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EXISTENCE OF SOLUTIONS OF FRACTIONAL PARTIAL INTEGRODIFFERENTIAL EQUATIONS WITH NEUMANN BOUNDARY CONDITION

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Abstract. The main purpose of this paper is to study the existence of solutions for the nonlinear fractional partial integrodifferential equations with Neumann boundary condition. Under suitable assumptions, the results are established by using the Leray-Schauder fixed point theorem and Arzela-Ascoli theorem. The examples are provided to illustrate the main result.

1. Introduction

Fractional calculus is a field of mathematics that extends the concepts of integer order differentiation and integration to an arbitrary order. Applications of fractional calculus can be observed in stochastic dynamical systems, plasma physics, image processing, controlled thermonuclear fusion and biological systems (for more applications see [18, 22, 23]). Fractional derivatives provide more accurate models of real world problems than integer order derivatives. Due to many applications of fractional differential equations in engineering

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and science, research in this area has grown significantly. Several authors including Chang and Nieto [7], among others [5, 6], have proved the existence and uniqueness results for fractional differential equations.

Integrodifferential equations arise as mathematical models in diverse disciplines. The origins of the study of integrodifferential equations trace back to the works of Lotka, Malthus, Abel, Volterra, Fredholm and Verhulst on problems in economics, mechanics and mathematical biology. From these beginnings, the theory and applications of integrodifferential equations have emerged as new areas of investigation. Fujita [9] have studied the integrodifferential equations with time fractional integral. Annapoorani et al. [2] showed the existence of solutions of neutral integrodifferential equations by using fixed point theorems. There are few articles available in the literature for the study of fractional integrodifferential equations. For example, Balachandran et al. [4, 3], Akilandeeswari et al. [1] and Shri Akiladevi et al. [25] studied the existence results for several kinds of fractional integrodifferential equations in Banach spaces using fixed point techniques.

Some partial differential equations of fractional order type like one-dimensional time-fractional diffusion-wave equation were used for modeling relevant physical processes [21]. Saxena et al. [24] gave closed form solutions to the fractional reaction and fractional diffusion equations in the form of Fox and Mittag-Leffler functions via asymptotic expansion. Nowadays some of the researchers started to consider fractional partial differential equations including well-posedness. Regarding fractional partial differential equations, Zhang et al. [27] proved the existence and uniqueness of variational solution of space-fractional partial differential equation and obtained a fully discrete approximating system by using the Galerkin finite element method and a backward difference method. The book [10] provides some efficient numerical methods for fractional partial differential equations. Jafari et al. [12] proposed a method called iterative Laplace transform method for solving a system of linear and nonlinear fractional partial differential equations. The solutions of system of fractional partial differential equations has been found by Parthiban and Balachandran [20] by using Adomain decomposition method. Joice Nirmala and Balachandran [13] determined the solution of time fractional telegraph equation by means of Adomain decomposition method and analysed the efficiency of this method. The existence and uniqueness of solution for an attractive fractional coupled system is established by Ibrahim and Jahangiri [11].

Motivated by these accomplishments, we extend the results of [19] to fractional order partial integrodifferential equation with Neumann boundary condition.

2. Preliminaries

Before looking at the existence result of fractional partial integrodifferential equation, we introduce some basic definitions and facts that are inherently tied to fractional calculus. Let Ω be a bounded domain in $\mathbb{R}$. Let $\Gamma(\cdot)$ denote the gamma function. For any arbitrary $0 < \alpha < 1$, the Riemann Liouville derivative and Caputo derivative are defined as follows:

Definition 2.1. [14] The partial Riemann-Liouville fractional integral operator of order $0 < \alpha < 1$ with respect to t of a function $f(x,t)$ is defined by

$$I^{\alpha} f(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(x,s)\, \mathrm{d}s.$$

