\lambda,\lambda), Hf^{-1}(x'_\lambda,\lambda))\\ &\le \phi(\max\{d(x_\lambda, y_\lambda), 0, 0, \frac{1}{2}[d(x_\lambda, x'_\lambda) + d(x'_\lambda, x_\lambda)]\})\\ &= \phi(d(x_\lambda, x'_\lambda)),\end{aligned}$$

which gives a contradiction. Further, let $f^{-1}x_\lambda = y_\lambda$ then since f is a surjective map we have a unique $y_\lambda \in G^{-1}(U)$ with $fy_\lambda = H_\lambda y_\lambda = Gy_\lambda$. □

If $Y = X$ and f is the identity map on X then our Theorem 2.7 reduces to the following result of O'Regan et al. [18, Theorem 2.4].

Corollary 2.8. *Let (X,d) be a complete metric space and U an open subset of X with $H : (\overline{U} \cup G^{-1}(U)) \times [0,1] \to X$ and $G : X \to X$ and for each $\lambda \in [0,1]$, H_λ and G are compatible on $\overline{U}$, and $H_\lambda(G^{-1}(U)) \subseteq G(X)$. Assume the following conditions hold:*

(i) *for $\lambda \in [0,1]$, $x = G(x) = H(x,\lambda)$ cannot occur for $x \in \partial U$ (the boundary of U in X);*

(ii) *G is continuous;*
(iii) *there exists a continuous, nondecreasing function $\phi : [0,\infty) \to [0,\infty)$ satisfying $\phi(z) < z$ for $z > 0$ such that for all $\lambda \in [0,1]$ and $x, y \in \overline{U} \cup G^{-1}(U)$ we have*

$$d(H(x,\lambda), H(y,\lambda)) \leq \phi(M(x,y,\lambda)),$$

where

$$M(x,y,\lambda) = \max\{d(G(x),G(y)), d(G(x),H(x,\lambda)), d(G(y),H(y,\lambda)), \frac{1}{2}[d(G(x),H(y,\lambda)) + d(G(y),H(x,\lambda))]\};$$

(iv) *$H(x,\lambda)$ is continuous in λ uniformly for $x \in \overline{U}$;*
(v) *$\phi(a+b) \leq \phi(a) + \phi(b)$ for $a \geq 0$, $b \geq 0$;*
(vi) *$H(U \times [0,1])$ is bounded.*

In addition assume H_0 and G have a coincidence point (i.e. there exists $x \in G^{-1}(U)$ with $H_0(x) = G(x)$). Then for each $\lambda \in [0,1]$, we have that H_λ and G have a coincidence point $x_\lambda \in G^{-1}(U)$ (here $H_\lambda(\cdot) = H(\cdot,\lambda)$). Moreover, for each $\lambda \in [0,1]$, H_λ and G have a unique common fixed point $G(x_\lambda)$.

Corollary 2.9. *Let (X,d) be a complete metric space with U an open subset of X with $H : \overline{U} \times [0,1] \to X$. Assume the following conditions hold:*

(i) *$x \neq H(x,\lambda)$ for $x \in \partial U$ (the boundary of U in X) and $\lambda \in [0,1]$;*
(ii) *There exists a continuous, nondecreasing function $\phi : [0,\infty) \to [0,\infty)$ satisfying $\phi(z) < z$ for $z > 0$ such that for all $\lambda \in [0,1]$ and $x, y \in \overline{U}$ we have*

$$d(H(x,\lambda), H(y,\lambda)) \leq \phi(M(x,y,\lambda)),$$

where

$$M(x,y,\lambda) = \max\{d(x,y), d(x,H(x,\lambda)), d(y,H(y,\lambda)), \frac{1}{2}[d(x,H(y,\lambda)) + d(y,H(x,\lambda))]\};$$

(iii) *$H(x,\lambda)$ is continuous in λ uniformly for $x \in \overline{U}$;*
(iv) *$\phi(a+b) \leq \phi(a) + \phi(b)$ for $a \geq 0$, $b \geq 0$;*
(v) *$H(U \times [0,1])$ is bounded.*

In addition assume H_0 has a fixed point. Then for each $\lambda \in [0,1]$, we have that H_λ has a fixed point $x_\lambda \in U$ (here $H_\lambda(\cdot) = H(\cdot,\lambda)$).

3. Fixed point theory for single valued maps in gauge spaces

In this section, we present some local and global common fixed point results for f-hybrid compatible maps. Let Y be an arbitrary space, $E = (E, \{d_\alpha\}_{\alpha\in\Lambda})$ be a gauge space endowed with a complete gauge structure $\{d_\alpha : \alpha \in \Lambda\}$; here Λ is a directed set (see [11, pp. 198, 308]). Let $f : Y \to E$ be a bijection map,$G : Y \to E$. For $r = \{r_\alpha\}_{\alpha\in\Lambda} \in (0,\infty)^\Lambda$ and $x_0 \in E$, we define

$$B(Gf^{-1}x_0, r) = \{y \in E : d_\alpha(Gf^{-1}x_0, y) < r_\alpha \text{ for all } \alpha \in \Lambda\}$$

and

$$\overline{B(Gf^{-1}x_0, r)} = \{y \in E : d_\alpha(Gf^{-1}x_0, y) \leq r_\alpha \text{ for all } \alpha \in \Lambda\}.$$

Let $F : f^{-1}(\overline{B(Gf^{-1}x_0, r)}) \cup G^{-1}(B(Gf^{-1}x_0, r))) \to E$ with

$$FG^{-1}(B(Gf^{-1}x_0, r)) \subseteq Gf^{-1}(E).$$

Then F and G are called f-hybrid compatible on $\overline{B(Gf^{-1}\ x_0, r)}$ if for each $\alpha \in \Lambda$,

$$\lim_{n\to\infty} d_\alpha(Ff^{-1}Gf^{-1}x_n, Gf^{-1}Ff^{-1}x_n) = 0$$

whenever $\{x_n\}$ is a sequence in $fG^{-1}(B(Gf^{-1}x_0, r))$ such that for each $\alpha \in \Lambda$,

$$\lim_{n\to\infty} d_\alpha(Ff^{-1}x_n, t) = \lim_{n\to\infty} d_\alpha(Gf^{-1}x_n, t) = 0$$

for some $t \in \overline{B(Gf^{-1}x_0,\ r)}$.

Remark 3.1. If F and G are f-hybrid compatible and $Ff^{-1}x = Gf^{-1}x$ for some $x \in fG^{-1}(B(Gf^{-1}x_0, r))$, then

$$Ff^{-1}Gf^{-1}x = Gf^{-1}Ff^{-1}x$$

(i.e. Ff^{-1} and Gf^{-1} commute at coincidence point). This is clear if we let $x_n = x$ for each n

A subset Ω of E is bounded if for each $\alpha \in \Lambda$, there exists $M_\alpha > 0$ with $d_\alpha(x, y) \leq M_\alpha$ for all $x, y \in \Omega$.

Theorem 3.1. *Let Y be an arbitrary space, E be a complete gauge space, $f : Y \to E$ be a bijection map, $x_0 \in E$, $r = \{r_\alpha\}_{\alpha\in\Lambda} \in (0,\infty)^\Lambda$ with $F : f^{-1}((\overline{B(Gf^{-1}x_0, r)}) \cup G^{-1}(B(Gf^{-1}x_0, r))) \to E$ and $G : Y \to E$, f-hybrid compatible on $\overline{B(Gf^{-1}x_0, r)}$ and*

$$FG^{-1}(B(Gf^{-1}x_0, r)) \subseteq Gf^{-1}(E).$$

Suppose f^{-1} and G are continuous and for each $\alpha \in \lambda$, there exists a continuous, nondecreasing function $\phi_\alpha : [0,\infty) \to [0,\infty)$ satisfying $\phi_\alpha(z) < z$ for

$z > 0$ *such that for* $x, y \in \overline{B(Gf^{-1}x_0, r)} \cup fG^{-1}(B(Gf^{-1}x_0, r))$ *we have*

$$d_\alpha(Ff^{-1}x, Ff^{-1}y) \leq \phi_\alpha(M_\alpha(x, y; f^{-1})), \tag{3.1}$$

where

$$\begin{aligned} &M_\alpha(x, y; f^{-1}) \\ =& \max\{d_\alpha(Gf^{-1}x, Gf^{-1}y), d_\alpha(Gf^{-1}x, Ff^{-1}x), d_\alpha(Gf^{-1}y, Ff^{-1}y), \\ &\frac{1}{2}[d_\alpha(Gf^{-1}x, Ff^{-1}y) + d_\alpha(Gf^{-1}y, Ff^{-1}x)]\}. \end{aligned}$$

Also suppose

$$\text{for each } \alpha \in \Lambda, \text{ we have } d_\alpha(Gf^{-1}x_0, Ff^{-1}x_0) < r_\alpha - \phi_\alpha(r_\alpha). \tag{3.2}$$

Then there exists a unique $x \in \overline{B(Gf^{-1}x_0, r)}$ *with* $x = Ff^{-1}x = Gf^{-1}x$. *Moreover, there exists a unique* $y \in f^{-1}\Big(\overline{B(Gf^{-1}x_0, r)}\Big)$ *with* $fy = Fy = Gy$.

Proof. Let $Gf^{-1}x_1 = Ff^{-1}x_0$ for some $x_1 \in X$ (This is possible since $Gf^{-1}x_0 \in B(Gf^{-1}x_0, r)$ and $FG^{-1}(B(Gf^{-1}x_0, r)) \subseteq Gf^{-1}(E)$). Then, by (3.2),

$$d_\alpha(Gf^{-1}x_1, Gf^{-1}x_0) < r_\alpha$$

for each $\alpha \in \Lambda$ and so

$$Gf^{-1}x_1 \in B(Gf^{-1}x_0, r).$$

Now let $Gf^{-1}x_2 = Ff^{-1}x_1$ (this is possible since $Gf^{-1}x_1 \in B(Gf^{-1}x_0, r)$ and $FG^{-1}(B(Gf^{-1}x_0, r)) \subseteq Gf^{-1}(E)$). Fix $\alpha \in \Lambda$. For $n \in \{2, 3, \cdots\}$, we let $Gf^{-1}x_{n+1} = Ff^{-1}x_n$. This is possible if we have

$$Gf^{-1}x_n \in B(Gf^{-1}x_0, r)$$

since $FG^{-1}(B(Gf^{-1}x_0, r)) \subseteq Gf^{-1}(E)$. Essentially the same reasoning as in Theorem 2.1 guarantees that

$$Gf^{-1}x_n \in B_\alpha(Gf^{-1}x_0, r_\alpha) = \{y : d_\alpha(Gx_0, y) < r_\alpha\}$$

and $\{Gf^{-1}x_n\}$ is a Cauchy sequence with respect to d_α. Since we can do this for any $\alpha \in \Lambda$, we have

$$Gf^{-1}x_n \in B(Gf^{-1}x_0, r)$$

and the sequence $\{Gf^{-1}x_n\}$ is Cauchy. Hence, there exists $x \in \overline{B(Gf^{-1}x_0, r)}$ such that $Gf^{-1}x_n \to x$. Also, $Ff^{-1}x_n = Gf^{-1}x_{n+1} \longrightarrow x$ as $n \to \infty$. Since $\lim_n Ff^{-1}x_n = x = \lim_n Gf^{-1}x_n$ and $Ff^{-1}x_n = Gf^{-1}x_{n+1} \in B(Gf^{-1}x_0, r)$ for $n \in \{1, 2, \cdots\}$, the continuity of f^{-1} and G and f-hybrid compatibility of F and G imply that

$$\lim_{n\to\infty} d_\alpha(Ff^{-1}Gf^{-1}x_n, Gf^{-1}x) = \lim_{n\to\infty} d_\alpha(Ff^{-1}Gf^{-1}x_n, Gf^{-1}Ff^{-1}x_n) = 0$$

for each $\alpha \in \Lambda$ since

$$d_\alpha(Ff^{-1}Gf^{-1}x_n, Gf^{-1}x) \leq d_\alpha(Ff^{-1}Gf^{-1}x_n, Gf^{-1}Ff^{-1}x_n) + d_\alpha(Gf^{-1}Ff^{-1}x_n, Gx).$$

Thus $Ff^{-1}Gf^{-1}x_n \to Gf^{-1}x$ as $n \to \infty$. Now fix $\alpha \in \Lambda$. Then

$$\begin{aligned}
&d_\alpha(Ff^{-1}x, Gf^{-1}x) \\
&\leq d_\alpha(Ff^{-1}x, Ff^{-1}Gf^{-1}x_n) + d_\alpha(Ff^{-1}Gf^{-1}x_n, Gf^{-1}x) \\
&\leq \phi_\alpha(\max\{d_\alpha(Gf^{-1}x, Gf^{-1}Gf^{-1}x_n), \\
&\quad d_\alpha(Gf^{-1}x, Ff^{-1}x), d_\alpha(Gf^{-1}Gf^{-1}x_n, Ff^{-1}Gf^{-1}x_n), \\
&\quad \frac{1}{2}[d_\alpha(Gf^{-1}x, Ff^{-1}Gf^{-1}x_n) + d_\alpha(Gf^{-1}Gf^{-1}x_n, Ff^{-1}x)]\}) \\
&\quad + d_\alpha(Ff^{-1}Gf^{-1}x_n, Gf^{-1}x).
\end{aligned}$$

Taking the limit as $n \to \infty$, we have (from above we know $Ff^{-1}Gf^{-1}x_n \to Gf^{-1}x$ and $Ff^{-1}x_n \to x$)

$$\begin{aligned}
d_\alpha(Ff^{-1}x, Gf^{-1}x) &\leq \phi_\alpha(\max\{0, d_\alpha(Gf^{-1}x, Ff^{-1}x), 0, \\
&\qquad \frac{1}{2}d_\alpha(Gf^{-1}x, Ff^{-1}x)\}) \\
&= \phi_\alpha(d_\alpha(Gf^{-1}x, Ff^{-1}x)).
\end{aligned}$$

This implies that

$$d_\alpha(Ff^{-1}x, Gf^{-1}x) = 0$$

for each $\alpha \in \Lambda$. As a result, $Fx = Gx$. Also (as in Theorem 2.1) one can show that $d_\alpha(x, Gf^{-1}x) = 0$ for each $\alpha \in \Lambda$ and so $x = Gf^{-1}x = Ff^{-1}x$. The uniqueness is easy to establish. Further, let $f^{-1}x = y$ then since f is a surjective map we have a unique $y \in G^{-1}(U)$ with $fy = Fy = Gy$. □

If $Y = E$ and f is the identity map on E then our Theorem 3.1 reduces to the following result of O'Regan et al. [18, Theorem 3.1].

Corollary 3.2. *Let E be a complete gauge space, $x_0 \in E$, $r = \{r_\alpha\}_{\alpha\in\Lambda} \in (0,\infty)^\Lambda$ with $F : (\overline{B(Gx_0, r)} \cup G^{-1}(B(Gx_0, r))) \to E$ and $G : E \to E$ compatible on $\overline{B(Gx_0, r)}$ and $FG^{-1}(B(Gx_0, r)) \subseteq G(E)$. Suppose G is continuous and for each $\alpha \in \lambda$, there exists a continuous, nondecreasing function $\phi_\alpha : [0,\infty) \to [0,\infty)$ satisfying $\phi_\alpha(z) < z$ for $z > 0$ such that for $x, y \in \overline{B(Gx_0, r)} \cup G^{-1}(B(Gx_0, r))$ we have*

$$d_\alpha(Fx, Fy) \leq \phi_\alpha(M_\alpha(x, y)),$$

where

$$M_\alpha(x, y) = \max\{d_\alpha(Gx, Gy), d_\alpha(Gx, Fx), d_\alpha(Gy, Fy),$$

$$\frac{1}{2}[d_\alpha(Gx, Fy) + d_\alpha(Gy, Fx)]\}.$$

Also suppose

$$\text{for each } \alpha \in \Lambda, \text{ we have } d_\alpha(Gx_0, Fx_0) < r_\alpha - \phi_\alpha(r_\alpha).$$

Then there exists a unique $x \in \overline{B(Gx_0, r)}$ *with* $x = Fx = Gx$ *.*

If $Y = E$, f and G are the identity maps on E then our Theorem 3.1 reduces to the following result of O'Regan et al. [18, Corollary 3.2].

Corollary 3.3. *Let* E *be a complete gauge space,* $x_0 \in E$ *,* $r \in (0,\infty)^\Lambda$ *with* $F : E \to E$ *and* $G : E \to E$ *compatible maps and* $F(E) \subseteq G(E)$*. Suppose* G *is continuous and for each* $\alpha \in \lambda$*, there exists a continuous, nondecreasing function* $\phi_\alpha : [0,\infty) \to [0,\infty)$ *satisfying* $\phi_\alpha(z) < z$ *for* $z > 0$ *such that for* $x, y \in E$ *we have*

$$d_\alpha(Fx, Fy) \leq \phi_\alpha(M_\alpha(x, y)),$$

where

$$M_\alpha(x, y) = \max\{d_\alpha(Gx, Gy), d_\alpha(Gx, Fx), d_\alpha(Gy, Fy), \frac{1}{2}[d_\alpha(Gx, Fy) + d_\alpha(Gy, Fx)]\}.$$

Then there exists a unique $x \in E$ *with* $x = Fx = Gx$ *.*

Theorem 3.4. *Let* Y *be an arbitrary space,* E *be a complete gauge space,* $f : Y \to E$ *be a bijection map, and* U *an open subset of* E *with* $H : (f^{-1}(\overline{U}) \cup G^{-1}(U)) \times [0,1] \to E$ *and* $G : Y \to E$ *and for each* $\lambda \in [0,1]$*,* H_λ *and* G *are* f*-hybrid compatible on* $\overline{U}$*, and* $H_\lambda(G^{-1}(U)) \subseteq Gf^{-1}(E)$*. Assume the following conditions hold:*

(i) *for* $\lambda \in [0,1]$*,* $f(y) = G(y) = H(y,\lambda)$ *cannot occur for* $y \in f^{-1}(\partial U)$ *(*∂U *denotes the boundary of* U *in* E*);*

(ii) f^{-1} *and* G *are continuous;*

(iii) *for each* $\alpha \in \Lambda$*, there exists a continuous, nondecreasing function* $\phi_\alpha : [0,\infty) \to [0,\infty)$ *satisfying* $\phi_\alpha(z) < z$ *for* $z > 0$ *such that for all* $\lambda \in [0,1]$ *and* $x, y \in \overline{U} \cup G^{-1}(U)$ *we have*

$$d_\alpha(Hf^{-1}(x,\lambda), Hf^{-1}(y,\lambda)) \leq \phi_\alpha(M_\alpha(x, y, \lambda; f^{-1})),$$

where

$$M_\alpha(x, y, \lambda; f^{-1}) = \max\{d_\alpha(Gf^{-1}(x), Gf^{-1}(y)), d_\alpha(Gf^{-1}(x), Hf^{-1}(x,\lambda)), d_\alpha(Gf^{-1}(y), Hf^{-1}(y,\lambda)), \frac{1}{2}[d_\alpha(Gf^{-1}(x), Hf^{-1}(y,\lambda)) + d_\alpha(Gf^{-1}(y), Hf^{-1}(x,\lambda))]\};$$

(iv) *for every* $\epsilon = \{\epsilon_\alpha\}_{\alpha\in\Lambda} \in (0,\infty)^\Lambda$, *there exists* $\delta = \delta(\epsilon) > 0$ *(which does not depend on* α*) such that* $t, s \in [0,1]$ *with* $|s-t| < \delta$ *and* $\alpha \in \Lambda$, *we have*

$$d_\alpha(Hf^{-1}(x,t), Hf^{-1}(x,s)) < \epsilon_\alpha$$

for $x \in \overline{U}$;

(v) *for each* $\alpha \in \Lambda$, $\phi_\alpha(a+b) \le \phi_\alpha(a) + \phi_\alpha(b)$ *for* $a \ge 0$, $b \ge 0$;

(vi) $H(f^{-1}(U) \times [0,1])$ *is bounded.*

In addition assume $H_0 f^{-1}$ *and* Gf^{-1} *have a coincidence point (i.e. there exists* $x \in \overline{U} \cup fG^{-1}(U)$ *with* $H_0 f^{-1}(x) = Gf^{-1}(x)$*). Then for each* $\lambda \in [0,1]$, *we have that* f, H_λ *and* G *have a unique coincidence point* $y_\lambda \in G^{-1}(U)$ *(here* $H_\lambda f^{-1}(.) = Hf^{-1}(.,\lambda)$*). Moreover, for each* $\lambda \in [0,1]$, $H_\lambda f^{-1}$ *and* Gf^{-1} *have a unique common fixed point* $x_\lambda \in fG^{-1}(U)$ *(i.e.* $x_\lambda = Hf^{-1}(x_\lambda,\lambda) = Gf^{-1}(x_\lambda)$*).*

Proof. Let

$$A = \{\lambda \in [0,1] : Hf^{-1}(x,\lambda) = Gf^{-1}(x) \text{ for some } x \in \overline{U} \cup fG^{-1}(U)\}.$$

Clearly A is nonempty. We will show that A is both open and closed in $[0,1]$ and so by the connectedness of $[0,1]$, for each $\lambda \in [0,1]$, $H_\lambda f^{-1}$ and Gf^{-1} have a coincidence point x_λ since $A = [0,1]$.

First we show A is open in $[0,1]$. Let $\lambda_0 \in A$. Then there exists $x_0 \in fG^{-1}(U)$ with

$$Hf^{-1}(x_0,\lambda_0) = Gf^{-1}(x_0).$$

Then $Gf^{-1}(x_0) \in U$. Since U is open, there exists δ_1, ..., δ_m in $(0,\infty)$ with

$$U(Gf^{-1}(x_0),\delta_1) \cap \cdots \cap U(Gf^{-1}(x_0),\delta_m) \subseteq U;$$

here $U(Gf^{-1}(x_0),\delta_i) = \{x : d_{\alpha_i}(x, Gf^{-1}(x_0)) \le \delta_i\}$ for $i \in \{1,2,\cdots,m\}$ (with $\alpha_i \in \Lambda$ for $i \in \{1,2,\cdots,\}$). Consequently, there exists $\delta = \{\delta_\alpha\}_{\alpha\in\Lambda} \in (0,\infty)$ with

$$\overline{B(Gf^{-1}(x_0),\delta)} \subseteq U.$$

Now, fix $\alpha \in \Lambda$. Then, by (iv), there exists $\eta = \eta(\delta) > 0$ with

$$\begin{aligned} d_\alpha(Gf^{-1}(x_0), Hf^{-1}(x_0,\lambda) &= d_\alpha(Hf^{-1}(x_0,\lambda_0), Hf^{-1}(x_0,\lambda)) \\ &< \delta_\alpha - \phi_\alpha(\delta_\alpha) \end{aligned}$$

for $\lambda \in [0,1]$ and $|\lambda - \lambda_0| < \eta$. Theorem 3.1 (here $r = \delta$, $F = H_\lambda$ and $G = G$) guarantees that there exists $x_\lambda \in \overline{B(Gf^{-1}(x_0),\delta)} \subseteq U$ with

$$x_\lambda = H_\lambda f^{-1}(x_\lambda) = Gf^{-1}(x_\lambda)$$

for $\lambda \in [0,1]$ and $|\lambda - \lambda_0| < \eta$. (note if $y_\lambda = f^{-1}(x_\lambda) \in f^{-1}(U)$ then

$$f(y_\lambda) = H_\lambda(y_\lambda) = G(y_\lambda) \in U$$

and $f(y_\lambda) = G(y_\lambda)$ implies $y_\lambda = f^{-1}(x_\lambda) \in G^{-1}(U)$,i.e., $x_\lambda \in fG^{-1}(U)$). Since $x_\lambda = G(x_\lambda)$ implies

$$x_\lambda \in fG^{-1}(U),$$

it follows that A is open in $[0, 1]$.

Next we show A is closed in $[0, 1]$. Let $\{\lambda_k\}$ be a sequence in A such that $\lambda_k \to \lambda \in [0, 1]$ as $k \to \infty$. Then for each k, there exists $x_k \in fG^{-1}(U)$ with

$$Hf^{-1}(x_k, \lambda_k) = Gf^{-1}(x_k).$$

Fix $\alpha \in \Lambda$. Essentially the same reasoning as in Theorem 2.7 guarantees that $\{Gf^{-1}(x_k)\} \subseteq U$ is a Cauchy sequence with respect to d_α and so there exists $x \in \overline{U}$ with $Gf^{-1}(x_k) \to x$ as $k \to \infty$. Also as in Theorem 2.7 we have that

$$d_\alpha(Gf^{-1}(x), Hf^{-1}(x, \lambda)) = 0$$

and

$$d_\alpha(x, Gf^{-1}(x)) = 0$$

for every $\alpha \in \Lambda$. Thus

$$x = Gf^{-1}(x) = Hf^{-1}(x, \lambda).$$

From (i), $x \in U$ and so $x = Gf^{-1}(x)$ implies $x \in fG^{-1}(U)$. It follows that $\lambda \in A$. Hence A is closed in $[0, 1]$. Thus, for each $\lambda \in [0, 1]$, $H_\lambda f^{-1}$ and Gf^{-1} have a coincidence point x_λ (i.e. $Hf^{-1}(x_\lambda, \lambda) = Gf^{-1}(x_\lambda)$). Now fix $\alpha \in \Lambda$ and $\lambda \in [0, 1]$. It remains to show the uniqueness. If $y_\lambda = Gf^{-1}(y_\lambda) = H_\lambda f^{-1}(y_\lambda)$ with $x_\lambda \neq x'_\lambda$, then

$$\begin{aligned} d_\alpha(x_\lambda, x'_\lambda) &= d_\alpha(Hf^{-1}(x_\lambda, \lambda), Hf^{-1}(x'_\lambda, \lambda)) \\ &\leq \phi(\max\{d_\alpha(x_\lambda, y_\lambda), 0, 0, \frac{1}{2}[d_\alpha(x_\lambda, x'_\lambda) + d_\alpha(x'_\lambda, x_\lambda)]\}) \\ &= \phi(d_\alpha(x_\lambda, x'_\lambda)), \end{aligned}$$

which gives a contradiction. Further, let $f^{-1}x_\lambda = y_\lambda$ then since f is a surjective map we have a unique $y_\lambda \in G^{-1}(U)$ with $fy_\lambda = H_\lambda y_\lambda = Gy_\lambda$. □

If $Y = E$ and f is the identity maps on E then our Theorem 3.4 reduces to the following result of O'Regan et al. [18, Theorem 3.3].

