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EXISTENCE RESULTS FOR FRACTIONAL NEUTRAL DIFFERENTIAL EQUATIONS WITH NONLOCAL AND NON-INSTANTANEOUS IMPULSIVE CONDITIONS

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Abstract. We consider the non-instantaneous impulsive fractional neutral differential equations with a nonlocal condition. Here the resolvent operator technique is used to derive the mild solution of the desired problem. The existence results are proved by using fixed point theorem and measure of noncompactness.

1. Introduction

Impulsive differential equations are considered most important in recent literature simulated by their numerous applications. These equations are suitable to model the natural description of several evolutionary processes subject to certain perturbations whose duration is negligible in comparison with the duration of the process. Many of the literature treated with the instantaneous impulsive differential equations and studied the existence and qualitative properties of these equations, see [1, 5, 6, 9, 23, 32, 33].

On the other hand, Hernández and O'Regan [18] introduced a new type of impulsive differential equations, in which the impulses are not instantaneous. In this case the impulses start abruptly at a certain point and their

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action continues on the finite time interval. This theory is involved with certain dynamics of evolution processes in pharmacotheraphy. In [28] Pierri et al. analyzed the existence results of non-instantaneous impulsive differential equations with values in fractional spaces. The relevant research topic is presented in some recent papers [8, 11, 21, 25, 29, 36, 38].

Hernández *et al.* [19] discussed the theory on non-instantaneous impulsive differential equations improving the results which were studied in [18]. They demonstrated this problem with an example of the hemodynamical equilibrium of a person. In the case of a decompensation (for example, high or low levels of glucose) one can prescribe some intravenous drugs (insulin). Since the introduction of the drugs in the bloodstream and the consequent absorption for the body are gradual and continuous processes, we can interpret the above situation as an impulsive action which starts abruptly and stays active on a finite time interval. In this example Hernández et al. mentioned the advantages of the new model presented in [19], that is the introduction of the drug at time t_i depend on the level of glucose observed in time intervals previous to the time t_i.

Fractional impulsive differential equations have received more attention in the recent studies. It is because of fractional derivatives have the memory and hereditary properties of various materials and processes. Also fractional differential equations can be found in many scientific applications like viscoelasticity, electrochemistry, porous media, control problem, polymer rheology and aerodynamics. For further information and detailed study about this theory refer [17, 22, 30, 39]. The nonlocal conditions have been studied in numerous scientific fields, since it has the nature to take additional information. Byszewski and Lakshmikandam [3], Anguraj and Karthikeyan [2], Hernández [17], Zhang *et al.* [39] and Chadha and Pandey [4] have investigated some existence and uniqueness results for nonlocal problem.

The neutral differential equations are depending on present and previous values but that involve derivatives with delays as well as function itself [14, 15]. These features direct to study fractional neutral impulsive differential equations in many real life applications. The existence of mild solution of fractional impulsive neutral integro-differential equations with infinite delay have been studied by using the solution operator in [10] and by using the analytic α-resolvent family in [12] for the non-instantaneous impulsive fractional neutral differential equations. The Riemann-Liouville fractional derivative was considered in [37] to study the non-instantaneous impulsive neutral functional differential equations with infinite delay. In [27] Pandey et al. established the existence results of second-order neutral non-instantaneous impulsive differential equations with state delay by using Darbo-Sadovskii fixed point theorem.

The concept of measure of noncompactness is used by many authors to study the existence results for impulsive fractional differential equations (see [4, 7, 13, 24, 29, 34, 35]). Here they used Darbo fixed point theorem or Mönch fixed point theorem to prove the existence theorems.

Here we study the existence results for non-instantaneous impulsive fractional neutral differential equations with nonlocal conditions of the form

$$\begin{aligned} &{}^{c}_{s_i}D^{\alpha}_{t}(u(t)+p(t,u(t),u_t)) \\ &= A\left(u(t)+\int_0^t k(t-s)u(s)ds\right) \\ &\quad + {}_{s_i}I^{1-\alpha}_{t} q(t,u(t),u_t), \;\; t\in(s_i,t_{i+1}], \;\; i=0,1,\cdots,m, \end{aligned} \tag{1.1}$$

$$u(t)=g_i(t,u_{|_{I_i(t)}}), \; t\in(t_i,s_i], \; i=1,\cdots,m, \tag{1.2}$$

$$u(t)=\phi+h(u), \; t\in(-\infty,0], \tag{1.3}$$

where A is the closed and unbounded operator in X with dense domain $\mathcal{D}(A)$, $0=t_0=s_0<t_1<s_1<\cdots<t_{i+1}$ are prefixed numbers. The functions $u_s:(-\infty,0]\to X, u_s(\theta)=u(s+\theta)$, and ϕ belongs to some abstract phase space $\mathcal{B}$ described axiomatically. Here $p,q:[0,a]\times X\times\mathcal{B}\to X,\; h:\mathcal{B}\to\mathcal{B}$ and $k\in L^1_{loc}(\mathbb{R}^+)$ are appropriate functions. The relation $t\to I_i(t)$ defines a $2^{[0,t]}$-set valued function, each function $g_i(t,\cdot)$ is a continuous function defined from a Banach space $\mathcal{C}_i(t)$ into X, the spaces $\mathcal{C}_i(t)$ are formed by function defined from $I_i(t)$ into X. ${}^{c}_{s_i}D^{\alpha}_{t}$ is a Caputo fractional derivative with lower limit s_i, where $0<\alpha<1$.

In this paper we first concentrate to derive the mild solution of the equations (1.1)-(1.3) in more appropriate way under the resolvent operator involved in [16, 31] with perturbation results. Next we prove the existence results for the problem (1.1)-(1.3) by using contraction mapping principle and fixed point theorem for condensing map. Further we relax the compactness of the resovent operator to prove the existence of mild solution by the concept of Hausdorff measure of noncompactness and Darbo-Sadovskii fixed point theorem.

2. Preliminaries

Let X and Y be Banach spaces and $\mathcal{L}(X,Y)$ denote the space of all bounded linear operators with norm $\|\cdot\|_{\mathcal{L}(X,Y)}$. The domain of A is endowed with the graph norm $\|\cdot\|_{\mathcal{D}(A)}=\|u\|+\|Au\|$. In addition, $B_r(u,X)$ represents the closed ball with center at u and radius r in X. The space $C([0,a];X)$ denotes the space of all continuous functions with norm $\|\cdot\|_{C([0,a];X)}=\sup_{t\in[0,a]}\|u(t)\|_X$ and $C^{\gamma}([0,a];X), \gamma\in(0,1)$ denotes the space formed by all the functions $u\in C([0,a];X)$ such that $[u]_{C^{\gamma}([0,a];X)}=\sup_{t,s\in[0,a],t\neq s}\frac{\|u(t)-u(s)\|_X}{(t-s)^{\gamma}}$ is finite, endowed with the norm $\|u\|_{C^{\gamma}([0,a];X)}=\|u\|_{C([0,a];X)}+[u]_{C^{\gamma}([0,a];X)}$.

We introduce the space $PC(X)$ which is formed by all the functions $u : [0,a] \to X$ such that $u(\cdot)$ is continuous at $t \neq t_i$, $u(t_i^-) = u(t_i)$ and $u(t_i^+)$ exists for all $i = 1, 2, \cdots, N$, is a Banach space with respect to the norm $\|u\|_{PC(X)} = \sup_{t\in[0,a]} \|u(t)\|$. For a function $u \in PC(X)$ and $i \in \{0, 1, \cdots, N\}$, we introduce the function $\tilde{u_i} \in C([t_i, t_{i+1}]; X)$ given by

$$\tilde{u_i}(t) = \begin{cases} u(t), \text{ for } t \in (t_i, t_{i+1}], \\ u(t_i^+), \text{ for } t = t_i \end{cases} \tag{2.1}$$

In addition, for $B \subseteq PC(X)$ and $i \in \{0, 1, \cdots, N\}$, we use the notation $\tilde{B}_i$ for the set $\tilde{B}_i = \{\tilde{u_i} : u \in B\}$. We note the following Ascoli-Arzela type criteria.

Lemma 2.1. ([18]) *A set $B \subseteq PC(X)$ is relatively compact in $PC(X)$ if and only if each set $\tilde{B}_i$ is relatively compact in $C([t_i, t_{i+1}]; X)$.*

We consider the notation $(\mathcal{C}_i(t), \|\cdot\|_{\mathcal{C}_i(t)})$, with $t \in (t_i, s_i]$ and $i \in \{1, \cdots, N\}$, to represent an abstract Banach space formed by functions defined from $I_i(t) \subset [0, s_i]$ into X. In addition, for a set $I \subset [0,a]$, we use the notation $PC(X)_{|_I}$ for the space $PC(X)_{|_I} = \{u_{|_{I_i(t)}} : u \in PC(X)\}$ endowed with the uniform norm.

We consider the phase space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$, is a linear space of function u_t mapping from $(-\infty, 0]$ into X with respect to the seminorm $\|\cdot\|_{\mathcal{B}}$, which is previously addressed in Hino et al., [20] to examine the infinite delay problem.

We assume the space $\mathcal{B}$ meets the axioms given below:

(1) If $u : (-\infty, \nu + a] \to X$, $\nu \in \mathbb{R}$, $a > 0$ such that $u_\nu \in \mathcal{B}$, and $u|_{[\nu,\nu+a]} \in PC([\nu, \nu+a]; X)$, then the subsequent conditions hold for all $t \in [\nu, \nu+a)$
   - (i) $u_t \in \mathcal{B}$,
   - (ii) $\|u(t)\|_X \leq \mathfrak{H}\|u\|_{\mathcal{B}}$,
   - (iii) $\|u_t\|_{\mathcal{B}} \leq \mathfrak{K}(t-\nu)\sup\{\|u(s)\|_X : \nu \leq s \leq t\} + \mathfrak{M}(t-\nu)\|u_\nu\|_{\mathcal{B}}$,

   where $\mathfrak{M}, \mathfrak{K} : [0,\infty) \to [1,\infty)$, is locally bounded and continuous respectively; $\mathfrak{H} > 0$ is a constant. $\mathfrak{K}, \mathfrak{H}, \mathfrak{M}$ are independent of $u(\cdot)$.

(2) The phase space $\mathcal{B}$ is complete.

The Caputo fractional derivative of order $\alpha > 0$ of a function u defined as follows:

$${}_a^cD_t^\alpha u(t) = {}_aI_t^{n-\alpha}\, {}_aD_t^n u(t), \; n = \lceil \alpha \rceil,$$

where ${}_aI_t^\alpha u(t) = (a_\alpha * u)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}u(s)ds$, $a_\alpha = \frac{1}{\Gamma(\alpha)}(t)^{\alpha-1}$, $t > 0$, $\alpha \geq 0$. Also, in general the Caputo derivative is a left inverse of Riemann-Liouville fractional integral I^α but not a right inverse, *i.e.*, we have ${}_a^cD_t^\alpha\, {}_aI_t^\alpha u = u$, and ${}_aI_t^\alpha\, {}_a^cD_t^\alpha u(t) = u(t) - u(a)$, for $0 < \alpha < 1$.