Definition 2.2. [14] The partial Riemann-Liouville fractional derivative of order $0 < \alpha < 1$ of a function $f(x,t)$ with respect to t is of the form

$$\frac{\partial^{\alpha} f(x,t)}{\partial t^{\alpha}} = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^t \frac{f(x,s)}{(t-s)^{\alpha}}\, \mathrm{d}s.$$

Definition 2.3. [14] The Caputo partial fractional derivative of order $0 < \alpha < 1$ with respect to t of a function $f(x,t)$ is defined as

$$\frac{{}^{C}\partial^{\alpha} f(x,t)}{\partial t^{\alpha}} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-s)^{\alpha}} \frac{\partial f(x,s)}{\partial s}\, \mathrm{d}s.$$

For a detailed study of the above operators refer the books [17, 8]. The Riemann Liouville and Caputo fractional derivatives are linked by the following relationship:

$$\frac{{}^{C}\partial^{\alpha} f(x,t)}{\partial t^{\alpha}} = \frac{\partial^{\alpha}}{\partial t^{\alpha}} f(x,t) - \frac{f(x,0)}{\Gamma(1-\alpha) t^{\alpha}}.$$

To our equation, we adopt Caputo fractional derivative which has the advantage of approaching initial value problems, since the initial conditions of fractional differential equations in terms of Caputo derivatives hold the same form of integer order differential equations. Next we present some tools which will be used to prove our main result.

In this paper, we consider the fractional partial integrodifferential equation

$$\begin{aligned} \frac{{}^{C}\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} =& a(t)\Delta u(x,t) + \int_0^t h(t-s) u(x,s) \mathrm{d}s \\ &+ f\left(t, u(x,t), \int_0^t g(t,s,u(x,s)) \mathrm{d}s\right),\ t \in J, \qquad (2.1) \end{aligned}$$

where $0 < \alpha < 1$, $h : J \to \mathbb{R}$, $g : J \times J \times \mathbb{R} \to \mathbb{R}$ and the nonlinear function $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. The initial and Neumann boundary conditions are

$$u(x,0) = \varphi(x), \; x \in \Omega, \tag{2.2}$$

$$\frac{\partial u}{\partial n} = 0, \; (x,t) \in \partial\Omega \times J. \tag{2.3}$$

In order to establish our result, assume the following conditions.

(H_1) $f(t, u_1, u_2)$ is continuous with respect to u_1, u_2, Lebesgue measurable with respect to t and satisfies

$$\frac{1}{\mathrm{vol}(\Omega)} \int_\Omega f(t, u_1, u_2)\, \mathrm{d}x \le f\left(t, \frac{\int_\Omega u_1(x,t)\, \mathrm{d}x}{\mathrm{vol}(\Omega)}, \frac{\int_\Omega u_2(x,t)\, \mathrm{d}x}{\mathrm{vol}(\Omega)}\right), t \in J.$$

(H_2) There exists an integrable function $m_1(t) : J \to [0, \infty)$ such that

$$\| f(t, u_1, u_2) \| \le m_1(t) \left(\|u_1\| + \|u_2\| \right), \quad t \in J,$$

where $m_1(t) \ge 0$ and $\left(\int\limits_0^t (m_1(s))^{\frac{1}{\beta}}\, \mathrm{d}s \right)^\beta \le l_1$, for some $\beta \in (0, \alpha)$ and $l_1 \ge 0$.

(H_3) $g(t, s, u)$ is continuous with respect to u, Lebesgue measurable with respect to t and also satisfies the inequality

$$\frac{1}{\mathrm{vol}(\Omega)} \int_\Omega g(t, s, u)\, \mathrm{d}x \le g\left(t, s, \frac{\int_\Omega u(x,t)\, \mathrm{d}x}{\mathrm{vol}(\Omega)}\right).$$

(H_4) There exists an integrable function $m_2(t, s) : J \times J \to [0, \infty)$ such that

$$\| g(t, s, u) \| \le m_2(t, s) \|u\|, \quad t, s \in J,$$

where $m_2(t, s) \ge 0$ and $\left(\int\limits_0^t m_2(s, \tau)\, \mathrm{d}s \right) \le l_2$, for $l_2 \ge 0$.