Corollary 3.5. *Let E be a complete gauge space and U an open subset of E with $H : (\overline{U} \cup G^{-1}(U)) \times [0, 1] \to E$ and $G : E \to E$ and for each $\lambda \in [0, 1]$, H_λ and G are compatible on $\overline{U}$, and $H_\lambda(G^{-1}(U)) \subseteq G(E)$. Assume the following conditions hold:*

(i) *for $\lambda \in [0, 1]$, $x = G(x) = H(x, \lambda)$ cannot occur for $x \in \partial U$ (the boundary of U in E);*

(ii) *G is continuous;*

(iii) *for each* $\alpha \in \Lambda$, *there exists a continuous, nondecreasing function* $\phi_\alpha : [0,\infty) \to [0,\infty)$ *satisfying* $\phi_\alpha(z) < z$ *for* $z > 0$ *such that for all* $\lambda \in [0,1]$ *and* $x, y \in \overline{U} \cup G^{-1}(U)$ *we have*

$$d_\alpha(H(x,\lambda), H(y,\lambda)) \leq \phi_\alpha(M_\alpha(x,y,\lambda)),$$

where

$$M_\alpha(x,y,\lambda) = \max\{d_\alpha(G(x),G(y)), d_\alpha(G(x),H(x,\lambda)), d_\alpha(G(y),H(y,\lambda)), \frac{1}{2}[d_\alpha(G(x),H(y,\lambda)) + d_\alpha(G(y),H(x,\lambda))]\};$$

(iv) *for every* $\epsilon = \{\epsilon_\alpha\}_{\alpha\in\Lambda} \in (0,\infty)^\Lambda$, *there exists* $\delta = \delta(\epsilon) > 0$ *(which does not depend on* α*) such that* $t, s \in [0,1]$ *with* $|s - t| < \delta$ *and* $\alpha \in \Lambda$, *we have*

$$d_\alpha(H(x,t), H(x,s)) < \epsilon_\alpha$$

for $x \in \overline{U}$*;*

(v) *for each* $\alpha \in \Lambda$, $\phi_\alpha(a + b) \leq \phi_\alpha(a) + \phi_\alpha(b)$ *for* $a \geq 0$, $b \geq 0$*;*

(vi) $H(U \times [0,1])$ *is bounded.*

In addition assume H_0 *and* G *have a coincidence point (i.e. there exists* $x \in G^{-1}(U)$ *with* $H_0(x) = G(x)$*). Then for each* $\lambda \in [0,1]$, *we have that* H_λ *and* G *have a coincidence point* $x_\lambda \in G^{-1}(U)$ *(here* $H_\lambda(.) = H(.,\lambda)$*). Moreover, for each* $\lambda \in [0,1]$, H_λ *and* G *have a unique common fixed point* $G(x_\lambda)$*.*

If $Y = E$, f and G are the identity maps on E then our Theorem 3.4 reduces to the following result of O'Regan et al. [18, Corollary 3.4].

Corollary 3.6. *Let* E *be a complete gauge space and* U *an open subset of* E *with* $H : \overline{U} \times [0,1] \to E$. *Assume the following conditions hold:*

(i) $x \neq H(x,\lambda)$ *for* $x \in \partial U$ *(the boundary of* U *in* E*) and* $\lambda \in [0,1]$*;*

(ii) *for each* $\alpha \in \Lambda$, *there exists a continuous, nondecreasing function* $\phi_\alpha : [0,\infty) \to [0,\infty)$ *satisfying* $\phi_\alpha(z) < z$ *for* $z > 0$ *such that for all* $\lambda \in [0,1]$ *and* $x, y \in \overline{U}$ *we have*

$$d_\alpha(H(x,\lambda), H(y,\lambda)) \leq \phi_\alpha(M_\alpha(x,y,\lambda)),$$

where

$$M_\alpha(x,y,\lambda) = \max\{d_\alpha(x,y), d_\alpha(x,H(x,\lambda)), d_\alpha(y,H(y,\lambda)), \frac{1}{2}[d_\alpha(x,H(y,\lambda)) + d_\alpha(y,H(x,\lambda))]\};$$

(iii) *for every* $\epsilon = \{\epsilon_\alpha\}_{\alpha\in\Lambda} \in (0,\infty)^\Lambda$, *there exists* $\delta = \delta(\epsilon) > 0$ *(which does not depend on* α*) such that* $t, s \in [0,1]$ *with* $|s - t| < \delta$ *and* $\alpha \in \Lambda$, *we have*

$$d_\alpha(H(x,t), H(x,s)) < \epsilon_\alpha$$

for $x \in \overline{U}$*;*

(iv) *for each* $\alpha \in \Lambda$, $\phi_\alpha(a+b) \leq \phi_\alpha(a) + \phi_\alpha(b)$ *for* $a \geq 0$, $b \geq 0$*;*

(v) $H(U \times [0,1])$ *is bounded.*

In addition assume H_0 *has a fixed point. Then for each* $\lambda \in [0,1]$*, we have that* H_λ *has a unique fixed point* $x_\lambda \in U$ *(here* $H_\lambda(\cdot) = H(\cdot, \lambda)$*).*

We end this section with a result motivated from ideas in [13, 17, 18].

Theorem 3.7. *Let* Y *be an arbitrary space,* E *be a complete gauge space,* $f : Y \to E$ *be a bijection map,* $x_0 \in E$*,* $r = \{r_\alpha\}_{\alpha\in\Lambda} \in (0,\infty)^\Lambda$ *with* $F : f^{-1}(\overline{B(Gf^{-1}x_0, r)}) \cup G^{-1}(B(Gf^{-1}x_0, r))) \to E$ *and* $G : Y \to E$*,* f*-hybrid compatible on* $\overline{B(Gf^{-1}x_0, r)}$ *and* $FG^{-1}(B(Gf^{-1}x_0, r)) \subseteq Gf^{-1}(E)$*. Suppose* f^{-1} *and* G *are continuous and for each* $\alpha \in \Lambda$*, there exists a continuous, nondecreasing function* $\phi_\alpha : [0,\infty) \to [0,\infty)$ *satisfying* $\phi_\alpha(z) < z$ *for* $z > 0$*. Also assume there exists functions* $\beta : \Lambda \to \Lambda$ *and* $\gamma : \Lambda \to \Lambda$ *such that for each* $\alpha \in \Lambda$ *and* $x, y \in \overline{B(Gf^{-1}x_0, r)} \cup fG^{-1}(B(Gf^{-1}x_0, r))$ *we have*

$$d_\alpha(Ff^{-1}x, Ff^{-1}y) \leq \phi_{\beta(\alpha)}(d_{\gamma(\alpha)}(Gf^{-1}x, Gf^{-1}y)). \tag{3.3}$$

Further suppose for each $\alpha \in \Lambda$ *that*

$$\sum_{n=1}^{\infty} \phi_{\beta(\alpha)}\phi_{\beta(\gamma(\alpha))}\cdots\phi_{\beta(\gamma^{n-1}(\alpha))}\ (d_{\gamma^n(\alpha)}(Gf^{-1}x_0, Ff^{-1}x_0)) \tag{3.4}$$

$$+ d_\alpha(Gf^{-1}x_0, Ff^{-1}x_0) < r_\alpha$$

holds; here $\gamma^0(\alpha) = \alpha$ *and* $\gamma^n(\alpha) = \gamma(\gamma^{n-1}(\alpha))$ *for* $n \in \{1, 2, \cdots\}$*. Then there exists* $x \in \overline{B(Gf^{-1}x_0, r)}$ *with* $x = Ff^{-1}x = Gf^{-1}x$ *.*

Remark 3.2. Suppose for each $\alpha \in \Lambda$ the following conditions hold:

$$d_\alpha(Gf^{-1}x_0, Ff^{-1}x_0) < r_\alpha - \phi_{\beta(\alpha)}(r_\alpha). \tag{3.5}$$

and

$$\sum_{n=1}^{\infty} \phi_{\beta(\alpha)}\phi_{\beta(\gamma(\alpha))}\cdots\phi_{\beta(\gamma^{n-1}(\alpha))}\ (r_{\gamma^n(\alpha)} - \phi_{\beta(\gamma^n(\alpha))}(r_{\gamma^n(\alpha)})) \leq \phi_{\beta(\alpha)}(r_\alpha). \tag{3.6}$$

Then (3.4) also holds.

Proof. Let $Gf^{-1}x_1 = Ff^{-1}x_0$ for some $x_1 \in E$ (this is possible since $Gf^{-1}x_0 \in B(Gf^{-1}x_0, r)$ and $FG^{-1}(B(Gf^{-1}x_0, r)) \subseteq Gf^{-1}(E)$). Then, by (3.4),

$$d_\alpha(Gf^{-1}x_1, Gf^{-1}x_0) < r_\alpha$$

for each $\alpha \in \Lambda$ and so $Gf^{-1}x_1 \in B(Gf^{-1}x_0, r)$. Now let $Gf^{-1}x_2 = Ff^{-1}x_1$ (this is possible since $Gf^{-1}x_1 \in B(Gf^{-1}x_0, r)$ and $FG^{-1}(B(Gf^{-1}x_0, r)) \subseteq$

$Gf^{-1}(E))$. For $n \in \{2, 3, \cdot\}$, we let

$$Gf^{-1}x_{n+1} = Ff^{-1}x_n.$$

This is possible if we show $Gf^{-1}x_n \in B(Gf^{-1}x_0, r)$ since $FG^{-1}(B\,(Gf^{-1}x_0, r)) \subseteq Gf^{-1}(E)$. Fix $\alpha \in \Lambda$. Notice for $n \in \{1, 2, \cdots\}$ that

$$\begin{aligned} d_\alpha(Gf^{-1}x_{n+1}, Gf^{-1}x_n) &= d_\alpha(Ff^{-1}x_n, Ff^{-1}x_{n-1}) \\ &\le \phi_{\beta(\alpha)}(d_{\gamma(\alpha)}(Gf^{-1}x_n, Gf^{-1}x_{n-1})) \end{aligned}$$

and so

$$\begin{aligned} & d_\alpha(Gf^{-1}x_{n+1}, Gf^{-1}x_n) \\ & \le \phi_{\beta(\alpha)}\phi_{\beta(\gamma(\alpha))}...\phi_{\beta(\gamma^{n-1}(\alpha))}(d_{\gamma^n(\alpha)}(Gf^{-1}x_1, Gf^{-1}x_0)) \\ & = \phi_{\beta(\alpha)}\phi_{\beta(\gamma(\alpha))}...\phi_{\beta(\gamma^{n-1}(\alpha))}(d_{\gamma^n(\alpha)}(Gf^{-1}x_0, Ff^{-1}x_0)). \end{aligned}$$

From (3.4) and the precessing inequality, it follows that

$$\begin{aligned} d_\alpha(Gf^{-1}x_{n+1}, Gf^{-1}x_0) &\le d_\alpha(Gf^{-1}x_0, Gf^{-1}x_1) + d_\alpha(Gf^{-1}x_1, Gf^{-1}x_2) \\ &\quad + \cdots + d_\alpha(Gf^{-1}x_n, Gf^{-1}x_{n+1}) \\ &\le \sum_{k=1}^{\infty} \phi_{\beta(\alpha)}\phi_{\beta(\gamma(\alpha))} \cdots \phi_{\beta(\gamma^{k-1}(\alpha))}(d_{\gamma^k(\alpha)}(Gf^{-1}x_0, \\ &\quad Ff^{-1}x_0)) + d_\alpha(Gf^{-1}x_0, Ff^{-1}x_0) \\ &< r_\alpha. \end{aligned}$$

This implies that

$$Gf^{-1}x_n \in B_\alpha(Gf^{-1}x_0, r_\alpha) = \{y : d_\alpha(Gf^{-1}x_0, y) < r_\alpha\}$$

for each $\alpha \in \Lambda$ and so

$$Gf^{-1}x_n \in B(Gf^{-1}x_0, r).$$

Again fix $\alpha \in \Lambda$. We claim

$$\{Gf^{-1}x_n\} \text{ is a Cauchy sequence with respect to } d_\alpha. \tag{3.7}$$

Let $n, p \in \{0, 1, \cdots\}$. Then we have

$$\begin{aligned} & d_\alpha(Gf^{-1}x_{n+p}, Gf^{-1}x_n) \\ & \le d_\alpha(Gf^{-1}x_{n+p}, Gf^{-1}x_{n+p-1}) + \cdots + d_\alpha(Gf^{-1}x_n, Gf^{-1}x_{n+1}) \\ & \le \sum_{k=n}^{\infty} \phi_{\beta(\alpha)}\phi_{\beta(\gamma(\alpha))} \cdots \phi_{\beta(\gamma^{k-1}(\alpha))}(d_{\gamma^k(\alpha)}(Gf^{-1}x_0, Ff^{-1}x_0)). \end{aligned}$$

This together with (3.4) guarantees that $\{Gf^{-1}x_n\}$ is a Cauchy sequence with respect to d_α. Thus (3.7) is true for each $\alpha \in \Lambda$. Consequently, the sequence

$\{Gf^{-1}x_n\}$ is Cauchy. So there exists $x \in \overline{B(Gf^{-1}x_0, r)}$ with $Gf^{-1}x_n \to x$. Also, $Ff^{-1}x_n = Gf^{-1}x_{n+1} \longrightarrow x$ as $n \to \infty$. Since

$$\lim_n Ff^{-1}x_n = x = \lim_n Gf^{-1}x_n$$

and

$$Ff^{-1}x_n = Gf^{-1}x_{n+1} \in B(Gf^{-1}x_0, r)$$

for $n \in \{1, 2, \cdots\}$, the continuity of f^{-1} and G and f-hybrid compatibility of F and G imply that

$$\lim_{n\to\infty} d_\alpha(Ff^{-1}Gf^{-1}x_n, Gf^{-1}x) = \lim_{n\to\infty} d_\alpha(Ff^{-1}Gf^{-1}x_n, Gf^{-1}Ff^{-1}x_n) = 0$$

for each $\alpha \in \Lambda$. Thus $Ff^{-1}Gf^{-1}x_n \to Gf^{-1}x$ as $n \to \infty$. Now fix $\alpha \in \Lambda$. Then

$$\begin{aligned} d_\alpha(Ff^{-1}x, Gf^{-1}x) &\leq d_\alpha(Ff^{-1}x, Ff^{-1}Gf^{-1}x_n) \\ &\quad + d_\alpha(Ff^{-1}Gf^{-1}x_n, Gf^{-1}x) \\ &\leq \phi_{\beta(\alpha)}(d_{\gamma(\alpha)}(Gf^{-1}Gf^{-1}x_n, Gf^{-1}x)) \\ &\quad + d_\alpha(Ff^{-1}Gf^{-1}x_n, Gf^{-1}x). \end{aligned}$$

Taking the limit as $n \to \infty$, we obtain (note $d_\delta(Ff^{-1}Gf^{-1}x_n, Gf^{-1}x) \to 0$, $d_\delta(Gf^{-1}Gf^{-1}x_n, Gf^{-1}x) \to 0$ and $\phi_{\beta(\delta)}(0) = 0$ for all $\delta \in \Lambda$)

$$d_\alpha(Ff^{-1}x, Gf^{-1}x) = 0.$$

Thus $d_\alpha(Ff^{-1}x, Gf^{-1}x) = 0$ for each $\alpha \in \Lambda$ and so we have

$$Ff^{-1}x = Gf^{-1}x.$$

Also, for each $\alpha \in \Lambda$,

$$d_\alpha(Ff^{-1}x_n, Ff^{-1}Gf^{-1}x_n) \leq \phi_{\beta(\alpha)}(d_{\gamma(\alpha)}(Gf^{-1}x_n, Gf^{-1}Gf^{-1}x_n)),$$

which on letting $n \to \infty$ gives $d_\alpha(x, Gf^{-1}x) = 0$ for each $\alpha \in \Lambda$. As a result $x = Gf^{-1}x = Ff^{-1}x$. □

If $Y = E$ and f is the identity maps on E then our Theorem 3.7 reduces to the following result of O'Regan et al. [18, Theorem 3.5].

Corollary 3.8. *Let E be a complete gauge space, $x_0 \in E$, $r = \{r_\alpha\}_{\alpha\in\Lambda} \in (0,\infty)^\Lambda$ with $F : (\overline{B(Gx_0, r)} \cup G^{-1}(B(Gx_0, r))) \to E$ and $G : E \to E$ compatible on $\overline{B(Gx_0, r)}$ and $FG^{-1}(B(Gx_0, r)) \subseteq G(E)$. Suppose G is continuous and for each $\alpha \in \Lambda$, there exists a continuous, nondecreasing function $\phi_\alpha : [0,\infty) \to [0,\infty)$ satisfying $\phi_\alpha(z) < z$ for $z > 0$. Also assume there exists functions $\beta : \Lambda \to \Lambda$ and $\gamma : \Lambda \to \Lambda$ such that for each $\alpha \in \Lambda$ and $x, y \in \overline{B(Gx_0, r)} \cup G^{-1}(B(Gx_0, r))$ we have*

$$d_\alpha(Fx, Fy) \leq \phi_{\beta(\alpha)}(d_{\gamma(\alpha)}(Gx, Gy)).$$

Further suppose for each $\alpha \in \Lambda$ *that*

$$\sum_{n=1}^{\infty} \phi_{\beta(\alpha)}\phi_{\beta(\gamma(\alpha))}...\phi_{\beta(\gamma^{n-1}(\alpha))}\ (d_{\gamma^n(\alpha)}(Gx_0, Fx_0)) + d_\alpha(Gx_0, Fx_0) < r_\alpha$$

holds; here $\gamma^0(\alpha) = \alpha$ *and* $\gamma^n(\alpha) = \gamma(\gamma^{n-1}(\alpha))$ *for* $n \in \{1, 2, \cdots\}$. *Then there exists* $x \in \overline{B(Gx_0, r)}$ *with* $x = Fx = Gx$ *.*

If $Y = E$, f and G are the identity maps on E then our Theorem 3.7 reduces to the following result of O'Regan et al. [18, Corollary 3.6].

Corollary 3.9. *Let* E *be a complete gauge space,* $x_0 \in E$ *,* $r = \{r_\alpha\}_{\alpha\in\Lambda} \in (0,\infty)^\Lambda$ *with* $F : \overline{B(x_0, r)} \to E$ *. Suppose for each* $\alpha \in \Lambda$*, there exists a continuous, nondecreasing function* $\phi_\alpha : [0,\infty) \to [0,\infty)$ *satisfying* $\phi_\alpha(z) < z$ *for* $z > 0$*. Also assume there exists functions* $\beta : \Lambda \to \Lambda$ *and* $\gamma : \Lambda \to \Lambda$ *such that for each* $\alpha \in \Lambda$ *and* $x, y \in \overline{B(x_0, r)}$ *we have*

$$d_\alpha(Fx, Fy) \le \phi_{\beta(\alpha)}(d_{\beta(\alpha)}(x, y)).$$

Further suppose for each $\alpha \in \Lambda$ *that*

$$\sum_{n=1}^{\infty} \phi_{\beta(\alpha)}\phi_{\beta(\gamma(\alpha))}...\phi_{\beta(\gamma^{n-1}(\alpha))}\ (d_{\gamma^n(\alpha)}(x_0, Fx_0)) + d_\alpha(x_0, Fx_0) < r_\alpha$$

holds; here $\gamma^0(\alpha) = \alpha$ *and* $\gamma^n(\alpha) = \gamma(\gamma^{n-1}(\alpha))$ *for* $n \in \{1, 2, \cdots\}$. *Then there exists* $x \in \overline{B(x_0, r)}$ *with* $x = Fx$ *.*

4. Fixed point theory for multivalued maps in metric spaces

This section presents fixed point, coincidence point, and homotopy results for multivalued generalized contractive maps. Let (X, d) be a metric space. Let $CD(X)$ be the family of all nonempty closed subsets of X. We set

$$B(C, r) = \cup_{x\in C} B(x, r)$$

where C is a subset of X and $r > 0$. For any $A, B \in CD(X)$, we define the generalized Hausdorff distance D to be

$$D(A, B) = \inf\{\epsilon > 0 : A \subseteq B(B, \epsilon), B \subseteq B(A, \epsilon)\} \in [0, \infty].$$

Let Y be an arbitrary space and $f : Y \to X$ be a bijection map. Let $F : f^{-1}(\overline{B(Gf^{-1}x_0, r)} \cup G^{-1}(B(Gf^{-1}x_0, r))) \to CD(X)$ and $G : Y \to X$ be a mapping with

$$FG^{-1}(B(Gf^{-1}x_0, r)) \subseteq Gf^{-1}(X).$$

Then F and G are called f-hybrid compatible on $\overline{B(Gf^{-1}x_0, r)}$ if

$$\lim_{n\to\infty} dist(Gf^{-1}y_n, Ff^{-1}Gf^{-1}x_n) = 0$$

whenever $\{x_n\}$ is a sequence in $fG^{-1}(B(Gf^{-1}x_0,r))$ and $\{y_n\}$ is a sequence in $B(Gf^{-1}x_0,r)$ such that

$$\lim_{n\to\infty} Gf^{-1}x_n = t = \lim_{n\to\infty} y_n$$

for some $t \in \overline{B(Gf^{-1}x_0,r)}$, where $y_n \in Ff^{-1}x_n$ for $n \in \{1,2,\cdots\}$.

Remark 4.1. If F and G are f-hybrid compatible and $Gf^{-1}x \in Ff^{-1}x$ for some $x \in fG^{-1}(B(Gf^{-1}x_0,r))$, then

$$Gf^{-1}Gf^{-1}x \in Ff^{-1}Gf^{-1}x.$$

This is clear if we set $x_n = x$ and $y_n = Gf^{-1}x$ for all n.

Theorem 4.1. *Let Y be an arbitrary space, (X,d) be a complete metric space, $f : Y \to X$ be a bijection map, $x_0 \in X$, $r > 0$ with $F : f^{-1}(\overline{B(Gf^{-1}x_0,r)} \cup G^{-1}(B(Gf^{-1}x_0,r))) \to CD(X)$ and $G : Y \to X$ compatible maps on $\overline{B(Gx_0,r)}$ and $FG^{-1}(B(Gf^{-1}x_0,r)) \subseteq Gf^{-1}(X)$. Suppose f^{-1} and G are continuous and there exists a continuous function $\phi : [0,\infty) \to [0,\infty)$ satisfying $\phi(z) < z$ for $z > 0$ and ϕ nondecreasing on $(0,r]$ such that for $x,y \in \overline{B(Gf^{-1}x_0,r)} \cup fG^{-1}(B(Gf^{-1}x_0,r))$ we have*

$$D(Ff^{-1}x, Ff^{-1}y) \le \phi(M(x,y;f^{-1})), \tag{4.1}$$

with strict inequality if $M(x,y;f^{-1}) \neq 0$; here

$$\begin{aligned} M(x,y;f^{-1}) = \max\{&d(Gf^{-1}x, Gf^{-1}y), dist(Gf^{-1}x, Ff^{-1}x), \\ &dist(Gf^{-1}y, Ff^{-1}y), \frac{1}{2}[dist(Gf^{-1}x, Ff^{-1}y) \\ &+ dist(Gf^{-1}y, Ff^{-1}x)]\}. \end{aligned}$$

Also suppose

$$dist(Gf^{-1}x_0, Ff^{-1}x_0) < r - \phi(r) \tag{4.2}$$

$$\sum_{i=0}^{\infty} \phi^i(t) < \infty \text{ for } t \in (0, r-\phi(r)] \tag{4.3}$$

and

$$\sum_{i=0}^{\infty} \phi^i(r - \phi(r)) \le \phi(r). \tag{4.4}$$

Then there exists $x \in \overline{B(Gf^{-1}x_0,r)}$ with $Gf^{-1}x \in Ff^{-1}x$ and Ff^{-1} and Gf^{-1} have a common fixed point $Gf^{-1}x$ provided $Gf^{-1}Gf^{-1}x = Gf^{-1}x$ and $Gf^{-1}x \in \overline{B(Gf^{-1}x_0,r)} \cup fG^{-1}(B(Gf^{-1}x_0,r))$. Moreover, there exists a unique $y \in f^{-1}\Big(\overline{B(Gf^{-1}x_0,r)}\Big)$ with $fy = Gy \in Fy$.

