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INTEGRAL MEAN ESTIMATES FOR THE POLYNOMIALS

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Abstract. In this paper certain integral inequalities for the polar derivative of a polynomial with restricted zeros are given, which generalize and refine some well-known polynomial inequalities due to Malik, Aziz and others.

1. Introduction

Let $\mathcal{P}_n$ denote the space of all complex polynomials $P(z)$ of degree n. For $P \in \mathcal{P}_n$ having all their zeros in $|z| \le 1$, it was shown by Turan [14] that

$$n \max_{|z|=1} |P(z)| \le 2 \max_{|z|=1} \left|P'(z)\right|. \tag{1.1}$$

The result is sharp and equality in (1.1) holds for $P(z) = az^n + b, |a| = |b|$.

As a generalization of (1.1), Malik [9] proved that if $P \in \mathcal{P}_n$ has all its zeros in $|z| \le k,\ k \le 1$, then

$$n \max_{|z|=1} |P(z)| \le (1+k) \max_{|z|=1} \left|P'(z)\right|, \tag{1.2}$$

and Govil [4] showed that if $P \in \mathcal{P}_n$ has all its zeros in $|z| \le k,\ k \ge 1$, then

$$n \max_{|z|=1} |P(z)| \le (1+k^n) \max_{|z|=1} \left|P'(z)\right|. \tag{1.3}$$

Both the estimates are sharp.

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Malik [10] obtained an extension of (1.1) in the sense that the left hand side of (1.1) is replaced by a factor involving the integral mean of $|P(z)|$ on $|z|=1$ by proving that if $P \in \mathcal{P}_n$ has all its zeros in $|z| \leq 1$, then for each $q > 0$,

$$n\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^q d\theta\right\}^{1/q} \leq \left\{\int_0^{2\pi}\left|1+e^{i\theta}\right|^q d\theta\right\}^{1/q} \max_{|z|=1}\left|P'(z)\right|. \tag{1.4}$$

Equality in (1.4) holds for $P(z) = az^n + b, |a| = |b| \neq 0$.

As generalizations of the inequalities (1.2), (1.3) and (1.4), Aziz [1] considered the class of polynomials $P \in \mathcal{P}_n$ having all their zeros in $|z| \leq k$ and proved for each $q > 0$,

$$n\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^q d\theta\right\}^{1/q} \leq \left\{\int_0^{2\pi}\left|1+ke^{i\theta}\right|^q d\theta\right\}^{1/q} \max_{|z|=1}\left|P'(z)\right|, k \leq 1 \tag{1.5}$$

and

$$n\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^q d\theta\right\}^{1/q} \leq \left\{\int_0^{2\pi}\left|1+k^n e^{i\theta}\right|^q d\theta\right\}^{1/q} \max_{|z|=1}\left|P'(z)\right|, k \geq 1. \tag{1.6}$$

The estimate (1.6) is sharp and equality in (1.6) holds for $P(z) = z^n + k^n$. In the limiting case when $q \to \infty$, the inequalities (1.5) and (1.6) reduce to inequalities (1.2) and (1.3), respectively.

Let $D_\alpha P(z)$ denote the polar derivative of a polynomial $P \in \mathcal{P}_n$ of degree n with respect to point $\alpha \in \mathbb{C}$. Then

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)$$

(see [8]). The polynomial $D_\alpha P(z)$ is of degree at most $n-1$ and it generalizes the ordinary $P'(z)$ of $P(z)$ in the sense that

$$\lim_{\alpha\to\infty}\frac{D_\alpha P(z)}{\alpha} = P'(z)$$

uniformly with respect z for $|z| \leq R$, $R > 0$.