(H_5) The integral kernel satisfies

$$\left(\int_0^t \left(\int_0^s h(s - \tau)\, \mathrm{d}\tau \right)^{\frac{1}{\beta}} \right)^\beta \le l_3,$$

where $l_3 \ge 0$.

It is easy to show that the initial value problem (2.1) is equivalent to the following integral equation:

$$\begin{aligned} u(x,t) &= \varphi(x) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} a(s) \Delta u(x,s) \, \mathrm{d}s \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big(\int_0^s h(s-\tau) u(x,\tau) \, \mathrm{d}\tau \Big) \, \mathrm{d}s \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(x,s), v(x,s)) \, \mathrm{d}s, \end{aligned} \tag{2.4}$$

where $v(x,s) = \int\limits_0^s g(s,\tau,u(x,\tau)) \, \mathrm{d}\tau$ and $t \in J$.

3. Existence Results

In this section, we are concerned with the existence of solutions of the problem (2.1)-(2.3). We define the function $W(t)$ as

$$W(t) = \frac{1}{\mathrm{vol}(\Omega)} \int_\Omega u(x,t) \, \mathrm{d}x. \tag{3.1}$$

Theorem 3.1. *Assume that there exists a $\beta \in (0,\alpha)$ for some $0 < \alpha < 1$ such that $(H_1) - (H_5)$ holds. For any constant $b > 0$, suppose that*

$$r = \min \left\{ T, \left[\frac{\Gamma(\alpha) b}{(\|W(0)\| + b)(l_1(1+l_2)+l_3)} \left(\frac{\alpha-\beta}{1-\beta} \right)^{1-\beta} \right]^{\frac{1}{\alpha-\beta}} \right\}. \tag{3.2}$$

Then there exists at least one solution for the initial value problem (2.1) on $\Omega \times [0,r]$.

Proof. First we have to prove the initial value problem (2.1) has a solution if and only if the equation

$$\begin{aligned} W(t) &= W(0) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big(\int_0^s h(s-\tau) W(\tau) \, \mathrm{d}\tau \Big) \, \mathrm{d}s \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, W(s), V(s)) \, \mathrm{d}s, \end{aligned} \tag{3.3}$$

where $V(t) = \int\limits_0^t g(t,s,W(s)) \, \mathrm{d}s$, has a solution.

Step 1. The proof of sufficiency is similar to that of Lemma 3.1 in [19]. To prove the necessary part, let $u(x,t)$ be a solution of (2.1). This implies $u(x,t)$

is a solution of (2.4). Now integrating both sides of equation (2.4) with respect to $x \in \Omega$, we get

$$\begin{aligned}\int_\Omega u(x,t)\ \mathrm{d}x = \int_\Omega \varphi(x)\ \mathrm{d}x + \frac{1}{\Gamma(\alpha)}\int_\Omega\int_0^t (t-s)^{\alpha-1}a(s)\Delta u(x,s)\ \mathrm{d}s\ \mathrm{d}x \\ + \frac{1}{\Gamma(\alpha)}\int_\Omega\int_0^t (t-s)^{\alpha-1}\Big(\int_0^s h(s-\tau)u(x,\tau)\ \mathrm{d}\tau\Big)\ \mathrm{d}s\ \mathrm{d}x \\ + \frac{1}{\Gamma(\alpha)}\int_\Omega\int_0^t (t-s)^{\alpha-1}f(s,u(x,s),v(x,s))\ \mathrm{d}s\ \mathrm{d}x. \qquad (3.4)\end{aligned}$$

Using Green's identity and the Neumann boundary condition, we obtain

$$\int_\Omega \Delta u(x,t)\ \mathrm{d}x = 0. \qquad (3.5)$$

Combining (3.5) and assumption (H_1), equation (3.4) implies

$$\begin{aligned}W(t) \leq W(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big(\int_0^s h(s-\tau)W(\tau)\ \mathrm{d}\tau\Big)\ \mathrm{d}s \\ +\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,W(s),V(s))\ \mathrm{d}s. \qquad (3.6)\end{aligned}$$