Note that the following perturbed convolution equation

$$u(t) \quad = \quad (a_\alpha + a_\alpha * k) * Au(t) + f(t),\ t \in [0,a], \tag{2.2}$$

has a corresponding resolvent operator $(S(t))_{t\geq 0}$ on X and $f \in C([0,a];X)$, see [16] and [31, Section 1.4].

Definition 2.2. ([31, Definition 1.3]) A family $(S(t))_{t\geq 0} \subset \mathcal{L}(X)$ of bounded linear operators in X is called resolvent for (2.2)(or solution operator for (2.2)), if the following conditions are satisfied

(S1) $S(t)$ is strongly continuous on $\mathbb{R}^+$ and $S(0) = I$,
(S2) $S(t)$ commutes with A, which means that $S(t)\mathcal{D}(A) \subset \mathcal{D}(A)$ and $AS(t)x = S(t)Ax$ for all $x \in \mathcal{D}(A)$ and $t \geq 0$;
(S3) The resolvent equation holds

$$S(t)x = x + a_\alpha A * S(t)x + k * a_\alpha A * S(t)x,$$

for all $x \in \mathcal{D}(A),\ t \geq 0$.

Definition 2.3. ([31, Definition 1.4]) A resolvent operator $S(t)$ for equation (2.2) is said to be differentiable, if $S(\cdot)u \in \mathcal{W}^{1,1}([0,\infty);X)$ for every $u \in \mathcal{D}(A)$ and there is $\varphi \in L^1_{loc}([0,\infty))$ with $\|S'(t)u\| \leq \varphi(t)\|u\|_{\mathcal{D}(A)}$, *a.e.* on $[0,\infty)$, for every $u \in \mathcal{D}(A)$.

Definition 2.4. ([31]) A function $u \in C([0,a];X)$ is called a mild solution of (2.2) on $[0,a]$ if $(a_\alpha + a_\alpha * k) * u \in \mathcal{D}(A)$ for all $t \in [0,a]$ and

$$u(t) = A(a_\alpha + a_\alpha * k) * u(t) + f(t),\ t \in [0,a].$$

The next result follows from [16, Lemma 1.1] and [31].

Lemma 2.5. *Suppose equation* (2.2) *admits a differentiable resolvent* $S(t)$*.*

(i) *If* $u(\cdot)$ *is a mild solution of* (2.2) *on* $[0,a]$*, then the function* $t \to \int_0^t S(t-s)f(s)ds$ *is continuously differentiable on* $[0,a]$*, and*

$$u(t) = \frac{d}{dt}\int_0^t S(t-s)f(s)ds,\ t \in [0,a],$$

in particular, mild solutions of (2.2) *are unique.*

(ii) *If* $f \in C([0,a];\mathcal{D}(A))$ *then the function* $u : [0,a] \to X$ *defined by*

$$u(t) = \int_0^t S'(t-s)f(s)ds + f(t),\ t \in [0,a],$$

is a mild solution of (2.2) *on* $[0,a]$*.*

To derive the mild solution of the problem (1.1)-(1.3), we first write the corresponding integral equation of the system (1.1)-(1.3) as follows.

$$u(t) = \begin{cases} \phi + h(u) - p(t, u(t), u_t) + p(0, \phi + h(u), u_0) \\ + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} A \left(u(s) + \int_0^s k(s-\tau) u(\tau) d\tau \right) ds \\ + \int_0^t q(s, u(s), u_s) ds, \ t \in [0, t_1], \\ g_i(t, u_{|_{I_i(t)}}), \ t \in (t_i, s_i], \ i = 1, \cdots, m, \\ g_i(s_i, u_{|_{I_i(s_i)}}) - p(t, u(t), u_t) + p(s_i, g_i(s_i, u_{|_{I_i(s_i)}}), u_{s_i}) \\ + \frac{1}{\Gamma(\alpha)} \int_{s_i}^t (t-s)^{\alpha-1} A \left(u(s) + \int_0^s k(s-\tau) u(\tau) d\tau \right) ds \\ + \int_{s_i}^t q(s, u(s), u_s) ds, \ \ t \in [s_i, t_{i+1}], \ i = 1, \cdots, m. \end{cases} \tag{2.3}$$

3. Existence and uniqueness results

In this section we define the mild solution of the problem (1.1)-(1.3) and we prove the existence and uniqueness results.

From the Definition 2.4 and from the integral equation (2.3), we introduce the mild solution of (1.1)-(1.3) as follows.

Definition 3.1. A function $u : (-\infty, a] \to X$ is called a mild solution of the equations (1.1)-(1.3), if $(a_\alpha + a_\alpha * k) * u \in C([0, a]; \mathcal{D}(A))$,

$$u(t) = \begin{cases} \phi + h(u), \ t \in (-\infty, 0] \\ g_i(t, u_{|_{I_i(t)}}), \ t \in (t_i, s_i], \ i = 1, \cdots, m, \end{cases}$$

and

$$u(t) = \begin{cases} \phi + h(u) - p(t, u(t), u_t) + p(0, \phi + h(u), u_0) \\ + A \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (u(s) + \int_0^s k(s-\tau) u(\tau)) d\tau ds \\ + \int_0^t q(s, u(s), u_s) ds, \ t \in [0, t_1], \\ g_i(s_i, u_{|_{I_i(s_i)}}) - p(t, u(t), u_t) + p(s_i, g_i(s_i, u_{|_{I_i(s_i)}}), u_{s_i}) \\ + A \frac{1}{\Gamma(\alpha)} \int_{s_i}^t (t-s)^{\alpha-1} (u(s) + \int_0^s k(s-\tau) u(\tau) d\tau) ds \\ + \int_{s_i}^t q(s, u(s), u_s) ds, \ t \in [s_i, t_{i+1}], \ i = 1, \cdots, m. \end{cases}$$

Now since the equation (2.3) has an associated resolvent operator, from Lemma 2.5, we have the following result.

Proposition 3.2. *Assume that $(S(t))_{t \geq 0}$ is differentiable resolvent operator and $\phi \in \mathcal{D}(A)$, $p, q \in C([0, a] \times X \times \mathcal{B}; \mathcal{D}(A))$, $h \in C(\mathcal{B}; \mathcal{D}(A))$, $g_i \in C((t_i, s_i] \times$*

$\mathcal{C}_i(t); \mathcal{D}(A))$, *then*

$$u(t) \tag{3.1}$$
$$= \begin{cases} \phi + h(u) - p(t, u(t), u_t) + p(0, \phi + h(u), u_0) + \int_0^t q(s, u(s), u_s)ds \\ + \int_0^t S'(t-s)\left(\phi + h(u) - p(s, u(s), u_s) + p(0, \phi + h(u), u_0)\right. \\ \left. + \int_0^s q(\tau, u(\tau), u_\tau)d\tau\right) ds, \quad t \in [0, t_1], \\ g_i(t, u_{|_{I_i(t)}}), \ t \in (t_i, s_i], \ i = 1, \cdots, m, \\ g_i(s_i, u_{|_{I_i(s_i)}}) - p(t, u(t), u_t) + p(s_i, g_i(s_i, u_{|_{I_i(s_i)}}), u_{s_i}) \\ + \int_{s_i}^t q(s, u(s), u_s)ds \\ + \int_{s_i}^t S'(t-s)\left(g_i(s_i, u_{|_{I_i(s_i)}}) - p(s, u(s), u_s) + p(s_i, g_i(s_i, u_{|_{I_i(s_i)}}), u_{s_i})\right. \\ \left. + \int_{s_i}^s q(\tau, u(\tau), u_\tau)d\tau\right) ds, \ t \in [s_i, t_{i+1}], \ i = 1, \cdots, m, \end{cases}$$

is a mild solution of (1.1)-(1.3).

We will adopt the subsequent hypotheses:

(H1) $q : [0, a] \times X \times \mathcal{B} \to \mathcal{D}(A)$ is a continuous function and let $L_q \in C([0, a]; \mathbb{R}^+)$ such that

$$\|q(t, u_1, v_1) - q(t, u_2, v_2)\|_{\mathcal{D}(A)} \leq L_q(t)(\|u_1 - u_2\| + \|v_1 - v_2\|_{\mathcal{B}}),$$

$t \in [0, a]$, u_1, $u_2 \in X$, v_1, $v_2 \in \mathcal{B}$.

(H2) The function m_q belongs to $C([0, a]; \mathbb{R}^+)$ and a non-decreasing function $W : [0, +\infty) \to (0, +\infty)$ such that

$$\|q(t, u, v)\|_{\mathcal{D}(A)} \leq m_q(t)W(\|u\| + \|v\|_{\mathcal{B}}),$$

$t \in [0, a]$, $(u, v) \in X \times \mathcal{B}$.

(H3) $p : [0, a] \times X \times \mathcal{B} \to \mathcal{D}(A)$ is a continuous function and $L_p \in C([0, a]; \mathbb{R}^+)$ with

$$\|p(t, u_1, v_1) - p(t, u_2, v_2)\|_{\mathcal{D}(A)} \leq L_p(t)(\|u_1 - u_2\| + \|v_1 - v_2\|_{\mathcal{B}}),$$

$t \in [0, a]$, u_1, $u_2 \in X$, v_1, $v_2 \in \mathcal{B}$.

(H4) $C_1 > 0$ and $C_2 > 0$ such that

$$\|p(t, u, v)\|_{\mathcal{B}} \leq C_1(\|u\| + \|v\|_{\mathcal{B}}) + \mathcal{C}_2,$$

$t \in [0, a]$, $u \in X$, $v \in \mathcal{B}$.

(H5) The function $h : \mathcal{B} \to \mathcal{D}(A)$ is a continuous and there exists $L_h \in C([0, a]; \mathbb{R}^+)$ such that

$$\|h(u) - h(v)\|_{\mathcal{D}(A)} \leq L_h(t)\|u - v\|_{\mathcal{B}},$$

for all $u, v \in \mathcal{B}$.

(H6) For all $i = 1, \cdots, N$ and each $t \in (t_i, s_i]$, the function $g_i(t, \cdot) \in C(\mathcal{C}_i(t); \mathcal{D}(A))$ and there is a bounded function $L_{g_i} \in C((t_i, s_i]; \mathbb{R}^+)$ such that

$$\|g_i(t,u) - g_i(t,v)\|_{\mathcal{D}(A)} \leq L_{g_i}(t)\|u - v\|_{\mathcal{C}_i(t)},$$

for all $u, v \in \mathcal{C}_i(t)$.

(H7) For all $i = 1, \cdots, N$ and each $u \in PC(X)$, the function

$$t \to g_i(t, u_{|_{I_i(t)}}) \in C((t_i, s_i]; X)$$

and

$$\lim_{t \downarrow t_i} g_i(t, u_{|_{I_i(t)}})$$

exists.