Aziz and Rather [2] proved several sharp results concerning the maximum modulus of the polar derivative of a polynomial with restricted zeros. Among other things, they extended inequalities (1.2) and (1.3) to the polar derivative of a polynomial by showing that if $P \in \mathcal{P}_n$ and $P(z) = 0$ in $|z| \leq k$, then for every $\alpha \in \mathbb{C}$ with $\alpha \geq k$,

$$n\left(|\alpha| - k\right)\max_{|z|=1}|P(z)| \leq (1+k)\max_{|z|=1}|D_\alpha P(z)|,\ k \leq 1 \tag{1.7}$$

and

$$n\left(|\alpha| - k\right)\max_{|z|=1}|P(z)| \leq (1+k^n)\max_{|z|=1}|D_\alpha P(z)|,\ k \geq 1. \tag{1.8}$$

Recently Rather *et.al.* [13] extended inequality (1.3) to the polar derivative of polynomial and proved that if $P \in \mathcal{P}_n$ and $P(z)$ has all its zeros in $|z| \leq k$ where $k \leq 1$, then

$$n\left(|\alpha|-k\right)\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^q d\theta\right\}^{1/q} \leq \left\{\int_0^{2\pi}\left|1+ke^{i\theta})\right|^q d\theta\right\}^{1/q}\max_{|z|=1}\left|D_\alpha P(z)\right|. \tag{1.9}$$

The main aim of this paper is to extends the inequality (1.4) to the polar derivative of a ploynomial and obtain a generalization of (1.6) in the sense that the left hand side of (1.8) is replaced by a factor involving the integral mean of $|P(z)|$ on $|z| = 1$.

For the proofs of our main results, we need the following lemmas. The first lemma is a simple deduction from Maximum Modulus Principle(see [5] or [11]).

Lemma 1.1. ([5],[11]) *If $P \in \mathcal{P}_n$, then for $R \geq 1$,*

$$\max_{|z|=R}\left|P(z)\right| \leq R^n \max_{|z|=1}\left|P(z)\right|. \tag{1.10}$$

The next lemma is a simple consequence of a well-known result of Hardy [6].

Lemma 1.2. ([6]) *If $P \in \mathcal{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then for $q > 0$, $R \geq 1$,*

$$\left\{\int_0^{2\pi}\left|P(Re^{i\theta})\right|^q d\theta\right\}^{1/q} \leq R^n\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^q d\theta\right\}^{1/q}. \tag{1.11}$$

We also require the following result due to Rahman and Schmeisser [12].

Lemma 1.3. ([12]) *If $P \in \mathcal{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then for $R \geq 1$ and $q > 0$,*

$$\left\{\int_0^{2\pi}\left|P(Re^{i\theta})\right|^q d\theta\right\}^{1/q} \leq C_q\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^q d\theta\right\}^{1/q}, \tag{1.12}$$

where

$$C_q = \frac{\left\{\int_0^{2\pi}\left|1+R^n e^{i\theta}\right|^q d\theta\right\}^{1/q}}{\left\{\int_0^{2\pi}\left|1+e^{i\theta}\right|^q d\theta\right\}^{1/q}}.$$

2. Main results

Now we prove our main theorems.

Theorem 2.1. *If $P \in \mathcal{P}_n$ and $P(z)$ has all its zeros in $|z| \leq k$ where $k \geq 1$, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \geq k$ and for each $q > 0$,*

$$n\left(|\alpha| - k\right)\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^q d\theta\right\}^{1/q}$$
$$\leq \left\{\int_0^{2\pi}\left|1 + k^n e^{i\theta}\right|^q d\theta\right\}^{1/q}\max_{|z|=1}\left|D_\alpha P(z)\right|. \tag{2.1}$$

Proof. Let $F(z) = P(kz)$. Since all the zeros of $P(z)$ lie in $|z| \leq k$, all the zeros of $F(z)$ lie in $|z| \leq 1$, the polynomial $G(z) = z^n\overline{F(1/\bar{z})}$ has all its zeros in $|z| \geq 1$ and

$$|G(z)| = |F(z)| \quad \text{for} \quad |z| = 1.$$

Hence, it follows by a result of De Bruijn (see [3, Theorem 1, p.1265]) that

$$\left|G'(z)\right| \leq \left|F'(z)\right| \quad \text{for} \quad |z| = 1. \tag{2.2}$$

Since $G(z) = z^n\overline{F(1/\bar{z})}$, $F(z) = z^n\overline{G(1/\bar{z})}$ and it can be easily seen that

$$\left|G'(z)\right| = \left|nF(z) - zF'(z)\right| \quad \text{and} \quad \left|F'(z)\right| = \left|nG(z) - zG'(z)\right| \tag{2.3}$$

for $|z| = 1$. Combining (2.2) and (2.3), we get

$$\left|G'(z)\right| \leq \left|nG(z) - zG'(z)\right| \quad for \quad |z| = 1. \tag{2.4}$$