Let $K = \{W : W \in C(J,\ \mathbb{R}),\ \| W(t) - W(0) \| \leq b\}$. Define an operator $F : K \to K$ as

$$\begin{aligned}FW(t) = W(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big(\int_0^s h(s-\tau)W(\tau)\ \mathrm{d}\tau\Big)\ \mathrm{d}s \\ +\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,W(s),V(s))\ \mathrm{d}s. \qquad (3.7)\end{aligned}$$

Clearly $W(0) \in K$. This means that K is nonempty. From our construction of K, we can say that K is closed and bounded. Now for any $W_1, W_2 \in K$ and for any $a_1, a_2 \geq 0$ such that $a_1 + a_2 = 1$,

$$\begin{aligned}\| a_1W_1 + a_2W_2 - W(0) \| &\leq a_1 \| W_1 - W(0) \| + a_2 \| W_2 - W(0) \| \\ &\leq a_1b + a_2b = b.\end{aligned}$$

Thus $a_1W_1 + a_2W_2 \in K$. Therefore K is nonempty closed convex set. Next we have to prove the operator F maps K into itself.

$$\begin{aligned}\| FW(t) - FW(0) \| \leq \frac{1}{\Gamma(\alpha)}\left(\|W(0)\| + b\right)\int_0^t (t-s)^{\alpha-1}\Big(\int_0^s h(s-\tau)\ \mathrm{d}\tau\Big)\ \mathrm{d}s \\ + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,W(s),V(s))\|\ \mathrm{d}s.\end{aligned}$$

Then by using Holder inequality and the assumptions $(H_1) - (H_5)$, we arrive

$$\begin{aligned}
\| FW(t) - FW(0) \| &\le \frac{1}{\Gamma(\alpha)} \left(\|W(0)\| + b\right) \left(\int_0^t \left((t-s)^{\alpha-1}\right)^{\frac{1}{1-\beta}} \ \mathrm{d}s \right)^{1-\beta} \\
&\quad \left(\int_0^t \left(\int_0^s h(s-\tau) \ \mathrm{d}\tau \right)^{\frac{1}{\beta}} \mathrm{d}s \right)^{\beta} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \big\| f(s, W(s), V(s)) \big\| \ \mathrm{d}s \\
&\le \frac{1}{\Gamma(\alpha)} \left(\|W(0)\| + b\right) \left(\int_0^t \left((t-s)^{\alpha-1}\right)^{\frac{1}{1-\beta}} \mathrm{d}s \right)^{1-\beta} \left(\int_0^t \left(\int_0^s h(s-\tau) \mathrm{d}\tau \right)^{\frac{1}{\beta}} \mathrm{d}s \right)^{\beta} \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t m_1(s)(t-s)^{\alpha-1} \left(\|W(s)\| + \|V(s)\|\right) \mathrm{d}s \\
&\le \frac{1}{\Gamma(\alpha)} \left(\|W(0)\| + b\right) \left(\int_0^t \left((t-s)^{\alpha-1}\right)^{\frac{1}{1-\beta}} \mathrm{d}s \right)^{1-\beta} \left(\int_0^t \left(\int_0^s h(s-\tau) \mathrm{d}\tau \right)^{\frac{1}{\beta}} \mathrm{d}s \right)^{\beta} \\
&\quad + \frac{1}{\Gamma(\alpha)} \left(\|W(0)\| + b\right) \left(\int_0^t \left((t-s)^{\alpha-1}\right)^{\frac{1}{1-\beta}} \mathrm{d}s \right)^{1-\beta} \left(\int_0^t (m_1(s))^{\frac{1}{\beta}} \ \mathrm{d}s \right)^{\beta} \\
&\quad + \frac{l_2}{\Gamma(\alpha)} \left(\|W(0)\| + b\right) \left(\int_0^t \left((t-s)^{\alpha-1}\right)^{\frac{1}{1-\beta}} \mathrm{d}s \right)^{1-\beta} \left(\int_0^t (m_1(s))^{\frac{1}{\beta}} \ \mathrm{d}s \right)^{\beta} \\
&\le \frac{\left(\|W(0)\| + b\right) l_1}{\Gamma(\alpha)} \left(\frac{1-\beta}{\alpha-\beta} \right)^{1-\beta} r^{\alpha-\beta} + \frac{\left(\|W(0)\| + b\right) l_1 l_2}{\Gamma(\alpha)} \left(\frac{1-\beta}{\alpha-\beta} \right)^{1-\beta} r^{\alpha-\beta} \\
&\quad \frac{\left(\|W(0)\| + b\right) l_3}{\Gamma(\alpha)} \left(\frac{1-\beta}{\alpha-\beta} \right)^{1-\beta} r^{\alpha-\beta} \\
&= \frac{\left(\|W(0)\| + b\right) \left(l_1(1+l_2) + l_3\right)}{\Gamma(\alpha)} \left(\frac{1-\beta}{\alpha-\beta} \right)^{1-\beta} r^{\alpha-\beta} \\
&\le b, \quad t \in [0, r].
\end{aligned}$$