(H8) For all $t \in (t_i, s_i]$ and $i \in \{1, \cdots, N\}$, the map

$$\Psi_i(t) : PC(X)_{|_{I_i(t)}} = \{u_{|_{I_i(t)}} : u \in PC(X)\} \to \mathcal{C}_i(t)$$

given by $\Psi_i(t)u = u_{|_{I_i(t)}}$ is a bounded linear operator and we always assume that the set of operators $\{\Psi_i(t) : t \in (t_i, s_i], i = 1, \cdots, N\}$ is bounded. For our simplicity, we use the notation

$$\tilde{\Psi}_i(s) = \|\Psi_i(s)\|_{\mathcal{L}(PC(X)_{|_{I_i(s)}}, \mathcal{C}_i(s))}.$$

Remark 3.3. We will use the notations K_h for $K_h = \sup\{\|h(u)\| : u \in \mathcal{B}\}$.

Let a function $x : (-\infty, a] \to X$ be defined by $x_0 = \phi + h(u)$ and

$$x(t) = \phi(0) + h(u) + \int_0^t S'(t-s)(\phi(0) + h(u))ds$$

for all $t \in [0, a]$. It is easily say that

$$\|x_t\| \leq (\mathfrak{K}_a \mathfrak{H}(1 + \|\varphi\|_{L^1([0,a];\mathbb{R}^+)}) + \mathfrak{M}_a)(\|\phi\|_{\mathcal{B}} + K_h),$$

where $\mathfrak{M}_a = \sup_{t\in[0,a]} \mathfrak{M}(t)$, $\mathfrak{K}_a = \sup_{t\in[0,a]} \mathfrak{K}(t)$.

Theorem 3.4. *Assume that* $(H1)$, $(H3)$, *and* $(H5)$-$(H8)$ *are satisfied and if*

$$\begin{aligned}\max_{i=1,\cdots,m} \Big\{ &[L_{g_i}(s_i)\tilde{\Psi}_i(s) + L_p((1+\mathfrak{K}_a) + L_{g_i}(s_i)\tilde{\Psi}_i(s) + \mathfrak{K}_a) \\ &+ t_{i+1}L_q(1+\mathfrak{K}_a)](1 + \|\varphi\|_{L^1([s_i,t_{i+1}];\mathbb{R}^+)}), \\ &[L_p(2L_h\mathfrak{K}_a + 1 + \mathfrak{K}_a) + t_1 L_q(1+\mathfrak{K}_a)](1 + \|\varphi\|_{L^1([0,t_1];\mathbb{R}^+)}), \\ &L_{g_i}(s_i)\tilde{\Psi}_i(s)\Big\} < 1,\end{aligned}$$

then (1.1)-(1.3) *has a unique mild solution.*

Proof. Let the space

$$\mathcal{E}(a)=\{u:(-\infty,a]\to X: u_0=0, u|_{[0,a]}\in PC(X)\}$$

endowed with the sup-norm. Now by Proposition 3.2, we consider the operator $\mathscr{Z}:\mathcal{E}(a)\to\mathscr{E}(a)$ by

$$\mathscr{Z}u(t)$$
$$=\begin{cases}0,\ t\in[-\infty,0],\\ p(0,\phi(0)+h(u)(0),u_0)-p(t,u(t)+x(t),u_t+x_t)\\ +\int_0^t q(s,u(s)+x(s),u_s+x_s)ds\\ +\int_0^t S'(t-s)\left(p(0,\phi(0)+h(u)(0),u_0)-p(s,u(s)+x(s),u_s+x_s)\right.\\ \left.+\int_0^s q(\tau,u(\tau)+x(\tau),u_\tau+x_\tau)d\tau\right)ds,\ t\in[0,t_1],\\ g_i(t,u|_{I_i(t)}),\ t\in(t_i,s_i],\ i=1,\cdots,m,\\ g_i(s_i,u|_{I_i(s_i)})-p(t,u(t)+x(t),u_t+x_t)+p(s_i,g_i(s_i,u|_{I_i(s_i)}),u_{s_i}+x_{s_i})\\ +\int_{s_i}^t q(s,u(s)+x(s),u_s+x_s)ds+\int_{s_i}^t S'(t-s)\Big(g_i(s_i,u|_{I_i(s_i)})\\ -p(s,u(s)+x(s),u_s+x_s)+p(s_i,g_i(s_i,u|_{I_i(s_i)}),u_{s_i}+x_{s_i})\\ +\int_{s_i}^s q(\tau,u(\tau)+x(\tau),u_\tau+x_\tau)d\tau\Big)ds,\quad t\in[s_i,t_{i+1}],\ i=1,\cdots,m.\end{cases}$$

It is easily seen that

$$\|u_t+x_t\|_{\mathcal{B}}\le\mathfrak{K}_a\|u\|_t+(\mathfrak{K}_a\mathfrak{H}(1+\|\varphi\|_{L^1([0,a];\mathbb{R}^+)})+\mathfrak{M}_a)(\|\phi\|_{\mathcal{B}}+K_h),$$

where $\|u\|_t=\sup_{0\le s\le t}\|u(s)\|$. Let $u\in\mathcal{E}(a)$ and from the assumption $(H1)$, $(H3)$, we get that

$$\begin{aligned}&\int_0^t\|S'(t-s)(p(0,\phi(0)+h(u)(0),u_0)-p(s,u(s)+x(s),u_s+x_s)\\ &\quad+\int_0^s q(\tau,u(\tau)+x(\tau),u_\tau+x_\tau)d\tau)\|ds\\ &\le\int_0^t\varphi(t-s)\left(\|p(0,\phi(0)+h(u)(0),u_0)\|+\|p(s,u(s)+x(s),u_s+x_s)\|\right.\\ &\quad\left.+t_1\|q(s,u(s)+x(s),u_s+x_s)\|\right)ds\\ &\le\left(\|p(0,\phi(0)+h(u)(0),u_0)\|+\|p(s,u(s)+x(s),u_s+x_s)\|\right.\\ &\quad\left.+t_1\|q(s,u(s)+x(s),u_s+x_s)\|\right)\|\varphi\|_{L^1([0,t_1];\mathbb{R}^+)},\end{aligned}$$

which follows that

$$\begin{aligned}s\ &\to\ S'(t-s)(p(0,\phi(0)+h(u)(0),u_0)-p(s,u(s)+x(s),u_s+x_s)\\ &\qquad+\int_0^t q(s,u(s)+x(s),u_s+x_s)ds)\end{aligned}$$

is integrable on $[0,t]$, for all $t\in[0,t_1]$.

In the similar way we will say that the function

$$s \rightarrow S'(t-s)(g_i(s_i, u_{|_{I_i(s_i)}}) - p(t, u(t)+x(t), u_t+x_t)$$
$$+ p(s_i, g_i(s_i, u_{|_{I_i(s_i)}}), u_{s_i}+x_{s_i}) + \int_{s_i}^{t} q(s, u(s)+x(s), u_s+x_s)ds)$$

is also integrable on $[0,t]$, for all $t \in [s_i, t_{i+1}]$, $i = 1, \cdots, m$. Then, the operator $\mathscr{Z}$ is well defined and $\mathscr{Z}$ have the values in $\mathcal{E}(a)$. Now, we consider $u, v \in \mathcal{E}(a)$ and for $t \in [s_i, t_{i+1}]$, $i = 1, \cdots, m$,

$$\begin{aligned}
&\|\mathscr{Z}u(t) - \mathscr{Z}v(t)\| \\
&\leq \|g_i(s_i, u_{|_{I_i(s_i)}}) - g_i(s_i, v_{|_{I_i(s_i)}})\|_{\mathcal{D}(A)} \\
&\quad + \|p(t, u(t)+x(t), u_t+x_t) - p(t, v(t)+x(t), v_t+x_t)\|_{\mathcal{D}(A)} \\
&\quad + \|p(s_i, g_i(s_i, u_{|_{I_i(s_i)}}), u_{s_i}+x_{s_i}) - p(s_i, g_i(s_i, v_{|_{I_i(s_i)}}), v_{s_i}+x_{s_i})\|_{\mathcal{D}(A)} \\
&\quad + \int_{s_i}^{t} \|q(s, u(s)+x(s), u_s+x_s) - q(s, v(s)+x(s), v_s+x_s)\|_{\mathcal{D}(A)} ds \\
&\quad + \int_{s_i}^{t} \|S'(t-s)\| \Big(\|g_i(s_i, u_{|_{I_i(s_i)}}) - g_i(s_i, v_{|_{I_i(s_i)}})\|_{\mathcal{D}(A)} \\
&\quad + \|p(s, u(s)+x(s), u_s+x_s) - p(s, v(s)+x(s), v_s+x_s)\|_{\mathcal{D}(A)} \\
&\quad + \|p(s_i, g_i(s_i, u_{|_{I_i(s_i)}}), u_{s_i}+x_{s_i}) - p(s_i, g_i(s_i, v_{|_{I_i(s_i)}}), v_{s_i}+x_{s_i})\|_{\mathcal{D}(A)} \\
&\quad + \int_{s_i}^{s} \|q(\tau, u(\tau)+x(\tau), u_\tau+x_\tau) - q(\tau, v(\tau)+x(\tau), v_\tau+x_\tau)\|_{\mathcal{D}(A)} d\tau \Big) ds \\
&\leq L_{g_i}(s_i)\|u_{|_{I_i(s_i)}} - v_{|_{I_i(s_i)}}\|_{\mathcal{C}_i(s_i)} + L_p(\|u-v\| + \|u_t - v_t\|) \\
&\quad + L_p(L_{g_i}(s_i)\|u_{|_{I_i(s_i)}} - v_{|_{I_i(s_i)}}\| + \|u_{s_i} - v_{s_i}\|) + t_{i+1}L_q(\|u-v\| + \|u_t - v_t\|) \\
&\quad + \int_{s_i}^{t} \varphi(t-s)(L_{g_i}(s_i)\|u_{|_{I_i(s_i)}} - v_{|_{I_i(s_i)}}\|_{\mathcal{C}_i(s_i)} + L_p(\|u-v\| + \|u_s - v_s\|) \\
&\quad + L_p(L_{g_i}(s_i)\|u_{|_{I_i(s_i)}} - v_{|_{I_i(s_i)}}\| + \|u_{s_i} - v_{s_i}\|) + t_{i+1}L_q(\|u-v\| + \|u_s - v_s\|))ds \\
&\leq [L_{g_i}(s_i)\|\Psi_i(s)\|_{\mathcal{L}(PC(X)_{|_{I_i(s)}}, \mathcal{C}_i(s))}\|u-v\| + L_p(\|u-v\| + \mathfrak{K}_a\|u-v\|_t) \\
&\quad + L_p(L_{g_i}(s_i)\|\Psi_i(s)\|_{\mathcal{L}(PC(X)_{|_{I_i(s)}}, \mathcal{C}_i(s))}\|u-v\| + \mathfrak{K}_a\|u-v\|_{s_i}) \\
&\quad + t_{i+1}L_q(\|u-v\| + \mathfrak{K}_a\|u-v\|_t)](1 + \|\varphi\|_{L^1([s_i, t_{i+1}]; \mathbb{R}^+)}) \\
&\leq [L_{g_i}(s_i)\tilde{\Psi}_i(s) + L_p((1+\mathfrak{K}_a) + L_{g_i}(s_i)\tilde{\Psi}_i(s) + \mathfrak{K}_a) \\
&\quad + t_{i+1}L_q(1+\mathfrak{K}_a)](1 + \|\varphi\|_{L^1([s_i, t_{i+1}]; \mathbb{R}^+)})\|u-v\|_{t_{i+1}},
\end{aligned}$$

