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EXTRAGRADIENT METHODS FOR GENERALIZED MIXED EQUILIBRIUM PROBLEMS AND FIXED POINT PROBLEMS IN HILBERT SPACES

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Abstract. In this paper, we suggest an iterative schemes on extragradient methods for finding a common element of the set of solutions of generalized mixed equilibrium problems and fixed points of a nonexpansive mappings, and the set of solutions of a variational inequality problems for inverse strongly monotone mappings. We prove the convergence theorems for the sequences generated by these iterative process in Hilbert spaces.

1. Introduction and preliminaries

There are various problems reduced to finding solutions of equilibrium problems, which cover variational inequalities, variational inclusions, complementarity problems, saddle point problems, noncooperative game theory, minimax theory, fixed point problems as special cases.

Equilibrium problems which was initiated by Blum and Oettli [5] has been extensively studied as an effective and powerful tools for a wide range of problems which arises in economics, finance, image reconstruction, ecology, transportation network, engineering and optimization problems [2, 8, 20, 35]. For the variational inequality problem, projection algorithm is efficient. However,

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they request the involving monotone mapping on inverse strongly monotone [24]. To relax the restriction on inverse strongly monotone extragradient algorithms, which have been extensively studied [9, 25] are considered for a variational inequality involving a continuous and monotone mapping in this works.

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$, respectively. Let C be a nonempty closed convex subsets of H. Let $B : C \to H$ be a nonlinear mapping, $F : C \times C \to \mathbb{R}$ be a bifunction and $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a function, where $\mathbb{R}$ is the set of real numbers.

We consider the *generalized mixed equilibrium problems* for finding $x \in C$ such that

$$F(x, y) + \varphi(y) + \langle Bx, y - x \rangle \geq \varphi(x), \ \forall y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $GMEP(F, B, \varphi)$. It is easy to see that x is a solution of problem (1.1) implies that $x \in dom\varphi = \{x \in C : \varphi(x) < +\infty\}$.

Special cases:

(i) If $B = 0$, then problem (1.1) is reduces to the following *mixed equilibrium problem* for finding $x \in C$ such that

$$F(x, y) + \varphi(y) - \varphi(x) \geq 0, \ \forall y \in C, \tag{1.2}$$

studied by Ceng and Yao [7] and also Peng and Yao [26, 27] and its solution set is denoted by $MEP(F, \varphi)$.

(ii) If $\varphi = 0$, then problem (1.1) is reduces to the following *generalized equilibrium problem* for finding $x \in C$ such that

$$F(x, y) + \langle Bx, y - x \rangle \geq 0 \ \forall y \in C, \tag{1.3}$$

studied by Takahashi and Takahashi [29].

(iii) If $\varphi = 0$ and $B = 0$ then problem (1.1) is equivalent to the following *equilibrium problem* for finding $x \in C$ such that

$$F(x, y) \geq 0, \ \forall y \in C, \tag{1.4}$$

studied by Blum and Oettli [5].

(iv) If $F(x, y) = 0$ for all $x, y \in C$, then problem (1.1) is equivalent to the following *generalized nonlinear variational inclusion problem* for finding $x \in C$ such that

$$\langle Bx, y - x \rangle + \varphi(y) - \varphi(x) \geq 0, \ \forall y \in C, \tag{1.5}$$

which is variant form of [3, 4, 28].

(v) If $\varphi = 0$ and $F(x, y) = 0$ for all $x, y \in C$, then problem (1.1) becomes the following *variational inequality problem* for finding $x \in C$ such that

$$\langle Bx, y - x\rangle \geq 0, \ \forall y \in C, \tag{1.6}$$

studied by Lions and Stampacchia [20]. The set of solutions of (1.6) is denoted by $VIP(B, C)$.

(vi) If $B = 0$ and $F(x, y) = 0$ for all $x, y \in C$, then problem (1.1) is reduces to the following *minimization problem* for finding $x \in C$ such that

$$\varphi(y) \geq \varphi(x), \ \forall y \in C. \tag{1.7}$$

In 1976, Korpelevich [16] introduced the extragradient method for the variational inequality problems in the finite dimensional Euclidean spaces as follows::

$$\begin{cases} x_1 = x \in C, \\ y_n = P_C(x_n - \lambda B x_n), \\ x_{n+1} = P_C(x_n - \lambda B y_n), \ \text{for every } n = 0, 1, \cdots, \ \lambda \in (0, \frac{1}{k}) \end{cases} \tag{1.8}$$

where C is a closed convex subset of $\mathbb{R}^n, B : C \to \mathbb{R}^n$ is a monotone and k-Lipschitz continuous mapping and P_C is the metric projection of $\mathbb{R}^n$ onto C. She showed that if $VIP(B, C)$ is nonempty, then the sequences $\{x_n\}$ and $\{y_n\}$ generated by (1.8), converge to the same point $x \in VIP(B, C)$.