Also since $F(z)$ has all its zeros in $|z| \leq 1$, by Gauss-Lucas theorem all the zeros of $F'(z)$ also lie in $|z| \leq 1$. This implies that the polynomial

$$z^{n-1}\overline{F'(1/\bar{z})} \equiv nG(z) - zG'(z)$$

does not vanish in $|z| < 1$. Therefore, it follows from (2.4) that the function

$$w(z) = \frac{zG'(z)}{nG(z) - zG'(z)}$$

is analytic for $|z| \leq 1$ and $|w(z)| \leq 1$ for $|z| \leq 1$. Furthermore, $w(0) = 0$. Thus the function $1 + w(z)$ is subordinate to the function $1 + z$ for $|z| \leq 1$. Hence by a well-known property of subordination [4, p.422], we have for each $q > 0$,

$$\int_0^{2\pi}\left|1 + w(e^{i\theta})\right|^q d\theta \leq \int_0^{2\pi}\left|1 + e^{i\theta}\right|^q d\theta. \tag{2.5}$$

Now

$$1 + w(z) = \frac{nG(z)}{nG(z) - zG'(z)},$$

which gives with the help of (2.3),

$$\begin{aligned} n\,|G(z)| &= |1+w(z)|\,\left|nG(z)-zG'(z)\right| \\ &= |1+w(z)|\,\left|F'(z)\right|, \quad \text{for } \ |z|=1. \end{aligned}$$

This implies for each $q>0$,

$$n^q \int_0^{2\pi} \left|G(e^{i\theta})\right|^q d\theta = \int_0^{2\pi} \left|1+w(e^{i\theta})\right|^q \left|F'(e^{i\theta})\right|^q d\theta. \tag{2.6}$$

Also, by using (2.2) and (2.3), we have for every $\alpha \in \mathbb{C}$ with $|\alpha| \geq k$ and for $|z|=1$,

$$\begin{aligned} \left|D_{\alpha/k}F(z)\right| &= \left|nF(z) + \left(\frac{\alpha}{k} - z\right) F'(z)\right| \\ &\geq \frac{|\alpha|}{k}\left|F'(z)\right| - \left|nF(z) - zF'(z)\right| \\ &= \frac{|\alpha|}{k}\left|F'(z)\right| - \left|G'(z)\right| \\ &\geq \frac{|\alpha|}{k}\left|F'(z)\right| - \left|F'(z)\right| = \left(\frac{|\alpha|}{k} - 1\right)\left|F'(z)\right|. \end{aligned} \tag{2.7}$$

Combining (2.6) and (2.7), we have for each $q>0$,

$$n^q\left(|\alpha|-k\right)^q \int_0^{2\pi} \left|G(e^{i\theta})\right|^q d\theta \leq \int_0^{2\pi} \left|1+w(e^{i\theta})\right|^q k^q \left|D_{\frac{\alpha}{k}}F(e^{i\theta})\right|^q d\theta. \tag{2.8}$$

Again since $G(z) \neq 0$ in $|z|<1$ and $k \geq 1$, by taking $R=k \geq 1$ in Lemma 1.3, we have for each $q>0$,

$$\int_0^{2\pi} \left|G(ke^{i\theta})\right|^q \leq B_q^q \int_0^{2\pi} \left|G(e^{i\theta})\right|^q d\theta, \tag{2.9}$$

where

$$B_q = \frac{\left\{\int_0^{2\pi} \left|1+k^n e^{i\theta}\right|^q d\theta\right\}^{1/q}}{\left\{\int_0^{2\pi} \left|1+e^{i\theta}\right|^q d\theta\right\}^{1/q}}. \tag{2.10}$$