which implies that

$$\begin{aligned}
&\|\mathscr{Z}u(t) - \mathscr{Z}v(t)\|_{C([s_i, t_{i+1}]; \mathcal{E}(a))} \\
&\leq [L_{g_i}(s_i)\tilde{\Psi}_i(s) + L_p((1+\mathfrak{K}_a) + L_{g_i}(s_i)\tilde{\Psi}_i(s) + \mathfrak{K}_a) \\
&\quad + t_{i+1}L_q(1+\mathfrak{K}_a)](1 + \|\varphi\|_{L^1([s_i, t_{i+1}]; \mathbb{R}^+)})\|u-v\|_{t_{i+1}}
\end{aligned}$$

for all $t \in [s_i, t_{i+1}],\ i = 1, \cdots, m$. We proceed for the interval $t \in [0, t_1]$,

$$\begin{aligned}
&\|\mathscr{Z}u(t) - \mathscr{Z}v(t)\|_{C([0,t_1];\mathcal{E}(a))} \\
&\leq \|p(0, \phi(0) + h(u)(0), u_0) - p(0, \phi(0) + h(v)(0), v_0)\| \\
&\quad + \|p(t, u(t) + x(t), u_t + x_t) - p(t, v(t) + x(t), v_t + x_t)\| \\
&\quad + \int_0^t \|q(s, u(s) + x(s), u_s + x_s) - q(s, v(s) + x(s), v_s + x_s)\| ds \\
&\quad + \int_0^t \|S'(t-s)\| \left(\|p(0, \phi(0) + h(u)(0), u_0) - p(0, \phi(0) + h(v)(0), v_0)\|\right. \\
&\quad + \|p(s, u(s) + x(s), u_s + x_s) - p(s, v(s) + x(s), v_s + x_s)\| \\
&\quad \left. + \int_0^s \|q(\tau, u(\tau) + x(\tau), u_\tau + x_\tau) - q(\tau, v(\tau) + x(\tau), v_\tau + x_\tau)\| d\tau \right) ds \\
&\leq L_p(2L_h)\|u - v\|_{\mathcal{B}} + L_p(\|u - v\| + \|u_t - v_t\|) \\
&\quad + t_1 L_q(\|u - v\| + \|u_t - v_t\|) + \int_0^t \varphi(t-s)(L_p(2L_h)\|u - v\|_{\mathcal{B}} \\
&\quad + L_p(\|u - v\| + \|u_s - v_s\|) + t_1 L_q(\|u - v\| + \|u_s - v_s\|))ds \\
&\leq [L_p(2L_h\mathfrak{K}_a + 1 + \mathfrak{K}_a)\|u - v\| + t_1 L_q(1 + \mathfrak{K}_a)\|u - v\|](1 + \|\varphi\|_{L^1([0,t_1];\mathbb{R}^+)}) \\
&\leq [L_p(2L_h\mathfrak{K}_a + 1 + \mathfrak{K}_a) + t_1 L_q(1 + \mathfrak{K}_a)](1 + \|\varphi\|_{L^1([0,t_1];\mathbb{R}^+)})\|u - v\|_{t_1}.
\end{aligned}$$

Similarly, we have for $t \in [t_i, s_i],\ i = 1, \cdots, m$,

$$\begin{aligned}
\|\mathscr{Z}u(t) - \mathscr{Z}v(t)\|_{C([t_i,s_i];\mathcal{E}(a))} &\leq \|g_i(t, u_{|_{I_i(t)}}) - g_i(t, v_{|_{I_i(t)}})\| \\
&\leq L_{g_i}(s_i)\|\Psi_i(s)\|_{\mathcal{L}(PC(X)_{|_{I_i(s)}}, \mathcal{C}_i(s))}\|u - v\| \\
&\leq L_{g_i}(t)\tilde{\Psi}_i(s)\|u - v\|_{s_i}.
\end{aligned}$$

It says form the above three inequalities that

$$\begin{aligned}
&\|\mathscr{Z}u(t) - \mathscr{Z}v(t)\|_{\mathcal{E}(a)} \\
&\leq \max_{i=1,\cdots,m} \Big\{ [L_{g_i}(s_i)\tilde{\Psi}_i(s) + L_p((1 + \mathfrak{K}_a) + L_{g_i}(s_i)\tilde{\Psi}_i(s) + \mathfrak{K}_a) \\
&\qquad\qquad + t_{i+1} L_q(1 + \mathfrak{K}_a)](1 + \|\varphi\|_{L^1([s_i,t_{i+1}];\mathbb{R}^+)}), \\
&\qquad\qquad [L_p(2L_h\mathfrak{K}_a + 1 + \mathfrak{K}_a) + t_1 L_q(1 + \mathfrak{K}_a)](1 + \|\varphi\|_{L^1([0,t_1];\mathbb{R}^+)}), \\
&\qquad\qquad L_{g_i}(t)\tilde{\Psi}_i(s) \Big\} \|u - v\|_{\mathcal{E}(A)}.
\end{aligned}$$

Hence we infer that $\mathscr{Z}(\cdot)$ is a contraction map and there exists a unique mild solution of the problem (1.1)-(1.3). □

In the subsequent part of the work, the existence results follows from the fixed point theorem for condensing map.

Theorem 3.5. *Suppose that the resolvent operator* $(S(t))_{t\geq 0}$ *is a compact, the functions* $g_i(\cdot,0),\ i=1,\cdots,m$ *are bounded and that the condition* $(H2)-(H8)$ *are satisfied. If*

$$\begin{aligned}
&L_{g_i}(s_i)\tilde{\Psi}_i(s)+C_1(1+\mathfrak{K}_a)+C_1(L_{g_i}(s_i)\tilde{\Psi}_i(s)+\mathfrak{K}_a)\\
&+t_{i+1}m_q(t)\limsup_{r\to\infty}W\bigg(r(1+\mathfrak{K}_a)+\bigg((1+\|\varphi\|)+\mathfrak{K}_a\mathfrak{H}(1+\|\varphi\|_{L^1([0,a];\mathbb{R}^+)})\\
&+\mathfrak{M}_a\bigg)(\|\phi\|_{\mathcal{B}}+K_h)\bigg)(1+\|\varphi\|_{C([s_i,t_{i+1}];\mathbb{R}^+)})<1,\\
&C_1(1+\mathfrak{K}_a)+t_1m_q(t)\limsup_{r\to\infty}W(r(1+\mathfrak{K}_a)+((1+\|\varphi\|)\\
&+\mathfrak{K}_a\mathfrak{H}(1+\|\varphi\|_{L^1([0,a];\mathbb{R}^+)})+\mathfrak{M}_a)\times(\|\phi\|_{\mathcal{B}}+K_h))(1+\|\varphi\|_{C([0,t_1];\mathbb{R}^+)})<1,\\
&\max_{i=1,\cdots,N}\Big\{[L_{g_i}(s_i)\tilde{\Psi}_i(s)+L_p((1+\mathfrak{K}_a)+(L_{g_i}(s_i)\tilde{\Psi}_i(s)+\mathfrak{K}_a)](1+\|\varphi\|_{L^1([s_i,t_{i+1}];\mathbb{R}^+)}),\\
&L_{g_i}(t)\tilde{\Psi}_i(s),[L_p(2L_h\mathfrak{K}_a+1+\mathfrak{K}_a)](1+\|\varphi\|_{L^1([0,t_1];\mathbb{R}^+)})\Big\}<1,
\end{aligned}$$

then the problem (1.1)-(1.3) *has at least one mild solution.*

Proof. We consider the fixed point operator $\mathscr{X}$ as presented in previous Theorem 3.4. Here we going to prove that $\mathscr{X}$ is a condensing map from $B_r(0,\mathcal{E}(A))$ into $B_r(0,\mathcal{E}(A))$. Choose $r>0$ such that

$$\begin{aligned}
&L_{g_i}(s_i)\tilde{\Psi}_i(s)r+\|g_i(s_i,0)\|+C_1(r(1+\mathfrak{K}_a)+((1+\|\varphi\|)\\
&+\mathfrak{K}_a\mathfrak{H}(1+\|\varphi\|_{L^1([0,a];\mathbb{R}^+)})+\mathfrak{M}_a)(\|\phi\|_{\mathcal{B}}+K_h))+2C_2+C_1(L_{g_i}(s_i)\tilde{\Psi}_i(s)r\\
&+\|g_i(s_i,0)\|+\mathfrak{K}_ar+(\mathfrak{K}_a\mathfrak{H}(1+\|\varphi\|_{L^1([0,a];\mathbb{R}^+)})+\mathfrak{M}_a)(\|\phi\|_{\mathcal{B}}+K_h))\\
&+t_{i+1}m_q(t)W(r(1+\mathfrak{K}_a)+\bigg((1+\|\varphi\|)+\mathfrak{K}_a\mathfrak{H}(1+\|\varphi\|_{L^1([0,a];\mathbb{R}^+)})+\mathfrak{M}_a\bigg)\\
&\times(\|\phi\|_{\mathcal{B}}+K_h))(1+\|\varphi\|_{C([s_i,t_{i+1}];\mathbb{R}^+)})<s,\\
&(2C_1(\|\phi\|_{\mathcal{B}}+K_h)+2C_2+C_1\bigg(r(1+\mathfrak{K}_a)+\bigg((1+\|\varphi\|)+\mathfrak{K}_a\mathfrak{H}(1+\|\varphi\|_{L^1([0,a];\mathbb{R}^+)})\\
&+\mathfrak{M}_a\bigg)(\|\phi\|_{\mathcal{B}}+K_h)\bigg)+t_1m_q(t)W(r(1+\mathfrak{K}_a)+((1+\|\varphi\|)\\
&+\mathfrak{K}_a\mathfrak{H}(1+\|\varphi\|_{L^1([0,a];\mathbb{R}^+)})+\mathfrak{M}_a)(\|\phi\|_{\mathcal{B}}+K_h))(1+\|\varphi\|_{C([0,t_1];\mathbb{R}^+)})<s,
\end{aligned}$$

for all $s\geq r$.