Recently, Zeng and Yao [37], and Nadezhkina and Takahashi [21] suggested some iterative schemes based on the extragradient techniques for finding the common point for the set of fixed points of nonexpansive mappings and the set of solutions of a variational inequality problems for a monotone, Lipschitz continuous mapping. Yao and Yao [34] defined the iterative schemes based on the extragradient techniques for finding the common point of the set of fixed points of nonexpansive mappings and the set of solutions of a variational inequality problems for a k-inverse strongly monotone mapping. Plubtieng and Punpaeng [23] introduced an iterative schemes based on the extragradient method for finding the common element of the set of fixed points of nonexpansive mappings, the set of solutions of an equilibrium problem and the set of solution of a variational inequality problems for α-inverse strongly monotone mappings.

In 2003, Takahashi and Toyoda [30] defined the following iterative scheme:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) SP_C(x_n - \lambda_n T x_n), \tag{1.9}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$. They show that if $F(S) \cap VIP(A, C) \neq \emptyset$, then the sequence $\{x_n\}$ defined by (1.9) converges weakly to some point $z \in F(S) \cap VIP(A, C)$, where $F(S)$ denote the fixed point set of the mapping S.

Recently Zeng and Yao [37] introduced the following iterative scheme:

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda_n x_n), \\ x_{n+1} = \alpha_n x_0 + (1-\alpha_n) S P_C(x_n - \lambda_n A y_n), \end{cases} \tag{1.10}$$

where$\{\lambda_n\}$ and $\{\alpha_n\}$ satisfy the following conditions:

(i) $\lambda_n k \subset (0, 1-\delta)$ for some $\delta \in (0,1)$;

(ii) $\alpha_n \subset (0,1), \sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n\to\infty} \alpha_n = 0$.

And, they proved that the sequence $\{x_n\}$ and $\{y_n\}$ generated by (1.10) converge strongly to the some point $P_{F(S)\cap VIP(C,B)} x_0$ provided that

$$\lim_{x\to\infty} \|x_{n+1} - x_n\| = 0.$$

In 2010, Noor and Rassias [22], Huang *et al.* [10], defined the set of projection residual function by

$$R_\lambda(x) = x - P_C(x - \lambda A x). \tag{1.11}$$

It is well known that $x \in C$ is a solution of variational inequality (1.6) if and only if $x \in C$ is a zero of the projection residual function (1.11). They proved the strong convergence result of the iterative scheme (1.9) using the error analysis techniques.

By the recent works [1, 2, 6, 10, 12, 13, 14, 15, 17, 19, 23, 11, 28], we define an iterative process based on the extragradient method for finding a common point of the set of solution of a *generalized mixed equilibrium problems.*

$$\begin{cases} x_1 = x \in C, \\ F(u_n, y) + \langle B x_n, y - u_n\rangle + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall y \in C, \\ y_n = P_C(u_n - \lambda_n A u_n), \\ x_{n+1} = \alpha_n x_n + \alpha_n^{'} S[\beta_n x_n + (1-\beta_n) P_C(y_n - \lambda_n A y_n)] + \alpha_n^{''} e_n, \end{cases}$$

where $\{e_n\}$ is a bounded sequence in C and $\{\alpha_n\}, \{\alpha_n^{'}\}, \{\alpha_n^{''}\}, \{\beta_n\}, \{r_n\}, \{\lambda_n\}$ satisfied some parameter control conditions.

2. Preliminaries

In this paper, we assume that C is a closed convex subset of a real Hilbert space H. Then there exists a unique nearest point in C denoted by $P_C(x)$ such that

$$\|x - P_C(x)\| \leq \|x - y\|, \ \forall y \in C.$$

P_C is called the metric projection of H onto C. It is well known that P_C is a nonexpansive mapping from H onto C and satisfies

$$\langle x - y, P_C(x) - P_C(y)\rangle \geq \|P_C(x) - P_C(y)\|^2, \ \forall x, y \in H.$$

Moreover, $P_C(x)$ is characterized by the following properties: $P_C(x) \in C$ and

$$\langle x - P_C(x), y - P_C(y)\rangle \leq 0, \tag{2.1}$$

$$\|x - y\|^2 \geq \|x - P_C(x)\|^2 + \|y - P_C(x)\|^2, \ \forall x \in H, y \in C.$$

Definition 2.1. A mapping $A : C \to H$ is called *monotone* if

$$\langle Ax - Ay, x - y\rangle \geq 0, \ \forall x, y \in C,$$

and *α-inverse strongly monotone* if there exists a positive real number α such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \forall x, y \in C.$$