From (2.5), (2.8) and (2.9), we deduce for each $q > 0$,

$$
\begin{aligned}
& n^q\left(|\alpha|-k\right)^q \int_0^{2\pi} \left|G(ke^{i\theta})\right|^q d\theta \\
& \leq B_q^q \int_0^{2\pi} \left|1+w(e^{i\theta})\right|^q k^q \left|D_{\frac{\alpha}{k}}F(e^{i\theta})\right|^q d\theta. \\
& \leq k^q B_q^q \int_0^{2\pi} \left|1+w(e^{i\theta})\right|^q d\theta \left(\max_{|z|=1}\left|D_{\frac{\alpha}{k}}F(z)\right|\right)^q \\
& \leq k^q B_q^q \int_0^{2\pi} \left|1+e^{i\theta}\right|^q d\theta \left(\max_{|z|=1}\left|D_{\frac{\alpha}{k}}F(z)\right|\right)^q \\
& = k^q \int_0^{2\pi} \left|1+k^n e^{i\theta}\right|^q d\theta \left(\max_{|z|=1}\left|D_{\frac{\alpha}{k}}F(z)\right|\right)^q. \qquad (2.11)
\end{aligned}
$$

Moreover, we have

$$G(z) = z^n\overline{F(1/\bar{z})} = z^n\overline{P(k/\bar{z})},$$

therefore, for $0 \leq \theta < 2\pi$,

$$\left|G(ke^{i\theta}\right| = \left|k^n e^{in\theta}\overline{P(e^{i\theta})}\right| = k^n\left|P(e^{i\theta})\right|. \qquad (2.12)$$

Using this in (2.11), we get

$$
\begin{aligned}
& n^q k^{nq}\left(|\alpha|-k\right)^q \int_0^{2\pi} \left|P(e^{i\theta})\right|^q d\theta \\
& \leq k^q \int_0^{2\pi} \left|1+k^n e^{i\theta}\right|^q d\theta \left(\max_{|z|=1}\left|D_{\frac{\alpha}{k}}F(z)\right|\right)^q. \qquad (2.13)
\end{aligned}
$$

Further, $D_\alpha P(z)$ being a polynomial of degree at most $n-1$, it follows from Lemma 1.1 with $R = k \geq 1$ that

$$\max_{|z|=1}\left|D_{\frac{\alpha}{k}}F(z)\right| = \max_{|z|=k}\left|D_\alpha P(z)\right| \leq k^{n-1}\max_{|z|=1}\left|D_\alpha P(z)\right|. \qquad (2.14)$$

This in conjuction with (2.13) yields,

$$n^q k^{nq}\left(|\alpha|-k\right)^q \int_0^{2\pi}\left|P(e^{i\theta})\right|^q d\theta \leq k^{nq}\int_0^{2\pi}\left|1+k^n e^{i\theta}\right|^q d\theta\left(\max_{|z|=1}\left|D_\alpha P(z)\right|\right)^q,$$

or equivalently,

$$n\left(|\alpha|-k\right)\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^q d\theta\right\}^{1/q} \leq \left\{\int_0^{2\pi}\left|1+k^n e^{i\theta}\right|^q d\theta\right\}^{1/q}\max_{|z|=1}\left|D_\alpha P(z)\right|.$$

This proves Theorem 2.1 □

Remark 2.2. If we divide the two sides of (2.1) by $|\alpha|$ and let $|\alpha| \to \infty$, we get inequality (1.6). Further if make $q \to \infty$ in (2.1), we get inequality (1.8).

Next, we have the following theorem.

Theorem 2.3. *If $P \in \mathcal{P}_n$, $P(z)$ as all its zeros in $|z| \leq k$ where $k \geq 1$ and $m = \min_{|z|=k} |P(z)|$, then for $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \geq k$, $|\beta| \leq 1$ and for each $q > 0$,*

$$n\left(|\alpha| - k\right) \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) + \beta m \right|^q d\theta \right\}^{1/q}$$
$$\leq \left\{ \int_0^{2\pi} \left| 1 + k^n e^{i\theta} \right|^q d\theta \right\}^{1/q} \left\{ \max_{|z|=1} |D_\alpha P(z)| - nm/k^{n-1} \right\}. \tag{2.15}$$