Proof. First we show there exists $x \in \overline{B(Gf^{-1}x_0, r)}$ with

$$Gf^{-1}x \in Ff^{-1}x\,.$$

We are finished if $M(x, yf^{-1}) = 0$ for some $x, y \in \overline{B(Gf^{-1}x_0, r)}$ since $dist(G f^{-1}x, Ff^{-1}x) \leq M(x, y; f^{-1})$ and so

$$Gf^{-1}x \in Ff^{-1}x = Ff^{-1}x$$

(also we obtain $Gf^{-1}y \in Ff^{-1}y$). By (4.2), there exists $z \in Ff^{-1}x_0$ with

$$d(Gf^{-1}x_0, z) < r.$$

Since $Gf^{-1}x_0 \in B(Gf^{-1}x_0, r)$ and $FG^{-1}(B(Gf^{-1}x_0, r)) \subseteq Gf^{-1}(X)$ and so $Ff^{-1}x_0 \subseteq Gf^{-1}(X)$, we have $z \in Gf^{-1}(X)$. Thus there exists $x_1 \in X$ with $z = Gf^{-1}x_1$. As a result, we have

$$Gf^{-1}x_1 \in Ff^{-1}x_0$$

and

$$d(Gf^{-1}x_1, Gf^{-1}x_0) < r.$$

Notice $Gf^{-1}x_1 \in B(Gf^{-1}x_0, r)$. We may assume $M(x_0, x_1; f^{-1}) \neq 0$ since otherwise we are finished. Since, from (4.1), we have that

$$D(Ff^{-1}x_0, Ff^{-1}x_1) < \phi(M(x_0, x_1; f^{-1})),$$

we may choose $\epsilon > 0$ with

$$D(Ff^{-1}x_0, Ff^{-1}x_1) + \epsilon \leq \phi(M(x_0, x_1; f^{-1})).$$

Thus we can choose $w \in Ff^{-1}x_1$ so that

$$d(Gf^{-1}x_1, w) \leq D(Ff^{-1}x_0, Ff^{-1}x_1) + \epsilon.$$

Since $Gf^{-1}x_1 \in B(Gf^{-1}x_0, r)$ and $FG^{-1}(B(Gf^{-1}x_0, r)) \subseteq Gf^{-1}(X)$ and so $Ff^{-1}x_1 \subseteq Gf^{-1}(X)$, we have

$$w \in Gf^{-1}(X).$$

Therefore, there exists $x_2 \in X$ with $w = Gf^{-1}x_2$. Consequently, we have

$$Gf^{-1}x_2 \in Ff^{-1}x_1$$

and

$$d(Gf^{-1}x_1, Gf^{-1}x_2) \leq D(Ff^{-1}x_0, Ff^{-1}x_1) + \epsilon$$

and so

$$d(Gf^{-1}x_1, Gf^{-1}x_2) \leq \phi(M(x_0, x_1; f^{-1})).$$

Now we show

$$d(Gf^{-1}x_1, Gf^{-1}x_2) \leq \phi(d(Gf^{-1}x_0, Gf^{-1}x_1)). \tag{4.5}$$

Notice

$$\begin{aligned} d(Gf^{-1}x_1, Gf^{-1}x_2) \le \phi(\max\{&d(Gf^{-1}x_0, Gf^{-1}x_1),\\ &dist(Gf^{-1}x_0, Ff^{-1}x_0), dist(Gf^{-1}x_1, Ff^{-1}x_1),\\ &\frac{1}{2}[dist(Gf^{-1}x_0, Ff^{-1}x_1) + dist(Gf^{-1}x_1, Ff^{-1}x_0)]\}). \end{aligned}$$

Let

$$\begin{aligned} \beta_1 = \max\{&d(Gf^{-1}x_0, Gf^{-1}x_1), dist(Gf^{-1}x_0, Ff^{-1}x_0),\\ &dist(Gf^{-1}x_1, Ff^{-1}x_1), \frac{1}{2}[dist(Gf^{-1}x_0, Ff^{-1}x_1)\\ &+ dist(Gf^{-1}x_1, Ff^{-1}x_0)]\}. \end{aligned}$$

If $\beta_1 = d(Gf^{-1}x_0, Gf^{-1}x_1)$, then we immediately have (4.5). If $\beta_1 = dist(Gf^{-1}x_0, Ff^{-1}x_0)$, then again (4.5) holds since $dist(Gf^{-1}x_0, Ff^{-1}x_0) \le d(Gf^{-1}x_0, Gf^{-1}x_1)$. Now assume $\beta_1 = dist(Gf^{-1}x_1, Ff^{-1}x_1)$. If $\beta_1 \neq 0$, then

$$\begin{aligned} d(Gf^{-1}x_1, Gf^{-1}x_2) &\le \phi(dist(Gf^{-1}x_1, Ff^{-1}x_1))\\ &< dist(Gf^{-1}x_1, Ff^{-1}x_1))\\ &\le d(Gf^{-1}x_1, Gf^{-1}x_2), \end{aligned}$$

a contradiction. Thus $\beta_1 = dist(Gf^{-1}x_1, Ff^{-1}x_1) = 0$ and (4.5) is true since

$$d(Gf^{-1}x_1, Gf^{-1}x_2) \le \phi(\beta_1) = \phi(0) = 0.$$

Finally assume

$$\beta_1 = \frac{1}{2}[dist(Gf^{-1}x_0, Ff^{-1}x_1) + dist(Gf^{-1}x_1, Ff^{-1}x_0)].$$

Then (4.5) is trivial if $\beta_1 = 0$. If $\beta_1 \neq 0$, then

$$\begin{aligned} d(Gf^{-1}x_1, Gf^{-1}x_2) &\le \phi(\beta_1) < \beta_1\\ &= \frac{1}{2}[dist(Gf^{-1}x_0, Ff^{-1}x_1) + dist(Gf^{-1}x_1, Ff^{-1}x_0)]\\ &\le \frac{1}{2}d(Gf^{-1}x_0, Gf^{-1}x_2)\\ &\le \frac{1}{2}[d(Gf^{-1}x_0, Gf^{-1}x_1) + d(Gf^{-1}x_1, Gf^{-1}x_2). \end{aligned}$$

This implies that

$$\frac{1}{2}d(Gf^{-1}x_1, Gf^{-1}x_2) < \frac{1}{2}d(Gf^{-1}x_0, Gf^{-1}x_1)$$

and so

$$\begin{aligned}\beta_1 &= \frac{1}{2}[dist(Gf^{-1}x_0, Ff^{-1}x_1) + dist(Gf^{-1}x_1, Ff^{-1}x_0)] \\ &\leq \frac{1}{2}d(Gf^{-1}x_0, Gf^{-1}x_2) \\ &\leq \frac{1}{2}[d(Gf^{-1}x_0, Gf^{-1}x_1) + d(Gf^{-1}x_1, Gf^{-1}x_2)] \\ &< d(Gf^{-1}x_0, Gf^{-1}x_1),\end{aligned}$$

which contradicts the definition of β_1. As a result (4.5) is true. Also

$$\begin{aligned}d(Gf^{-1}x_0, Gf^{-1}x_2) &\leq d(Gf^{-1}x_0, Gf^{-1}x_1) + d(Gf^{-1}x_1, Gf^{-1}x_2) \\ &\leq d(Gf^{-1}x_0, Gf^{-1}x_1) + \phi(d(Gf^{-1}x_0, Gf^{-1}x_1)) \\ &< [r - \phi(r)] + \phi(r - \phi(r)) \\ &\leq [r - \phi(r)] + \phi(r) \\ &= r\end{aligned}$$

and hence $Gf^{-1}x_2 \in B(Gf^{-1}x_0, r)$. If $M(x_1, x_2; f^{-1}) = 0$, then, as before, we are finished. So we assume $M(x_1, x_2; f^{-1}) \neq 0$. Choose $\delta > 0$ with

$$D(Ff^{-1}x_1, Ff^{-1}x_2) + \delta \leq \phi(M(x_1, x_2; f^{-1})).$$

As above we can choose $x_3 \in X$ with $Gf^{-1}x_3 \in Ff^{-1}x_2$ and

$$d(Gf^{-1}x_2, Gf^{-1}x_3) \leq \phi(M(x_1, x_2; f^{-1})).$$

We now show that

$$d(Gf^{-1}x_2, Gf^{-1}x_3) \leq \phi(d(Gf^{-1}x_1, Gf^{-1}x_2)). \tag{4.6}$$

To see this notice

$$\begin{aligned}d(Gf^{-1}x_2, Gf^{-1}x_3) \leq \phi(\max\{&d(Gf^{-1}x_1, Gf^{-1}x_2), \\ &dist(Gf^{-1}x_1, Ff^{-1}x_1), dist(Gf^{-1}x_2, Ff^{-1}x_2), \\ &\frac{1}{2}[dist(Gf^{-1}x_1, Ff^{-1}x_2) + dist(Gf^{-1}x_2, Ff^{-1}x_1)]\}).\end{aligned}$$

Let

$$\begin{aligned}\beta_2 = \max\{&d(Gf^{-1}x_1, Gf^{-1}x_2), dist(Gf^{-1}x_1, Ff^{-1}x_1), dist(Gf^{-1}x_2, Ff^{-1}x_2), \\ &\frac{1}{2}[dist(Gf^{-1}x_1, Ff^{-1}x_2) + dist(Gf^{-1}x_2, Ff^{-1}x_1)]\}.\end{aligned}$$

If $\beta_2 = d(Gf^{-1}x_1, Gf^{-1}x_2)$, then clearly (4.6) holds. If $\beta_2 = dist(Gf^{-1}x_1, F f^{-1}x_1)$, then (4.6) is true since $dist(Gf^{-1}x_1, Ff^{-1}x_1) \leq d(Gf^{-1}x_1, Gf^{-1}x_2)$.

If $\beta_2 = dist(Gf^{-1}x_2, Ff^{-1}x_2)$, then if $\beta_2 \neq 0$ we have

$$\begin{aligned} d(Gf^{-1}x_2, Gf^{-1}x_3) \leq \phi(\beta_2) < \beta_2 & \\ &= dist(Gf^{-1}x_2, Ff^{-1}x_2) \\ &\leq d(Gf^{-1}x_2, Gf^{-1}x_3), \end{aligned}$$

which is a contradiction. Thus $\beta_2 = dist(Gf^{-1}x_2, Ff^{-1}x_2) = 0$, so

$$d(Gf^{-1}x_2, Gf^{-1}x_3) \leq \phi(\beta_2) = \phi(0) = 0$$

and (4.6) is true. Finally assume

$$\beta_2 = \frac{1}{2}[dist(Gf^{-1}x_1, Ff^{-1}x_2) + dist(Gf^{-1}x_2, Ff^{-1}x_1)].$$

If $\beta_2 = 0$, then $d(Gf^{-1}x_2, Gf^{-1}x_3) \leq \phi(\beta_2) = \phi(0) = 0$, so (4.6) is immediate. If $\beta_2 \neq 0$, then

$$\begin{aligned} d(Gf^{-1}x_2, Gf^{-1}x_3) \leq \phi(\beta_2) < \beta_2 & \\ &= \frac{1}{2}[dist(Gf^{-1}x_1, Ff^{-1}x_2) + dist(Gf^{-1}x_2, Ff^{-1}x_1)] \\ &\leq \frac{1}{2}d(Gf^{-1}x_1, Gf^{-1}x_3) \\ &\leq \frac{1}{2}[d(Gf^{-1}x_1, Gf^{-1}x_2) + d(Gf^{-1}x_2, Gf^{-1}x_3), \end{aligned}$$

so

$$\frac{1}{2}d(Gf^{-1}x_2, Gf^{-1}x_3) < \frac{1}{2}d(Gf^{-1}x_1, Gf^{-1}x_2).$$

Consequently

$$\begin{aligned} \beta_2 &= \frac{1}{2}[dist(Gf^{-1}x_1, Ff^{-1}x_2) + dist(Gf^{-1}x_2, Ff^{-1}x_1)] \\ &\leq \frac{1}{2}d(Gf^{-1}x_1, Gf^{-1}x_3) \\ &\leq \frac{1}{2}[d(Gf^{-1}x_1, Gf^{-1}x_2) + d(Gf^{-1}x_2, Gf^{-1}x_3)] \\ &< d(Gf^{-1}x_1, Gf^{-1}x_2), \end{aligned}$$

which contradicts the definition of β_2. As a result (4.6) holds. and so

$$\begin{aligned} d(Gf^{-1}x_2, Gf^{-1}x_3) &\leq \phi(d(Gf^{-1}x_1, Gf^{-1}x_2)) \\ &\leq \phi^2(d(Gf^{-1}x_0, Gf^{-1}x_1)). \end{aligned}$$

From (4.4), we obtain

$$\begin{aligned} d(Gf^{-1}x_0, Gf^{-1}x_3) &\le d(Gf^{-1}x_0, Gf^{-1}x_1) + d(Gf^{-1}x_1, Gf^{-1}x_2) \\ &\quad + d(Gf^{-1}x_2, Gf^{-1}x_3) \\ &\le [r - \phi(r)] + \phi(d(Gf^{-1}x_0, Gf^{-1}x_1)) \\ &\quad + \phi^2(d(Gf^{-1}x_0, Gf^{-1}x_1)) \\ &< [r - \phi(r)] + \phi(r - \phi(r)) + \phi^2(r - \phi(r)) \\ &\le r + [\sum_{j=1}^{\infty} \phi^j(r - \phi(r)) - \phi(r)] \\ &\le r. \end{aligned}$$

As a result $Gf^{-1}x_3 \in B(Gf^{-1}x_0, r)$. Continuing inductively we can construct a sequence $\{x_n\}$ in X with

$$Gf^{-1}x_{n+1} \in Ff^{-1}x_n$$

for $n \in \{3, 4, \cdots\}$ and

$$d(Gf^{-1}x_{n+1}, Gf^{-1}x_n) \le \phi(M(x_{n-1}, x_n; f^{-1}))$$

(here we assumed without loss of generality that $M(x_{n-1}, x_n; f^{-1}) \neq 0$). Essentially the same argument as above guarantees that

$$\begin{aligned} d(Gf^{-1}x_n, Gf^{-1}x_{n+1}) &\le \phi(d(Gf^{-1}x_{n-1}, Gf^{-1}x_n)) \\ &\quad \cdots \\ &\le \phi^n(d(Gf^{-1}x_0, Gf^{-1}x_1)) \end{aligned} \tag{4.7}$$

and

$$Gf^{-1}x_{n+1} \in B(Gf^{-1}x_0, r) \text{ for } n \in \{3, 4, \cdots\}.$$

The sequence $\{Gf^{-1}x_n\}$ is Cauchy. To see this, notice for $n \in \{1, 2, \cdots\}$ and $p \in \{1, 2, \cdots\}$ that (4.7) gives

$$\begin{aligned} d(Gf^{-1}x_{n+p}, Gf^{-1}x_n) &\le d(Gf^{-1}x_{n+p}, Gf^{-1}x_{n+p-1}) \\ &\quad + \cdots \\ &\quad + d(Gf^{-1}x_{n+1}, Gf^{-1}x_n) \\ &\le \phi^{n+p-1}(d(Gf^{-1}x_0, Gf^{-1}x_1)) \\ &\quad + \cdots \\ &\quad + \phi^n(d(Gf^{-1}x_0, Gf^{-1}x_1) \\ &\le \sum_{j=n}^{\infty} \phi^j(d(Gf^{-1}x_0, Gf^{-1}x_1)). \end{aligned}$$

The preceeding inequality together with (4.3) guarantees that $\{Gf^{-1}x_n\}$ is a Cauchy sequence. Since (X,d) is complete, there exists $x \in \overline{B(Gf^{-1}x_0, r)}$ with $Gf^{-1}x_n \longrightarrow x$ as $n \to \infty$. Now since $Gf^{-1}x_{n+1} \in Ff^{-1}x_n$ for $n \in \{1, 2, \cdots\}$, it follows that

$$dist(x, Ff^{-1}x_n) \leq d(x, Gf^{-1}x_{n+1}) \to 0$$

as $n \to \infty$. The continuity of f^{-1} and G and f-hybrid compatibility of F and G imply that

$$\lim_{n\to\infty} dist(Gf^{-1}x, Ff^{-1}Gf^{-1}x_n) = 0$$

since

$$\begin{aligned} dist(Gf^{-1}x, Ff^{-1}Gf^{-1}x_n) &\leq d(Gf^{-1}x, Gf^{-1}Gf^{-1}x_{n+1}) \\ &\quad + dist(Gf^{-1}Gf^{-1}x_{n+1}, Ff^{-1}Gf^{-1}x_n). \end{aligned}$$

We claim that $Gf^{-1}x \in Ff^{-1}x$. Notice (here we use the inequality: $|dist(w, A) - dist(w, B)| \leq D(A, B)$ for $w \in X$ and $A, B \in CD(X)$)

$$\begin{aligned} dist(Gf^{-1}x, Ff^{-1}x) &\leq dist(Gf^{-1}x, Ff^{-1}Gf^{-1}x_n) \\ &\quad + D(Ff^{-1}Gf^{-1}x_n, Ff^{-1}x) \\ &\leq dist(Gf^{-1}x, Ff^{-1}Gf^{-1}x_n) \\ &\quad + \phi(\max\{d(Gf^{-1}x, Gf^{-1}Gf^{-1}x_n), \\ &\quad dist(Gf^{-1}x, Ff^{-1}x), \\ &\quad dist(Gf^{-1}Gf^{-1}x_n, Ff^{-1}Gf^{-1}x_n), \\ &\quad \frac{1}{2}[dist(Gf^{-1}x, Ff^{-1}Gf^{-1}x_n) \\ &\quad + dist(Gf^{-1}Gf^{-1}x_n, Ff^{-1}x)]\}). \end{aligned}$$

Taking the limit as $n \to \infty$, we have (notice that $dist(Gf^{-1}x, Ff^{-1}Gf^{-1}x_n) \to 0$, and also that $dist(Gf^{-1}Gf^{-1}x_n, Ff^{-1}Gf^{-1}x_n) \leq d(Gf^{-1}Gf^{-1}x_n, Gf^{-1}Gf^{-1}x_{n+1}) + dist(Gf^{-1}Gf^{-1}x_{n+1}, Ff^{-1}Gf^{-1}x_n) \to 0$ and $|dist(Gf^{-1}Gf^{-1}x_n, Ff^{-1}x) - dist(Gf^{-1}x, Ff^{-1}x)| \leq d(Gf^{-1}Gf^{-1}x_n, Gf^{-1}x) \to 0$),

$$\begin{aligned} dist(Ff^{-1}x, Gf^{-1}x) &\leq \phi(\max\{0, dist(Gf^{-1}x, Ff^{-1}x), 0, \\ &\quad \frac{1}{2}dist(Gf^{-1}x, Ff^{-1}x)\}) \\ &= \phi(dist(Gf^{-1}x, Ff^{-1}x)). \end{aligned}$$

Thus $dist(Gf^{-1}x, Ff^{-1}x) = 0$ and so

$$Gf^{-1}x \in \overline{Ff^{-1}x} = Ff^{-1}x.$$

Finally, we show that F and G have a common fixed point provided $Gf^{-1}x = Gf^{-1}Gf^{-1}x$ and $Gf^{-1}x \in \overline{B(Gf^{-1}x_0, r)} \cup G^{-1}(B(Gf^{-1}x_0, r))$. To see this, let $z = Gf^{-1}x = Gf^{-1}Gf^{-1}x$. Now $z = Gf^{-1}z$ and $z \in \overline{B(Gf^{-1}x_0, r)} \cup fG^{-1}(B(Gf^{-1}x_0, r))$. We now consider two cases, namely when $z \in fG^{-1}(B(Gf^{-1}x_0, r))$ and when $z \in \overline{B(Gf^{-1}x_0, r)}$. Suppose first $z \in fG^{-1}(B(Gf^{-1} x_0, r))$. Then

$$Gf^{-1}z \in B(Gf^{-1}x_0, r)$$

and so

$$z \in B(Gf^{-1}x_0, r).$$

This implies that

$$x \in fG^{-1}(B(Gf^{-1}x_0, r))$$

Notice $z = Gf^{-1}x = Gf^{-1}z = Gf^{-1}Gf^{-1}x$. By Remark 4.1, we have

$$Gf^{-1}Gf^{-1}x \in Ff^{-1}Gf^{-1}x$$

and so

$$z = Gf^{-1}z \in Ff^{-1}z.$$

Now suppose $z \in \overline{B(Gf^{-1}x_0, r)}$. Then

$$\begin{aligned}
&dist(Gf^{-1}Gf^{-1}x_{n+1}, Ff^{-1}z)\\
&\leq dist(Gf^{-1}Gf^{-1}x_{n+1}, Ff^{-1}Gf^{-1}x_n) + D(Ff^{-1}Gf^{-1}x_n, Ff^{-1}z)\\
&\leq dist(Gf^{-1}Gf^{-1}x_{n+1}, Ff^{-1}Gf^{-1}x_n) + \phi(\max\{d(Gf^{-1}Gf^{-1}x_n, Gf^{-1}z),\\
&\quad dist(Gf^{-1}Gf^{-1}x_n, Ff^{-1}Gf^{-1}x_n), dist(Gf^{-1}z, Ff^{-1}z),\\
&\quad \frac{1}{2}[dist(Gf^{-1}z, Ff^{-1}Gf^{-1}x_n) + dist(Gf^{-1}Gf^{-1}x_n, Ff^{-1}z)]\})\\
&\leq dist(Gf^{-1}Gf^{-1}x_{n+1}, Ff^{-1}Gf^{-1}x_n) + \phi(\max\{d(Gf^{-1}Gf^{-1}x_n, z),\\
&\quad dist(Gf^{-1}Gf^{-1}x_n, Ff^{-1}Gf^{-1}x_n), dist(z, Ff^{-1}z),\\
&\quad \frac{1}{2}[dist(z, Ff^{-1}Gf^{-1}x_n) + dist(Gf^{-1}Gf^{-1}x_n, Ff^{-1}z)]\}).
\end{aligned}$$

Letting $n \to \infty$, we obtain, as before,

$$dist(z, Ff^{-1}z) \leq \phi(\max\{0, 0, dist(z, Ff^{-1}z), \frac{1}{2}[dist(z, Ff^{-1}z)]\}).$$

This implies that

$$dist(z, Ff^{-1}z) = 0.$$

Hence z is a common fixed point of Gf^{-1} and Ff^{-1}. Further, let $f^{-1}z = y$ then since f is a surjective map we have a unique $y \in f^{-1}\Big(\overline{B(Gf^{-1}x_0, r)}\Big)$ with $fy = Gy \in Fy$. □

Remark 4.2. Let f, G and F be as in Theorem 4.1. Also suppose

$$Gf^{-1}(C(Gf^{-1}, Ff^{-1})) \subseteq B(Gf^{-1}x_0, r);$$

here $C(Gf^{-1}, Ff^{-1}) = \{x \in \overline{B(Gf^{-1}x_0, r)} : Gf^{-1}x \in Ff^{-1}x, Gf^{-1}Gf^{-1}x = Gf^{-1}x\}$. Then there exists $x \in \overline{B(Gf^{-1}x_0, r)}$ with

$$Gf^{-1}x \in Ff^{-1}x.$$

Moreover, Ff^{-1} and Gf^{-1} have a common fixed point $Gf^{-1}x$ provided $Gf^{-1}Gf^{-1}x = Gf^{-1}x$.

Remark 4.3. Let f, G and F be as in Theorem 4.1. In addition, assume that

$$\begin{aligned} d(Gf^{-1}x, Gf^{-1}Gf^{-1}x) &\leq dist(Gf^{-1}Gf^{-1}x, Ff^{-1}Gf^{-1}x) \\ &\quad + dist(Gf^{-1}x, Ff^{-1}x) \\ &\quad + D(Ff^{-1}x, Ff^{-1}Gf^{-1}x) \end{aligned}$$

for all $x \in fG^{-1}(B(Gf^{-1}x_0, r))$. Then Gf^{-1} and Ff^{-1} have a common fixed point.