In this sequel first we prove that $\mathscr{X}$ has values in $B_r(0,\mathcal{E}(A))$, *i.e.*,

$$\mathscr{X}B_r(0,\mathcal{E}(A))\subset B_r(0,\mathcal{E}(A)).$$

Consider $u\in B_r(0,\mathcal{E}(A))$. For $t\in[s_i,t_{i+1}],\ i=1,\cdots,m$, we get

$$
\begin{aligned}
&\|\mathscr{Z}u(t)\| \\
&\leq \|g_i(s_i, u_{|_{I_i(s_i)}}) - g_i(s_i, 0)\| + \|g_i(s_i, 0)\| + \|p(t, u(t) + x(t), u_t + x_t)\| \\
&\quad + \|p(s_i, g_i(s_i, u_{|_{I_i(s_i)}}), u_{s_i} + x_{s_i})\| + \int_{s_i}^{t} \|q(s, u(s) + x(s), u_s + x_s)\| ds \\
&\quad + \int_{s_i}^{t} \|S'(t-s)\|(\|g_i(s_i, u_{|_{I_i(s_i)}}) - g_i(s_i, 0)\| + \|g_i(s_i, 0)\| + \|p(s, u(s) + x(s), u_s + x_s)\| \\
&\quad + \|p(s_i, g_i(s_i, u_{|_{I_i(s_i)}}), u_{s_i} + x_{s_i})\| + \int_{s_i}^{s} \|q(\tau, u(\tau) + x(\tau), u_\tau + x_\tau)\| d\tau) ds \\
&\leq L_{g_i}(s_i)\tilde{\Psi}_i(s)\|u\|_{\mathcal{E}(A)} + \|g_i(s_i, 0)\| + C_1(\|u\| + \|x\| + \|u_t + x_t\|) + C_2 \\
&\quad + C_1(\|g_i(s_i, u_{|_{I_i(s_i)}})\| + \|u_{s_i} + x_{s_i}\|) + C_2 + \int_{s_i}^{t} m_q(t) W(\|u\| + \|x\| + \|u_s + x_s\|) ds \\
&\quad + \int_{s_i}^{t} \varphi(t-s)(L_{g_i}(s_i)\tilde{\Psi}_i(s)\|u\|_{\mathcal{E}(A)} + \|g_i(s_i, 0)\| + C_1(\|u\| + \|x\| + \|u_s + x_s\|) + C_2 \\
&\quad + C_1(\|g_i(s_i, u_{|_{I_i(s_i)}})\| + \|u_{s_i} + x_{s_i}\|) + C_2 + \int_{s_i}^{s} m_q(\tau) W(\|u\| + \|x\| + \|u_\tau + x_\tau\|) d\tau) ds \\
&\leq L_{g_i}(s_i)\tilde{\Psi}_i(s)\|u\|_{\mathcal{E}(A)} + \|g_i(s_i, 0)\| + C_1(\|u\| + (\|\phi\|_{\mathcal{B}} + K_h)(1 + \|\varphi\|) + \mathfrak{K}_a\|u\|_t \\
&\quad + (\mathfrak{K}_a\mathfrak{H}(1 + \|\varphi\|_{L^1([0,a];\mathbb{R}^+)}) + \mathfrak{M}_a)(\|\phi\|_{\mathcal{B}} + K_h)) + 2C_2 \\
&\quad + C_1(L_{g_i}(s_i)\tilde{\Psi}_i(s)\|u\|_{\mathcal{E}(A)} + \|g_i(s_i, 0)\| + \mathfrak{K}_a\|u\|_{s_i} + (\mathfrak{K}_a\mathfrak{H}(1 + \|\varphi\|_{L^1([0,a];\mathbb{R}^+)}) + \mathfrak{M}_a) \\
&\quad \times (\|\phi\|_{\mathcal{B}} + K_h)) + t_{i+1} m_q(t) W(\|u\| + (\|\phi\|_{\mathcal{B}} + K_h)(1 + \|\varphi\|) + \mathfrak{K}_a\|u\|_t \\
&\quad + (\mathfrak{K}_a\mathfrak{H}(1 + \|\varphi\|_{L^1([0,a];\mathbb{R}^+)}) + \mathfrak{M}_a)(\|\phi\|_{\mathcal{B}} + K_h))(1 + \|\varphi\|_{C([s_i,t_{i+1}];\mathbb{R}^+)}) \\
&\leq L_{g_i}(s_i)\tilde{\Psi}_i(s) r + \|g_i(s_i, 0)\| + C_1(r(1 + \mathfrak{K}_a) \\
&\quad + ((1 + \|\varphi\|) + \mathfrak{K}_a\mathfrak{H}(1 + \|\varphi\|_{L^1([0,a];\mathbb{R}^+)}) + \mathfrak{M}_a)(\|\phi\|_{\mathcal{B}} + K_h)) + 2C_2 \\
&\quad + C_1(L_{g_i}(s_i)\tilde{\Psi}_i(s) r + \|g_i(s_i, 0)\| + \mathfrak{K}_a r + (\mathfrak{K}_a\mathfrak{H}(1 + \|\varphi\|_{L^1([0,a];\mathbb{R}^+)}) + \mathfrak{M}_a) \\
&\quad \times (\|\phi\|_{\mathcal{B}} + K_h)) + t_{i+1} m_q(t) W(r(1 + \mathfrak{K}_a) \\
&\quad + ((1 + \|\varphi\|) + \mathfrak{K}_a\mathfrak{H}(1 + \|\varphi\|_{L^1([0,a];\mathbb{R}^+)}) + \mathfrak{M}_a)(\|\phi\|_{\mathcal{B}} + K_h))(1 + \|\varphi\|_{C([s_i,t_{i+1}];\mathbb{R}^+)}) \\
&\leq r.
\end{aligned}
$$

Proceeding for the interval $t \in [0, t_1]$,

$$
\begin{aligned}
&\|\mathscr{Z}u(t)\| \\
&\leq \|p(0, \phi(0) + h(u)(0), u_0)\| + \|p(t, u(t) + x(t), u_t + x_t)\| \\
&\quad + \int_0^t \|q(s, u(s) + x(s), u_s + x_s)\| ds + \int_0^t \|S'(t-s)\|(\|p(0, \phi(0) + h(u)(0), u_0)\| \\
&\quad + \|p(s, u(s) + x(s), u_s + x_s)\| + \int_0^s \|q(\tau, u(\tau) + x(\tau), u_\tau + x_\tau)\| d\tau) ds \\
&\leq 2C_1(\|\phi\|_{\mathcal{B}} + \|h(u)\|) + 2C_2 + C_1(\|u\| + \|x\| + \|u_t + x_t\|)
\end{aligned}
$$

$$
\begin{aligned}
&+ t_1 m_q(t) W(\|u\| + \|x\| + \|u_t + x_t\|) + \int_0^t \varphi(t-s)(2C_1(\|\phi\|_{\mathcal{B}} + \|h(u)\|) + 2C_2 \\
&+ C_1(\|u\| + \|x\| + \|u_s + x_s\|) + t_1 m_q(s) W(\|u\| + \|x\| + \|u_s + x_s\|))ds \\
\le\ & (2C_1(\|\phi\|_{\mathcal{B}} + K_h) + 2C_2 + C_1(r + (\|\phi\|_{\mathcal{B}} + K_h)(1 + \|\varphi\|) + \mathfrak{K}_a \|u\|_t \\
&+ (\mathfrak{K}_a \mathfrak{H}(1 + \|\varphi\|_{L^1([0,a];\mathbb{R}^+)}) + \mathfrak{M}_a)(\|\phi\|_{\mathcal{B}} + K_h)) \\
&+ t_1 m_q(t) W(r + (\|\phi\|_{\mathcal{B}} + K_h)(1 + \|\varphi\|) \\
&+ \mathfrak{K}_a \|u\|_t + (\mathfrak{K}_a \mathfrak{H}(1 + \|\varphi\|_{L^1([0,a];\mathbb{R}^+)}) + \mathfrak{M}_a)(\|\phi\|_{\mathcal{B}} + K_h)))(1 + \|\varphi\|_{C([0,t_1];\mathbb{R}^+)}) \\
\le\ & (2C_1(\|\phi\|_{\mathcal{B}} + K_h) + 2C_2 + C_1(r(1 + \mathfrak{K}_a) \\
&+ ((1 + \|\varphi\|) + \mathfrak{K}_a \mathfrak{H}(1 + \|\varphi\|_{L^1([0,a];\mathbb{R}^+)}) + \mathfrak{M}_a)(\|\phi\|_{\mathcal{B}} + K_h)) \\
&+ t_1 m_q(t) W(r(1 + \mathfrak{K}_a) + ((1 + \|\varphi\|) + \mathfrak{K}_a \mathfrak{H}(1 + \|\varphi\|_{L^1([0,a];\mathbb{R}^+)}) + \mathfrak{M}_a) \\
&\times (\|\phi\|_{\mathcal{B}} + K_h)))(1 + \|\varphi\|_{C([0,t_1];\mathbb{R}^+)}) \\
\le\ & r.
\end{aligned}
$$

Likewise we get for $t \in (t_i, s_i],\ i = 1, \cdots, m$,

$$
\begin{aligned}
\|\mathscr{X} u(t)\| &\le \|g_i(t, u_{|_{I_i(t)}}) - g_i(t, 0)\| + \|g_i(\cdot, 0)\|_{C((t_i,s_i];\mathcal{D}(A))} \\
&\le L_{g_i}(t) \tilde{\Psi}_i(s) \|u\| + \|g_i(\cdot, 0)\|_{C((t_i,s_i];\mathcal{D}(A))} \\
&\le L_{g_i}(t) \tilde{\Psi}_i(s) r + \|g_i(\cdot, 0)\|_{C((t_i,s_i];\mathcal{D}(A))} \\
&\le r.
\end{aligned}
$$

Hence from the above find outs that

$$
\|\mathscr{X} u(t)\|_{\mathcal{E}(A)} \quad \le \quad r.
$$

Thus we get the operator $\mathscr{X}$ has a values in $B_r(0, \mathcal{E}(A))$. To continue the remaining proof we divide the operator $\mathscr{X}$ like follows $\mathscr{X} = \mathscr{X}_1 + \mathscr{X}_2 + \mathscr{X}_3$, where

$$
\mathscr{X}_1 u(t) = \begin{cases}
0,\ t \in [-\infty, 0], \\
p(0, \phi(0) + h(u)(0), u_0) - p(t, u(t) + x(t), u_t + x_t) \\
+ \int_0^t S'(t-s)(p(0, \phi(0) + h(u)(0), u_0) \\
-p(s, u(s) + x(s), u_s + x_s)ds,\ t \in [0, t_1], \\
g_i(t, u_{|_{I_i(t)}}),\ t \in (t_i, s_i],\ i = 1, \cdots, m, \\
g_i(s_i, u_{|_{I_i(s_i)}}) - p(t, u(t) + x(t), u_t + x_t) + p(s_i, g_i(s_i, u_{|_{I_i(s_i)}}), u_{s_i} + x_{s_i}) \\
+ \int_{s_i}^t S'(t-s)(g_i(s_i, u_{|_{I_i(s_i)}}) - p(s, u(s) + x(s), u_s + x_s) \\
+p(s_i, g_i(s_i, u_{|_{I_i(s_i)}}), u_{s_i} + x_{s_i}))ds,\ t \in [s_i, t_{i+1}],\ i = 1, \cdots, m,
\end{cases}
$$

$$
\mathscr{X}_2 u(t) = \int_{s_i}^t q(s, u(s) + x(s), u_s + x_s)ds,\ t \in [s_i, t_{i+1}],\ i = 0, \cdots, m,
$$

$$
\mathscr{X}_3 u(t) = \int_{s_i}^t S'(t-s)(\int_{s_i}^s q(\tau, u(\tau) + x(\tau), u_\tau + x_\tau)d\tau)ds,\ t \in [s_i, t_{i+1}],\ i = 0, \cdots, m.
$$

In the first step we prove that the operator $\mathscr{Z}_1$ is a contraction.