Therefore F maps K into itself. Now we define a sequence $\{W_k(t)\}$ in K such that

$$W_0(t) = W(0) \ \text{ and } \ W_{k+1}(t) = FW_k(t), \quad k = 0, 1, 2, \ldots$$

Since K is closed, there exists a subsequence $\{W_{k_i}(t)\}$ of $W_k(t)$ and $\widetilde{W}(t) \in K$ such that

$$\lim_{k_i \to \infty} W_{k_i}(t) = \widetilde{W}(t).$$

Then Lebesgue's dominated convergence theorem yields that

$$\begin{aligned}
\widetilde{W}(t) &= \widetilde{W}(0) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big(\int_0^s h(s-\tau) \widetilde{W}(\tau) \ \mathrm{d}\tau \Big) \ \mathrm{d}s \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, \widetilde{W}(s), \widetilde{V}(s)) \ \mathrm{d}s,
\end{aligned}$$

where $\widetilde{V}(t)=\int\limits_0^t g(t,s,\widetilde{W}(t))\ \mathrm{d}s$. Next we claim that F is completely continuous.

Step 2. For that first we prove $F:K\to K$ is continuous. Let $\{W_m(t)\}$ be a convergence sequence in K to $W(t)$. Then for any $\epsilon>0$, we have

$$\|W_m(t)-W(t)\|\leq \frac{\Gamma(\alpha)\epsilon}{2l_3r^{\alpha-\beta}}\left(\frac{\alpha-\beta}{1-\beta}\right)^{1-\beta}. \tag{3.8}$$

By assumption (H_1),

$$f\left(t,W_m(t),\int_0^t g(t,s,W_m(s))\ \mathrm{d}s\right)\longrightarrow f\left(t,W(t),\int_0^t g(t,s,W(s))\ \mathrm{d}s\right),$$

for each $t\in[0,r]$. Therefore for any $\epsilon>0$, we can take

$$\left\|f\Big(t,W_m(t),V_m(t)\Big)\ -\ f\Big(t,W(t),V(t)\Big)\right\|\leq\frac{\alpha\Gamma(\alpha)\epsilon}{2r^{\alpha}}, \tag{3.9}$$

where $V_m(t)=\int\limits_0^t g(t,s,W_m(s))\ \mathrm{d}s$. Using (3.8) and (3.9) and simplifying, we have

$$\begin{aligned}\|FW_m(t)-FW(t)\|&\leq\frac{l_3}{\Gamma(\alpha)}\left(\frac{1-\beta}{\alpha-\beta}\right)^{1-\beta}r^{\alpha-\beta}\|W_m(t)-W(t)\|+\frac{r^{\alpha}}{\alpha\Gamma(\alpha)}\\&\quad\times\left\|f\left(s,W_m(s),V_m(s)\right)-f\left(s,W(s),V(s)\right)\right\|\\&\leq\epsilon.\end{aligned}$$

Taking the limit $m\to\infty$, the right hand side of the above inequality tends to zero, since ϵ can be arbitrary small. Therefore F is continuous.