Proof. Let $F(z) = P(kz)$. Then

$$m = \min_{|z|=k} |P(z)| = \min_{|z|=1} |P(kz)| = \min_{|z|=1} |F(z)|.$$

This gives

$$m \leq |F(z)| \quad \text{for} \quad |z| = 1.$$

Since $P(z)$ has all its zeros in $|z| \leq k$, all the zeros of $F(z) = P(kz)$ lie in $|z| \leq 1$ and therefore, it follows from maximun modulus theorem that

$$m < |F(z)| \quad for \quad |z| > 1. \tag{2.16}$$

We show all the zeros of polynomial $f(z) = F(z) + \beta m$ lie in $|z| \leq 1$ for every β with $|\beta| \leq 1$. This is obvious if $m = 0$. For $m \neq 0$, if there is a point $z = z_0$ with $|z_0| > 1$ such that $f(z_0) = F(z_0) + \beta m = 0$, then $|f(z_0)| = \beta|m \leq m$, a contradiction to (2.17). Therefore, all the zeros of $f(z)$ lie in $|z| \leq 1$ for every β with $|\beta| \leq 1$. If $G(z) = z^n\overline{F(1/\bar{z})}$, then

$$g(z) = z^n\overline{f(1/\bar{z})} = z^n\overline{F(1/\bar{z})} + \bar{\beta} m z^n = G(z) + \bar{\beta} m z^n$$

and

$$|g(z)| = |f(z)| \quad \text{for} \quad |z| = 1.$$

Since $g(z) \neq 0$ in $|z| < 1$, by a result of De Bruijn [3], it follows that

$$\left|g'(z)\right| \leq \left|f'(z)\right| \quad \text{for} \quad |z| = 1.$$

Equivalently,

$$\left|G'(z) + nm\bar{\beta}z^{n-1}\right| \leq \left|F'(z)\right| \quad \text{for} \quad |z| = 1. \tag{2.17}$$

Since $G(z) = z^n\overline{F(1/\bar{z})}$, $F(z) = z^n\overline{G(1/\bar{z})}$ and it can be easily verified that

$$\left|F'(z)\right| = \left|nG(z) - zG'(z)\right| \quad \text{and} \quad \left|G'(z)\right| = \left|nF(z) - zF'(z)\right| \tag{2.18}$$

for $|z| = 1$. Using this in (2.17), we get

$$\left|G'(z) + nm\bar{\beta}z^{n-1}\right| \leq \left|nG(z) - zG'(z)\right| \quad \text{for} \quad |z| = 1. \tag{2.19}$$

Now choosing the argument of β with $|\beta| = 1$ in the left hand side of (2.19) suitably, we get

$$\left|G'(z)\right| + nm \leq \left|nG(z) - zG'(z)\right| \quad \text{for } |z| = 1. \tag{2.20}$$

Now consider the function

$$w(z) = \frac{z(G'(z) + nm\bar{\beta}z^{n-1})}{nG(z) - zG'(z)}.$$

Since all the zeros of $f(z)$ lie in $|z| \leq 1$, by Gauss-Lucas theorem all the zeros of $F'(z)$ also lie in $|z| \leq 1$. Therefore, all the zeros of polynomial $z^{n-1}\overline{F'(1/\bar{z})} \equiv nG(z) - zG'(z)$ lie in $|z| \geq 1$. Hence the function $w(z)$ is analytic in $|z| \leq 1$ and $|w(z)| \leq 1$. Moreover, $w(0) = 0$. Thus the function $1 + w(z)$ is subordinate to the function $1 + z$ for $|z| \leq 1$. Therefore by well-known property of subordination [4, p.422], we have for each $q > 0$

$$\int_0^{2\pi} \left|1 + w(e^{i\theta})\right|^q d\theta \leq \int_0^{2\pi} \left|1 + e^{i\theta}\right|^q d\theta. \tag{2.21}$$

Further,

$$1 + w(z) = \frac{nG(z) + nm\bar{\beta}z^n}{nG(z) - zG'(z)}$$

so that for $|z| = 1$, we have

$$\left|nG(z) + nm\bar{\beta}z^n\right| = |1 + w(z)| \left|nG(z) - zG'(z)\right| = |1 + w(z)| \left|F'(z)\right|$$

which implies for each $q > 0$,

$$n^q \int_0^{2\pi} \left|G(e^{i\theta}) + m\bar{\beta}e^{in\theta}\right|^q d\theta = \int_0^{2\pi} \left|1 + w(e^{i\theta})\right|^q \left|F'(e^{i\theta})\right|^q d\theta. \tag{2.22}$$