Step 1. Let $u, v \in B_r(0, \mathcal{E}(A))$. For $t \in [s_i, t_{i+1}],\ i = 1, \cdots, m$,

$$\begin{aligned}
&\|\mathscr{Z}_1 u(t) - \mathscr{Z}_1 v(t)\| \\
&\le L_{g_i}(s_i)\|u_{|_{I_i(s_i)}} - v_{|_{I_i(s_i)}}\|_{\mathcal{C}_i(s_i)} + L_p(\|u - v\| + \|u_t - v_t\|) \\
&\quad + L_p(L_{g_i}(s_i)\|u_{|_{I_i(s_i)}} - v_{|_{I_i(s_i)}}\| + \|u_{s_i} - v_{s_i}\|) \\
&\quad + \int_{s_i}^{t} \varphi(t-s)(L_{g_i}(s_i)\|u_{|_{I_i(s_i)}} - v_{|_{I_i(s_i)}}\|_{\mathcal{C}_i(s_i)} + L_p(\|u - v\| + \|u_s - v_s\|) \\
&\quad + L_p(L_{g_i}(s_i)\|u_{|_{I_i(s_i)}} - v_{|_{I_i(s_i)}}\| + \|u_{s_i} - v_{s_i}\|) \\
&\le [L_{g_i}(s_i)\|\Psi_i(s)\|_{\mathcal{L}(PC(X)_{|_{I_i(s)}}, \mathcal{C}_i(s))}\|u - v\| + L_p(\|u - v\| + \mathfrak{K}_a\|u - v\|_t) \\
&\quad + L_p(L_{g_i}(s_i)\|\Psi_i(s)\|_{\mathcal{L}(PC(X)_{|_{I_i(s)}}, \mathcal{C}_i(s))}\|u - v\| + \mathfrak{K}_a\|u - v\|_{s_i})] \\
&\quad \times (1 + \|\varphi\|_{L^1([s_i, t_{i+1}];\mathbb{R}^+)}) \\
&\le [L_{g_i}(s_i)\tilde{\Psi}_i(s) + L_p((1 + \mathfrak{K}_a) + L_{g_i}(s_i)\tilde{\Psi}_i(s) + \mathfrak{K}_a)] \\
&\quad \times (1 + \|\varphi\|_{L^1([s_i, t_{i+1}];\mathbb{R}^+)})\|u - v\|_{t_{i+1}}
\end{aligned}$$

which implies that

$$\begin{aligned}
&\|\mathscr{Z}_1 u(t) - \mathscr{Z}_1 v(t)\|_{C([s_i, t_{i+1}];\mathcal{E}(a))} \\
&\le [L_{g_i}(s_i)\tilde{\Psi}_i(s) + L_p((1 + \mathfrak{K}_a) + L_{g_i}(s_i)\tilde{\Psi}_i(s) + \mathfrak{K}_a)] \\
&\quad \times (1 + \|\varphi\|_{L^1([s_i, t_{i+1}];\mathbb{R}^+)})\|u - v\|_{t_{i+1}}
\end{aligned}$$

Similarly, for $t \in [t_i, s_i],\ i = 1, \cdots, m$,

$$\|\mathscr{Z}_1 u(t) - \mathscr{Z}_1 v(t)\|_{C([t_i, s_i];\mathcal{E}(a))} \le L_{g_i}(t)\tilde{\Psi}_i(s)\|u - v\|_{s_i},$$

and for $t \in [0, t_1]$, we get

$$\begin{aligned}
&\|\mathscr{Z}_1 u(t) - \mathscr{Z}_1 v(t)\|_{C([0, t_1];\mathcal{E}(a))} \\
&\le [L_p(2L_h\mathfrak{K}_a + 1 + \mathfrak{K}_a)](1 + \|\varphi\|_{L^1([0, t_1];\mathbb{R}^+)})\|u - v\|_{t_1},
\end{aligned}$$

which implies that

$$\begin{aligned}
&\|\mathscr{Z}_1 u(t) - \mathscr{Z}_1 v(t)\|_{\mathcal{E}(a)} \\
&\le \max_{i=1,\cdots,N} \Big\{ [L_{g_i}(s_i)\tilde{\Psi}_i(s) + L_p((1 + \mathfrak{K}_a) + L_{g_i}(s_i)\tilde{\Psi}_i(s) + \mathfrak{K}_a)] \\
&\qquad \times (1 + \|\varphi\|_{L^1([s_i, t_{i+1}];\mathbb{R}^+)}),\ \ L_{g_i}(t)\tilde{\Psi}_i(s), \\
&\qquad [L_p(2L_h\mathfrak{K}_a + 1 + \mathfrak{K}_a)](1 + \|\varphi\|_{L^1([0, t_1];\mathbb{R}^+)}) \Big\} \|u - v\|_{\mathcal{E}(A)}.
\end{aligned}$$

Thus we say that $\mathscr{Z}_1$ is a contraction on $B_r(0, \mathcal{E}(A))$.

Next, we prove the operators $\mathscr{Z}_2$ and $\mathscr{Z}_3$ are completely continuous.

Step 2. The map $\mathscr{Z}_2$ is completely continuous. We can easily see that $\mathscr{Z}_2$ is continuous. So the remaining we only prove that $\mathscr{Z}_2$ is a compact operator.

For $i = 1, \cdots, N$ and $s_i < \epsilon < t \leq t_{i+1}$. From the mean value theorem for the Bochner integral [26], for $u \in B_r(0, \mathcal{E}(A))$, we see that

$$\begin{aligned}\mathscr{Z}_2 u(t) &= \int_{s_i}^{\epsilon} q(s, u(s) + x(s), u_s + x_s)ds + \int_{\epsilon}^{t} q(s, u(s) + x(s), u_s + x_s)ds \\ &\in B_{c\epsilon}(0, X) + (t-\epsilon)\overline{co(\{q(s, u(s) + x(s), u_s + x_s); s \in [\epsilon, t]\})}\end{aligned}$$

where

$$c = \|m_q\| W(r(1+\mathfrak{K}_a) + ((1+\|\varphi\|) + \mathfrak{K}_a \mathfrak{H}(1+\|\varphi\|_{L^1([0,a];\mathbb{R}^+)}) + \mathfrak{M}_a)(\|\phi\|_{\mathcal{B}} + K_h)).$$

Since the map i_c is compact ([16, Lemma 2.2]) and $q \in C([0,a] \times X \times \mathcal{B}; \mathcal{D}(A))$, from the above result we obtain that

$$\{\mathscr{Z}_2 : u \in B_r(0, \mathcal{E}(A))\} \subset B_{c\epsilon}(0, X) + K_\epsilon,$$

where K_ϵ is compact and $\mathrm{diam}(B_{c\epsilon}(0, X)) \to 0$ as $\epsilon \to 0$, it follows that the set $\{\mathscr{Z}_2 : u \in B_r(0, \mathcal{E}(A))\}$ is relatively compact in X for all $t \in (s_i, t_{i+1}]$, $i = 1, \cdots, N$.

On the other hand, for $t \in (s_i, t_{i+1})$ and $h > 0$ such that $t + h \in (s_i, t_{i+1}]$, we get

$$\begin{aligned}\|\mathscr{Z}_2\widetilde{u(t+h)} - \widetilde{\mathscr{Z}_2 u(t)}\| &= \|\mathscr{Z}_2 u(t+h) - \mathscr{Z}_2 u(t)\| \\ &\leq \|m_q(t)\| W(r(1+\mathfrak{K}_a) + ((1+\|\varphi\|) \\ &\quad + \mathfrak{K}_a \mathfrak{H}(1+\|\varphi\|_{L^1([0,a];\mathbb{R}^+)}) + \mathfrak{M}_a)(\|\phi\|_{\mathcal{B}} + K_h))h\end{aligned}$$

for all $u \in B_r(0, \mathcal{E}(a))$, which asserts that the set of functions $[\mathscr{Z}_2 \widetilde{B_r(0, \mathcal{E}(A))}]$ is an equicontinuous subset of $C((s_i, t_{i+1}]; X)$. This completes the proof that $\mathscr{Z}_2$ is completely continuous.

Step 3. The map $\mathscr{Z}_3$ is completely continuous. First, we prove that the set $\{\mathscr{Z}_3 u(t) : u \in B_r(0, \mathcal{E}(A))\}$ is relatively compact in X for all $t \in (s_i, t_{i+1}]$, $i = 1, \cdots, N$. Let $s_i < \epsilon < t \leq t_{i+1}$. We know that the set $S = \{\mathscr{Z}_2 u(t) : u \in B_r(0, \mathcal{E}(A)), t \in (s_i, t_{i+1}]\}$ is relatively compact in X from the above Step 2. If $u \in B_r(0, \mathcal{E}(A))$, from the mean value theorem for the Bochner integral we get

$$\begin{aligned}\mathscr{Z}_3 u(t) &= \int_{s_i}^{\epsilon} S'(t-s)\mathscr{Z}_2 u(s)ds + \int_{\epsilon}^{t} S'(t-s)\mathscr{Z}_2 u(s)ds \\ &\in B_{C_\epsilon}(0, X) + (t-\epsilon)\overline{co(\{S'(t-s)x : s \in [\epsilon, t], x \in \bar{S}\})}\end{aligned}$$

where

$$C_\epsilon = \|m_q\| W(r(1+\mathfrak{K}_a) + ((1+\|\varphi\|) + \mathfrak{K}_a \mathfrak{H}(1+\|\varphi\|_{L^1([0,a];\mathbb{R}^+)}) + \mathfrak{M}_a)$$
$$\times (\|\phi\|_{\mathcal{B}} + K_h))\|\varphi\|_{L^1((s_i,\epsilon])}.$$

Hence, $\{\mathscr{Z}_3 u(t) : u \in B_r(0, \mathcal{E}(A))\} \subset B_{C_\epsilon}(0, X) + K_\epsilon$. Since K_ϵ is compact and $\mathrm{diam}(B_{C_\epsilon}(0,X)) \to 0$ as $\epsilon \to 0$, from the above we infer that the set $\{\mathscr{Z}_3 u(t) : u \in B_r(0, \mathcal{E}(A))\}$ is relatively compact in X.

Next, we prove that the set of functions $[\mathscr{Z}_3 \widetilde{B_r(0, \mathcal{E}(A))}]$, $i = 1, \cdots, N$ is an equicontinuous subset of $C((s_i, t_{i+1}]; X)$.