Step 3. Moreover, for $W\in K$,

$$\begin{aligned}\|\ FW(t)\ \|&\leq\|W(0)\|+\frac{l_1+l_2+l_3}{\Gamma(\alpha)}(\|W(0)\|+b)\left(\frac{1-\beta}{\alpha-\beta}\right)^{1-\beta}r^{\alpha-\beta}\\&\leq\|W(0)\|+b.\end{aligned}$$

Hence FK is uniformly bounded. Now it remains to show that F maps K into an equicontinuous family.

Step 4. Now let $U \in K$ and $t_1, t_2 \in J$. Then for $0 < t_1 < t_2 \le r$, by the assumptions $(H_1) - (H_5)$ we obtain

$$\begin{aligned}
\| FW(t_1) - FW(t_2) \| \le{} & \frac{1}{\Gamma(\alpha)} (\|W(0)\| + b) \int_0^{t_1} \left((t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}\right) \\
& \Big(\int_0^s h(s-\tau)\mathrm{d}\tau\Big)\mathrm{d}s + \frac{1}{\Gamma(\alpha)} (\|W(0)\| + b) \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \Big(\int_0^s h(s-\tau)\mathrm{d}\tau\Big)\mathrm{d}s \\
& + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \Big((t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}\Big) \big\|f(s, W(s), V(s))\big\| \,\mathrm{d}s \\
& + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \big\|f(s, W(s), V(s))\big\| \,\mathrm{d}s \\
\le{} & \frac{l_3}{\Gamma(\alpha)} (\|W(0)\| + b) \left(\int_0^{t_1} \left((t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}\right)^{\frac{1}{1-\beta}} \,\mathrm{d}s \right)^{1-\beta} \\
& + \frac{l_3}{\Gamma(\alpha)} (\|W(0)\| + b) \left(\int_{t_1}^{t_2} ((t_2 - s)^{\alpha-1})^{\frac{1}{1-\beta}} \,\mathrm{d}s \right)^{1-\beta} \\
& + \frac{1}{\Gamma(\alpha)} \left(\int_0^{t_1} \Big((t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}\Big)^{\frac{1}{1-\beta}} \mathrm{d}s \right)^{1-\beta} \left(\int_0^t (m_1(s))^{\frac{1}{\beta}} \,\mathrm{d}s \right)^{\beta} \\
& + \frac{1}{\Gamma(\alpha)} (\|W(0)\| + b) \left(\int_{t_1}^{t_2} ((t_2 - s)^{\alpha-1})^{\frac{1}{1-\beta}} \mathrm{d}s \right)^{1-\beta} \left(\int_0^t (m_1(s))^{\frac{1}{\beta}} \,\mathrm{d}s \right)^{\beta} \\
& + \frac{l_2}{\Gamma(\alpha)} \left(\int_0^{t_1} \Big((t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}\Big)^{\frac{1}{1-\beta}} \mathrm{d}s \right)^{1-\beta} \left(\int_0^t (m_1(s))^{\frac{1}{\beta}} \,\mathrm{d}s \right)^{\beta} \\
& + \frac{l_2}{\Gamma(\alpha)} (\|W(0)\| + b) \left(\int_{t_1}^{t_2} ((t_2 - s)^{\alpha-1})^{\frac{1}{1-\beta}} \mathrm{d}s \right)^{1-\beta} \left(\int_0^t (m_1(s))^{\frac{1}{\beta}} \,\mathrm{d}s \right)^{\beta} \\
\le{} & \frac{l_1 + l_1 l_2 + l_3}{\Gamma(\alpha)} (\|W(0)\| + b) \left(\int_0^{t_1} ((t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1})^{\frac{1}{1-\beta}} \,\mathrm{d}s \right)^{1-\beta} \\
& + \frac{l_1 + l_1 l_2 + l_3}{\Gamma(\alpha)} (\|W(0)\| + b) \left(\int_{t_1}^{t_2} ((t_2 - s)^{\alpha-1})^{\frac{1}{1-\beta}} \,\mathrm{d}s \right)^{1-\beta}.
\end{aligned}$$