As in Theorem 4.1, there exists $x \in \overline{B(Gf^{-1}x_0, r)}$ such that

$$Gf^{-1}x \in Ff^{-1}x.$$

We claim $x = Gf^{-1}x$. Suppose that $d(x, Gf^{-1}x) = s$ for some $s > 0$. Since

$$\begin{aligned} d(Gf^{-1}x_n, Gf^{-1}Gf^{-1}x_n) &\leq dist(Gf^{-1}Gf^{-1}x_n, Ff^{-1}Gf^{-1}x_n) \\ &\quad + dist(Gf^{-1}x_n, Ff^{-1}x_n) \\ &\quad + D(Ff^{-1}x_n, Ff^{-1}Gf^{-1}x_n) \\ &\leq dist(Gf^{-1}Gf^{-1}x_n, Ff^{-1}Gf^{-1}x_n) \\ &\quad + dist(Gf^{-1}x_n, Ff^{-1}x_n) \\ &\quad + \phi(\max\{d(Gf^{-1}x_n, Gf^{-1}Gf^{-1}x_n), \\ &\quad dist(Gf^{-1}x_n, Ff^{-1}x_n), \\ &\quad dist(Gf^{-1}Gf^{-1}x_n, Ff^{-1}Gf^{-1}x_n), \\ &\quad \frac{1}{2}[dist(Gf^{-1}x_n, Ff^{-1}Gf^{-1}x_n) \\ &\quad + dist(Gf^{-1}Gf^{-1}x_n, Ff^{-1}x_n)]\}), \end{aligned}$$

$$\begin{aligned}&\leq dist(Gf^{-1}Gf^{-1}x_n, Ff^{-1}Gf^{-1}x_n)\\&\quad + d(Gf^{-1}x_n, Gf^{-1}x_{n+1})\\&\quad + \phi(\max\{d(Gf^{-1}x_n, Gf^{-1}Gf^{-1}x_n),\\&\quad d(Gf^{-1}x_n, Gf^{-1}x_{n+1}),\\&\quad dist(Gf^{-1}Gf^{-1}x_n, Ff^{-1}Gf^{-1}x_n),\\&\quad \frac{1}{2}[d(Gf^{-1}x_n, Gf^{-1}Gf^{-1}x_{n+1})\\&\quad + dist(Gf^{-1}Gf^{-1}x_{n+1}, Ff^{-1}Gf^{-1}x_n)\\&\quad + d(Gf^{-1}Gf^{-1}x_n, Gf^{-1}x_{n+1})]\}),\end{aligned}$$

which on letting $n \to \infty$ gives (recall from Theorem 5.1 that $Gf^{-1}x_n \to x$, $dist(Gf^{-1}Gf^{-1}x_{n+1}, Ff^{-1}Gf^{-1}x_n) \to 0$ and $dist(Gf^{-1}Gf^{-1}x_n, Ff^{-1}Gf^{-1} x_n) \leq d(Gf^{-1}Gf^{-1}x_n, Gf^{-1}Gf^{-1}x_{n+1}) + dist(Gf^{-1}Gf^{-1}x_{n+1}, Ff^{-1}Gf^{-1} x_n) \to 0$)

$$\begin{aligned}s = d(x, Gf^{-1}x) &\leq \phi(\max\{d(x, Gf^{-1}x), 0, 0,\\&\quad \frac{1}{2}[d(x, Gf^{-1}x) + d(Gf^{-1}x, x)]\})\\&= \phi(d(x, Gf^{-1}x)) = \phi(s) < s,\end{aligned}$$

a contradiction. Hence $x = Gf^{-1}x \in Ff^{-1}x$ and so x is a common fixed point of Gf^{-1} and Ff^{-1}.

Remark 4.4. Let $F : f^{-1}(\overline{B(Gf^{-1}x_0, r)} \cup G^{-1}(B(Gf^{-1}x_0, r))) \to CD(X)$ and $G : Y \to X$ with $FG^{-1}(B(Gf^{-1}x_0, r)) \subseteq Gf^{-1}(X)$. Theorem 4.1 remains valid if we use the following notion of compatibility which is slightly different from the above definition. F and G are said to be f-hybrid compatible on $\overline{B(Gf^{-1}x_0, r)}$ if for all $x \in fG^{-1}(B(Gf^{-1}x_0, r))$,

$$Gf^{-1}Ff^{-1}x \in CD(X)$$

and

$$\lim_{n\to\infty} D(Gf^{-1}Ff^{-1}x_n, Ff^{-1}Gf^{-1}x_n) = 0,$$

whenever $\{x_n\}$ is a sequence in $fG^{-1}(B(Gf^{-1}x_0, r))$ such that

$$\lim_{n\to\infty} Gf^{-1}x_n = t \in M = \lim_{n\to\infty} Ff^{-1}x_n$$

for some $t \in \overline{B(Gf^{-1}x_0, r)}$ and $M \in CD(X)$. Here we must mention that if F and G are f-hybrid compatible and $Gf^{-1}x \in Ff^{-1}x$ for some $x \in fG^{-1}(B(Gf^{-1}x_0, r))$, then Ff^{-1} and Gf^{-1} commute at coincidence points. To see this, let $x_n = x$ for each n. Then

$$Gf^{-1}x_n = Gf^{-1}x \to Gf^{-1}x$$

and $Ff^{-1}x_n \to Fx$. Put $M = Ff^{-1}x$. Then $M \in CD(X)$ and $Gf^{-1}x \in M$. From the f-hybrid compatibility of G and F it follows that

$$D(Gf^{-1}Ff^{-1}x, Ff^{-1}Gf^{-1}x) = D(Gf^{-1}Ff^{-1}x_n, Ff^{-1}Gf^{-1}x_n) \to 0,$$

that is,

$$D(Gf^{-1}Ff^{-1}x, Ff^{-1}Gf^{-1}x) = 0.$$

This implies that

$$Gf^{-1}Ff^{-1}x = Ff^{-1}Gf^{-1}x.$$

We now sketch the proof of Theorem 4.1 with this notion. As in the proof of Theorem 4.1, we may obtain that $Gf^{-1}x_{n+1} \in Ff^{-1}x_n$ with

$$\begin{aligned} d(Gf^{-1}x_n, Gf^{-1}x_{n+1}) &\le \phi(d(Gf^{-1}x_{n-1}, Gf^{-1}x_n)) \\ &(\le \phi^n(d(Gf^{-1}x_0, Gf^{-1}x_1))) \end{aligned}$$

and that $\{Gf^{-1}x_n\}$ is a Cauchy sequence. Since (X, d) is complete, there exists $x \in \overline{B(Gf^{-1}x_0, r)}$ with $Gf^{-1}x_n \longrightarrow x$ as $n \to \infty$. We claim the sequence $\{Ff^{-1}x_n\}$ is Cauchy in the space $(CD(X), D)$. Using (4.1), we obtain

$$\begin{aligned} D(Ff^{-1}x_{n-1}, Ff^{-1}x_n) \le \phi(\max\{&d(Gf^{-1}x_{n-1}, Gf^{-1}x_n), \\ &dist(Gf^{-1}x_{n-1}, Ff^{-1}x_{n-1}), \\ &dist(Gf^{-1}x_n, Ff^{-1}x_n), \\ &\frac{1}{2}[dist(Gf^{-1}x_{n-1}, Ff^{-1}x_n) \\ &+ dist(Gf^{-1}x_n, Ff^{-1}x_{n-1})]\}) \\ \le \phi(\max\{&d(Gf^{-1}x_{n-1}, Gf^{-1}x_n), \\ &d(Gf^{-1}x_{n-1}, Gf^{-1}x_n), \\ &d(Gf^{-1}x_n, Gf^{-1}x_{n+1}), \\ &\frac{1}{2}[d(Gf^{-1}x_{n-1}, Gf^{-1}x_{n+1})]\}) \\ \le \phi(\max\{&d(Gf^{-1}x_{n-1}, Gf^{-1}x_n), \\ &d(Gf^{-1}x_n, Gf^{-1}x_{n+1}), \\ &\frac{1}{2}[d(Gf^{-1}x_{n-1}, Gf^{-1}x_n) \\ &+ d(Gf^{-1}x_n, Gf^{-1}x_{n+1})]\}) \\ \le \phi(\max\{&d(Gf^{-1}x_{n-1}, Gf^{-1}x_n), \\ &d(Gf^{-1}x_n, Gf^{-1}x_{n+1})]\}). \end{aligned}$$

If

$$\max\{d(Gf^{-1}x_{n-1}, Gf^{-1}x_n), d(Gf^{-1}x_n, Gf^{-1}x_{n+1})]\}$$
$$= d(Gf^{-1}x_{n-1}, Gf^{-1}x_n),$$

then

$$\begin{aligned} D(Ff^{-1}x_{n-1}, Ff^{-1}x_n) &\leq \phi(d(Gf^{-1}x_{n-1}, Gf^{-1}x_n)) \\ &\leq \phi^n((d(Gf^{-1}x_0, Gf^{-1}x_1)). \end{aligned}$$

If

$$\max\{d(Gf^{-1}x_{n-1}, Gf^{-1}x_n), d(Gf^{-1}x_n, Gf^{-1}x_{n+1})]\}$$
$$= d(Gf^{-1}x_n, Gf^{-1}x_{n+1}),$$

then

$$\begin{aligned} D(Ff^{-1}x_{n-1}, Ff^{-1}x_n) &\leq \phi(d(Gf^{-1}x_n, Gf^{-1}x_{n+1})) \\ &\leq \phi^{n+1}(d(Gf^{-1}x_0, Gf^{-1}x_1)). \end{aligned}$$

But $\phi^{n+1}(d(Gf^{-1}x_0, Gf^{-1}x_1)) \leq \phi^n(d(Gf^{-1}x_0, Gf^{-1}x_1))$ since $\phi(z) \leq z$ and ϕ is nondecreasing. Thus

$$D(Ff^{-1}x_{n-1}, Ff^{-1}x_n) \leq \phi^n(d(Gf^{-1}x_0, Gf^{-1}x_1))$$

in all cases. Notice for $n \in \{1, 2, \cdots\}$ and $p \in \{1, 2, \cdots\}$, we have

$$\begin{aligned} D(Ff^{-1}x_{n+p}, Ff^{-1}x_n) &\leq D(Ff^{-1}x_{n+p}, Ff^{-1}x_{n+p-1}) \\ &\quad + \cdots \\ &\quad + D(Ff^{-1}x_{n+1}, Ff^{-1}x_n) \\ &\leq \phi^{n+p}(d(Gf^{-1}x_0, Gf^{-1}x_1)) \\ &\quad + \cdots \\ &\quad + \phi^{n+1}(d(Gf^{-1}x_0, Gf^{-1}x_1) \\ &\leq \sum_{j=n+1}^{\infty} \phi^j(d(Gf^{-1}x_0, Gf^{-1}x_1)). \end{aligned}$$

The preceeding inequality together with (4.3) guarantees that $\{Ff^{-1}x_n\}$ is a Cauchy sequence in the complete metric space $(CD(X), D)$.
Now let $Ff^{-1}x_n \to M$. Then

$$\begin{aligned} dist(x, M) &\leq d(x, Gf^{-1}x_n) + dist(Gf^{-1}x_n, M) \\ &\leq d(x, Gf^{-1}x_n) + D(Ff^{-1}x_{n-1}, M), \end{aligned}$$

which on taking $n \to \infty$ yields $x \in M$ since M is closed. Since $\{x_n\}$ is a sequence in $fG^{-1}(B(Gf^{-1}x_0, r))$ such that $Ff^{-1}x_n \to M \in CD(X)$ and $Gf^{-1}x_n \to x \in M$, the f-hybrid compatibility of F and G implies that

$$\lim_{n\to\infty} D(Gf^{-1}Ff^{-1}x_n, Ff^{-1}Gf^{-1}x_n) = 0.$$

Now from the continuity of f^{-1} and G, we have

$$\lim_{n\to\infty} dist(Gf^{-1}x, Ff^{-1}Gf^{-1}x_n) = 0$$

since

$$\begin{aligned} dist(Gf^{-1}x, Ff^{-1}Gf^{-1}x_n) &\leq d(Gf^{-1}x, Gf^{-1}Gf^{-1}x_{n+1}) \\ &\quad + dist(Gf^{-1}Gf^{-1}x_{n+1}, Ff^{-1}Gf^{-1}x_n) \\ &\leq d(Gf^{-1}x, Gf^{-1}Gf^{-1}x_{n+1}) \\ &\quad + D(Gf^{-1}Ff^{-1}x_n, Ff^{-1}Gf^{-1}x_n). \end{aligned}$$

Next we show $Gf^{-1}x \in Ff^{-1}x$. Notice

$$\begin{aligned} dist(Gf^{-1}x, Ff^{-1}x) &\leq d(Gf^{-1}x, Gf^{-1}Gf^{-1}x_{n+1}) \\ &\quad + dist(Gf^{-1}Gf^{-1}x_{n+1}, Ff^{-1}x) \\ &\leq d(Gf^{-1}x, Gf^{-1}Gf^{-1}x_{n+1}) \\ &\quad + D(Gf^{-1}Ff^{-1}x_n, Ff^{-1}x) \\ &\leq d(Gf^{-1}x, Gf^{-1}Gf^{-1}x_{n+1}) \\ &\quad + D(Gf^{-1}Ff^{-1}x_n, Ff^{-1}Gf^{-1}x_n) \\ &\quad + D(Ff^{-1}Gf^{-1}x_n, Ff^{-1}x) \\ &\leq d(Gf^{-1}x, Gf^{-1}Gf^{-1}x_{n+1}) \\ &\quad + D(Gf^{-1}Ff^{-1}x_n, Ff^{-1}Gf^{-1}x_n) \\ &\quad + \phi(\max\{d(Gf^{-1}Gf^{-1}x_n, Gf^{-1}x), \\ &\quad dist(Gf^{-1}Gf^{-1}x_n, Ff^{-1}Gf^{-1}x_n), \\ &\quad dist(Gf^{-1}x, Ff^{-1}x), \\ &\quad \frac{1}{2}[dist(Gf^{-1}Gf^{-1}x_n, Ff^{-1}x) \\ &\quad + dist(Gf^{-1}x, Ff^{-1}Gf^{-1}x_n)]\}). \end{aligned}$$

Taking the limit as $n \to \infty$ and using the continuity of f^{-1} and G, we have (notice that $dist(Gf^{-1}x, Ff^{-1}Gf^{-1}x_n) \to 0$, $dist(Gf^{-1}Gf^{-1}x_{n+1}, Ff^{-1}G f^{-1}x_n) \leq D(Gf^{-1}Ff^{-1}x_n, Ff^{-1}Gf^{-1}x_n) \to 0$, and also that $dist(Gf^{-1}G f^{-1}x_n, Ff^{-1}Gf^{-1}x_n) \leq d(Gf^{-1}Gf^{-1}x_n, Gf^{-1}Gf^{-1}x_{n+1}) + dist(Gf^{-1}G f^{-1}x_{n+1}, Ff^{-1}Gf^{-1}x_n) \to 0$ and $|dist(Gf^{-1}Gf^{-1}x_n, Ff^{-1}x) - dist(Gf^{-1}x,$

$Ff^{-1}x)| \leq d(Gf^{-1}Gf^{-1}x_n, Gf^{-1}x) \to 0)$,

$$\begin{aligned} dist(Ff^{-1}x, Gf^{-1}x) &\leq \phi(\max\{0, 0, dist(Gf^{-1}x, Ff^{-1}x), \\ &\quad \frac{1}{2} dist(Gf^{-1}x, Ff^{-1}x)\}) \\ &= \phi(dist(Gf^{-1}x, Ff^{-1}x)). \end{aligned}$$

Thus

$$dist(Gf^{-1}x, Ff^{-1}x) = 0$$

and so

$$Gf^{-1}x \in \overline{Ff^{-1}x} = Ff^{-1}x.$$

Using a previous argument, it can be seen that Ff^{-1} and Gf^{-1} have a common fixed point $x \in \overline{B(Gf^{-1}x_0, r)}$ provided $Gf^{-1}x = Gf^{-1}Gf^{-1}x$ and $Gf^{-1}x \in \overline{B(Gf^{-1}x_0, r)} \cup fG^{-1}(B(Gf^{-1}x_0, r))$. Further, let $f^{-1}x = y$ then since f is a surjective map we have a unique $y \in f^{-1}\Big(\overline{B(Gf^{-1}x_0, r)}\Big)$ with $fy = Gy \in Fy$.

If $Y = X$ and f is the identity map on X then our theorem 4.1 reduces to the following result of O'Regan et al. [18, Theorem 4.1].

Corollary 4.2. *Let (X, d) be a complete metric space, $x_0 \in X$, $r > 0$ with $F : (\overline{B(Gx_0, r)} \cup G^{-1}(B(Gx_0, r))) \to CD(X)$ and $G : X \to X$ compatible maps on $\overline{B(Gx_0, r)}$ and $FG^{-1}(B(Gx_0, r)) \subseteq G(X)$. Suppose G is continuous and there exists a continuous function $\phi : [0, \infty) \to [0, \infty)$ satisfying $\phi(z) < z$ for $z > 0$ and ϕ nondecreasing on $(0, r]$ such that for $x, y \in \overline{B(Gx_0, r)} \cup G^{-1}(B(Gx_0, r))$ we have*

$$D(Fx, Fy) \leq \phi(M(x, y)),$$

with strict inequality if $M(x, y) \neq 0$; here

$$M(x, y) = \max\{d(Gx, Gy), dist(Gx, Fx), dist(Gy, Fy), \\ \frac{1}{2}[dist(Gx, Fy) + dist(Gy, Fx)]\}.$$

Also suppose

$$dist(Gx_0, Fx_0) < r - \phi(r)$$

$$\sum_{i=0}^{\infty} \phi^i(t) < \infty \text{ for } t \in (0, r - \phi(r)]$$

and

$$\sum_{i=0}^{\infty} \phi^i(r - \phi(r)) \leq \phi(r).$$

Then there exists $x \in \overline{B(Gx_0, r)}$ *with* $Gx \in Fx$. *Moreover,* F *and* G *have a common fixed point* Gx *provided* $GGx = Gx$ *and* $Gx \in \overline{B(Gx_0, r)} \cup G^{-1}(B(Gx_0, r))$.

Corollary 4.3. *Let* Y *be an arbitrary space,* (X, d) *be a complete metric space,* $f : Y \to X$ *be a bijection map,* $x_0 \in X$, $r > 0$ *with* $F : f^{-1}(\overline{B(x_0, r)}, r))) \to CD(X)$ *and* $Ff^{-1}(B(x_0, r)) \subseteq f^{-1}(X)$. *Suppose* f^{-1} *is continuous and there exists a continuous function* $\phi : [0, \infty) \to [0, \infty)$ *satisfying* $\phi(z) < z$ *for* $z > 0$ *and* ϕ *nondecreasing on* $(0, r]$ *such that for* $x, y \in f^{-1}\Big(\overline{B(x_0, r)}\Big)$ *we have*

$$D(Ff^{-1}x, Ff^{-1}y) \leq \phi(M(x, y; f^{-1})),$$

with strict inequality if $M(x, y; f^{-1}) \neq 0$*; here*

$$M(x, y; f^{-1}) = \max\{d(f^{-1}x, f^{-1}y), dist(f^{-1}x, Ff^{-1}x), dist(f^{-1}y, Ff^{-1}y), \frac{1}{2}[dist(f^{-1}x, Ff^{-1}y) + dist(f^{-1}y, Ff^{-1}x)]\}.$$

Also suppose

$$dist(f^{-1}x_0, Ff^{-1}x_0) < r - \phi(r)$$

$$\sum_{i=0}^{\infty} \phi^i(t) < \infty \text{ for } t \in (0, r - \phi(r)]$$

and

$$\sum_{i=0}^{\infty} \phi^i(r - \phi(r)) \leq \phi(r).$$

Then there exists $x \in f^{-1}\Big(\overline{B(x_0, r)}\Big)$ *with* $f^{-1}x \in Ff^{-1}x$.

Corollary 4.4. *Let* (X, d) *be a complete metric space,* $x_0 \in X$, $r > 0$ *with* $F : \overline{B(x_0, r)} \to CD(X)$. *Suppose there exists a continuous function* $\phi : [0, \infty) \to [0, \infty)$ *satisfying* $\phi(z) < z$ *for* $z > 0$ *and* ϕ *nondecreasing on* $(0, r]$ *such that for* $x, y \in \overline{B(x_0, r)}$ *we have*

$$D(Fx, Fy) \leq \phi(M(x, y)),$$

with strict inequality if $M(x, y) \neq 0$*; here*

$$M(x, y) = \max\{d(x, y), dist(x, Fx), dist(y, Fy), \frac{1}{2}[dist(x, Fy) + dist(y, Fx)]\}.$$

Also suppose

$$dist(x_0, Fx_0) < r - \phi(r)$$

$$\sum_{i=0}^{\infty} \phi^i(t) < \infty \text{ for } t \in (0, r - \phi(r)]$$

and

$$\sum_{i=0}^{\infty} \phi^i(r - \phi(r)) \le \phi(r).$$

Then there exists $x \in \overline{B(x_0, r)}$ *with* $x \in Fx$.

Next we derive a global result.

Theorem 4.5. *Let* Y *be an arbitrary space,* (X, d) *be a complete metric space,* $f : Y \to X$ *be a bijection map,* $F : Y \to X$ *and* $G : Y \to X$ *be* f*-hybrid compatible maps and* $Ff^{-1}(X) \subseteq Gf^{-1}(X)$*. Suppose* f^{-1} *and* G *are continuous and there exists* $r_0 > 0$ *and a continuous, nondecreasing function* $\phi : [0, \infty) \to [0, \infty)$ *satisfying* $\phi(z) < z$ *for* $z > 0$ *and* ϕ *nondecreasing on* $(0, r_0]$ *such that for* $x, y \in X$ *we have*

$$d(Ff^{-1}x, Ff^{-1}y) \le \phi(M(x, y; f^{-1})), \tag{4.8}$$

with strict inequality if $M(x, y; f^{-1}) \ne 0$*; here*

$$M(x, y; f^{-1}) = \max\{d(Gf^{-1}x, Gf^{-1}y), d(Gf^{-1}x, Ff^{-1}x), d(Gf^{-1}y, Ff^{-1}y), \frac{1}{2}[d(Gf^{-1}x, Ff^{-1}y) + d(Gf^{-1}y, Ff^{-1}x)]\}.$$

Also suppose

$$\sum_{i=0}^{\infty} \phi^i(t) < \infty \text{ for } t \in (0, r_0]. \tag{4.9}$$

Then there exists $x \in X$ *with*

$$Gf^{-1}x \in Ff^{-1}x.$$

Moreover, Ff^{-1} *and* Gf^{-1} *have a common fixed point* $Gf^{-1}x$ *provided*

$$Gf^{-1}Gf^{-1}x = Gf^{-1}x.$$

Proof. First we claim

$$\inf_{x \in X} dist(Gf^{-1}x, Ff^{-1}x) = 0. \tag{4.10}$$

Suppose

$$\inf_{x \in X} dist(Gf^{-1}x, Ff^{-1}x) = \delta > 0.$$

Note $\phi(\delta) < \delta$. Since ϕ is continuous, we can find $\epsilon > 0$ with

$$\phi(t) < \delta \text{ for } t \in [\delta, \delta + \epsilon). \tag{4.11}$$

Choose $w \in X$ with

$$\delta \leq dist(Gf^{-1}w, Ff^{-1}w) < \delta + \epsilon. \tag{4.12}$$

Since $Ff^{-1}(X) \subseteq Gf^{-1}(X)$, we have

$$Ff^{-1}w \subseteq Gf^{-1}(X)$$

and so there exists $z \in X$ so that

$$Gf^{-1}z \in Ff^{-1}w$$

and

$$\delta \leq d(Gf^{-1}w, Gf^{-1}z) < \delta + \epsilon. \tag{4.13}$$

Also

$$\begin{aligned} dist(Gf^{-1}z, Ff^{-1}z) &\leq D(Ff^{-1}w, Ff^{-1}z) \\ &\leq \phi(\max\{d(Gf^{-1}w, Gf^{-1}z), \\ &\quad dist(Gf^{-1}w, Ff^{-1}w), dist(Gf^{-1}z, Ff^{-1}z), \\ &\quad \frac{1}{2}[dist(Gf^{-1}w, Ff^{-1}z) + dist(Gf^{-1}z, Ff^{-1}w)]\}). \end{aligned}$$

We claim

$$dist(Gf^{-1}z, Ff^{-1}z) < \delta. \tag{4.14}$$

Let

$$\begin{aligned} \beta = \max\{&d(Gf^{-1}w, Gf^{-1}z), dist(Gf^{-1}w, Ff^{-1}w), dist(Gf^{-1}z, Ff^{-1}z), \\ &\frac{1}{2}[dist(Gf^{-1}w, Ff^{-1}z) + dist(Gf^{-1}z, Ff^{-1}w)]\}). \end{aligned}$$

If $\beta = d(Gf^{-1}w, Gf^{-1}z)$, then, by (4.11) and (4.13), we have

$$dist(Gf^{-1}z, Ff^{-1}z) \leq \phi(d(Gf^{-1}w, Gf^{-1}z)) < \delta,$$

so (4.14) holds. If $\beta = dist(Gf^{-1}w, Ff^{-1}w)$, then, using (4.11) and (4.12), we obtain

$$dist(Gf^{-1}z, Ff^{-1}z) \leq \phi(dist(Gf^{-1}w, Gf^{-1}w)) < \delta,$$

so (4.14) holds. If $\beta = dist(Gf^{-1}z, Ff^{-1}z)$, then $\beta = 0$ since if $\beta \neq 0$, then

$$\begin{aligned} dist(Gf^{-1}z, Ff^{-1}z) &\leq \phi(dist(Gf^{-1}z, Ff^{-1}z)) \\ &< dist(Gf^{-1}z, Ff^{-1}z), \end{aligned}$$

a contradiction. As a result, (4.14) holds. Finally, assume

$$\beta = \frac{1}{2}[dist(Gf^{-1}w, Ff^{-1}z) + dist(Gf^{-1}z, Ff^{-1}w)].$$

If $\beta = 0$, then

$$dist(Gf^{-1}z, Ff^{-1}z) \leq \phi(\beta) = \phi(0) = 0,$$

so (4.14) holds. If $\beta \neq 0$, then

$$\begin{aligned} dist(Gf^{-1}z, Ff^{-1}z) \leq \phi(\beta) < \beta \\ &= \frac{1}{2}[dist(Gf^{-1}w, Ff^{-1}z) + dist(Gf^{-1}z, Ff^{-1}w)] \\ &\leq \frac{1}{2}[d(Gf^{-1}w, Gf^{-1}z) + dist(Gf^{-1}z, Ff^{-1}z) \\ &\quad + d(Gf^{-1}z, Gf^{-1}z)] \\ &= \frac{1}{2}[d(Gf^{-1}w, Gf^{-1}z) + dist(Gf^{-1}z, Ff^{-1}z)]. \end{aligned}$$