Definition 2.2. A mapping $S : C \to C$ is called *nonexpansive* if

$$\|Sx - Sy\| \leq \|x - y\|, \forall x, y \in C,$$

and *pseudocontractive* if

$$\langle Sx - Sy, x - y\rangle \leq \|x - y\|^2, \forall x, y \in C,$$

and also called *k-strictly pseudocontractive* if there exists a constant $k \in [0, 1)$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + \kappa\|(I - S)x - (I - S)y\|^2, \forall x, y \in C.$$

Let $A : C \to H$ be a monotone mapping. The variational inequality problem has the characterization by projection (2.1) as follows:

$$u \in VIP(A, C) \Leftrightarrow u = P_C(u - \lambda Au), \forall \lambda > 0. \tag{2.2}$$

For solving the generalized mixed equilibrium problems, let us give the following assumptions for the bifunction F, function φ and the set C:

(A1) $F(x, x) = 0, \forall x \in C$;

(A2) F is monotone *i.e.,*

$$F(x, y) + F(y, x) \leq 0, \forall x, y \in C;$$

(A3) for any $y \in C$, $x \vdash F(x, y)$ is weakly upper semi-continuous;

(A4) for each $x \in C$, $y \vdash F(x, y)$ is convex;

(A5) for any $x \in C$, $y \vdash F(x, y)$ is lower semi-continuous;

(B1) for each $x \in H$ and $r > 0$, there exists a bounded subset $D_x \subset C$ and $y_x \in C \cap dom\varphi$ such that for every $z \in C - D_x$,

$$F(z, y_n) + \varphi(y_x) + \langle Bz, y_x - z\rangle + \frac{1}{r}\langle y_n - z, z - x\rangle \leq \varphi(z);$$

(B2) C is a bounded set.

Lemma 2.3. [26] *Let C be a nonempty closed convex subset of a Hilbert space H. Let $F : C \times C \to \mathbb{R}$ be a function satisfying $(A1)$-$(A5)$ and $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper convex lower-semicontinuous. Assume that either $(B1)$ or $(B2)$ holds. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r = \big\{z \in C : F(z,y) + \varphi(y) + \langle Bz, y - z\rangle + \frac{1}{r}\langle y - z, z - x\rangle \leq \varphi(z), \forall y \in C\big\}.$$

Then the following statements hold:

(1) *for each $x \in H, T_r(x) \neq \emptyset$;*
(2) *T_r is single-valued;*
(3) *T_r is firmly nonexpansive, i.e., for any $x, y \in H$*

$$\|T_r(x) - T_r(y)\|^2 \leq \langle T_r(x) - T_r(y), x - y\rangle;$$

(4) *$F(T_r(I - rB)) = GMEP(F, B, \varphi)$;*
(5) *$GMEP(F, B, \varphi)$ is closed and convex.*

Lemma 2.4. [6] *If $A : C \to H$ is α-inverse strongly monotone, then for any $\lambda \in [0, 4\alpha]$, $R_\lambda(x)$ is $(1 - \frac{\lambda}{4\alpha})$-inverse strongly monotone and for $x^* \in VIP(A, C)$,*

$$\langle x - x^*, R_\lambda(x)\rangle \geq (1 - \frac{\lambda}{4\alpha})\|R_\lambda(x)\|^2,$$

where $R_\lambda(x) = x - P_C(x - \lambda Ax)$.

Lemma 2.5. [10] *For all $x \in H$ and $\lambda' \geq \lambda > 0$, we have*

$$\|R_\lambda(x)\| \leq \|R_{\lambda'}(x)\|$$

where $R_\lambda(x) = x - P_C(x - \lambda Ax)$.

Lemma 2.6. [32] *Let $\{a_n\}_{n=1}^\infty$ be a sequence of real numbers in $[0, 1]$ such that $\sum_{n=1}^\infty a_n = 1$. Then we have*

$$\|\sum_{i=1}^\infty a_i x_i\|^2 \leq \sum_{i=1}^\infty a_i\|x_i\|^2,$$

for any given bounded sequence $\{x_n\}_{n=1}^\infty$ in H.

Lemma 2.7. [18] *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative numbers such that $\sum_{n=1}^\infty b_n < +\infty$ and*

$$a_{n+1} \leq a_n + b_n, \ \forall n \in N.$$

If there exists a convergent subsequence of $\{a_n\}$ to 0, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.8. [33] *Assume that* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\rho_n)a_n + \rho_n\delta_n, \ n \geq 1,$$

where $\{\rho_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence in* $\mathbb{R}$ *such that*

(i) $\sum_{n=1}^{\infty} \rho_n = \infty$;

(ii) $\limsup_{n\to\infty} \delta_n \leq 0$, *or* $\sum_{n=1}^{\infty} \mid \rho_n\delta_n \mid < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.9. [36] *Let* $0 < p \leq t_n \leq q < 1$ *for all* $n \geq 1$. *Suppose that* $\{x_n\}$ *and* $\{y_n\}$ *are sequences in* H *such that*

$$\limsup_{n\to\infty} \|x_n\| \leq d, \ \limsup_{n\to\infty} \|y_n\| \leq d$$

and

$$\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = d,$$

for some $d \geq 0$. *Then*

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

It is also known that H satisfies *Opial's condition*, that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$.