Also, from (2.18) and (2.20), we have for every $\alpha \in \mathbb{C}$ with $|\alpha| \geq k$ and for $|z| = 1$,

$$\begin{aligned}
\left|D_{\alpha/k}F(z)\right| &= \left|nF(z) - zF'(z) + \frac{\alpha}{k}F'(z)\right| \\
&\geq \frac{|\alpha|}{k}\left|F'(z)\right| - \left|nF(z) - zF'(z)\right| \\
&= \frac{|\alpha|}{k}\left|F'(z)\right| - \left|G'(z)\right| \\
&\geq \frac{|\alpha|}{k}\left|F'(z)\right| - \left|F'(z)\right| + nm \\
&= \left(\frac{|\alpha|}{k} - 1\right)\left|F'(z)\right| + nm.
\end{aligned}$$

This gives for every $\alpha \in \mathbb{C}$ with $|\alpha| \geq k$ and for $|z| = 1$, we get

$$(|\alpha| - k)\left|F'(z)\right| \leq k(\left|D_{\alpha/k}F(z)\right| - nm).$$

Using this and (2.21) in (2.22), we get

$$\begin{aligned} &n^q\left(|\alpha|-k\right)^q\int_0^{2\pi}\left|G(e^{i\theta})+m\bar{\beta}e^{in\theta}\right|^q d\theta\\ &\le k^q\int_0^{2\pi}\left|1+w(e^{i\theta})\right|^q\left\{\left|D_{\frac{\alpha}{k}}F(e^{i\theta})\right|-nm\right\}^q d\theta\\ &\le k^q\int_0^{2\pi}\left|1+w(e^{i\theta})\right|^q d\theta\left\{\max_{|z|=1}\left|D_{\frac{\alpha}{k}}F(z)\right|-nm\right\}^q\\ &\le k^q\int_0^{2\pi}\left|1+e^{i\theta}\right|^q\left\{\max_{|z|=1}\left|D_{\frac{\alpha}{k}}F(z)\right|-nm\right\}^q d\theta. \end{aligned} \tag{2.23}$$

Also since $D_\alpha P(z)$ is a polynomial of degree at most $n-1$, by Lemma 1.1 for $R=k\ge 1$, we have

$$\max_{|z|=1}\left|D_{\frac{\alpha}{k}}F(z)\right|=\max_{|z|=k}|D_\alpha P(z)|\le k^{n-1}\max_{|z|=1}|D_\alpha P(z)|. \tag{2.24}$$

Further, $g(z)=G(z)+\bar{\beta}z^n m=z^n\overline{F(1/\bar{z})}+\bar{\beta}z^n m$, therefore, for $|z|=1$, we have

$$|g(kz)|=\left|k^n z^n\overline{F(1/k\bar{z})}+\bar{\beta}k^n z^n m\right|=k^n\left|F(z/k)+\beta m\right|=k^n\left|P(z)+\beta m\right|$$

and since $g(z)$ does not vanish in $|z|<1$, by Lemma 1.3, we have for $q>0$,

$$\begin{aligned} k^{nq}\int_0^{2\pi}\left|P(e^{i\theta})+\beta m\right|^q d\theta &= \int_0^{2\pi}\left|g(ke^{i\theta})\right|^q d\theta\\ &\le B_q^q\int_0^{2\pi}\left|g(e^{i\theta})\right|^q d\theta\\ &= B_q^q\int_0^{2\pi}\left|G(e^{i\theta})+\beta m\right|^q d\theta, \end{aligned} \tag{2.25}$$

where B_q is given by (2.10). From (2.23), (2.24) and (2.25), we deduce for each $q>0$,

$$\begin{aligned} &n\left(|\alpha|-k\right)\left\{\int_0^{2\pi}\left|P(e^{i\theta})+\beta m\right|^q d\theta\right\}^{1/q}\\ &\le B_q\left\{\int_0^{2\pi}\left|1+e^{i\theta}\right|^q d\theta\right\}^{1/q}\left\{\max_{|z|=1}\left|D_\alpha P(e^{i\theta})\right|-nm/k^{n-1}\right\}\\ &=\left\{\int_0^{2\pi}\left|1+k^n e^{i\theta}\right|^q d\theta\right\}^{1/q}\left\{\max_{|z|=1}|D_\alpha P(z)|-nm/k^{n-1}\right\}, \end{aligned}$$

which proves Theorem 2.3. □

The following result which is a refinement of (2.1) follows from Theorem 2.3 by setting $\beta = 0$ in (2.15).