Therefore,

$$\frac{1}{2}dist(Gf^{-1}z, Ff^{-1}z) < \frac{1}{2}d(Gf^{-1}w, Gf^{-1}z)$$

and so

$$\begin{aligned} \beta &= \frac{1}{2}[dist(Gf^{-1}w, Ff^{-1}z) + dist(Gf^{-1}z, Ff^{-1}w)] \\ &\leq \frac{1}{2}[d(Gf^{-1}w, Gf^{-1}z) + dist(Gf^{-1}z, Ff^{-1}z)] \\ &< \frac{1}{2}d(Gf^{-1}w, Gf^{-1}z) + \frac{1}{2}d(Gf^{-1}w, Gf^{-1}z) \\ &= d(Gf^{-1}w, Gf^{-1}z), \end{aligned}$$

which contradicts the definition of β. Hence, in all cases, (4.14) holds and so (4.10) holds. As a result, there exists $x_0 \in X$ with

$$dist(Gf^{-1}x_0, Ff^{-1}x_0) < r_0$$

and, therefore, there exists $y \in Ff^{-1}x_0$ with

$$d(Gf^{-1}x_0, y) < r_0.$$

Since $Ff^{-1}(X) \subseteq Gf^{-1}(X)$, we have

$$Ff^{-1}x_0 \subseteq Gf^{-1}(X)$$

so $y \in Gf^{-1}(X)$. Consequently, there exists $x_1 \in X$ with $y = Gf^{-1}x_1$. Thus,

$$d(Gf^{-1}x_0, Gf^{-1}x_1) < r_0.$$

As in Theorem 4.1, we can construct a sequence $\{x_n\}$ so that

$$Gf^{-1}x_{n+1} \in Ff^{-1}x_n$$

for $n \in \{1, 2, \cdots\}$ with

$$d(Gf^{-1}x_n, Gf^{-1}x_{n+1}) \leq \phi(M(x_n, x_{n-1}; f^{-1})).$$

The same reasoning as in Theorem 4.1 guarantees that $\{Gf^{-1}x_n\}$ is a Cauchy sequence, so there exists $x \in X$ with $Gf^{-1}x_n \to x$. Essentially the same reasoning as in Theorem 4.1 guarantees that $dist(Gf^{-1}x, Ff^{-1}x) = 0$ so

$Gf^{-1}x \in Ff^{-1}x$. It remains to show Ff^{-1} and Gf^{-1} have a common fixed point provided $Gf^{-1}x = Gf^{-1}Gf^{-1}x$. Let $z = Gf^{-1}x = Gf^{-1}Gf^{-1}x$. Then, by Remark 4.1, we have that

$$z = Gf^{-1}x \in Ff^{-1}Gf^{-1}x = Ff^{-1}x.$$

Further, let $f^{-1}x = y$ then since f is a surjective map we have a unique $y \in f^{-1}\left(\overline{B(Gf^{-1}x_0, r)}\right)$ with $fy = Gy \in Fy$. □

Corollary 4.6. *Let (X, d) be a complete metric space with $F : X \to CD(X)$. Suppose there exists $r_0 > 0$ and a continuous function $\phi : [0,\infty) \to [0,\infty)$ satisfying $\phi(z) < z$ for $z > 0$ and ϕ nondecreasing on $(0, r_0]$ such that for $x, y \in X$ we have*

$$D(Fx, Fy) \le \phi(M(x, y)),$$

with strict inequality if $M(x, y) \neq 0$; here

$$M(x, y) = \max\{d(x, y), dist(x, Fx), dist(y, Fy),$$

$$\frac{1}{2}[dist(x, Fy) + dist(y, Fx)]\}.$$

Also suppose

$$\sum_{i=0}^{\infty} \phi^i(t) < \infty \text{ for } t \in (0, r_0].$$

Then there exists $x \in X$ with $x \in Fx$.

Next we prove a homotopy result.

Theorem 4.7. *Let Y be an arbitrary space, (X, d) be a complete metric space, $f : Y \to X$ be a bijection map, and U an open subset of X with $H : (f^{-1}(\overline{U}) \cup G^{-1}(U)) \times [0, 1] \to X$ and $G : Y \to X$ and for each $\lambda \in [0, 1]$, H_λ and G are f-hybrid compatible on $\overline{U}$, and $H_\lambda(G^{-1}(U)) \subseteq Gf^{-1}(X)$. Assume the following conditions hold:*

- (i) *$Gf^{-1}(U) \subseteq U$ (i.e. U is invariant under Gf^{-1});*
- (ii) *f^{-1} and G are continuous;*
- (iii) *there exists a continuous nondecreasing function $\phi : [0,\infty) \to [0,\infty)$ satisfying $\phi(z) < z$ for $z > 0$ such that for all $\lambda \in [0, 1]$ and $x, y \in \overline{U} \cup fG^{-1}(U)$ we have*

$$D(Hf^{-1}(x, \lambda), Hf^{-1}(y, \lambda)) \le \phi(M(x, y, \lambda; f^{-1})),$$

with strict inequality if $M(x, y, \lambda; f^{-1}) \neq 0$*; here*

$$M(x, y, \lambda; f^{-1}) = \max\{d(Gf^{-1}(x), Gf^{-1}(y)), dist(Gf^{-1}(x), Hf^{-1}(x, \lambda)), \\ dist(Gf^{-1}(y), Hf^{-1}(y, \lambda)), \frac{1}{2}[dist(Gf^{-1}(x), Hf^{-1}(y, \lambda)) \\ + dist(Gf^{-1}(y), Hf^{-1}(x, \lambda))]\};$$

(iv) *for any* $\epsilon > 0$*, there exists* $\delta = \delta(\epsilon) > 0$ *such that when* $t, s \in [0, 1]$ *with* $|t - s| < \delta$ *then*

$$D(Hf^{-1}(x, t), Hf^{-1}(x, s)) < \epsilon$$

for $x \in \overline{U}$*;*

(v) *there exists* $r_0 > 0$ *such that*

$$\sum_{i=0}^{\infty} \phi^i(t) < \infty$$

for $t \in (0, r_0 - \phi(r_0)]$;

(vi) $\sum_{i=0}^{\infty} \phi^i(r - \phi(r)) \leq \phi(r)$ *for any* $r \in (0, r_0]$; *and*

(vii) $\inf\{dist(Gf^{-1}(x), H_\lambda f^{-1}(x)) : x \in \partial U, \ \lambda \in [0, 1]\} > 0$*; here* $H_\lambda f^{-1}(\cdot) = Hf^{-1}(\cdot, \lambda)$.

In addition assume $H_0 f^{-1}$ *and* Gf^{-1} *have a coincidence point (i.e. there exists* $x \in fG^{-1}(U)$ *with* $Gf^{-1}(x) \in H_0 f^{-1}(x)$*). Then for each* $\lambda \in [0, 1]$*, we have that* $H_\lambda f^{-1}$ *and* Gf^{-1} *have a coincidence point* $x_\lambda \in fG^{-1}(U)$*.*

Remark 4.5. In Theorem 4.7 notice

$$\text{for } \lambda \in [0, 1],\ Gf^{-1}(x) \notin Hf^{-1}(x, \lambda) \text{ for } x \in \partial U.$$

This is implicitly implied by the other assumptions. To see this, suppose there exist some $x_0 \in \partial U$ and $\lambda_0 \in [0, 1]$ such that

$$Gf^{-1}(x_0) \in Hf^{-1}(x_0, \lambda_0).$$

Then

$$dist(Gf^{-1}(x_0), Hf^{-1}(x_0, \lambda_0)) = 0.$$

From condition (vii), we have

$$0 < \inf\{dist(Gf^{-1}(x), H_\lambda f^{-1}(x)) : x \in \partial U, \ \lambda \in [0, 1]\} \\ \leq dist(Gf^{-1}(x_0), Hf^{-1}(x_0, \lambda_0)) = 0.$$

This is a contradiction. As a result, for $\lambda \in [0, 1]$, $Gf^{-1}(x) \notin Hf^{-1}(x, \lambda)$ for $x \in \partial U$.

Proof. First, we shall show that for each $\lambda \in [0,1]$, $H_\lambda f^{-1}$ and Gf^{-1} have a common fixed point x_λ. Let

$$A = \{\lambda \in [0,1] : Gf^{-1}(x) \in Hf^{-1}(x,\lambda) \text{ for some } x \in \overline{U} \cup fG^{-1}(U)\}.$$

Since H_0 and G have a coincidence point, $0 \in A$ and so A is nonempty. It is enough to show that A is both open and closed in $[0,1]$ since by the connectedness of $[0,1]$, we have $A = [0,1]$.

First we show A is open in $[0,1]$. Let $\lambda_0 \in A$. Then there exists $x_0 \in fG^{-1}(U)$ with

$$Gf^{-1}(x_0) \in Hf^{-1}(x_0, \lambda_0).$$

Then $Gf^{-1}(x_0) \in U$ and since U is open, there exists a ball $B(Gf^{-1}(x_0), \delta)$, $\delta > 0$ (choose also $\delta < r_0$), with

$$\overline{B(Gf^{-1}(x_0), \delta)} \subseteq U.$$

Now, by (iv), there exists $\eta(\delta) > 0$ with

$$dist(Gf^{-1}(x_0), Hf^{-1}(x_0,\lambda) \le D(Hf^{-1}(x_0,\lambda_0), Hf^{-1}(x_0,\lambda)) < \delta - \phi(\delta)$$

for $\lambda \in [0,1]$ and $|\lambda - \lambda_0| < \eta$. Now Theorem 4.1 (here $r = \delta$, $F = H_\lambda$ and $G = G$) guarantees that there exists $x_\lambda \in \overline{B(Gf^{-1}(x_0), \delta)} \subseteq U$ with

$$Gf^{-1}(x_\lambda) \in H_\lambda f^{-1}(x_\lambda)$$

for $\lambda \in [0,1]$ and $|\lambda - \lambda_0| < \eta$ (note $x_\lambda \in U$ and $Gf^{-1}(x_\lambda) \in U$ since $Gf^{-1}(U) \subseteq U$, so $x_\lambda \in fG^{-1}(U)$ for $\lambda \in [0,1]$ and $|\lambda - \lambda_0| < \eta$). Consequently A is open in $[0,1]$.

Next we show A is closed in $[0,1]$. Let $\{\lambda_k\}$ be a sequence in A with $\lambda_k \to \lambda \in [0,1]$ as $k \to \infty$. Then for each k, there exists $x_k \in fG^{-1}(U)$ with

$$Gf^{-1}(x_k) \in Hf^{-1}(x_k, \lambda_k).$$

We claim

$$\inf_{k \ge 1} dist(Gf^{-1}(x_k), \partial U) > 0. \tag{4.15}$$

Suppose it is not true. Fix $i \in \{1, 2, \cdots\}$. Then there exist $n_i \in \{1, 2, \cdots\}$ and a $y_{n_i} \in \partial U$ such that

$$d(Gf^{-1}(x_{n_i}), y_{n_i}) < \frac{1}{l(n_i)}$$

(with $l(n_i) \to \infty$ if $n_i \to \infty$) and since f^{-1} and G are continuous, we may assume

$$d(Gf^{-1}(Gf^{-1}(x_{n_i})), Gf^{-1}(y_{n_i})) < \frac{1}{i}.$$

Therefore, there exist a subsequence S of $\{1, 2, \cdots\}$ and a sequence $\{y_i\} \subseteq \partial U$ (for $i \in S$) such that

$$d(Gf^{-1}(Gf^{-1}(x_i)), Gf^{-1}(y_i)) < \frac{1}{i} \text{ for } i \in S. \tag{4.16}$$

This together with (vii) gives

$$\begin{aligned} 0 &< \inf\{dist(Gf^{-1}(x), H_\lambda f^{-1}(x)) : x \in \partial U, \lambda \in [0,1]\} \\ &\leq \liminf_{i\to\infty \text{ in } S} dist(Gf^{-1}(y_i), H_{\lambda_i} f^{-1}(y_i)). \end{aligned}$$

If

$$\liminf_{i\to\infty \text{ in } S} dist(Gf^{-1}(y_i), H_{\lambda_i} f^{-1}(y_i)) = 0 \tag{4.17}$$

is true, then we obtain a contradiction from the preceeding inequality and so (4.15) is true. To prove (4.17), notice

$$\begin{aligned} &\liminf_{i\to\infty \text{ in } S} dist(Gf^{-1}(y_i), H_{\lambda_i} f^{-1}(y_i)) \\ &\leq \liminf_{i\to\infty \text{ in } S} [d(Gf^{-1}(y_i), Gf^{-1}(Gf^{-1}(x_i))) \\ &\quad + dist(Gf^{-1}(Gf^{-1}(x_i)), H_{\lambda_i} f^{-1}(y_i))] \\ &\leq \liminf_{i\to\infty \text{ in } S} [\frac{1}{i} + D(H_{\lambda_i} f^{-1}(Gf^{-1}(x_i)), H_{\lambda_i} f^{-1}(y_i))] \\ &= \liminf_{i\to\infty \text{ in } S} D(H_{\lambda_i} f^{-1}(Gf^{-1}(x_i)), H_{\lambda_i} f^{-1}(y_i)) \\ &\leq \liminf_{i\to\infty \text{ in } S} \phi(\max\{d(Gf^{-1}(Gf^{-1}(x_i)), Gf^{-1}(y_i)), \\ &\quad dist(Gf^{-1}(Gf^{-1}(x_i)), Hf^{-1}(Gf^{-1}(x_i), \lambda_i)), dist(Gf^{-1}(y_i), Hf^{-1}(y_i, \lambda_i)), \\ &\quad \frac{1}{2}[dist(Gf^{-1}(Gf^{-1}(x_i)), Hf^{-1}(y_i, \lambda_i)) \\ &\quad + dist(Gf^{-1}(y_i), Hf^{-1}(Gf^{-1}(x_i), \lambda_i))]\}) \end{aligned}$$

(here we used the fact that $Gf^{-1}(x_i) \in H_{\lambda_i} f^{-1}(x_i)$ implies $Gf^{-1}(Gf^{-1}(x_i)) \in H_{\lambda_i} f^{-1}(Gf^{-1}(x_i))$ (see Remark 4.1). Let

$$\begin{aligned} \eta = \max\{&d(Gf^{-1}(Gf^{-1}(x_i)), Gf^{-1}(y_i)), dist(Gf^{-1}(Gf^{-1}(x_i)), \\ &Hf^{-1}(Gf^{-1}(x_i), \lambda_i)), dist(Gf^{-1}(y_i), Hf^{-1}(y_i, \lambda_i)), \\ &\frac{1}{2}[dist(Gf^{-1}(Gf^{-1}(x_i)), Hf^{-1}(y_i, \lambda_i)) + dist(Gf^{-1}(y_i), Hf^{-1}(Gf^{-1}(x_i), \lambda_i))]\}. \end{aligned}$$

] If $\eta = d(Gf^{-1}(Gf^{-1}(x_i)), Gf^{-1}(y_i))$, then

$$\begin{aligned} &\liminf_{i\to\infty \text{ in } S} dist(Gf^{-1}(y_i), H_{\lambda_i} f^{-1}(y_i)) \\ &\leq \liminf_{i\to\infty \text{ in } S} \phi(d(Gf^{-1}(Gf^{-1}(x_i)), Gf^{-1}(y_i))) \\ &\leq \liminf_{i\to\infty \text{ in } S} d(Gf^{-1}(Gf^{-1}(x_i)), Gf^{-1}(y_i)) \\ &\leq \liminf_{i\to\infty \text{ in } S} (\frac{1}{i}) = 0, \end{aligned}$$

so (4.17) is true. If $\eta = dist(Gf^{-1}(Gf^{-1}(x_i)), Hf^{-1}(Gf^{-1}(x_i), \lambda_i))$, then obviously $\eta = 0$, so (4.17) is true. If $\eta = dist(Gf^{-1}(y_i), Hf^{-1}(y_i, \lambda_i))$, then

$$\begin{aligned}&\liminf_{i\to\infty \text{ in } S} dist(Gf^{-1}(y_i), H_{\lambda_i}f^{-1}(y_i))\\ &\leq \liminf_{i\to\infty \text{ in } S} \phi(dist(Gf^{-1}(y_i), Hf^{-1}(y_i, \lambda_i))\\ &\leq \phi(\liminf_{i\to\infty \text{ in } S} dist(Gf^{-1}(y_i)), Hf^{-1}(y_i, \lambda_i)))\end{aligned}$$

which gives $\liminf_{i\to\infty \text{ in } S} dist(Gf^{-1}(y_i), H_{\lambda_i}f^{-1}(y_i)) = 0$ since $\phi(z) < z$ if $z > 0$, so (4.17) is true (here we used the fact that if $\{t_n\}$ is a sequence of nonnegative real numbers, then $\liminf \phi(t_n) \leq \phi(\liminf t_n)$). Finally, if $\eta = \frac{1}{2}[dist(Gf^{-1}(Gf^{-1}(x_i)), H(y_i, \lambda_i)) + dist(Gf^{-1}(y_i), Hf^{-1}(Gf^{-1}(x_i), \lambda_i))]\}$, then since $\phi(\eta) \leq \eta$, we have

$$\begin{aligned}&\liminf_{i\to\infty \text{ in } S} dist(Gf^{-1}(y_i), H_{\lambda_i}f^{-1}(y_i))\\ &\leq \liminf_{i\to\infty \text{ in } S} \frac{1}{2}[dist(Gf^{-1}(Gf^{-1}(x_i)), Hf^{-1}(y_i, \lambda_i))\\ &\quad + dist(Gf^{-1}(y_i), Hf^{-1}(Gf^{-1}(x_i), \lambda_i))]\\ &\leq \liminf_{i\to\infty \text{ in } S} [\frac{1}{2}d(Gf^{-1}(Gf^{-1}(x_i)), Gf^{-1}(y_i))\\ &\quad + \frac{1}{2}dist(Gf^{-1}(y_i), Hf^{-1}(y_i, \lambda_i)) + \frac{1}{2}d(Gf^{-1}(Gf^{-1}(x_i)), Gf^{-1}(y_i))\\ &\quad + \frac{1}{2}dist(Gf^{-1}(Gf^{-1}(x_i)), Hf^{-1}(Gf^{-1}(x_i), \lambda_i))]\\ &\leq \liminf_{i\to\infty \text{ in } S} [\frac{1}{i} + \frac{1}{2}dist(Gf^{-1}(y_i), Hf^{-1}(y_i, \lambda_i)) + 0],\end{aligned}$$

which implies

$$\liminf_{i\to\infty \text{ in } S} dist(Gf^{-1}(y_i), H_{\lambda_i}f^{-1}(y_i)) \leq \frac{1}{2} \liminf_{i\to\infty \text{ in } S} dist(Gf^{-1}(y_i), H_{\lambda_i}f^{-1}(y_i)),$$

so (4.17) is immediate (here we used Remark 4.1). Thus there exists $\delta > 0$ (choose also $\delta < r_0$) with

$$d(Gf^{-1}(x_k), z) > \delta$$

for all $k \geq 1$ and for all $z \in \partial U$. Since $Gf^{-1}(x_k) \in U$ for each k, we have

$$\overline{B(Gf^{-1}(x_k), \delta)} \subseteq U \text{ for } k \geq 1.$$

As a result, by (iv) we have

$$\begin{aligned}dist(Gf^{-1}(x_{n_0}), H_\lambda f^{-1}(x_{n_0})) &\leq D(Hf^{-1}(x_{n_0}, \lambda_{n_0}), Hf^{-1}(x_{n_0}, \lambda))\\ &< \delta - \phi(\delta).\end{aligned}$$

Now Theorem 4.1 guarantees that Gf^{-1} and $H_\lambda f^{-1}$ have a coincidence point $x_{\lambda,n_0} \in \overline{B(Gf^{-1}(x_{n_0}),\delta)} \subset U$. As before, we have that

$$x_{\lambda,n_0} \in fG^{-1}(U).$$

Consequently $\lambda \in A$ and so A is closed in $[0,1]$. Hence we can deduce that $A = [0,1]$ and so for each $\lambda \in [0,1]$, $H_\lambda f^{-1}$ and Gf^{-1} have a coincidence point $x_\lambda \in fG^{-1}(U)$ (i.e. $Gf^{-1}(x_\lambda) \in Hf^{-1}(x_\lambda,\lambda)$). □

If $Y = X$ and f is the identity map on X then our Theorem 4.7 reduces to the following result of O'Regan et al. [18, Theorem 4.5].

Corollary 4.8. *Let (X,d) be a complete metric space and U an open subset of X with $H : (\overline{U} \cup G^{-1}(U)) \times [0,1] \to CD(X)$ and $G : X \to X$ and for each $\lambda \in [0,1]$, H_λ and G are compatible on $\overline{U}$, and $H_\lambda(G^{-1}(U)) \subseteq G(X)$. Assume the following conditions hold:*

(i) *$G(U) \subseteq U$ (i.e. U is invariant under G);*

(ii) *G is continuous;*

(iii) *there exists a continuous nondecreasing function $\phi : [0,\infty) \to [0,\infty)$ satisfying $\phi(z) < z$ for $z > 0$ such that for all $\lambda \in [0,1]$ and $x,y \in \overline{U} \cup G^{-1}(U)$ we have*

$$D(H(x,\lambda),H(y,\lambda)) \le \phi(M(x,y,\lambda)),$$

with strict inequality if $M(x,y,\lambda) \neq 0$; here

$$M(x,y,\lambda) = \max\{d(G(x),G(y)), dist(G(x),H(x,\lambda)), dist(G(y),H(y,\lambda)), \frac{1}{2}[dist(G(x),H(y,\lambda)) + dist(G(y),H(x,\lambda))]\};$$

(iv) *for any $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that when $t,s \in [0,1]$ with $|t-s| < \delta$ then*

$$D(H(x,t),H(x,s)) < \epsilon$$

for $x \in \overline{U}$;

(v) *there exists $r_0 > 0$ such that*

$$\sum_{i=0}^{\infty} \phi^i(t) < \infty$$

for $t \in (0, r_0 - \phi(r_0)]$;

(vi) *$\sum_{i=0}^{\infty} \phi^i(r - \phi(r)) \le \phi(r)$ for any $r \in (0,r_0]$; and*

(vii) *$\inf\{dist(G(x),H_\lambda(x)) : x \in \partial U,\ \lambda \in [0,1]\} > 0$; here $H_\lambda(.) = H(.,\lambda)$.*

In addition assume H_0 and G have a coincidence point (i.e. there exists $x \in G^{-1}(U)$ with $G(x) \in H_0(x)$). Then for each $\lambda \in [0,1]$, we have that H_λ and G have a coincidence point $x_\lambda \in G^{-1}(U)$.

O'Regan et al.[18, Theorem 4.6] obtained the following homotopy result via Zorn's Lemma.

Theorem 4.9. *Let (X,d) be a complete metric space and U an open subset of X with $H : (\overline{U} \cup G^{-1}(U)) \times [0,1] \to CD(X)$ and $G : X \to X$ and for each $\lambda \in [0,1]$, H_λ and G are compatible on $\overline{U}$, and $H_\lambda(G^{-1}(U)) \subseteq G(X)$. Assume the following conditions hold:*

(i) *for $\lambda \in [0,1]$, $G(x) \notin H(x,\lambda)$ for $x \in \partial U$ (the boundary of U in X) and $G(U) \subseteq U$;*

(ii) *H is closed (i.e. has closed graph), G is continuous and*
$$d(G(x),G(y)) \leq d(G(x),y)$$
for all $x \in G^{-1}(U)$ and $y \in U$;

(iii) *there exists a continuous nondecreasing function $\phi : [0,\infty) \to [0,\infty)$ satisfying $\phi(z) < z$ for $z > 0$ and all $\lambda \in [0,1]$ and $x,y \in \overline{U} \cup G^{-1}(U)$ we have*
$$D(H(x,\lambda),H(y,\lambda)) \leq \phi(M(x,y,\lambda)),$$
with strict inequality if $M(x,y,\lambda) \neq 0$; here
$$M(x,y,\lambda) = \max\{d(G(x),G(y)), dist(G(x),H(x,\lambda)), dist(G(y),H(y,\lambda)), \frac{1}{2}[dist(G(x),H(y,\lambda)) + dist(G(y),H(x,\lambda))]\};$$

(iv) *there exists a continuous increasing function $\psi : [0,1] \to \mathbf{R}$ such that*
$$D(H(x,t),H(x,s)) \leq |\psi(t) - \psi(s)|$$
for all $t,s \in [0,1]$ and $x \in \overline{U}$;

(v) *there exists $r_0 > 0$ such that*
$$\sum_{i=0}^{\infty} \phi^i(t) < \infty$$
for $t \in (0, r_0 - \phi(r_0)]$;

(vi) *$\sum_{i=0}^{\infty} \phi^i(r - \phi(r)) \leq \phi(r)$ for any $r \in (0,r_0]$;*

(vii) *$\Phi : [0,\infty) \to [0,\infty)$ is strictly increasing (here $\Phi(x) = x - \phi(x)$); and*

(viii) *$\Phi^{-1}(a) + \Phi^{-1}(b) \leq \Phi^{-1}(a+b)$ for $a \geq 0$ and $b \geq 0$.*

In addition assume H_0 and G have a coincidence point (i.e. there exists $x \in G^{-1}(U)$ with $G(x) \in H_0(x)$). Then H_1 and G have a coincidence point.