Lemma 2.10. [31] *Let* C *be a nonempty closed convex subset of a real Hilbert space* H *and* $S : C \to C$ *be a strictly pseudocontractive mapping. If* $\{x_n\}$ *is a sequence in* C *such that* $x_n \rightharpoonup x$ *and* $\lim_{n\to\infty} \|x_n - Sx_n\| = 0$, *then* $x = Sx$.

Lemma 2.11. [31] *Let* $S : C \to C$ *be a* k*-strictly pseudocontractive mapping. Define* $S_t : C \to C$ *by*

$$S_t x = tx + (1-t)Sx$$

for each $x \in C$. *Then for* $t \in [k,1)$, S_t *is nonexpansive such that* $F(S_t) = F(S)$.

Lemma 2.12. [30] *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* $\{x_n\}$ *be a sequence in* H. *Suppose that for any* $x^* \in C$

$$\|x_{n+1} - x^*\| \leq \|x_n - x^*\|.$$

Then

$$\lim_{n\to\infty} P_C(x_n) = z \in C.$$

3. Main results

Theorem 3.1. *Let C be a closed convex subset of a real Hilbert space H. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A1)$-$(A5)$, and $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let $A : C \to H$ be an α-inverse strongly monotone mapping and $B : C \to H$ be a β-inverse strongly monotone mapping. Let $S : C \to C$ be a nonexpansive and k-strictly pseudocontractive mapping such that*

$$\Omega := F(S) \cap VIP(A, C) \cap GMEP(F, B, \varphi) \neq \emptyset.$$

Let $\{\alpha_n\}, \{\alpha_n^{'}\}, \{\alpha_n^{''}\}$ and $\{\beta_n\}$ be the sequences of real numbers in $(0, 1)$. Assume that either $(B1)$ or $(B2)$ holds. Let $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ be the sequences generated by

$$\begin{cases} x_1 = x \in C, \\ F(u_n, y) + \langle Bx_n, y - u_n\rangle + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall y \in C, \\ y_n = P_C(u_n - \lambda_n A u_n), \\ x_{n+1} = \alpha_n x_n + \alpha_n^{'} S[\beta_n x_n + (1 - \beta_n) P_C(y_n - \lambda_n A y_n)] + \alpha_n^{''} e_n, \end{cases} \tag{3.1}$$

where $\{e_n\}$ is a bounded sequence in C. Suppose that the following conditions hold:

(i) $\alpha_n + \alpha_n^{'} + \alpha_n^{''} = 1,\ 0 < a \leq \alpha_n \leq b < 1;$
(ii) $0 < r_n < 2\beta,\ \{\lambda_n\} \subset [a, b]$ *for some* $a, b \in (0, 2\alpha);$
(iii) $\{\alpha\} \subset [c, d],\ \{\beta\} \subset [e, f]$ *for some* $c, d, e, f \in (0, 1);$
(iv) $\liminf_{n\to\infty} r_n > 0,\ \sum_{n=1}^{\infty} \mid \alpha_n^{''} \mid < \infty.$

Then $\{x_n\}$ converges strongly to $p^* \in \Omega$, where $p^* = \lim_{n\to\infty} P_\Omega(x_n)$.

Proof. We divide the proof into five steps:

Step 1. We claim that $\{x_n\}$ is bounded and

$$\lim_{n\to\infty} R_a(u_n) = \lim_{n\to\infty} R_{\lambda_n}(u_n) = 0.$$

Put $v_n = P_C(y_n - \lambda_n A y_n)$ and $w_n = \beta_n x_n + (1 - \beta_n) v_n$, then we have

$$R_{\lambda_n}(u_n) = u_n - P_C(u_n - \lambda_n A u_n)$$

and

$$R_{\lambda_n}(y_n) = y_n - P_C(y_n - \lambda_n A y_n),$$

for every $n = 1, 2, \cdots$.