Let $t \in (s_i, t_{i+1})$, $\epsilon \in (s_i, t)$ and $\|\varphi\|_{L^1([t,t+h];\mathbb{R}^+)} < \epsilon$ for every $0 < h < \epsilon$, $S'(\cdot) \in C((s_i, t_{i+1}]; \mathcal{L}(X))$. For $u \in B_r(0, \mathcal{E}(A))$ and $\epsilon > h > 0$ such that $t + h \leq t_{i+1}$, we get

$$\begin{aligned}
&\|\mathscr{Z}_3 \widetilde{u(t+h)} - \widetilde{\mathscr{Z}_3 u(t)}\| = \|\mathscr{Z}_3 u(t+h) - \mathscr{Z}_3 u(t)\| \\
&= \| \int_{s_i}^{t+h} S'(t+h-s) \int_{s_i}^{s} q(\tau, u(\tau) + x(\tau), u_\tau + x_\tau) d\tau ds \\
&\quad - \int_{s_i}^{t} S'(t-s) \int_{s_i}^{s} q(\tau, u(\tau) + x(\tau), u_\tau + x_\tau) d\tau ds\| \\
&\leq \int_{s_i}^{t} \|S'(t+h-s) - S'(t-s)\| \int_{s_i}^{s} \|q(\tau, u(\tau) + x(\tau), u_\tau + x_\tau)\| d\tau ds \\
&\quad + \int_{t}^{t+h} \|S'(t+h-s)\| \int_{s_i}^{s} \|q(\tau, u(\tau) + x(\tau), u_\tau + x_\tau)\| d\tau ds \\
&\leq \epsilon \|m_q\| W(r(1+\mathfrak{K}_a) + ((1+\|\varphi\|) + \mathfrak{K}_a \mathfrak{H}(1+\|\varphi\|_{L^1([0,a];\mathbb{R}^+)}) + \mathfrak{M}_a) \\
&\quad \times (\|\phi\|_{\mathcal{B}} + K_h)) \int_{s_i}^{t} (s - s_i) ds + t_{i+1} \|m_q\| W(r(1+\mathfrak{K}_a) + ((1+\|\varphi\|) \\
&\quad + \mathfrak{K}_a \mathfrak{H}(1+\|\varphi\|_{L^1([0,a];\mathbb{R}^+)}) + \mathfrak{M}_a)(\|\phi\|_{\mathcal{B}} + K_h)) \int_{t}^{t+h} \varphi(t+h-s) ds \\
&\leq t_{i+1} \|m_q\| W(r(1+\mathfrak{K}_a) + ((1+\|\varphi\|) + \mathfrak{K}_a \mathfrak{H}(1+\|\varphi\|_{L^1([0,a];\mathbb{R}^+)}) + \mathfrak{M}_a) \\
&\quad \times (\|\phi\|_{\mathcal{B}} + K_h))(\frac{t_{i+1}}{2}\epsilon + \|\phi_A\|_{L^1([t,t+h];\mathbb{R}^+)})
\end{aligned}$$

which implies that $[\mathscr{Z}_3 \widetilde{B_r(0, \mathcal{E}(A))}]$ is right equicontinuous at t.

Proceeding as above, for $t = s_i$ and $h > 0$ with $s_i + h < t_{i+1}$, we have that

$$\begin{aligned}
&\|\mathscr{Z}_3 \widetilde{u(t+h)} - \widetilde{\mathscr{Z}_3 u(t)}\| \\
&= \left\| \int_{s_i}^{t+h} S'(t+h-s) \int_{s_i}^{s} q(\tau, u(\tau) + x(\tau), u_\tau + x_\tau) d\tau ds \right\|
\end{aligned}$$

$$\leq t_{i+1}\|m_q\|W(r(1+\mathfrak{K}_a)+((1+\|\varphi\|)+\mathfrak{K}_a\mathfrak{H}(1+\|\varphi\|_{L^1([0,a];\mathbb{R}^+)}) \\ +\mathfrak{M}_a)(\|\phi\|_{\mathcal{B}}+K_h))\|\varphi\|_{L^1((s_i,s_i+h];\mathbb{R}^+)},$$

which implies that $[\mathscr{Z}_3\widetilde{B_r(0,\mathcal{E}(A))}]$ is right equicontinuous at s_i. A similar argument permit us to show that $[\mathscr{Z}_3\widetilde{B_r(0,\mathcal{E}(A))}]$ is left equicontinuous at $t\in(s_i,t_{i+1}]$. This completes the proof that the set $[\mathscr{Z}_3\widetilde{B_r(0,\mathcal{E}(A))}]$ is equicontinuous.

From the above steps and Lemma 2.1, we have that $\mathscr{Z}_1$ is a contraction and $\mathscr{Z}_j,\ j=2,3$ is completely continuous. We conclude from these results $\mathscr{Z}$ is a condensing operator from $B_r(0,\mathcal{E}(A))$ into $B_r(0,\mathcal{E}(A))$. Finally, from [26, Theorem 4.3.2] we infer that there exists a mild solution of (1.1)-(1.3). □

In the next section we analyze the existence result when the resolvent operator $(S(t))_{t\geq 0}$ is not compact and the nonlinear growth function does not have the Lipschitz type condition. The result is proved based on the concept of measure of noncompactness.

4. Existence results by the concept of measure of noncompactness

First we recollect the concept of measure of noncompactness and their properties to prove the existence results for problem (1.1)-(1.3).

Here we consider the Hausdorff measure of noncompactness due to its applicability in the real field.

Definition 4.1. The Hausdorff measure of noncompactness β of the set E in a Banach space X is the greatest lower bound of those $\epsilon>0$ for which the set E has in the space X a finite ϵ-net, that is

$$\beta(E)=\inf\{\epsilon>0: E \text{ can be covered by finite number of balls with radii } \epsilon\},$$

for every bounded subset E in a Banach space X.

Lemma 4.2. *For any bounded set $U,V\subset X$, where X is a Banach space. Then, we have the following results:*

(1) $\beta(U)=0$ *iff U is precompact;*
(2) $\beta(U)=\beta(coU)=\beta(\bar{U})$, *where coU and $\bar{U}$ denotes the convex hull and closure of U, respectively;*
(3) $\beta(U)\subset\beta(V)$, *when $U\subset V$;*
(4) $\beta(U+V)\leq\beta(U)+\beta(V)$, *where $U+V=\{u+v; u\in U, v\in V\}$;*
(5) $\beta(U\cup V)\leq\max\{\beta(U),\beta(V)\}$;
(6) $\beta(\lambda U)=\lambda\beta(U)$, *for any $\lambda\in\mathbb{R}$;*

(7) *If the map* $Q : E(Q) \subset X \to Y$ *is Lipschitz continuous with constant* k *then* $\beta_Y(QU) \leq k\beta(U)$ *for any bounded subset* $U \subset E(Q), X$ *and* Y *are Banach spaces.*

Lemma 4.3. *Let* $E \subset X$ *be bounded closed and convex. If the continuous map* $Q : E \to E$ *is a* β*-contraction, then the map* Q *has a fixed point in* E*.*

We note that β be the Hausdorff measure of noncompactness of X, β_c be the Hausdorff measure of noncompactness of $C([0,a];X)$, and β_{pc} be the Hausdorff measure of noncompactness of $PC([0,a];X)$.

Lemma 4.4. *We have the following statements:*

(1) *If* $E \subseteq C([0,a];X)$ *is bounded, then* $\beta(E(t)) \leq \beta_c(E),\ \forall\ t \in [0,a]$, *where* $E(t) = \{u(t); u \in E\} \subseteq X$.

(2) *If* E *is equicontinuous on* $[0,a]$, *then* $\beta(E(t))$ *is continuous on* $[0,a]$ *and*
$$\beta_c(E) = \sup_{t\in[0,a]} \{\beta(E(t))\}.$$

(3) *If* $E \subseteq C([0,a];X)$ *is bounded and equicontinuous, then* $\beta(E(t))$ *is continuous and*
$$\beta\left(\int_0^t E(s)ds\right) \leq \int_0^t \beta(E(s))ds,\ \forall\ t \in [0,a],$$
where $\int_0^t E(s)ds = \{\int_0^t u(s)ds, u \in E\}$.

Lemma 4.5. *We have the following statements:*

(1) *If* $E \subseteq PC([0,a];X)$ *is bounded, then* $\beta(E(t)) \leq \beta_{pc}(E),\ \forall\ t \in [0,a]$, *where* $E(t) = \{u(t); u \in E\} \subseteq X$.

(2) *If* E *is piecewise equicontinuous on* $[0,a]$, *then* $\beta(E(t))$ *is piecewise continuous for* $t \in [0,a]$ *and*
$$\beta_{pc}(E) = \sup\{\beta(E(t)),\ t \in [0,a]\}.$$

Subsequent hypotheses is considered.

(H11) The nonlinear function $f : [0,a] \times X \times \mathcal{B} \to X$ is satisfies the caratheodary condition; *i.e.*, $f(t,\cdot,\cdot)$ is continuous for *a.e.* $t \in [0,a]$ and for each $u \in X,\ u_t \in \mathcal{B},\ f(\cdot,u,u_t) : [0,a] \to X$ is strongly measurable.

(H12) There exists a function $\eta \in L^1_{loc}([0,a];\mathbb{R}^+)$ such that
$$\beta(f(t,D,D_1)) \leq \eta(\beta(D) + \sup_{-\infty<\theta\leq 0} \beta(D_1(\theta))),\ t \in [0,a],$$
every bounded subset $D \subseteq X$ and $D_1 \subseteq PC((-\infty,0];X)$, where $D_1(\theta) = \{u(\theta) : u \in D_1\}$.

Theorem 4.6. *Assume that* $(H2)$-$(H8)$, $(H11)$ *and* $(H12)$ *are satisfied. Then the problem* (1.1)-(1.3) *has at least one mild solution.*

Proof. Here we consider the fixed point operator $\mathscr{X}$ as in Theorem 3.4 which is well defined. At first we can easily shown that $\mathscr{X}$ is continuous from the assumption of the function p, q, and g_i.

Next we claim that $\mathscr{X} B_r([0,a];\mathcal{E}(a)) \subset B_r([0,a];\mathcal{E}(a))$, from the previous Theorem 3.5.

We consider the decomposition operator $\mathscr{X} = \mathscr{X}_1 + \mathscr{X}_2 + \mathscr{X}_3$, where

$$\mathscr{X}_1 u(t) = \begin{cases} 0,\ t \in [-\infty, 0], \\ p(0, \phi(0) + h(u)(0), u_0) - p(t, u(t) + x(t), u_t + x_t) \\ + \int_0^t S'(t-s)(p(0, \phi(0) + h(u)(0), u_0) \\ -p(s, u(s) + x(s), u_s + x_s),\ t \in [0, t_1], \\ g_i(t, u_{|_{I_i(t)}}),\ t \in (t_i, s_i],\ i = 1, \cdots, m, \\ g_i(s_i, u_{|_{I_i(s_i)}}) - p(t, u(t) + x(t), u_t + x_t) + p(s_i, g_i(s_i, u_{|_{I_i(s_i)}}), u_{s_i} + x_{s_i}) \\ + \int_{s_i}^t S'(t-s)(g_i(s_i, u_{|_{I_i(s_i)}}) - p(s, u(s) + x(s), u_s + x_s) \\ + p(s_i, g_i(s_i, u_{|_{I_i(s_i)}}), u_{s_i} + x_{s_i}))ds,\ t \in [s_i, t_{i+1}],\ i = 1, \cdots, m. \end{cases}$$

$$\mathscr{X}_2 u(t) = \int_{s_i}^t q(s, u(s) + x(s), u_s + x_s) ds,\ t \in [s_i, t_{i+1}],\ i = 0, \cdots, m.$$

$$\mathscr{X}_3 u(t) = \int_{s_i}^t S'(t-s)\left(\int_{s_i}^s q(\tau, u(\tau) + x(\tau), u_\tau + x_\tau) d\tau\right) ds,\ t \in [s_i, t_{i+1}],\ i = 0, \cdots, m.$$