Corollary 2.4. *For $P \in \mathcal{P}_n$, if $P(z)$ has all its zeros in $|z| \le k$ where $k \ge 1$ and $m = \min_{|z|=k} |P(z)|$, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \ge k$ and for each $q > 0$,*

$$n\left(|\alpha| - k\right)\left\{\int_0^{2\pi} \left|P(e^{i\theta})\right|^q d\theta\right\}^{1/q} \le \left\{\int_0^{2\pi} \left|1 + k^n e^{i\theta}\right|^q d\theta\right\}^{1/q} \left\{\max_{|z|=1} |D_\alpha P(z)| - nm/k^{n-1}\right\}. \tag{2.26}$$

Letting $q \to \infty$ in (2.15) and choosing the argument of β with $|\beta| = 1$ suitably, we obtain the following refinement of inequality (1.8).

Corollary 2.5. *For $P \in \mathcal{P}_n$, if $P(z)$ has all its zeros in $|z| \le k$, where $k \ge 1$ and $m = \min_{|z|=k} |P(z)|$, then for every $\alpha \in \mathbb{C}$ with $\alpha \ge k$,*

$$n\left(|\alpha| - k\right)\max_{|z|=1} |P(z)| + n\left(|\alpha| + 1/k^{n-1}\right) m \le (1 + k^n) \max_{|z|=1} |D_\alpha P(z)|. \tag{2.27}$$

Finally we use Holder's inequality to establish a generalization of (2.1) in the sense that the maximum of $|D_\alpha P(z)|$ on $|z| = 1$ in the right hand side of (2.1) is replaced by factor involving the integral mean of $|D_\alpha P(z)|$ on $|z| = 1$.

In fact, we have the following:

Theorem 2.6. *If $P \in \mathcal{P}_n$ and $P(z)$ has all its zeros in $|z| \le k$ where $k \ge 1$, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \ge k$ and for $q > 0, r > 1, s > 1$ with $r^{-1} + s^{-1} = 1$,*

$$n\left(|\alpha| - k\right)\left\{\int_0^{2\pi} \left|P(e^{i\theta})\right|^q d\theta\right\}^{1/q} \le B_q \left\{\int_0^{2\pi} \left|1 + e^{i\theta}\right|^{qr} d\theta\right\}^{1/qr} \left\{\int_0^{2\pi} \left|D_\alpha P(e^{i\theta})\right|^{qs} d\theta\right\}^{1/qs}, \tag{2.28}$$

where

$$B_q = \frac{\left\{\int_0^{2\pi} \left|1 + k^n e^{i\theta}\right|^q d\theta\right\}^{1/q}}{\left\{\int_0^{2\pi} \left|1 + e^{i\theta}\right|^q d\theta\right\}^{1/q}}. \tag{2.29}$$

Proof. Let $F(z) = P(kz)$ and $G(z) = z^n \overline{F(1/\bar{z})}$. Since $P(z)$ has all its zeros in $|z| < k$ where $k \ge 1$, proceeding similarily as in the proof of Theorem 2.1, we have from (2.5) and (2.8) for each $q>0$ and $|\alpha| \geq k$,

$$\int_0^{2\pi} \left|1+w(e^{i\theta})\right|^q d\theta \leq \int_0^{2\pi} \left|1+e^{i\theta}\right|^q d\theta \tag{2.30}$$

and

$$\begin{aligned} n^q\left(|\alpha|-k\right)^q & \int_0^{2\pi} \left|G(ke^{i\theta})\right|^q d\theta \\ & \leq B_q^q \int_0^{2\pi} \left|1+w(e^{i\theta})\right|^q k^q \left|D_{\frac{\alpha}{k}} F(e^{i\theta})\right|^q d\theta. \end{aligned} \tag{2.31}$$

where B_q is defined by (2.10). This gives with the help of Holder's inequality for $r>1, s>1$ with $r^{-1}+s^{-1}=1$ and $q>0$,

$$\begin{aligned} & n^q\left(|\alpha|-k\right)^q \int_0^{2\pi} \left|G(ke^{i\theta})\right|^q d\theta \\ & \leq k^q B_q^q \left\{\int_0^{2\pi} \left|1+w(e^{i\theta})\right|^{qr} d\theta\right\}^{1/r} \left\{\int_0^{2\pi} \left|D_{\frac{\alpha}{k}} F(e^{i\theta})\right|^{qs} d\theta\right\}^{1/s}. \end{aligned} \tag{2.32}$$