Let $p \in \Omega$. Then, for the sequence of mappings $\{T_{r_n}\}$ defined in Lemma 2.3, we have

$$p = P_C(p - \lambda_n A p) = T_{r_n}(p - r_n B p).$$

From $u_n = T_{r_n}(x_n - r_n B x_n) \in C$, the β-inverse strongly monotonicity of B and $0 < r_n < 2\beta$, we have

$$\begin{aligned}\|u_n - p\|^2 &= \|T_{r_n}(x_n - r_n B x_n) - T_{r_n}(p - r_n Bp)\|^2 \\ &\le \|(x_n - r_n B x_n) - (p - r_n Bp)\|^2 \\ &\le \|x_n - p\|^2 - 2r_n\langle x_n - p, Bx_n - Bp\rangle + r_n^2\|Bx_n - Bp\|^2 \\ &\le \|x_n - p\|^2 - 2r_n\beta\|Bx_n - Bp\|^2 + r_n^2\|Bx_n - Bp\|^2 \\ &\le \|x_n - p\|^2 + r_n(r_n - 2\beta)\|Bx_n - Bp\|^2 \\ &\le \|x_n - p\|^2. \end{aligned} \tag{3.2}$$

Hence, from Lemma 2.4, we have

$$\begin{aligned}\|y_n - p\|^2 &= \|u_n - R_{\lambda_n}(u_n) - p\|^2 \\ &= \|u_n - p\|^2 - 2\langle u_n - p, R_{\lambda_n}(u_n)\rangle + \|R_{\lambda_n}(u_n)\|^2 \\ &\le \|u_n - p\|^2 - 2(1 - \frac{\lambda_n}{4\alpha})\|R_{\lambda_n}(u_n)\|^2 + \|R_{\lambda_n}(u_n)\|^2 \\ &\le \|u_n - p\|^2 - (1 - \frac{\lambda_n}{2\alpha})\|R_{\lambda_n}(u_n)\|^2, \end{aligned} \tag{3.3}$$

which implies from (3.2) that

$$\|y_n - p\|^2 \le \|x_n - p\|^2 - (1 - \frac{\lambda_n}{2\alpha})\|R_{\lambda_n}(u_n)\|^2. \tag{3.4}$$

By the same process an in (3.3), we also have from (3.4) that

$$\begin{aligned}\|v_n - p\|^2 &\le \|y_n - p\|^2 - (1 - \frac{\lambda_n}{2\alpha})\|R_{\lambda_n}(y_n)\|^2 \\ &\le \|y_n - p\|^2 - (1 - \frac{\lambda_n}{2\alpha})\|R_{\lambda_n}(y_n)\|^2 \\ &\le \|x_n - p\|^2 - (1 - \frac{\lambda_n}{2\alpha})\|R_{\lambda_n}(u_n)\|^2 \\ &\quad - (1 - \frac{\lambda_n}{2\alpha})\|R_{\lambda_n}(y_n)\|^2. \end{aligned} \tag{3.5}$$

Further from (3.1) and (3.5), we have

$$\begin{aligned}\|w_n - p\|^2 &= \beta_n^2\|x_n - p\|^2 + 2\beta_n(1-\beta_n)\langle x_n - p, v_n - p\rangle \\ &\quad + (1-\beta_n)^2\|v_n - p\|^2 \\ &= \beta_n^2\|x_n - p\|^2 + 2\beta_n(1-\beta_n)\|x_n - p\|^2 \\ &\quad + (1-\beta_n)^2\|x_n - p\|^2 \\ &\quad - (1-\beta_n)^2(1-\frac{\lambda_n}{2\alpha})\|R_{\lambda_n}(u_n)\|^2 \qquad (3.6) \\ &\quad - (1-\beta_n)^2(1-\frac{\lambda_n}{2\alpha})\|R_{\lambda_n}(y_n)\|^2 \\ &\le \|x_n - p\|^2 - (1-\beta_n)^2(1-\frac{\lambda_n}{2\alpha})\|R_{\lambda_n}(u_n)\|^2 \\ &\quad - (1-\beta_n)^2(1-\frac{\lambda_n}{2\alpha})\|R_{\lambda_n}(y_n)\|^2.\end{aligned}$$