Next we obtain a simplified proof of a homotopy result via Zorn's Lemma by dropping condition (viii), replacing condition (ii)with more general condition and modifying conditions (iii) and (iv) of Theorem 4.9. In fact, we prove the following.

Theorem 4.10. *Let Y be an arbitrary space, (X, d) be a complete metric space, $f : Y \to X$ be a bijection map, and U an open subset of X with $H : (f^{-1}(\overline{U}) \cup G^{-1}(U)) \times [0,1] \to CD(X)$ and $G : Y \to X$ and for each $\lambda \in [0,1]$, H_λ and G are f-hybrid compatible on $\overline{U}$, and $H_\lambda(G^{-1}(U)) \subseteq Gf^{-1}(X)$. Assume the following conditions hold:*

(i) *for $\lambda \in [0,1]$, $Gf^{-1}(x) \notin Hf^{-1}(x, \lambda)$ for $x \in \partial U$ (the boundary of U in X) and $Gf^{-1}(U) \subseteq U$;*

(ii) *H is closed (i.e. has closed graph), f^{-1} and G are continuous and*

$$d(Gf^{-1}(x), Gf^{-1}(y)) \leq d(Gf^{-1}(x), y)$$

for all $x \in fG^{-1}(U)$ and $y \in U$;

(iii) *there exists a continuous nondecreasing function $\phi : [0, \infty) \to [0, \infty)$ satisfying $\phi(z) < z$ for $z > 0$ and all $\lambda \in [0,1]$ and $x, y \in \overline{U} \cup fG^{-1}(U)$ we have*

$$D(Hf^{-1}(x, \lambda), Hf^{-1}(y, \lambda)) \leq \phi(M(x, y, \lambda; f^{-1})),$$

with strict inequality if $M(x, y, \lambda) \neq 0$; here

$$\begin{aligned} M(x, y, \lambda; f^{-1}) = \max\{&d(Gf^{-1}(x), Gf^{-1}(y)), dist(Gf^{-1}(x), Hf^{-1}(x, \lambda)), \\ &dist(Gf^{-1}(y), Hf^{-1}(y, \lambda)), \frac{1}{2}[dist(Gf^{-1}(x), Hf^{-1}(y, \lambda)) \\ &+ dist(Gf^{-1}(y), Hf^{-1}(x, \lambda))]\}; \end{aligned}$$

(iv) *there exists a continuous increasing function $\psi : [0,1] \to \mathbf{R}$ such that*

$$|D(Hf^{-1}(x, t), Hf^{-1}(y, s)) - d(x, y)| \leq \frac{1}{2}|\psi(t) - \psi(s)|$$

for all $t, s \in [0,1]$ and $x \in \overline{U}$;

(v) *there exists $r_0 > 0$ such that*

$$\sum_{i=0}^{\infty} \phi^i(t) < \infty$$

for $t \in (0, r_0 - \phi(r_0)]$;

(vi) $\sum_{i=0}^{\infty} \phi^i(r - \phi(r)) \leq \phi(r)$ *for any $r \in (0, r_0]$; and*

(vii) $\Phi : [0, \infty) \to [0, \infty)$ *is monotone increasing (here $\Phi(x) = x - \phi(x)$);*

In addition assume $H_0 f^{-1}$ and Gf^{-1} have a coincidence point (i.e. there exists $x \in fG^{-1}(U)$ with

$$Gf^{-1}(x) \in H_0 f^{-1}(x)).$$

Then $H_1 f^{-1}$ and Gf^{-1} have a coincidence point. Moreover, H_1 and G have a coincidence point $y \in f^{-1}\overline{B(Gf^{-1}x_0, r)}$.

Remark 4.6. Condition (ii) of Theorem 4.10 (here we also assume $Gf^{-1}(U) \subseteq U$) includes the class of maps satisfying the following condition:

$(ii)^\star$ H is closed, $(Gf^{-1})^2 = Gf^{-1}$ on $fG^{-1}(U)$ and Gf^{-1} is nonexpansive on U (i.e. $d(Gf^{-1}(x), Gf^{-1}(y)) \leq d(x, y)$ for all $x, y \in U$).

To see this, let G be a map satisfying condition $(ii)^\star$ and let $x \in fG^{-1}(U)$ and $y \in U$. Then

$$d(Gf^{-1}(Gf^{-1}(x)), Gf^{-1}(y)) \leq d(Gf^{-1}(x), y).$$

Since $(Gf^{-1})^2 = Gf^{-1}$ on $fG^{-1}(U)$, it follows that

$$\begin{aligned} d(Gf^{-1}(x), Gf^{-1}(y)) &= d(Gf^{-1}(Gf^{-1}(x)), Gf^{-1}(y)) \\ &\leq d(Gf^{-1}(x), y). \end{aligned}$$

Clearly Gf^{-1} is continuous. So Gf^{-1} satisfies (ii).

Let U be an open convex subset of a Hilbert space X. Then the metric projection P is a nonexpansive mapping from X to $\overline{U}$ with $P^2 = P$ (see [10, pp. 72, 73]). Therefore, the class of maps Gf^{-1} satisfying condition $(ii)^\star$ includes the class of metric projections.

Proof. Let

$$Q = \{(t, x) \in [0, 1] \times fG^{-1}(U) : Gf^{-1}(x) \in Hf^{-1}(x, t)\}.$$

Then Q is nonempty since $H_0 f^{-1}$ and Gf^{-1} have a coincidence point. We now define the partial order on Q (see (vii) for transitivity) as follows:

$$(t, x) \leq (s, y) \text{ iff } t \leq s \text{ and } d(Gf^{-1}(x), Gf^{-1}(y)) \leq \psi(s) - \psi(t).$$

Let P be a totally ordered subset of Q and let $t^* = \sup\{t : (t, x) \in P\}$. Consider a sequence $\{(t_n, x_n)\} \subseteq P$ such that $(t_n, x_n) \leq (t_{n+1}, x_{n+1})$ and $t_n \to t^*$. Then

$$d(Gf^{-1}(x_n), Gf^{-1}(x_m)) \leq \psi(t_n) - \psi(t_m) \text{for all } m > n.$$

This implies that $\{Gf^{-1}(x_n)\}$ is a Cauchy sequence and so converges to $x^\star \in U$. Since $Gf^{-1}(x_n) \in H_{t_n} f^{-1}(x_n)$, $x_n \in fG^{-1}(U)$ for each n, it follows by the f-hybrid compatibility of G and H_{t_n} that

$$Gf^{-1}(Gf^{-1}(x_n)) \in Hf^{-1}(Gf^{-1}(x_n), t_n)$$

(see Remark 4.1). Since H is closed and G is continuous, we have $(t^\star, x^\star) \in Q$ (note $Gf^{-1}(x^\star) \in Hf^{-1}(x^\star, t^\star)$ and from (i) $x^\star \in U$ and so $Gf^{-1}(x^\star) \in U$ i.e. $x^\star \in fG^{-1}(U)$). Since P is totally ordered, it follows from the definition of $t^\star$ that

$$(t, x) \leq (t^\star, x^\star) \text{ for every } (t, x) \in P.$$

Therefore $(t^\star, x^\star)$ is an upper bound of P. By Zorn's lemma Q admits a maximal element $(t_0, x_0) \in Q$. Note $x_0 \in fG^{-1}(U)$ (i.e. $Gf^{-1}(x_0) \in U$) and

$$Gf^{-1}(x_0) \in Hf^{-1}(x_0, t_0).$$

We claim $t_0 = 1$. Suppose it is not true. Then, choose $r > 0$ (with $r \leq r_0$) and $t \in (t_0, 1]$ with $\overline{B(Gf^{-1}(x_0), r)} \subseteq U$ and

$$r - \phi(r) = \psi(t) - \psi(t_0) \leq \psi(1) - \psi(t_0) = r_0 - \phi(r_0).$$

Notice

$$\begin{aligned} dist(Gf^{-1}(x_0), Hf^{-1}(x_0, t)) &\leq dist(Gf^{-1}(x_0), Hf^{-1}(x_0, t_0)) \\ &\quad + D(Hf^{-1}(x_0, t_0), Hf^{-1}(x_0, t)) \\ &\leq 0 + \frac{1}{2}[\psi(t) - \psi(t_0)] \\ &= \frac{1}{2}(r - \phi(r)) \\ &= \frac{1}{2}\Phi(r) < \Phi(r) = r - \phi(r). \end{aligned}$$

Now Theorem 4.1 guarantees that $H_t f^{-1}$ and Gf^{-1} have a coincidence point $x \in \overline{B(Gf^{-1}x_0, r)}$. Note $x \in U$ and $Gf^{-1}(x) \in U$ (from (i)), so $x \in fG^{-1}(U)$. Hence $(t, x) \in Q$. From (ii) and above, we have

$$d(Gf^{-1}(x_0), Gf^{-1}(x)) \leq d(Gf^{-1}(x_0), x) \leq r = \psi(t) - \psi(t_0) \text{ and } t_0 < t.$$

Therefore, $(t_0, x_0) < (t, x)$. This contradicts the maximality of (t_0, x_0). Consequently, $t_0 = 1$ and so we are finished. □

5. Fixed point theory for multivalued maps in gauge spaces

In this section, we discuss analogue of some of the results of section 4 in gauge spaces. For this section, $E = (E, \{d_\alpha\}_{\alpha \in \Lambda})$ will denote a gauge space endowed with a complete gauge structure $\{d_\alpha : \alpha \in \Lambda\}$. For any $A, B \subset E$, we define the generalized Hausdorff pseudometric induced by d_α to be

$$D_\alpha(A, B) = \inf\{\epsilon > 0 : \ \forall x \in A, \forall y \in B, \exists x^* \in A, \exists y^* \in B$$

$$\text{such that } d_\alpha(x, y^*) < \epsilon, d_\alpha(x^*, y) < \epsilon\},$$

with the convention that $\inf(\emptyset) = \infty$. Let Y be an arbitrary space, $f : Y \to E$ be a bijection map, $F : f^{-1}(\overline{B(Gf^{-1}x_0, r)} \cup G^{-1}(B(Gf^{-1}x_0, r))) \to CB(E)$ and $G : Y \to E$ with

$$FG^{-1}(B(Gf^{-1}x_0, r)) \subseteq Gf^{-1}(E).$$

Then F and G are said to be f-hybrid compatible on $\overline{B(Gf^{-1}x_0, r)}$ if for each $\alpha \in \Lambda$,

$$\lim_{n\to\infty} dist_\alpha(Gf^{-1}y_n, Ff^{-1}Gf^{-1}x_n) = 0$$

whenever $\{x_n\}$ is a sequence in $G^{-1}(B(Gf^{-1}x_0, r))$ and $\{y_n\}$ is a sequence in $B(Gf^{-1}x_0, r)$ such that for each $\alpha \in \Lambda$,

$$\lim_{n\to\infty} d_\alpha(y_n, t) = \lim_{n\to\infty} d_\alpha(Gf^{-1}x_n, t) = 0$$

for some $t \in \overline{B(Gf^{-1}x_0, r)}$, where $y_n \in Ff^{-1}x_n$ for $n \in \{1, 2, \cdots\}$.

Remark 5.1. If Ff^{-1} and Gf^{-1} are f-hybrid compatible and $Gf^{-1}x \in Ff^{-1}x$ for some $x \in fG^{-1}(B(Gf^{-1}x_0, r))$, then

$$Gf^{-1}Gf^{-1}x \in Ff^{-1}Gf^{-1}x.$$

Theorem 5.1. *Let Y be an arbitrary space, E be a complete gauge space, $f : Y \to E$ be a bijection map, $x_0 \in E$, $r = \{r_\alpha\}_{\alpha\in\Lambda} \in (0,\infty)^\Lambda$ with $F : f^{-1}(\overline{B(Gf^{-1}x_0, r)} \cup fG^{-1}(B(Gf^{-1}x_0, r))) \to CD(E)$ and $G : Y \to E$ be f-hybrid compatible maps on $\overline{B(Gf^{-1}x_0, r)}$ and*

$$FG^{-1}(B(Gf^{-1}x_0, r)) \subseteq Gf^{-1}(E).$$

Suppose f^{-1} and G are continuous and for each $\alpha \in \Lambda$, there exists a continuous function $\phi_\alpha : [0,\infty) \to [0,\infty)$ satisfying $\phi_\alpha(z) < z$ for $z > 0$ and ϕ_α strictly increasing on $(0, r_\alpha]$ such that for $x, y \in \overline{B(Gf^{-1}x_0, r)} \cup fG^{-1}(B(Gf^{-1}x_0, r))$ we have

$$D_\alpha(Ff^{-1}x, Ff^{-1}y) \le \phi_\alpha(M_\alpha(x, y; f^{-1})); \tag{5.1}$$

here

$$M(x, y; f^{-1}) = \max\{d_\alpha(Gf^{-1}x, Gf^{-1}y), dist_\alpha(Gf^{-1}x, Ff^{-1}x), dist_\alpha(Gf^{-1}y, Ff^{-1}y), \frac{1}{2}[dist_\alpha(Gf^{-1}x, Ff^{-1}y) + dist_\alpha(Gf^{-1}y, Ff^{-1}x)]\}.$$

Also suppose for each $\alpha \in \Lambda$ that

$$\Phi_\alpha \text{ is strictly increasing on } [0,\infty);\ \text{here } \Phi_\alpha(x) = x - \phi_\alpha(x) \tag{5.2}$$

$$dist_\alpha(Gf^{-1}x_0, Ff^{-1}x_0) < r_\alpha - \phi_\alpha(r_\alpha) \tag{5.3}$$

$$\sum_{i=0}^{\infty} \phi_\alpha^i(t) < \infty \ \textit{for}\ t \in (0, r_\alpha - \phi_\alpha(r_\alpha)] \tag{5.4}$$

and

$$\sum_{i=0}^{\infty} \phi_\alpha^i(r_\alpha - \phi_\alpha(r_\alpha)) \le \phi_\alpha(r_\alpha). \tag{5.5}$$

Finally assume the following condition holds:

$$\begin{cases} \textit{for every } x \in fG^{-1}(B(Gf^{-1}x_0, r)) \textit{ and every } \epsilon = \{\epsilon_\alpha\}_{\alpha\in\Lambda} \in (0,\infty)^\Lambda \\ \textit{there exists } y \in Ff^{-1}x \textit{ with } d_\alpha(Gf^{-1}x, y) \le dist_\alpha(Gf^{-1}x, Ff^{-1}x) + \epsilon_\alpha \\ \textit{for every } \alpha \in \Lambda. \end{cases} \tag{5.6}$$

Then there exists $x \in \overline{B(Gf^{-1}x_0, r)}$ *with* $Gf^{-1}x \in Ff^{-1}x$ *and* Ff^{-1} *and* Gf^{-1} *have a common fixed point* $Gf^{-1}x$ *provided* $Gf^{-1}Gf^{-1}x = Gf^{-1}x$ *and* $Gx \in \overline{B(Gf^{-1}x_0, r)} \cup fG^{-1}(B(Gf^{-1}x_0, r))$*. Moreover, there exists a unique* $y \in f^{-1}\Big(\overline{B(Gf^{-1}x_0, r)}\Big)$ *with* $fy = Gy \in Fy$ *.*

Proof. From (5.3) and (5.6), we may choose $z \in Ff^{-1}x_0$ with

$$d_\alpha(Gf^{-1}x_0, z) < r_\alpha - \phi_\alpha(r_\alpha) \text{ for every } \alpha \in \Lambda.$$

Since $Gf^{-1}x_0 \in B(Gf^{-1}x_0, r)$ and $FG^{-1}(B(Gf^{-1}x_0, r)) \subseteq Gf^{-1}(E)$ and so $Ff^{-1}(x_0) \subseteq Gf^{-1}(E)$, we have

$$z \in Gf^{-1}(E).$$

Therefore, there exists $x_1 \in E$ with $z = Gf^{-1}x_1$. Consequently, we have

$$Gf^{-1}x_1 \in Ff^{-1}x_0$$

and

$$d(Gf^{-1}x_1, Gf^{-1}x_0) < r_\alpha - \phi_\alpha(r_\alpha) \text{ for every } \alpha \in \Lambda. \tag{5.7}$$

Notice $Gf^{-1}x_1 \in B(Gf^{-1}x_0, r)$. Now for $\alpha \in \Lambda$, choose $\epsilon_\alpha > 0$ with $\epsilon_\alpha \le \phi_\alpha(r_\alpha)$ and $\Phi_\alpha^{-1}(\epsilon_\alpha) < r_\alpha$ such that

$$\phi_\alpha(d_\alpha(Gf^{-1}x_0, Gf^{-1}x_1) + \epsilon_\alpha) + \epsilon_\alpha + \phi_\alpha(\Phi_\alpha^{-1}(\epsilon_\alpha)) < \phi_\alpha(r_\alpha - \phi_\alpha(r_\alpha)) \tag{5.8}$$

(this is possible from (5.7) and the fact that ϕ_α is strictly increasing on $(0, r_\alpha]$). From (5.6) we can find $y \in Ff^{-1}x_1$ such that for every $\alpha \in \Lambda$, we have

$$d_\alpha(Gf^{-1}x_1, y) < dist_\alpha(Gf^{-1}x_1, Ff^{-1}x_1) + \epsilon_\alpha \le D_\alpha(Ff^{-1}x_0, Ff^{-1}x_1) + \epsilon_\alpha.$$

Since $Gf^{-1}x_1 \in B(Gf^{-1}x_0, r)$ and $FG^{-1}(B(Gx_0, r)) \subseteq G(E)$ we have

$$Ff^{-1}x_1 \subseteq Gf^{-1}(E),$$

and so $y \in Gf^{-1}(E)$. Thus there exists $x_2 \in E$ with

$$y = Gf^{-1}x_2.$$

As a result,

$$\begin{aligned} d_\alpha(Gf^{-1}x_1, Gf^{-1}x_2) &< dist_\alpha(Gf^{-1}x_1, Ff^{-1}x_1) + \epsilon_\alpha \\ &\le D_\alpha(Ff^{-1}x_0, Fvx_1) + \epsilon_\alpha. \end{aligned} \tag{5.9}$$

We claim

$$\begin{aligned} d_\alpha(Gf^{-1}x_1, Gf^{-1}x_2) &< \phi_\alpha(d_\alpha(Gf^{-1}x_0, Gf^{-1}x_1) + \epsilon_\alpha) \\ &\quad + \epsilon_\alpha + \phi_\alpha(\Phi_\alpha^{-1}(\epsilon_\alpha)). \end{aligned} \tag{5.10}$$

To see this notice

$$\left\{ \begin{array}{c} D_\alpha(Ff^{-1}x_0, Ff^{-1}x_1) + \epsilon_\alpha \le \phi_\alpha(\max\{d_\alpha(Gf^{-1}x_0, Gf^{-1}x_1), \\ dist_\alpha(Gf^{-1}x_0, Ff^{-1}x_0), dist_\alpha(Gf^{-1}x_1, Ff^{-1}x_1), \\ \frac{1}{2}[dist_\alpha(Gf^{-1}x_0, Ff^{-1}x_1) + dist_\alpha(Gf^{-1}x_1, Ff^{-1}x_0)]\}) + \epsilon_\alpha. \end{array} \right. \tag{5.11}$$

Let

$$\begin{aligned} \eta_\alpha = \max\{&d_\alpha(Gf^{-1}x_0, Gf^{-1}x_1), dist_\alpha(Gf^{-1}x_0, Ff^{-1}x_0), \\ &dist_\alpha(Gf^{-1}x_1, Ff^{-1}x_1), \frac{1}{2}[dist_\alpha(Gf^{-1}x_0, Ff^{-1}x_1) \\ &+ dist_\alpha(Gf^{-1}x_1, Ff^{-1}x_0)]\}. \end{aligned}$$

If $\eta_\alpha = d_\alpha(Gf^{-1}x_0, Gf^{-1}x_1)$, then (5.10) holds.
If $\eta_\alpha = dist_\alpha(Gf^{-1}x_0, Ff^{-1}x_0)$, then

$$\eta_\alpha \le d_\alpha(Gf^{-1}x_0, Gf^{-1}x_1)(\le r_\alpha)$$

so (5.10) holds. If $\eta_\alpha = dist_\alpha(Gf^{-1}x_1, Ff^{-1}x_1)$, then (5.9) gives

$$\begin{aligned} dist_\alpha(Gf^{-1}x_1, Ff^{-1}x_1) &\le d_\alpha(Gf^{-1}x_1, Gf^{-1}x_2) \\ &\le \phi_\alpha(dist_\alpha(Gf^{-1}x_1, Ff^{-1}x_1)) + \epsilon_\alpha, \end{aligned}$$

so $dist_\alpha(Gf^{-1}x_1, Ff^{-1}x_1) \le \Phi_\alpha^{-1}(\epsilon_\alpha)$. Therefore,

$$d_\alpha(Gf^{-1}x_1, Gf^{-1}x_2) \le \phi_\alpha(\Phi_\alpha^{-1}(\epsilon_\alpha)) + \epsilon_\alpha,$$

so (5.10) holds. Finally if

$$\eta_\alpha = \frac{1}{2}[dist_\alpha(Gf^{-1}x_0, Ff^{-1}x_1) + dist_\alpha(Gf^{-1}x_1, Ff^{-1}x_0)],$$

then

$$\begin{aligned} d_\alpha(Gf^{-1}x_1, Gf^{-1}x_2) &\le \frac{1}{2}[dist_\alpha(Gf^{-1}x_0, Ff^{-1}x_1) \\ &\quad + dist_\alpha(Gf^{-1}x_1, Ff^{-1}x_0)] + \epsilon_\alpha \\ &\le \frac{1}{2}[d_\alpha(Gf^{-1}x_0, Gf^{-1}x_1) \\ &\quad + d_\alpha(Gf^{-1}x_1, Gf^{-1}x_2)] + \epsilon_\alpha, \end{aligned}$$

so

$$\frac{1}{2}d_\alpha(Gf^{-1}x_0, Gf^{-1}x_2) \le \frac{1}{2}d_\alpha(Gf^{-1}x_0, Gf^{-1}x_1) + \epsilon_\alpha.$$

Consequently

$$\begin{aligned} d_\alpha(Gf^{-1}x_1, Gf^{-1}x_2) &\le \phi_\alpha(\frac{1}{2}[dist_\alpha(Gf^{-1}x_0, Ff^{-1}x_1) \\ &\quad + dist_\alpha(Gf^{-1}x_1, Ff^{-1}x_0)]) + \epsilon_\alpha \\ &\le \phi_\alpha(\frac{1}{2}[d_\alpha(Gf^{-1}x_0, Gf^{-1}x_1) \\ &\quad + d_\alpha(Gf^{-1}x_1, Gf^{-1}x_2)]) + \epsilon_\alpha \\ &\le \phi_\alpha(d_\alpha(Gf^{-1}x_0, Gf^{-1}x_1) + \epsilon_\alpha) + \epsilon_\alpha, \end{aligned}$$

(note $d_\alpha(Gf^{-1}x_0, Gf^{-1}x_1) + \epsilon_\alpha \le r_\alpha$) so (5.10) holds. As a result, (5.10) is true in all cases. Therefore, it follows from (5.8) and (5.10) that

$$d_\alpha(Gf^{-1}x_1, Gf^{-1}x_2) < \phi_\alpha(r_\alpha - \phi_\alpha(r_\alpha)). \tag{5.12}$$

Notice for all $\alpha \in \Lambda$

$$\begin{aligned} d_\alpha(Gf^{-1}x_0, Gf^{-1}x_2) &\le d_\alpha(Gf^{-1}x_0, Gf^{-1}x_1) + d_\alpha(Gf^{-1}x_1, Gf^{-1}x_2) \\ &< [r_\alpha - \phi_\alpha(r_\alpha)] + \phi_\alpha(r_\alpha - \phi_\alpha(r_\alpha)) \\ &\le [r_\alpha - \phi_\alpha(r_\alpha)] + \phi_\alpha(r_\alpha) = r_\alpha. \end{aligned}$$

This implies that

$$Gf^{-1}x_2 \in B(Gf^{-1}x_0, r).$$

Next for $\alpha \in \Lambda$, choose $\delta_\alpha > 0$ with $\phi_\alpha(r_\alpha - \phi_\alpha(r_\alpha)) + \delta_\alpha \le r_\alpha$ and $\Phi^{-1}(\delta_\alpha) < r_\alpha$ such that

$$\phi_\alpha(d_\alpha(Gf^{-1}x_1, Gf^{-1}x_2) + \delta_\alpha) + \delta_\alpha + \phi_\alpha(\Phi_\alpha^{-1}(\delta_\alpha)) < \phi_\alpha^2(r_\alpha - \phi_\alpha(r_\alpha)) \tag{5.13}$$

(this is possible from (5.12)). From (5.6), we may choose $x_3 \in E$ so that $Gf^{-1}x_3 \in Ff^{-1}x_2$ and

$$\begin{aligned} d_\alpha(Gf^{-1}x_2, Gf^{-1}x_3) &\le dist_\alpha(Gf^{-1}x_2, Ff^{-1}x_2) + \delta_\alpha \\ &\le D_\alpha(Ff^{-1}x_1, Ff^{-1}x_2) + \delta_\alpha. \end{aligned}$$