Our aim is to prove that $\mathscr{X}$ is $\beta-$contraction. From Step 1 of Theorem 3.5, we have that

$$\begin{aligned} &\|\mathscr{X}_1 u(t) - \mathscr{X}_1 v(t)\|_{\mathcal{E}(a)} \\ &\leq \max\{[L_{g_i}(s_i)\tilde{\Psi}_i(s) + L_p((1 + \mathfrak{K}_a) + L_{g_i}(s_i)\tilde{\Psi}_i(s) + \mathfrak{K}_a)] \\ &\qquad \times (1 + \|\varphi\|_{L^1([s_i, t_{i+1}];\mathbb{R}^+)}),\ \ L_{g_i}(t)\tilde{\Psi}_i(s), \\ &\qquad [L_p(2L_h\mathfrak{K}_a + 1 + \mathfrak{K}_a)](1 + \|\varphi\|_{L^1([0,t_1];\mathbb{R}^+)})\}\|u - v\|_{\mathcal{E}(A)}, \end{aligned}$$

it follows that $\mathscr{X}$ is Lipschitz continuous with Lipschitz constant

$$\begin{aligned} T = \max_{1,\cdots,N} \Big\{ &[L_{g_i}(s_i)\tilde{\Psi}_i(s) + L_p((1 + \mathfrak{K}_a) + (L_{g_i}(s_i)\tilde{\Psi}_i(s) + \mathfrak{K}_a)] \\ &\times (1 + \|\varphi\|_{L^1([s_i, t_{i+1}];\mathbb{R}^+)}),\ \ L_{g_i}(t)\tilde{\Psi}_i(s), \\ &[L_p(2L_h\mathfrak{K}_a + 1 + \mathfrak{K}_a)](1 + \|\varphi\|_{L^1([0,t_1];\mathbb{R}^+)}) \Big\} \|u - v\|_{\mathcal{E}(A)}. \end{aligned}$$

Let E be an arbitrary subset of $B_r(0, \mathcal{E}(a))$. From the hypothesis $(H2)$ and $(H11)$, we get $q(t, u + x, u_t + x_t)$ is equicontinuous.

Now consider,

$$\begin{aligned}
\beta(\mathscr{Z}_2(E(t))) &\leq \beta(\int_{s_i}^{t} q(s, E + x, E_s + x_s)ds) \\
&\leq \int_{s_i}^{t} \beta(q(s, E + x, E_s + x_s))ds \\
&\leq \int_{s_i}^{t} \eta(s)[\beta(E(s)) + \sup_{-\infty<\theta\leq 0} \beta(E(s+\theta) + x(s+\theta))]ds \\
&\leq \int_{s_i}^{t} \eta(s)(\beta(E(s)) + \sup_{0<s\leq a} \beta(E(s)))ds, \\
\beta_{pc}(\mathscr{Z}_2(E)) &\leq 2\beta_{pc}(E) \int_{s_i}^{t} \eta(s)ds.
\end{aligned}$$

Since $S'(t-s)q(s, u(s)+x(s), u_s+x_s)$ is equicontinuous, see Step 3 in Theorem 3.5. Then we consider

$$\begin{aligned}
\beta(\mathscr{Z}_3(E(t))) &\leq \beta(\int_{s_i}^{t} S'(t-s)q(s, E + x, E_s + x_s)ds) \\
&\leq \int_{s_i}^{t} \varphi(t-s)\beta(q(s, E + x, E_s + x_s))ds \\
&\leq \int_{s_i}^{t} \varphi(t-s)\eta(s)(\beta(E(s)) + \sup_{0<s<a} \beta(E(s)))ds, \\
\beta_{pc}(\mathscr{Z}_3(E)) &\leq 2\beta_{pc}(E) \int_{s_i}^{t} \varphi(t-s)\eta(s)ds.
\end{aligned}$$

Finally, we see that

$$\begin{aligned}
\beta_{pc}(\mathscr{Z}E) &= \beta_{pc}(\mathscr{Z}_1E + \mathscr{Z}_2E + \mathscr{Z}_3E) \\
&\leq \beta_{pc}(\mathscr{Z}_1E) + \beta_{pc}(\mathscr{Z}_2E) + \beta_{pc}(\mathscr{Z}_3E) \\
&\leq \left(T + \int_{s_i}^{t} \eta(s)ds + \int_{s_i}^{t} \varphi(t-s)\eta(s)ds\right)\beta_{pc}(E).
\end{aligned}$$

Thus, we get $\mathscr{Z}$ is β-contraction. Follows from the Darbo-Sadovskii fixed point theorem, we conclude that $\mathscr{Z}$ has atleast one fixed point in $B_r(0, \mathcal{E}(a))$ and the equation (1.1)-(1.3) has at least one mild solution. The proof is completed. □

5. Application

As an application of the abstract fractional non-instantaneous impulsive integro-differential problem (1.1)-(1.3), we consider the following example.

$$
\begin{aligned}
&{}^{c}_{s_i}D^{\alpha}_t \left(x(t,\xi) + a_1(t,\xi,x(t,\xi)) + \gamma_1(t,\xi)\int_{-\infty}^{t}\int_0^{\pi} \mu_1(s,\theta)a_2(\theta, x(t-s,\theta))d\theta ds \right) \\
&= \frac{\partial^2}{\partial x^2}\left[x(t,\xi) + \int_0^t k(t-s,\xi)x(s,\xi)ds \right] + \frac{1}{\Gamma(1-\alpha)}\int_{s_i}^{t}(t-s)^{-\alpha}\left(b_1(t,\xi,x(t,\xi))\right. \\
&\quad \left. +\gamma_2(t,\xi)\int_{-\infty}^{t}\int_0^{\pi}\mu_2(\tau,\varepsilon)b_2(\varepsilon, x(t-\tau,\varepsilon))d\varepsilon d\tau \right) ds, \qquad (5.1)
\end{aligned}
$$

for $(t,\xi) \in \cup_{i=1}^{N}[s_i, t_{i+1}] \times [0,\pi]$,

$$x(t,0) = x(t,\pi) = 0, \ t \in [0,a], \qquad (5.2)$$

$$x(0,\xi) = \int_0^{\pi} c_1(\xi,\sigma)x(t,\sigma)d\sigma + z(\xi), \ t \in (-\infty, 0), \ 0 \le \xi \le \pi, \qquad (5.3)$$

$$x(\tau,\xi) = G_i(t,\xi, \int_{t_i}^{t}\zeta_i(s)x(s,\xi)ds), \ \xi \in [0,\pi], \ t \in (t_i, s_i], \ i = 1,\cdots,N, \qquad (5.4)$$

where $\alpha \in (0,1)$, $z \in \mathcal{B}$, the numbers $0 < t_1 < s_1 < t_2 < \cdots < t_N$ are prefixed. The functions $a_1, b_1 : [0,a] \times [0,\pi] \times \mathbb{R} \to \mathbb{R}$ are continuous, $\gamma_j : [0,a] \times [0,\pi] \to \mathbb{R}$, $\mu_j : (\infty, 0] \times [0,\pi] \to \mathbb{R}, j = 1,2,$ $a_2, b_2 \in [0,\pi] \times \mathbb{R} \to \mathbb{R}$, $c_1 : [0,\pi] \times [0,\pi] \to \mathbb{R}$, $G_i : (t_i, s_i] \times [0,\pi] \times \mathbb{R} \to \mathbb{R}$ and $\zeta_i : (t_i, s_i] \to \mathbb{R}$ for all $i = 1,\cdots,N$ are continuous.

Here we consider in the space $X = L^2([0,\pi])$ and the phase space $\mathcal{B} = C_0 \times L^2(l;X)$. Take the operator A as $A\xi = \xi''$ with

$$\mathcal{D}(A) := \{\xi \in X : \xi'' \in L^2([0\pi]), \ \ \xi(0) = \xi(\pi) = 0\}.$$

The analytic semigroup $(T(t))_{t\ge 0}$ is generated by an operator A and this operator A has a discrete spectrum with eigenvalues of the form $-n^2$, $n \in \mathbb{N}$. The set $\{z_n(\xi) := (\frac{2}{\pi})^{1/2}\sin(n\xi)\}$ is a normalized eigenfunctions corresponding to A and it is an orthonormal basis for X.

$$T(t)\xi = \sum_{n=1}^{\infty} e^{-n^2 t}\langle \xi, z_n\rangle z_n \quad \text{for} \quad \xi \in X,$$

$$A\xi = -\sum_{n=1}^{\infty} n^2 \langle \xi, z_n\rangle z_n \quad \text{for} \quad x \in \mathcal{D}(A).$$

From the results in [31] that the integral equation

$$u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(1+(1*k))(t-s)Au(s)ds, \ t \ge 0,$$

has an associated analytic resolvent operator $(S(t))_{t\ge 0}$ on X which is given by

$$S(t) = \begin{cases} \frac{1}{2\pi i}\int_{\Gamma_{r,\theta}} e^{\lambda t}(\frac{1}{\lambda}(1-(1+\hat{k}(\lambda))\lambda^{-\alpha}A))^{-1}d\lambda, \ t \ge 0, \\ I, \ t = 0, \end{cases}$$

where $\Gamma_{r,\theta}$ denotes a contour consisting of the rays $\{re^{i\nu} : r \ge 0\}$ and $\{re^{-i\nu} : r \ge 0\}$ for some $\nu \in (\pi, \pi/2)$.

Let the maps $p, q : [0,a] \times X \times \mathcal{B} \to X,\ h : C([0,a];X) \to X,\ I_i : (t_i, s_i] \to 2^{(t_i,s_i]}$ and $g_i(t,\cdot) : \mathcal{C}_i(t) \to X$ given by $I_i(t) = (t_i, t]$,

$$
\begin{aligned}
g_i(t,u)(\xi) &= G_i(t,\xi, \int_{t_i}^{t} \zeta_i(s)x(s,\xi)ds),\\
p(t,u,w)(\xi) &= a_1(t,\xi,x(t,\xi)) + \gamma_1(t,\xi)\int_{-\infty}^{0}\int_0^{\pi} \mu_1(s,\theta)a_2(\theta, x(-s,\theta))d\theta ds,\\
q(t,u,w)(\xi) &= \frac{1}{\Gamma(1-\alpha)}\int_{s_i}^{t}(t-s)^{-\alpha}\bigg(b_1(t,\xi,x(t,\xi))\\
&\qquad + \gamma_2(t,\xi)\int_{-\infty}^{0}\int_0^{\pi}\mu_2(\tau,\varepsilon)b_2(\varepsilon, x(-\tau,\varepsilon))d\varepsilon d\tau\bigg)ds,\\
h(u)(\xi) &= \int_0^{\pi} c_1(\xi,\sigma)x(t,\sigma)d\sigma + z(\xi).
\end{aligned}
$$

From the above observation the problem (5.1)-(5.4) represents the abstract form (1.1)-(1.3). The considered functions p, q, g_i and h satisfy the required hypotheses of Theorem 3.5. Hence the equations (5.1)-(5.4) has at least one mild solution.

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