Let $S_n = \beta_n I + (1-\beta_n)S$. Then S_n is nonexpansive from the nonexpansivity of S, for each $n \in N$, and using Lemma 2.11, we find that $F(S_n) = F(S)$. Since $0 < \lambda_n < 2\alpha$ and from (3.6), we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le \alpha_n\|x_n - p\|^2 + \alpha_n^{'}\|S_n w_n - p\|^2 + \alpha_n^{''}\|e_n - p\| \\ &\le \alpha_n\|x_n - p\|^2 + \alpha_n^{'}\|w_n - p\|^2 + \alpha_n^{''}\|e_n - p\| \\ &\le \alpha_n\|x_n - p\|^2 + \alpha_n^{'}[\|x_n - p\|^2 - (1-\beta_n)^2(1-\frac{\lambda_n}{2\alpha})\|R_{\lambda_n}(u_n)\|^2 \\ &\quad - (1-\beta_n)^2(1-\frac{\lambda_n}{2\alpha})\|R_{\lambda_n}(y_n)\|^2] + \alpha_n^{''}\|e_n - p\| \\ &\le (\alpha_n + \alpha_n^{'})\|x_n - p\|^2 - \alpha_n^{'}(1-\beta_n)^2(1-\frac{\lambda_n}{2\alpha})\|R_{\lambda_n}(u_n)\|^2 \\ &\quad - \alpha_n^{'}(1-\beta_n)^2(1-\frac{\lambda_n}{2\alpha})\|R_{\lambda_n}(y_n)\|^2 + \alpha_n^{''}\|e_n - p\| \\ &\le (1-\alpha_n^{''})\|x_n - p\|^2 - \alpha_n^{'}(1-\beta_n)^2(1-\frac{\lambda_n}{2\alpha})\|R_{\lambda_n}(u_n)\|^2 \qquad (3.7) \\ &\quad - \alpha_n^{'}(1-\beta_n)^2(1-\frac{\lambda_n}{2\alpha})\|R_{\lambda_n}(y_n)\|^2 + \alpha_n^{''}\|e_n - p\| \\ &\le (1-\alpha_n^{''})\|x_n - p\|^2 + \alpha_n^{''}\{\|e_n - p\| \\ &\quad - \alpha_n^{'}(1-\beta_n)^2(1-\frac{\lambda_n}{2\alpha})\|R_{\lambda_n}(u_n)\|^2 \\ &\quad - \alpha_n^{'}(1-\beta_n)^2(1-\frac{\lambda_n}{2\alpha})\|R_{\lambda_n}(y_n)\|^2\} \\ &\le \|x_n - p\|^2 + \|e_n - p\|.\end{aligned}$$

Hence $\{\|x_n - p\|\}$ is a bounded and nonincreasing sequence, so $\lim_{n\to\infty} \|x_n - p\|$ exists. Hence $\{x_n\}$ is bounded. Consequently the sets $\{u_n\}, \{v_n\}, \{w_n\}, \{y_n\}$ are also bounded.

By inequality (3.7), we have

$$\alpha_n'(1-\beta_n)^2(1-\frac{\lambda_n}{2\alpha})\|R_{\lambda_n}(u_n)\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \|e_n - p\|.$$

From the conditions (i) and (ii), there exists a constant $M_1 > 0$ such that

$$\begin{aligned} M_1\|R_{\lambda_n}(u_n)\|^2 &\leq \alpha_n'(1-\beta_n)^2(1-\frac{\lambda_n}{2\alpha})\|R_{\lambda_n}(u_n)\|^2 + \|e_n - p\| \\ &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \|e_n - p\|. \end{aligned}$$

It follows that

$$\begin{aligned} M_1\sum_{n=1}^{\infty}\|R_{\lambda_n}(u_n)\|^2 &\leq \sum_{n=1}^{\infty}[\|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \|e_n - p\|] \\ &= \|x_1 - p\|^2 + \|e_1 - p\| \\ &< \infty. \end{aligned}$$

Hence, $\lim_{n\to\infty}\|R_{\lambda_n}(u_n)\| = 0$. And so, we have

$$\lim_{n\to\infty} R_{\lambda_n}(u_n) = 0.$$

Since

$$R_{\lambda_n}(u_n) = u_n - P_C(u_n - \lambda_n A u_n) = u_n - y_n,$$

we have

$$\lim_{n\to\infty}\|u_n - y_n\| = 0.$$

Notice that $\lambda_n \geq a$, then by Lemma 2.5,

$$\|R_a(u_n)\| \leq \|R_{\lambda_n}(u_n)\|.$$

Therefore, we have

$$\lim_{n\to\infty} R_a(u_n) = \lim_{n\to\infty} R_{\lambda_n}(u_n) = 0. \tag{3.8}$$

By the same way, we also obtain

$$\lim_{n\to\infty}\|R_{\lambda_n}(y_n)\| = \lim_{n\to\infty}\|y_n - v_n\| = 0,$$

and thus, we have

$$\lim_{n\to\infty}\|u_n - v_n\| = 0. \tag{3.9}$$

Step 2. We show that

$$\lim_{n\to\infty}\|x_n - u_n\| = \lim_{n\to\infty}\|S_n x_n - x_n\| = 0.$$

Indeed, for any $p \in \Omega$, it follows from (3.1) and (3.5) that

$$\begin{aligned}\|w_n - p\|^2 &= \beta_n\|x_n - p\|^2 + (1-\beta_n)\|v_n - p\|^2 - \beta_n(1-\beta_n)\|x_n - v_n\|^2\\ &\leq \|x_n - p\|^2 - \beta_n(1-\beta_n)\|x_n - v_n\|^2,\end{aligned}$$