Now using (2.30) and the fact that $\left|G(ke^{i\theta})\right| = k^n \left|P(e^{i\theta})\right|$ in (2.32), we get

$$\begin{aligned} & n^q k^{qn}\left(|\alpha|-k\right)^q \int_0^{2\pi} \left|P(e^{i\theta})\right|^q d\theta \\ & \leq k^q B_q^q \left\{\int_0^{2\pi} \left|1+e^{i\theta}\right|^{qr} d\theta\right\}^{1/r} \left\{\int_0^{2\pi} \left|D_{\frac{\alpha}{k}} F(e^{i\theta})\right|^{qs} d\theta\right\}^{1/s}. \end{aligned} \tag{2.33}$$

Further, since

$$D_{\frac{\alpha}{k}} F(z) = nF(z) + \left(\frac{\alpha}{k} - z\right) F'(z) = nP(kz) + (\alpha - kz)P'(kz)$$

is a polynomial of degree at most $n-1$, it follows from Lemma 1.2 by replacing q by qs and R by k, that

$$\begin{aligned} \int_0^{2\pi} \left|D_{\frac{\alpha}{k}} F(e^{i\theta})\right|^{qs} d\theta &= \int_0^{2\pi} \left|nP(ke^{i\theta}) + (\alpha - ke^{i\theta})P'(ke^{i\theta})\right|^{qs} d\theta \\ & \leq k^{n-1} \int_0^{2\pi} \left|nP(e^{i\theta}) + (\alpha - e^{i\theta})P'(e^{i\theta})\right|^{qs} d\theta \\ &= k^{n-1} \int_0^{2\pi} \left|D_\alpha P(e^{i\theta})\right|^{qs} d\theta. \end{aligned}$$

Combining this with (2.33), we get

$$n^q k^{qn} (|\alpha| - k)^q \int_0^{2\pi} \left|P(e^{i\theta})\right|^q d\theta$$
$$\leq k^q k^{q(n-1)} B_q^q \left\{ \int_0^{2\pi} \left|1 + e^{i\theta}\right|^{qr} d\theta \right\}^{1/r} \left\{ \int_0^{2\pi} \left|D_\alpha P(e^{i\theta})\right|^{qs} d\theta \right\}^{1/s}.$$

Equivalently,

$$n (|\alpha| - k) \left\{ \int_0^{2\pi} \left|P(e^{i\theta})\right|^q d\theta \right\}^{1/q}$$
$$\leq B_q \left\{ \int_0^{2\pi} \left|1 + e^{i\theta}\right|^{qr} d\theta \right\}^{1/qr} \left\{ \int_0^{2\pi} \left|D_\alpha P(e^{i\theta})\right|^{qs} d\theta \right\}^{1/qs}.$$

This completes the proof of Theorem 2.6. □

Remark 2.7. By letting $s \to \infty$ (so that $r \to 1$) in (2.28), we get inequality (2.1).

The following result is an immediate consequence of Theorem 2.6.

Corollary 2.8. *If $P \in \mathcal{P}_n$ and $P(z)$ has all its zeros in $|z| \leq k$ where $k \geq 1$, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \geq k$ and for each $q > 0$,*

$$n (|\alpha| - k) \left\{ \int_0^{2\pi} \left|P(e^{i\theta})\right|^q d\theta \right\}^{1/q} \leq (1 + k^n) \left\{ \int_0^{2\pi} \left|D_\alpha P(e^{i\theta})\right|^q d\theta \right\}^{1/q}. \quad (2.34)$$

Remark 2.9. Making $q \to \infty$ in (2.34), we get inequality (1.8).

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