As before, we have

$$\begin{aligned} d_\alpha(Gf^{-1}x_2, Gf^{-1}x_3) &\le \phi_\alpha(d_\alpha(Gf^{-1}x_2, Gf^{-1}x_3) + \delta_\alpha) \\ &\quad + \delta_\alpha + \phi_\alpha(\Phi_\alpha^{-1}(\delta_\alpha)) \end{aligned} \tag{5.14}$$

and this together with (5.13) gives

$$d_\alpha(Gf^{-1}x_2, Gf^{-1}x_3) < \phi_\alpha^2(r_\alpha - \phi_\alpha(r_\alpha)). \tag{5.15}$$

Now for each $\alpha \in \Lambda$, we have

$$\begin{aligned} d_\alpha(Gf^{-1}x_0, Gf^{-1}x_3) &\le d_\alpha(Gf^{-1}x_0, Gf^{-1}x_1) + d_\alpha(Gf^{-1}x_1, Gf^{-1}x_2) \\ &\quad + d_\alpha(Gf^{-1}x_2, Gf^{-1}x_3) \\ &< [r_\alpha - \phi_\alpha(r_\alpha)] + \phi_\alpha(r_\alpha - \phi_\alpha(r_\alpha)) \\ &\quad + \phi_\alpha^2(r_\alpha - \phi_\alpha(r_\alpha)) \\ &\le r_\alpha + [\sum_{j=1}^{\infty} \phi_\alpha^j(r_\alpha - \phi_\alpha(r_\alpha)) - \phi_\alpha(r_\alpha)] \\ &\le r_\alpha. \end{aligned}$$

Proceed inductively to obtain $Gf^{-1}x_n \in Ff^{-1}x_{n-1}$ for $n \in \{3, 4, \cdots\}$ with

$$Gf^{-1}x_n \in B(Gf^{-1}x_0, r)$$

and

$$d_\alpha(Gf^{-1}x_n, Gf^{-1}x_{n+1}) < \phi_\alpha^n(r_\alpha - \phi_\alpha(r_\alpha))$$

for each $\alpha \in \Lambda$. Now (5.4) implies that $\{Gf^{-1}x_n\}$ is Cauchy with respect to d_α for each $\alpha \in \Lambda$. Consequently $\{Gf^{-1}x_n\}$ is a Cauchy sequence in E. Since E is complete, there exists $x \in \overline{B(Gf^{-1}x_0, r)}$ with $Gf^{-1}x_n \to x$. Now since $Gf^{-1}x_{n+1} \in Ff^{-1}x_n$ for $n \in \{1, 2, \cdots\}$, it follows that from the continuity of f^{-1} and G and f-hybrid compatibility of F and G that

$$\lim_{n\to\infty} dist_\alpha(Gf^{-1}x, Ff^{-1}Gf^{-1}x_n) = 0$$

for each $\alpha \in \Lambda$. Now fix $\alpha \in \Lambda$. Then

$$\begin{aligned} dist_\alpha(Gf^{-1}x, Ff^{-1}x) &\le dist_\alpha(Gf^{-1}x, Ff^{-1}Gf^{-1}x_n) \\ &\quad + D_\alpha(Ff^{-1}Gf^{-1}x_n, Ff^{-1}x) \\ &\le dist_\alpha(Gf^{-1}x, Ff^{-1}Gf^{-1}x_n) \\ &\quad + \phi_\alpha(\max\{d_\alpha(Gf^{-1}x, Gf^{-1}Gf^{-1}x_n), \\ &\quad dist_\alpha(Gf^{-1}x, Ff^{-1}x), \\ &\quad dist_\alpha(Gf^{-1}Gf^{-1}x_n, Ff^{-1}Gf^{-1}x_n), \\ &\quad \frac{1}{2}[dist_\alpha(Gf^{-1}x, Ff^{-1}Gf^{-1}x_n) \\ &\quad + dist_\alpha(Gf^{-1}Gf^{-1}x_n, Ff^{-1}x)]\}). \end{aligned}$$

Letting $n \to \infty$, we obtain (as in Theorem 4.1) that

$$\begin{aligned} dist_\alpha(Ff^{-1}x, Gf^{-1}x) &\le \phi_\alpha(\max\{0, dist_\alpha(Gf^{-1}x, Ff^{-1}x), 0, \\ &\quad \frac{1}{2}dist_\alpha(Gf^{-1}x, Ff^{-1}x)\}) \\ &= \phi_\alpha(dist_\alpha(Gf^{-1}x, Ff^{-1}x)). \end{aligned}$$

This implies that

$$dist_\alpha(Gf^{-1}x, Ff^{-1}x) = 0$$

for each $\alpha \in \Lambda$. Thus

$$Gf^{-1}x \in \overline{Ff^{-1}x} = Ff^{-1}x.$$

As in Theorem 4.1 it is easy to check that Ff^{-1} and Gf^{-1} have a common fixed point provided $Gf^{-1}x = Gf^{-1}Gf^{-1}x$ and $Gf^{-1}x \in \overline{B(Gf^{-1}x_0, r)} \cup fG^{-1}(B(Gf^{-1}x_0, r))$. Further, let $f^{-1}x = y$ then since f is a surjective map, we have a unique $y \in f^{-1}\Big(\overline{B(Gf^{-1}x_0, r)}\Big)$ with $fy = Gy \in Fy$. □

If $Y = E$ and f is the identity map on E then our theorem 5.1 reduces to the following result of O'Regan et al. [18, Theorem 5.1].

Corollary 5.2. *Let E be a complete gauge space, $x_0 \in E$, $r = \{r_\alpha\}_{\alpha\in\Lambda} \in (0,\infty)^\Lambda$ with $F : (\overline{B(Gx_0, r)} \cup G^{-1}(B(Gx_0, r))) \to CD(E)$ and $G : E \to E$ compatible maps on $\overline{B(Gx_0, r)}$ and $FG^{-1}(B(Gx_0, r)) \subseteq G(E)$. Suppose G is continuous and for each $\alpha \in \Lambda$, there exists a continuous function $\phi_\alpha : [0,\infty) \to [0,\infty)$ satisfying $\phi_\alpha(z) < z$ for $z > 0$ and ϕ_α strictly increasing on $(0, r_\alpha]$ such that for $x, y \in \overline{B(Gx_0, r)} \cup G^{-1}(B(Gx_0, r))$ we have*

$$D_\alpha(Fx, Fy) \le \phi_\alpha(M_\alpha(x, y)); \tag{5.1'}$$

here

$$M(x, y) = \max\{d_\alpha(Gx, Gy), dist_\alpha(Gx, Fx), dist_\alpha(Gy, Fy), \frac{1}{2}[dist_\alpha(Gx, Fy) + dist_\alpha(Gy, Fx)]\}.$$

Also suppose for each $\alpha \in \Lambda$ that

$$\Phi_\alpha \text{ is strictly increasing on } [0,\infty);\ \text{here } \Phi_\alpha(x) = x - \phi_\alpha(x) \tag{5.2'}$$

$$dist_\alpha(Gx_0, Fx_0) < r_\alpha - \phi_\alpha(r_\alpha) \tag{5.3'}$$

$$\sum_{i=0}^{\infty} \phi_\alpha^i(t) < \infty \text{ for } t \in (0, r_\alpha - \phi_\alpha(r_\alpha)] \tag{5.4'}$$

and

$$\sum_{i=0}^{\infty} \phi_\alpha^i(r_\alpha - \phi_\alpha(r_\alpha)) \le \phi_\alpha(r_\alpha). \tag{5.5'}$$

Finally assume the following condition holds:

$$\left\{\begin{array}{c} \text{for every } x \in G^{-1}(B(Gx_0, r)) \text{ and every } \epsilon = \{\epsilon_\alpha\}_{\alpha\in\Lambda} \in (0,\infty)^\Lambda \\ \text{there exists } y \in Fx \text{ with } d_\alpha(Gx, y) \le dist_\alpha(Gx, Fx) + \epsilon_\alpha \\ \text{for every } \alpha \in \Lambda. \end{array}\right. \tag{5.6'}$$

Then there exists $x \in \overline{B(Gx_0, r)}$ *with* $Gx \in Fx$. *Moreover,* F *and* G *have a common fixed point* Gx *provided* $GGx = Gx$ *and* $Gx \in \overline{B(Gx_0, r)} \cup G^{-1}(B(Gx_0, r))$.

Theorem 5.3. *Let* Y *be an arbitrary space,* E *be a complete gauge space,* $f : Y \to E$ *be a bijection map, and* U *an open subset of* E *with* $H : f^{-1}(\overline{U} \cup fG^{-1}(U)) \times [0, 1] \to CD(E)$ *and* $G : Y \to E$ *and for each* $\lambda \in [0, 1]$, H_λ *and* G *are* f*-hybrid compatible on* $\overline{U}$, *and* $H_\lambda(G^{-1}(U)) \subseteq Gf^{-1}(E)$. *Assume the following conditions hold:*

(i) $Gf^{-1}(U) \subseteq U$;

(ii) Hf^{-1} *is closed,* G *is continuous;*

(iii) *for each* $\alpha \in \Lambda$, *there exists a continuous strictly increasing function* $\phi_\alpha : [0, \infty) \to [0, \infty)$ *satisfying* $\phi_\alpha(z) < z$ *for* $z > 0$ *and for all* $\lambda \in [0, 1]$ *and* $x, y \in \overline{U} \cup fG^{-1}(U)$ *we have*

$$D_\alpha(Hf^{-1}(x, \lambda), Hf^{-1}(y, \lambda)) \le \phi_\alpha(M_\alpha(x, y, \lambda; f^{-1})),$$

$$\begin{aligned} M_\alpha(x, y, \lambda; f^{-1}) = \max\{&d_\alpha(Gf^{-1}(x), Gf^{-1}(y)), dist_\alpha(Gf^{-1}(x), Hf^{-1}(x, \lambda)), \\ &dist_\alpha(Gf^{-1}(y), Hf^{-1}(y, \lambda)), \frac{1}{2}[dist_\alpha(Gf^{-1}(x), Hf^{-1}(y, \lambda)) \\ &+ dist_\alpha(Gf^{-1}(y), Hf^{-1}(x, \lambda))]\}; \end{aligned}$$

(iv) *for every* $\epsilon = \{\epsilon_\alpha\}_{\alpha \in \Lambda} \in (0, \infty)^\Lambda$, *there exists* $\delta = \delta(\epsilon) > 0$ *(which does not depend on* α*) such that when* $t, s \in [0, 1]$ *with* $|t - s| < \delta$, *then*

$$D_\alpha(Hf^{-1}(x, t), Hf^{-1}(x, s)) \le \epsilon_\alpha$$

for all $x \in \overline{U}$ *and all* $\alpha \in \Lambda$;

(v) *for each each* $\alpha \in \Lambda$ *and for any* $s_\alpha \in (0, \infty)$,

$$\sum_{i=0}^{\infty} \phi_\alpha^i(t) < \infty$$

for $t \in (0, s_\alpha - \phi_\alpha(s_\alpha)]$ *and*

$$\sum_{i=0}^{\infty} \phi_\alpha^i(s_\alpha - \phi_\alpha(s_\alpha)) \le \phi_\alpha(s_\alpha);$$

(vi) *for each* $\alpha \in \Lambda$, $\Phi_\alpha : [0, \infty) \to [0, \infty)$ *is monotone increasing (here* $\Phi_\alpha(x) = x - \phi_\alpha(x)$*);*
and

(vii) *for every* $\lambda \in [0, 1]$ *and every* $\epsilon = \{\epsilon_\alpha\}_{\alpha \in \Lambda} \in (0, \infty)^\Lambda$ *there exists* $y \in fG^{-1}(U)$ *with* $Gf^{-1}y \in Ff^{-1}x$ *with*

$$d_\alpha(Gf^{-1}x, Gf^{-1}y) \le dist_\alpha(Gf^{-1}x, Ff^{-1}x) + \epsilon_\alpha$$

for every $\alpha \in \Lambda$.

(viii) *there exists* $\alpha_0 \in \Lambda$ *with*

$$\inf\{dist_{\alpha_0}(Gf^{-1}(x), H_t f^{-1}(x)) : x \in \partial U, t \in [0,1]\} > 0;$$

here $H_t f^{-1}(\cdot) = Hf^{-1}(\cdot, t)$.

In addition assume $H_0 f^{-1}$ *and* Gf^{-1} *have a coincidence point (i.e. there exists* $x \in fG^{-1}(U)$ *with*

$$Gf^{-1}(x) \in H_0 f^{-1}(x)).$$

Then $H_1 f^{-1}$ *and* Gf^{-1} *have a coincidence point. Moreover, there exists a unique* $y \in f^{-1}\Big(\overline{B(Gf^{-1}x_0, r)}\Big)$ *with* $fy = Gy \in Fy$.

Remark 5.2. Note

$$\text{for } \lambda \in [0,1],\ Gf^{-1}(x) \notin Hf^{-1}(x,\lambda) \text{ for } x \in \partial U$$

is implicitly implied by the other assumptions.

Proof. Let

$$A = \{\lambda \in [0,1] : Gf^{-1}(x) \in Hf^{-1}(x,\lambda) \text{ for some } x \in fG^{-1}(U)\}.$$

Since $H_0 f^{-1}$ and Gf^{-1} have a coincidence point, A is nonempty.

First we show A is open in $[0,1]$. Let $\lambda_0 \in A$. Then there exists $x_0 \in fG^{-1}(U)$ with

$$Gf^{-1}(x_0) \in Hf^{-1}(x_0, \lambda_0).$$

Then $Gf^{-1}(x_0) \in U$. Since U is open, there exists $\delta = \{\delta_\alpha\}_{\alpha \in \Lambda} \in (0,\infty)^\Lambda$ with

$$\overline{B(Gf^{-1}(x_0), \delta)} \subseteq U.$$

Now fix $\alpha \in \Lambda$. Then by (iv), there exists $\eta(\delta) > 0$ with

$$dist_\alpha(Gf^{-1}(x_0), Hf^{-1}(x_0,\lambda) \le D_\alpha(Hf^{-1}(x_0,\lambda_0), Hf^{-1}(x_0,\lambda)) < \delta_\alpha - \phi_\alpha(\delta_\alpha)$$

for $\lambda \in [0,1]$ and $|\lambda - \lambda_0| < \eta$. Now Theorem 5.1 guarantees that there exists $x_\lambda \in \overline{B(Gf^{-1}(x_0), \delta)} \subseteq U$ with

$$Gf^{-1}(x_\lambda) \in H_\lambda f^{-1}(x_\lambda)$$

for $\lambda \in [0,1]$ and $|\lambda - \lambda_0| < \eta$. As a result A is open in $[0,1]$.

Next we show A is closed in $[0,1]$. Let $\{\lambda_k\}$ be a sequence in A with $\lambda_k \to \lambda \in [0,1]$ as $k \to \infty$. By definition, for each k, there exists $x_k \in fG^{-1}(U)$ with

$$Gf^{-1}(x_k) \in Hf^{-1}(x_k, \lambda_k).$$

We claim

$$\inf_{k \ge 1} dist_{\alpha_0}(Gf^{-1}(x_k), \partial U) > 0.$$

Suppose it is not true. Fix $i \in \{1, 2, \cdots\}$. Then there exist $n_i \in \{1, 2, \cdots\}$ and a $y_{n_i} \in \partial U$ such that

$$d(Gf^{-1}(x_{n_i}), y_{n_i}) < \frac{1}{l(i)} (\text{with } \ l(i) \to \infty \ \text{ if } \ i \to \infty).$$

Since f^{-1} and G is continuous, we may assume

$$d(Gf^{-1}(Gf^{-1}(x_{n_i})), Gf^{-1}(y_{n_i})) < \frac{1}{i}.$$

Therefore, there exist a subsequence S_{α_0} of $\{1, 2, \cdots\}$ and a sequence $\{y_i\} \subseteq \partial U$ (for $i \in S_{\alpha_0}$) such that

$$d(Gf^{-1}(Gf^{-1}(x_i)), Gf^{-1}(y_i)) < \frac{1}{i} \text{ for } i \in S.$$

This together with (viii) implies

$$\begin{aligned} 0 &< \inf\{dist_{\alpha_0}(Gf^{-1}(x), H_\lambda f^{-1}(x)) : x \in \partial U, \lambda \in [0,1]\} \\ &\leq \liminf_{i \to \infty \text{ in } S_{\alpha_0}} dist_{\alpha_0}(Gf^{-1}(y_i), H_{\lambda_i} f^{-1}(y_i)). \end{aligned}$$

Essentially the same argument as in Theorem 4.5 guarantees that

$$\liminf_{i \to \infty \text{ in } S_{\alpha_0}} dist_{\alpha_0}(Gf^{-1}(y_i), H_{\lambda_i} f^{-1}(y_i)) = 0,$$

so this contradicts (5.17). Consequently (5.16) is true. So, there exists $\epsilon_{\alpha_0} > 0$ with

$$d_{\alpha_0}(Gf^{-1}(x_k), z) > \epsilon_{\alpha_0}$$

for all $k \geq 1$ and for all $z \in \partial U$. Since $Gf^{-1}(x_k) \in U$ for each k, there exists $\epsilon = \{\epsilon_\alpha\}_{\alpha \in \Lambda} \in (0, \infty)^\Lambda$ such that

$$\overline{B(Gf^{-1}(x_k), \epsilon)} \subseteq U$$

for $k \geq 1$. Fix $\alpha \in \Lambda$. Then by (iv) there exists an integer n_0 (which does not depend on α) such that

$$\begin{aligned} dist_\alpha(Gf^{-1}(x_{n_0}), H_\lambda f^{-1}(x_{n_0})) &\leq D_\alpha(Hf^{-1}(x_{n_0}, \lambda_{n_0}), Hf^{-1}(x_{n_0}, \lambda)) \\ &< \epsilon_\alpha - \phi_\alpha(\epsilon_\alpha). \end{aligned}$$

Now Theorem 5.1 guarantees that Gf^{-1} and $H_\lambda f^{-1}$ have a coincidence point $x_{\lambda, n_0} \in \overline{B(Gf^{-1}(x_{n_0}), \delta)} \subset U$. As a result, $\lambda \in A$. Hence A is closed in $[0, 1]$. This completes the proof. □

If $Y = E$ and f is the identity map on E then our theorem 5.3 reduces to the following result of O'Regan et al. [18, Theorem 5.2].

Corollary 5.4. *Let E be a complete guage space and U an open subset of E with $H : (\overline{U} \cup G^{-1}(U)) \times [0, 1] \to CD(E)$ and $G : E \to E$ and for each*

$\lambda \in [0,1]$, H_λ and G are compatible on $\overline{U}$, and $H_\lambda(G^{-1}(U)) \subseteq G(E)$. Assume the following conditions hold:

(i) *$G(U) \subseteq U$;*

(ii) *H is closed, G is continuous;*

(iii) *for each $\alpha \in \Lambda$, there exists a continuous strictly increasing function $\phi_\alpha : [0,\infty) \to [0,\infty)$ satisfying $\phi_\alpha(z) < z$ for $z > 0$ and for all $\lambda \in [0,1]$ and $x, y \in \overline{U} \cup G^{-1}(U)$ we have*

$$D_\alpha(H(x,\lambda), H(y,\lambda)) \leq \phi_\alpha(M_\alpha(x,y,\lambda)),$$

$$\begin{aligned} M_\alpha(x,y,\lambda) = \max\{&d_\alpha(G(x),G(y)), dist_\alpha(G(x),H(x,\lambda)),\\ &dist_\alpha(G(y),H(y,\lambda)), \frac{1}{2}[dist_\alpha(G(x),H(y,\lambda))\\ &+ dist_\alpha(G(y),H(x,\lambda))]\};\end{aligned}$$

(iv) *for every $\epsilon = \{\epsilon_\alpha\}_{\alpha\in\Lambda} \in (0,\infty)^\Lambda$, there exists $\delta = \delta(\epsilon) > 0$ (which does not depend on α) such that when $t, s \in [0,1]$ with $|t-s| < \delta$, then*

$$D_\alpha(H(x,t), H(x,s)) \leq \epsilon_\alpha$$

for all $x \in \overline{U}$ and all $\alpha \in \Lambda$;

(v) *for each each $\alpha \in \Lambda$ and for any $s_\alpha \in (0,\infty)$,*

$$\sum_{i=0}^{\infty} \phi_\alpha^i(t) < \infty$$

for $t \in (0, s_\alpha - \phi_\alpha(s_\alpha)]$ and

$$\sum_{i=0}^{\infty} \phi_\alpha^i(s_\alpha - \phi_\alpha(s_\alpha)) \leq \phi_\alpha(s_\alpha);$$

(vi) *for each $\alpha \in \Lambda$, $\Phi_\alpha : [0,\infty) \to [0,\infty)$ is strictly increasing (here $\Phi_\alpha(x) = x - \phi_\alpha(x)$);*
and

(vii) *for every $\lambda \in [0,1]$ and every $\epsilon = \{\epsilon_\alpha\}_{\alpha\in\Lambda} \in (0,\infty)^\Lambda$ there exists $y \in G^{-1}(U)$ with $Gy \in Fx$ with*

$$d_\alpha(Gx, Gy) \leq dist_\alpha(Gx, Fx) + \epsilon_\alpha$$

for every $\alpha \in \Lambda$.

(viii) *there exists $\alpha_0 \in \Lambda$ with*

$$\inf\{dist_{\alpha_0}(G(x), H_t(x)) : x \in \partial U, t \in [0,1]\} > 0;$$

here $H_t(.) = H(.,t)$.

In addition assume H_0 *and* G *have a coincidence point (i.e. there exists* $x \in G^{-1}(U)$ *with* $G(x) \in H_0(x)$*). Then* H_1 *and* G *have a coincidence point.*

O'Regan et al. [18, Theorem 5.3] proved the following result:

Theorem 5.5. *Let* E *be a complete gauge space and* U *an open subset of* E *with* $H : (\overline{U} \cup G^{-1}(U)) \times [0,1] \to CD(E)$ *and* $G : E \to E$ *and for each* $\lambda \in [0,1]$, H_λ *and* G *are compatible on* $\overline{U}$, *and* $H_\lambda(G^{-1}(U)) \subseteq G(E)$. *Assume the following conditions hold:*

(i) *for* $\lambda \in [0,1]$, $G(x) \notin H(x,\lambda)$ *for* $x \in \partial U$ *and* $G(U) \subseteq U$;

(ii) H *is closed,* G *is continuous and for each* $\alpha \in \Lambda$,

$$d_\alpha(G(x), G(y)) \le d_\alpha(G(x), y)$$

for all $x \in G^{-1}(U)$ *and* $y \in U$;

(iii) *for each* $\alpha \in \Lambda$, *there exists a continuous strictly increasing function* $\phi_\alpha : [0,\infty) \to [0,\infty)$ *satisfying* $\phi_\alpha(z) < z$ *for* $z > 0$ *and for all* $\lambda \in [0,1]$ *and* $x, y \in \overline{U} \cup G^{-1}(U)$ *we have*

$$D_\alpha(H(x,\lambda), H(y,\lambda)) \le \phi_\alpha(M_\alpha(x,y,\lambda)),$$

$$\begin{aligned} M_\alpha(x,y,\lambda) = \max\{&d_\alpha(G(x),G(y)), dist_\alpha(G(x),H(x,\lambda)),\\ &dist_\alpha(G(y),H(y,\lambda)), \frac{1}{2}[dist_\alpha(G(x),H(y,\lambda))\\ &+ dist_\alpha(G(y),H(x,\lambda))]\};\end{aligned}$$

(iv) *there exists* $M = \{M_\alpha\}_{\alpha\in\Lambda} \in (0,\infty)^\Lambda$ *and there exists a continuous increasing function* $\psi : [0,1] \to \mathbf{R}$ *such that for every* $\alpha \in \Lambda$,

$$D_\alpha(H(x,t), H(x,s)) \le M_\alpha|\psi(t) - \psi(s)|$$

for all $t, s \in [0,1]$ *and* $x \in \overline{U}$;

(v) *for each each* $\alpha \in \Lambda$ *and for any* $s_\alpha \in (0,\infty)$,

$$\sum_{i=0}^{\infty} \phi_\alpha^i(t) < \infty$$

for $t \in (0, s_\alpha - \phi_\alpha(s_\alpha)]$ *and*

$$\sum_{i=0}^{\infty} \phi_\alpha^i(s_\alpha - \phi_\alpha(s_\alpha)) \le \phi_\alpha(s_\alpha);$$

(vi) *for each* $\alpha \in \Lambda$, $\Phi_\alpha : [0,\infty) \to [0,\infty)$ *is strictly increasing (here* $\Phi_\alpha(x) = x - \phi_\alpha(x)$*) and*

$$\Phi_\alpha^{-1}(a) + \Phi_\alpha^{-1}(b) \le \Phi_\alpha^{-1}(a+b)$$

for $a \ge 0$ *and* $b \ge 0$;

and

(vii) *for every* $\lambda \in [0,1]$ *and every* $\epsilon = \{\epsilon_\alpha\}_{\alpha\in\Lambda} \in (0,\infty)^\Lambda$ *there exists* $y \in G^{-1}(U)$ *with* $Gy \in Fx$ *with*

$$d_\alpha(Gx, Gy) \le dist_\alpha(Gx, Fx) + \epsilon_\alpha$$

for every $\alpha \in \Lambda$.