which implies that

$$\begin{aligned}\|x_{n+1} - p\|^2 &\leq \alpha_n\|x_n - p\|^2 + \alpha_n^{'}\|w_n - p\|^2\\ &\quad - \alpha_n\alpha_n^{'}\|S_nw_n - x_n\|^2 + \alpha_n^{''}\|e_n - p\|\\ &\leq \alpha_n\|x_n - p\|^2 + \alpha_n^{'}\|x_n - p\|^2\\ &\quad - \alpha_n^{'}\beta_n(1-\beta_n)\|x_n - v_n\|^2\\ &\quad - \alpha_n\alpha_n^{'}\|S_nw_n - x_n\|^2 + \alpha_n^{''}\|e_n - p\|\\ &\leq (\alpha + \alpha^{'})\|x_n - p\|^2 - \alpha_n^{'}\beta_n(1-\beta_n)\|x_n - v_n\|^2\\ &\quad - \alpha_n\alpha_n^{'}\|S_nw_n - x_n\|^2 + \alpha_n^{''}\|e_n - p\|.\end{aligned} \tag{3.10}$$

Thus, it follow from (3.10) that

$$\alpha_n\alpha_n^{'}\|S_nw_n - x_n\|^2 \leq (1-\alpha_n^{''})\|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n^{''}\|e_n - p\|.$$

From the condition (ii), there exists a constant $M_2 > 0$ such that

$$\begin{aligned}M_2\|S_nw_n - x_n\|^2 &\leq \alpha_n\alpha_n^{'}\|S_nw_n - x_n\|^2\\ &\leq (1-\alpha_n^{''})\|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n^{''}\|e_n - p\|.\end{aligned}$$

Hence, we have

$$\begin{aligned}M_2\sum_{n=1}^{\infty}\|S_nw_n - x_n\|^2 &\leq \sum_{n=1}^{\infty}\left[(1-\alpha_n^{''})\|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n^{''}\|e_n - p\|\right]\\ &= \|x_1 - p\|^2 + \|e_1 - p\|\\ &< \infty.\end{aligned}$$

Hence

$$\lim_{n\to\infty}\|S_nw_n - x_n\| = 0. \tag{3.11}$$

From (3.10), we also get

$$\alpha_n^{'}\beta_n(1-\beta_n)\|x_n - v_n\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n^{''}\|e_n - p\|.$$

Similarly, we obtain

$$\lim_{n\to\infty}\|x_n - v_n\| = 0. \tag{3.12}$$

This combinine with (3.9), then we have

$$\lim_{n\to\infty}\|x_n - u_n\| = 0. \tag{3.13}$$

Since

$$\begin{aligned} \|S_n x_n - x_n\| &\leq \|S_n x_n - S_n v_n\| + \|S_n v_n - S_n w_n\| + \|S_n w_n - x_n\| \\ &\leq \|x_n - v_n\| + \|v_n - w_n\| + \|S_n w_n - x_n\| \\ &\leq \|x_n - v_n\| + \beta_n \|x_n - v_n\| + \|S_n w_n - x_n\|, \end{aligned}$$

it implies from (3.11)-(3.12) that

$$\lim_{n\to\infty} \|S_n x_n - x_n\| = 0. \tag{3.14}$$

Step 3. Next, we prove that there exists a convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{k\to\infty} x_{n_k} = p^*$ for some $p^* \in C$. Moreover

$$p^* \in \Omega = F(S) \cap VIP(A, C) \cap GMEP(F, B, \varphi).$$

Since $\{x_n\}$ is a bounded sequence, there exists a weakly convergent subsequence $\{x_{n_k}\}$ such that $x_{n_k} \rightharpoonup p^*$. It implies from (3.11) and (3.13) that $S_n w_{n_k} \rightharpoonup p^*(k \to \infty)$ and $u_{n_k} \rightharpoonup p^*(k \to \infty)$. Since A is inverse strongly monotone with the positive constant $\alpha > 0$, we have

$$\alpha \|Ax - Ay\|^2 \leq \langle Ax - Ay, x - y\rangle \leq \|Ax - Ay\| \|x - y\|.$$

Hence $\|Az - Ay\| \leq \frac{1}{\alpha}\|x - y\|$, it means that A is $\frac{1}{\alpha}$-Lipschitz continuous.

From the $\frac{1}{\alpha}$-Lipschitz continuity of A and the continuity of P_C, it follows that $R_a(x) = x - P_C[x - aAx]$ is also continuous. Notice that $\rho_n \geq a$, then by Lemma 2.5, $\|R_a(x_n)\| \leq \|R_{\rho_n}(x_n)\|$. Then from Step 1,

$$\lim_{k\to\infty} \|R_a(x_{n_k})\| = \lim_{k\to\infty} \|R_{\rho_n}(x_{n_k})\| = 0.$$