The right hand side is independent of $W \in K$. Since $0 < \beta < \alpha < 1$, the right hand side of the above inequality goes to zero as $t_1 \to t_2$. Thus, F maps K into an equicontinuous family of functions. In the view of Ascoli-Arzela theorem, F is completely continuous. Then applying Leray-Schauder fixed point theorem, we deduce that F has a fixed point in K, which is a solution of (2.1). □

4. Examples

Example 4.1. Consider the partial fractional integrodifferential equation

$$\begin{aligned}\frac{{}^C\partial^{\frac{1}{2}}u(x,t)}{\partial t^\alpha} &= t^2\Delta u(x,t) + \int_0^t (t-s)^{\frac{1}{2}}u(x,s)\ \mathrm{d}s + tu^2 \\ &\quad + tu\int_0^t \big(s + u^{2/3}\exp(-s)\big)\ \mathrm{d}s, \quad (x,t)\in\Omega\times J \qquad (4.1)\end{aligned}$$

with the initial condition

$$u(x,0) = u_0, \quad x\in\Omega$$

and the boundary condition

$$\frac{\partial u}{\partial n} = 0, \quad (x,t)\in\partial\Omega\times J,$$

where $J = [0,1]$ and $\Omega = [0,\pi/2]$. Here $a(t) = t^2$, $h(t-s) = (t-s)^{\frac{1}{2}}$, $\int_0^t g(t,s,u(x,s))\ \mathrm{d}s = \int_0^t (s + u^{2/3}\exp(-s))\ \mathrm{d}s$ and

$$f\Big(t, u(x,t), \int_0^t g(t,s,u(x,s))\ \mathrm{d}s\Big) = tu^2 + tu\int_0^t \big(s + u^{2/3}\exp(-s)\big)\ \mathrm{d}s. \qquad (4.2)$$

We note that the assumptions (H1)-(H5) of Theorem 3.1 are satisfied for some $\beta\in(0,1/2)$. Hence the problem (4.1) has a solution.

Example 4.2. Consider the partial fractional intgrodifferential equation

$$\frac{{}^C\partial^{\frac{2}{3}}u(x,t)}{\partial t^\alpha} = \Delta u(x,t) + u(x,t) + \frac{1}{1+t^2}\int_0^t su(x,s)\ \mathrm{d}s,\ (x,t)\in\Omega\times J \qquad (4.3)$$

with the initial condition

$$u(x,0) = u_0,\ x\in\Omega$$

and the boundary condition

$$\frac{\partial u}{\partial n} = 0,\ (x,t)\in\partial\Omega\times J,$$

where $J = [0,1]$ and $\Omega = [0,\pi/2]$. Here $a(t) = 1$,

$$\int_0^t g(t,s,u(x,s))\ \mathrm{d}s = \frac{1}{1+t^2}\int_0^t su(x,s)\ \mathrm{d}s,$$

$h(t-s)=0$ and

$$f\Big(t, u(x,t), \int_0^t g(t,s,u(x,s))\ \mathrm{d}s\Big) = u(x,t) + \frac{1}{1+t^2}\int_0^t su(x,s)\ \mathrm{d}s. \quad (4.4)$$

The assumptions (H1)-(H5) of Theorem (3.2) are satisfied for some $\beta \in (0, 2/3)$. Thus the problem (4.3) has a solution.

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