In addition assume H_0 *and* G *have a coincidence point (i.e. there exists* $x \in G^{-1}(U)$ *with* $G(x) \in H_0(x)$*). Then* H_1 *and* G *have a coincidence point.*

We now extend and improve the above result in the following:

Theorem 5.6. *Let* Y *be an arbitrary space,* E *be a complete gauge space,* $f : Y \to E$ *be a bijection map, and* U *an open subset of* E *with* $H : f^{-1}(\overline{U} \cup fG^{-1}(U)) \times [0,1] \to CD(E)$ *and* $G : Y \to E$ *and for each* $\lambda \in [0,1]$, H_λ *and* G *are* f*-hybrid compatible on* $\overline{U}$, *and* $H_\lambda(G^{-1}(U)) \subseteq Gf^{-1}(E)$. *Assume the following conditions hold:*

(i) *for* $\lambda \in [0,1]$, $Gf^{-1}(x) \notin Hf^{-1}(x,\lambda)$ *for* $x \in \partial U$ *and* $Gf^{-1}(U) \subseteq U$;

(ii) Hf^{-1} *is closed,* f^{-1} *and* G *are continuous and for each* $\alpha \in \Lambda$,

$$d_\alpha(Gf^{-1}(x), Gf^{-1}(y)) \le d_\alpha(Gf^{-1}(x), y)$$

for all $x \in fG^{-1}(U)$ *and* $y \in U$;

(iii) *for each* $\alpha \in \Lambda$, *there exists a continuous strictly increasing function* $\phi_\alpha : [0,\infty) \to [0,\infty)$ *satisfying* $\phi_\alpha(z) < z$ *for* $z > 0$ *and for all* $\lambda \in [0,1]$ *and* $x, y \in \overline{U} \cup fG^{-1}(U)$ *we have*

$$D_\alpha(Hf^{-1}(x,\lambda), Hf^{-1}(y,\lambda)) \le \phi_\alpha(M_\alpha(x,y,\lambda; f^{-1})),$$

$$\begin{aligned} M_\alpha(x,y,\lambda; f^{-1}) = \max\{&d_\alpha(Gf^{-1}(x), Gf^{-1}(y)), dist_\alpha(Gf^{-1}(x), Hf^{-1}(x,\lambda)),\\ &dist_\alpha(Gf^{-1}(y), Hf^{-1}(y,\lambda)), \frac{1}{2}[dist_\alpha(Gf^{-1}(x), Hf^{-1}(y,\lambda))\\ &+ dist_\alpha(Gf^{-1}(y), Hf^{-1}(x,\lambda))]\}; \end{aligned}$$

(iv) *there exists* $M = \{M_\alpha\}_{\alpha\in\Lambda} \in (0,\infty)^\Lambda$ *and there exists a continuous increasing function* $\psi : [0,1] \to \mathbf{R}$ *such that for every* $\alpha \in \Lambda$,

$$D_\alpha(Hf^{-1}(x,t), Hf^{-1}(x,s)) \le \frac{1}{2} M_\alpha |\psi(t) - \psi(s)|$$

for all $t, s \in [0,1]$ *and* $x \in \overline{U}$;

(v) *for each each* $\alpha \in \Lambda$ *and for any* $s_\alpha \in (0,\infty)$,

$$\sum_{i=0}^{\infty} \phi_\alpha^i(t) < \infty$$

for $t \in (0, s_\alpha - \phi_\alpha(s_\alpha)]$ *and*

$$\sum_{i=0}^{\infty} \phi_\alpha^i(s_\alpha - \phi_\alpha(s_\alpha)) \le \phi_\alpha(s_\alpha);$$

(vi) *for each* $\alpha \in \Lambda$, $\Phi_\alpha : [0,\infty) \to [0,\infty)$ *is monotone increasing (here* $\Phi_\alpha(x) = x - \phi_\alpha(x)$*) ;*
and

(vii) *for every* $\lambda \in [0,1]$ *and every* $\epsilon = \{\epsilon_\alpha\}_{\alpha\in\Lambda} \in (0,\infty)^\Lambda$ *there exists* $y \in fG^{-1}(U)$ *with* $Gf^{-1}y \in Ff^{-1}x$ *with*

$$d_\alpha(Gf^{-1}x, Gf^{-1}y) \le dist_\alpha(Gf^{-1}x, Ff^{-1}x) + \epsilon_\alpha$$

for every $\alpha \in \Lambda$*.*

In addition assume H_0f^{-1} *and* Gf^{-1} *have a coincidence point (i.e. there exists* $x \in fG^{-1}(U)$ *with* $Gf^{-1}(x) \in H_0f^{-1}(x)$*). Then* H_1f^{-1} *and* Gf^{-1} *have a coincidence point.*

Proof. Let

$$Q = \{(t,x) \in [0,1] \times fG^{-1}(U) : Gf^{-1}(x) \in Hf^{-1}(x,t)\}.$$

Then Q is nonempty since H_0 and G have a coincidence point. On Q define the partial order

$$(t,x) \le (s,y) \text{ iff } t \le s \text{ and } d_\alpha(Gf^{-1}(x), Gf^{-1}(y)) \le M_\alpha[\psi(s) - \psi(t)]$$

for every $\alpha \in \Lambda$. Let P be a totally ordered subset of Q and let

$$t^* = \sup\{t : (t,x) \in P\}.$$

Take a sequence $\{(t_n, x_n)\} \subseteq P$ such that

$$(t_n, x_n) \le (t_{n+1}, x_{n+1})$$

and $t_n \to t^*$. Then, as in Theorem 4.6, $\{Gf^{-1}(x_n)\}$ is Cauchy with respect to d_α for each $\alpha \in \Lambda$ and so $(t^\star, x^\star) \in Q$ with

$$(t,x) \le (t^\star, x^\star) \text{ for every } (t,x) \in P.$$

Thus $(t^\star, x^\star)$ is an upper bound of P. By Zorn's lemma Q admits a maximal element $(t_0, x_0) \in Q$. Note $x_0 \in fG^{-1}(U)$ and $Gf^{-1}(x_0) \in Hf^{-1}(x_0, t_0)$.

We claim $t_0 = 1$. Suppose our claim is false. Note since U is open, there exist $\delta_1, \cdots, \delta_m \in (0,\infty)$ with

$$U(Gf^{-1}x_0, \delta_1) \cap \cdots \cap U(Gf^{-1}x_0, \delta_m) \subseteq U;$$

here $U(Gf^{-1}x_0, \delta_i) = \{x : d_{\alpha_i}(x, Gf^{-1}x_0) < \delta_i\}$ for $i \in \{1,2,\cdots,m\}$ and $\alpha_i \in \Lambda$ for $i \in \{1,2,\cdots,m\}$. Choose $r = \{r_\alpha\}_{\alpha\in\Lambda} \in (0,\infty)^\Lambda$ and $t \in (t_0, 1]$ with

$$\overline{B(Gf^{-1}x_0, r)} \subseteq U \text{ and } r_\alpha - \phi_\alpha(r_\alpha) = M_\alpha[\psi(t) - \psi(t_0)].$$

Notice for every $\alpha \in \Lambda$ that

$$\begin{aligned} dist_\alpha(Gf^{-1}(x_0), Hf^{-1}(x_0,t)) &\leq dist_\alpha(Gf^{-1}(x_0), Hf^{-1}(x_0,t_0)) \\ &\quad + D_\alpha(Hf^{-1}(x_0,t_0), Hf^{-1}(x_0,t)) \\ &\leq \frac{1}{2}M_\alpha[\psi(t)-\psi(t_0)] \\ &= \frac{1}{2}(r_\alpha - \phi_\alpha(r_\alpha)) < r_\alpha - \phi_\alpha(r_\alpha). \end{aligned}$$

Now Theorem 5.1 guarantees that $H_t f^{-1}$ and Gf^{-1} have a coincidence point $x \in \overline{B(Gf^{-1}x_0, r)}$. Note $x \in U$ and $Gf^{-1}(x) \in U$ (from (i)), so $x \in fG^{-1}(U)$. Hence $(t,x) \in Q$ and from (ii), we have

$$\begin{aligned} d_\alpha(Gf^{-1}(x_0), Gf^{-1}(x)) &\leq d_\alpha(Gf^{-1}(x_0), x) \\ &\leq r_\alpha \leq M_\alpha[\psi(t)-\psi(t_0)] \text{ and } t_0 < t \end{aligned}$$

for every $\alpha \in \Lambda$. Therefore, $(t_0, x_0) < (t, x)$. This contradicts the maximality of (t_0, x_0). As a result, $t_0 = 1$ and so we are finished. □

Our final result was motivated by a result in [17].

Theorem 5.7. *Let Y be an arbitrary space, E be a complete gauge space, $f : Y \to E$ be a bijection map, $x_0 \in E$, $r = \{r_\alpha\}_{\alpha\in\Lambda} \in (0,\infty)^\Lambda$ with $F : f^{-1}(\overline{B(Gf^{-1}x_0, r)} \cup fG^{-1}(B(Gx_0, r))) \to CD(E)$ and $G : Y \to E$ compatible maps on $\overline{B(Gf^{-1}x_0, r)}$ and $FG^{-1}(B(Gf^{-1}x_0, r)) \subseteq Gf^{-1}(E)$. Suppose f^{-1} and G are continuous and for each $\alpha \in \Lambda$, there exists a continuous monotone increasing function $\phi_\alpha : [0,\infty) \to [0,\infty)$ satisfying $\phi_\alpha(z) < z$ for $z > 0$. Also assume there exists functions $\beta : \Lambda \to \Lambda$ and $\gamma : \Lambda \to \Lambda$ such that for each $\alpha \in \Lambda$ and $x, y \in \overline{B(Gf^{-1}x_0, r)} \cup fG^{-1}(B(Gf^{-1}x_0, r))$ we have*

$$D_\alpha(Ff^{-1}x, Ff^{-1}y) \leq \phi_{\beta(\alpha)}(d_{\gamma(\alpha)}(Gf^{-1}x, Gf^{-1}y)). \tag{5.16}$$

Further suppose the following conditions hold:

$$\text{for each } \alpha \in \Lambda \text{ we have } dist_\alpha(Gf^{-1}x_0, Ff^{-1}x_0) < r_\alpha - \phi_{\beta(\alpha)}(r_\alpha) \tag{5.17}$$

and

$$\left\{ \begin{array}{c} \text{for every } x \in fG^{-1}(B(Gf^{-1}x_0, r)) \text{ and every } \epsilon = \{\epsilon_\alpha\}_{\alpha\in\Lambda} \in (0,\infty)^\Lambda \\ \text{there exists } y \in Fx \text{ with } d_\alpha(Gf^{-1}x, y) \leq dist_\alpha(Gf^{-1}x, Ff^{-1}x) + \epsilon_\alpha \\ \text{for every } \alpha \in \Lambda. \end{array} \right. \tag{5.18}$$

Finally assume for each $\alpha \in \Lambda$ that

$$\sum_{n=1}^{\infty} \phi_{\beta(\alpha)}\phi_{\beta(\gamma(\alpha))}...\phi_{\beta(\gamma^{n-1}(\alpha))}\ (r_{\gamma^n(\alpha)} - \phi_{\beta(\gamma^n(\alpha))}(r_{\gamma^n(\alpha)})) \tag{5.19}$$

$$\leq \phi_{\beta(\alpha)}(r_\alpha).$$

Then there exists $x \in \overline{B(Gf^{-1}x_0, r)}$ with $Gf^{-1}x \in Ff^{-1}x$ and Ff^{-1} and Gf^{-1} have a common fixed point $Gf^{-1}x$ provided $Gf^{-1}Gf^{-1}x = Gf^{-1}x$ and $Gf^{-1}x \in \overline{B(Gf^{-1}x_0, r)} \cup fG^{-1}(B(Gf^{-1}x_0, r))$. Moreover, there exists a unique $y \in f^{-1}\Big(\overline{B(Gf^{-1}x_0, r)}\Big)$ with $fy = Gy \in Fy$.

Proof. From (5.17) and (5.18), we may choose $z \in Ff^{-1}x_0$ with

$$d_\alpha(Gf^{-1}x_0, z) < r_\alpha - \phi_{\beta(\alpha)}(r_\alpha) \text{ for every } \alpha \in \Lambda.$$

Since $Gf^{-1}x_0 \in B(Gf^{-1}x_0, r)$ and $FG^{-1}(B(Gf^{-1}x_0, r)) \subseteq Gf^{-1}(E)$ and so $Ff^{-1}(x_0) \subseteq Gf^{-1}(E)$, we have $z \in Gf^{-1}(E)$. Therefore, there exists $x_1 \in E$ with $z = Gf^{-1}x_1$. As a result, we have

$$Gf^{-1}x_1 \in Ff^{-1}x_0$$

and

$$d(Gf^{-1}x_1, Gf^{-1}x_0) < r_\alpha - \phi_{\beta(\alpha)}(r_\alpha) \text{ for every } \alpha \in \Lambda. \tag{5.20}$$

Notice $Gf^{-1}x_1 \in B(Gf^{-1}x_0, r)$. Now for $\alpha \in \Lambda$, choose $\epsilon_\alpha > 0$ with

$$\phi_{\beta(\alpha)}(d_{\gamma(\alpha)}(Gf^{-1}x_0, Gf^{-1}x_1)) + \epsilon_\alpha < \phi_{\beta(\alpha)}(r_{\gamma(\alpha)} - \phi_{\beta(\gamma(\alpha))}(r_{\gamma(\alpha)})) \tag{5.21}$$

(this is possible from (5.20) and the fact that ϕ_α is monotone increasing). From (5.18) we can choose $y \in Ff^{-1}x_1$ such that for every $\alpha \in \Lambda$, we have

$$d_\alpha(Gf^{-1}x_1, y) \leq dist_\alpha(Gf^{-1}x_1, Ff^{-1}x_1) + \epsilon_\alpha \leq D_\alpha(Ff^{-1}x_0, Ff^{-1}x_1) + \epsilon_\alpha.$$

Since $Gf^{-1}x_1 \in B(Gf^{-1}x_0, r)$ and $FG^{-1}(B(Gf^{-1}x_0, r)) \subseteq Gf^{-1}(E)$ we have

$$Ff^{-1}x_1 \subseteq Gf^{-1}(E),$$

and so $y \in Gf^{-1}(E)$. Thus $y = Gf^{-1}x_2$ for some $x_2 \in E$. As a result,

$$\begin{aligned} d_\alpha(Gf^{-1}x_1, Gf^{-1}x_2) &\leq dist_\alpha(Gf^{-1}x_1, Ff^{-1}x_1) + \epsilon_\alpha \\ &\leq D_\alpha(Ff^{-1}x_0, Ff^{-1}x_1) + \epsilon_\alpha \\ &\leq \phi_{\beta(\alpha)}(d_{\gamma(\alpha)}(Gf^{-1}x_0, Gf^{-1}x_1)) + \epsilon_\alpha \end{aligned}$$

and this together with (5.21) gives for each $\alpha \in \Lambda$ that

$$d_\alpha(Gf^{-1}x_1, Gf^{-1}x_2) < \phi_{\beta(\alpha)}(r_{\gamma(\alpha)} - \phi_{\beta(\gamma(\alpha))}(r_{\gamma(\alpha)})). \tag{5.22}$$

Notice for each $\alpha \in \Lambda$

$$\begin{aligned} d_\alpha(Gf^{-1}x_2, Gf^{-1}x_0) &\leq d_\alpha(Gf^{-1}x_0, Gf^{-1}x_1) + d_\alpha(Gf^{-1}x_1, Gf^{-1}x_2) \\ &< [r_\alpha - \phi_{\beta(\alpha)}(r_\alpha)] + \phi_{\beta(\alpha)}(r_{\gamma(\alpha)} \\ &\quad -\phi_{\beta(\gamma(\alpha))}(r_{\gamma(\alpha)})) \\ &\leq [r_\alpha - \phi_{\beta(\alpha)}(r_\alpha)] + \phi_{\beta(\alpha)}(r_\alpha) = r_\alpha \end{aligned}$$

and so $Gf^{-1}x_2 \in B(Gf^{-1}x_0, r)$. Now fix $\alpha \in \Lambda$ and choose $\delta_\alpha > 0$ so that

$$\phi_{\beta(\alpha)}(d_{\gamma(\alpha)}(Gf^{-1}x_1, Gf^{-1}x_2)) + \delta_\alpha < \phi_{\beta(\alpha)}\phi_{\beta(\gamma(\alpha))}(r_{\gamma^2(\alpha)} - \phi_{\beta(\gamma^2(\alpha))}(r_{\gamma^2(\alpha)})). \tag{5.23}$$

Again from (5.18) we can choose $z \in Ff^{-1}x_2$ such that for every $\alpha \in \Lambda$, we have

$$d_\alpha(Gf^{-1}x_2, z) \leq dist_\alpha(Gf^{-1}x_2, Ff^{-1}x_2) + \delta_\alpha \leq D_\alpha(Ff^{-1}x_1, Ff^{-1}x_2) + \delta_\alpha.$$

Since $Gf^{-1}x_2 \in B(Gf^{-1}x_0, r)$ and $FG^{-1}(B(Gf^{-1}x_0, r)) \subseteq Gf^{-1}(E)$ we have

$$Ff^{-1}x_2 \subseteq Gf^{-1}(E),$$

and so $z \in Gf^{-1}(E)$. Thus $z = Gf^{-1}x_2$ for some $x_2 \in E$. Consequently,

$$\begin{aligned} d_\alpha(Gf^{-1}x_2, Gf^{-1}x_3) &\leq dist_\alpha(Gf^{-1}x_2, Ff^{-1}x_2) + \delta_\alpha \\ &\leq D_\alpha(Ff^{-1}x_1, Ff^{-1}x_2) + \delta_\alpha \\ &\leq \phi_{\beta(\alpha)}(d_{\gamma(\alpha)}(Gf^{-1}x_1, Gf^{-1}x_2)) + \delta_\alpha \end{aligned}$$

and this together with (5.23) yields for each $\alpha \in \Lambda$ that

$$d_\alpha(Gf^{-1}x_2, Gf^{-1}x_3) < \phi_{\beta(\alpha)}\phi_{\beta(\gamma(\alpha))}(r_{\gamma^2(\alpha)} - \phi_{\beta(\gamma^2(\alpha))}(r_{\gamma^2(\alpha)})). \tag{5.24}$$

Notice $Gf^{-1}x_3 \in B(Gf^{-1}x_0, r)$. Proceed inductively to obtain $Gf^{-1}x_{n+1} \in Ff^{-1}x_n$ for $n \in \{2, 3, \cdots\}$ such that

$$\begin{aligned} d_\alpha(Gf^{-1}x_n, Gf^{-1}x_{n+1}) < \phi_{\beta(\alpha)}\phi_{\beta(\gamma(\alpha))} \cdots \phi_{\beta(\gamma^{n-1}(\alpha))}(r_{\gamma^n(\alpha)} \\ - \phi_{\beta(\gamma^n(\alpha))}(r_{\gamma^n(\alpha)})) \end{aligned} \tag{5.25}$$

for each $\alpha \in \Lambda$. Notice $Gf^{-1}x_{n+1} \in B(Gf^{-1}x_0, r)$ for each $n \in \{2, 3, \cdots\}$ since for $\alpha \in \Lambda$ we have

$$\begin{aligned}
&d_\alpha(Gf^{-1}x_{n+1}, Gf^{-1}x_0)\\
&\leq d_\alpha(Gf^{-1}x_0, Gf^{-1}x_1) + d_\alpha(Gf^{-1}x_1, Gf^{-1}x_2) + \cdots\\
&\quad + d_\alpha(Gf^{-1}x_n, Gf^{-1}x_{n+1})\\
&\leq \sum_{k=1}^{\infty} \phi_{\beta(\alpha)}\phi_{\beta(\gamma(\alpha))}...\phi_{\beta(\gamma^{k-1}(\alpha))}(r_{\gamma^k(\alpha)} - \phi_{\beta(\gamma^k(\alpha))}(r_{\gamma^k(\alpha)}))\\
&\quad + d_\alpha(Gf^{-1}x_0, Gf^{-1}x_1)\\
&\leq d_\alpha(Gf^{-1}x_0, Gf^{-1}x_1) + \phi_{\beta(\alpha)}(r_\alpha)\\
&< [r_\alpha - \phi_{\beta(\alpha)}(r_\alpha)] + \phi_{\beta(\alpha)}(r_\alpha)\\
&= r_\alpha.
\end{aligned}$$

Also for each $\alpha \in \Lambda$ and $n, p \in \{0, 1, \cdots\}$, we have

$$\begin{aligned}
d_\alpha(Gf^{-1}x_{n+p}, Gf^{-1}x_n) \quad \leq \quad & \sum_{k=n}^{\infty} \phi_{\beta(\alpha)}\phi_{\beta(\gamma(\alpha))}...\phi_{\beta(\gamma^{k-1}(\alpha))}(r_{\gamma^k(\alpha)} -\\
& \phi_{(\beta(\gamma^k(\alpha))}(r_{\gamma^k(\alpha)})).
\end{aligned}$$

This together with (5.19) guarantees that $\{Gf^{-1}x_n\}$ is a Cauchy sequence with respect to d_α. Consequently, there exists $x \in \overline{B(Gf^{-1}x_0, r)}$ with $Gf^{-1}x_n \to x$. Now since $Gf^{-1}x_{n+1} \in Ff^{-1}x_n$ for $n \in \{1, 2, \cdots\}$, we have from the continuity of f^{-1} and G and f-hybrid compatibility of F and G that

$$\lim_{n\to\infty} dist_\alpha(Gf^{-1}x, Ff^{-1}Gf^{-1}x_n) = 0$$

for each $\alpha \in \Lambda$. Now fix $\alpha \in \Lambda$. Then

$$\begin{aligned}
dist_\alpha(Gf^{-1}x, Ff^{-1}x) \quad \leq \quad & dist_\alpha(Gf^{-1}x, Ff^{-1}Gf^{-1}x_n)\\
& + D_\alpha(Ff^{-1}Gf^{-1}x_n, Ff^{-1}x)\\
\leq \quad & dist_\alpha(Gx, FGx_n) +\\
& \phi_{\beta(\alpha)}(d_{\gamma(\alpha)}(Gf^{-1}Gf^{-1}x_n, Gf^{-1}x)).
\end{aligned}$$

Taking the limit as $n \to \infty$, we obtain $dist_\alpha(Ff^{-1}x, Gf^{-1}x) = 0$ for each $\alpha \in \Lambda$. Thus

$$Gf^{-1}x \in \overline{Ff^{-1}x} = Ff^{-1}x.$$

As in Theorem 4.1 it can be seen that Ff^{-1} and Gf^{-1} have a common fixed point provided $Gf^{-1}x = Gf^{-1}Gf^{-1}x$ and $Gf^{-1}x \in \overline{B(Gf^{-1}x_0, r)} \cup fG^{-1}(B(Gf^{-1}x_0, r))$. Further, let $f^{-1}x = y$ then since f is a surjective map we have a unique $y \in f^{-1}\Big(\overline{B(Gf^{-1}x_0, r)}\Big)$ with $fy = Gy \in Fy$. $\square$

Corollary 5.8. *Let E be a complete gauge space, $x_0 \in E$, $r = \{r_\alpha\}_{\alpha\in\Lambda} \in (0,\infty)^\Lambda$ with $F : (\overline{B(Gx_0, r)} \cup G^{-1}(B(Gx_0, r))) \to CD(E)$ and $G : E \to E$ compatible maps on $\overline{B(Gx_0, r)}$ and $FG^{-1}(B(Gx_0, r)) \subseteq G(E)$. Suppose G is continuous and for each $\alpha \in \Lambda$, there exists a continuous strictly increasing function $\phi_\alpha : [0,\infty) \to [0,\infty)$ satisfying $\phi_\alpha(z) < z$ for $z > 0$. Also assume there exists functions $\beta : \Lambda \to \Lambda$ and $\gamma : \Lambda \to \Lambda$ such that for each $\alpha \in \Lambda$ and $x, y \in \overline{B(Gx_0, r)} \cup G^{-1}(B(Gx_0, r))$ we have*

$$D_\alpha(Fx, Fy) \le \phi_{\beta(\alpha)}(d_{\gamma(\alpha)}(Gx, Gy)).$$

Further suppose the following conditions hold:

$$\text{for each } \alpha \in \Lambda \text{ we have } dist_\alpha(Gx_0, Fx_0) < r_\alpha - \phi_{\beta(\alpha)}(r_\alpha)$$

and

$$\begin{cases} \text{for every } x \in G^{-1}(B(Gx_0, r)) \text{ and every } \epsilon = \{\epsilon_\alpha\}_{\alpha\in\Lambda} \in (0,\infty)^\Lambda \\ \text{there exists } y \in Fx \text{ with } d_\alpha(Gx, y) \le dist_\alpha(Gx, Fx) + \epsilon_\alpha \\ \text{for every } \alpha \in \Lambda. \end{cases}$$

Finally assume for each $\alpha \in \Lambda$ that

$$\sum_{n=1}^{\infty} \phi_{\beta(\alpha)}\phi_{\beta(\gamma(\alpha))}\dots\phi_{\beta(\gamma^{n-1}(\alpha))}\ (r_{\gamma^n(\alpha)} - \phi_{\beta(\gamma^n(\alpha))}(r_{\gamma^n(\alpha)})) \le \phi_{\beta(\alpha)}(r_\alpha).$$

Then there exists $x \in \overline{B(Gx_0, r)}$ with $Gx \in Fx$. Moreover, F and G have a common fixed point Gx provided $GGx = Gx$ and $Gx \in \overline{B(Gx_0, r)} \cup G^{-1}(B(Gx_0, r))$.

Acknowledgement. The first author was supported by University Grants Commission, New Delhi, India (MRP-2005).

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