Therefore from the continuity of $R_a(x)$,

$$R_a(p^*) = \lim_{k\to\infty} R_a(x_{n_k}) = 0.$$

This show that p^* is a solution of the variational inequality (1.6), that is $p^* \in VIP(A, C)$. From (3.12), $\lim_{k\to\infty} \|x_{n_k} - p^*\| = 0$, and the property of the nonexpansivity of the mapping S, it follows that $p^* = Sp^*$, that is, $p^* \in F(S)$. Finally, from Theorem 3.1 in [5], we prove that $p^* \in GMEP(F, B, \varphi)$. Thus, we have

$$p^* \in \Omega := F(S) \cap VIP(A, C) \cap GMEP(F, B, \varphi).$$

Next, we will prove that $x_{n_k} \to p^*(k \to \infty)$. From (3.1), (3.6) and (3.7) we can calculate that

$$\begin{aligned}
\|x_{n+1} - p^*\|^2 &= \langle \alpha_n x_n + \alpha_n^{'} S_n w_n + \alpha_n^{''} e_n - p^*, x_{n+1} - p^* \rangle \\
&= \alpha_n \langle x_n - p^*, x_{n+1} - p^* \rangle + \alpha_n^{'} \langle S_n w_n - p^*, x_{n+1} - p^* \rangle \\
&\quad + \alpha_n^{''} \langle e_n - p^*, x_{n+1} - p^* \rangle \\
&\leq \alpha_n \|x_n - p^*\|^2 + \alpha_n^{'} \langle S_n w_n - p^*, x_{n+1} - p^* \rangle \\
&\quad + \alpha_n^{''} \langle e_n - p^*, x_{n+1} - p^* \rangle \\
&\leq \alpha_n \|x_n - p^*\|^2 + \alpha_n^{'} \langle S_n w_n - p^*, x_{n+1} - x_n \rangle \\
&\quad + \alpha_n^{'} \langle S_n w_n - p^*, x_n - p^* \rangle + \alpha_n^{''} \langle e_n - p^*, x_{n+1} - p^* \rangle \\
&\leq \alpha_n \|x_n - p^*\|^2 + \alpha_n^{'} \|x_n - p^*\|^2 + \alpha_n^{'} \langle S_n w_n - p^*, x_{n+1} - x_n \rangle \\
&\quad + \alpha_n^{''} \langle e_n - p^*, x_{n+1} - p^* \rangle \\
&\leq (\alpha_n + \alpha_n^{'}) \|x_n - p^*\|^2 + \alpha_n^{'} \langle S_n w_n - p^*, x_{n+1} - x_n \rangle \\
&\quad + \alpha_n^{''} \langle e_n - p^*, x_{n+1} - p^* \rangle \\
&\leq (1 - \alpha_n^{''}) \|x_n - p^*\|^2 + \alpha_n^{''} \Big(\frac{\alpha_n^{'}}{\alpha_n^{''}} \Big) \langle S_n w_n - p^*, x_{n+1} - x_n \rangle \\
&\quad + \langle e_n - p^*, x_{n+1} - p^* \rangle.
\end{aligned}$$

Since $S_n w_{n_k} \rightharpoonup p^*$ and $x_{n_k} \rightharpoonup p^*$ as $k \to \infty$, from Lemma 2.8, we conclude that

$$\|x_{n_k} - p^*\| \to 0, (k \to \infty).$$

Using the Kadec-Klee property of H, we obtain that $\lim_{k\to\infty} x_{n_k} = p^*$.

Step 4. We claim that the sequence $\{x_n\}$ generated by algorithm (3.1) converges strongly to $p^* \in \Omega := F(S) \cap VIP(A, C) \cap GMEP(F, B, \varphi)$. From the result of Step 3, we know that $p^* \in \Omega$. Let $p = p^*$ in (3.7). Consequently,

$$\|x_{n+1} - p^*\| \leq \|x_n - p^*\| + \|e_n - P^*\|.$$

And also, we know that, $\lim_{k\to\infty} \|x_{n_k} - p^*\| = 0$ from Step 3. Then from Lemma 2.7, we have

$$\lim_{n\to\infty} \|x_n - p^*\| = 0.$$

Therefore $\lim_{n\to\infty} x_n = p^*$.

Step 5. Finally, We claim that $p^* = \lim_{n\to\infty} P_\Omega x_n$. From (2.1), we have

$$\langle x_n - P_\Omega x_n \, , \, p^* - P_\Omega x_n \rangle \leq 0. \tag{3.15}$$

By (3.7) and Lemma 2.12, $\lim_{n\to\infty} P_\Omega x_n = q^*$ for some $q^* \in \Omega$. Since $\lim_{n\to\infty} x_n = p^*$ from Step 4, taking the limit in (3.15), we have

$$\langle p^* - q^* \, , \, p^* - q^* \rangle \leq 0,$$

and this means that $p^* = q^*$. Hence

$$\lim_{n\to\infty} P_\Omega x_n = p^*.$$

This completes the proof □

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