Nonlinear Functional Analysis and Applications
Vol. 22, No. 4 (2017), pp. 763-772
ISSN: 1229-1595(print), 2466-0973(online)

http://nfaa.kyungnam.ac.kr/journal-nfaa

NEW COMMON FIXED POINT RESULTS UNDER IMPLICIT RELATION IN CONE METRIC SPACES WITH APPLICATIONS

S. Poonkundran[1] and K. M. Dharmalingam[2]

[1]PG & Research Department of Mathematics
Cardamom Planters' Association College
Bodinayakanur - 625513, Theni, Tamil Nadu, India
e-mail: poonkundran16@gmail.com

[2]Department of Mathematics, The Madura College
Madurai, Tamil Nadu, India

Abstract. The purpose of this paper is to establish the existence and uniqueness of common fixed point of a family of self-mappings satisfying implicit relation in a cone metric space. As an application of the main results, the well-posedness of the common fixed point problem is proved. Our results generalize and improve many known results in the literature.

1. Introduction

The well known Banach contraction mapping principle is widely recognized as the source of metric fixed point theory. A mapping $T : X \to X$, where (X, d) is a metric space, is said to be a contraction mapping if, for all $x, y \in X$,

$$d(T(x), T(y)) \leq \alpha d(x, y), \quad \text{for } 0 < \alpha < 1. \tag{1.1}$$

In the aspect of the Banach contraction principle, any mapping T satisfying (1.1) in a complete metric space has a unique fixed point. This principle has been generalized in different directions in different spaces by mathematicians over the years. Also, in the contemporary research, it remains a heavily investigated branch. The studies noted in [6, 7] [10]-[16], [18, 19, 21] are some examples from this line of research.

[0]Received December 21, 2016. Revised June 5, 2017.
[0]2010 Mathematics Subject Classification: 47H10, 54H25.
[0]Keywords: Cone metric space, common fixed point, well-posedness.

Cone metric spaces are generalizations of metric spaces, where each pair of points is assigned to a member of a real Banach space with a cone. This cone naturally induces a partial order in the Banach space. The concept of cone metric space was introduced by Huang and Zhang [8] where they also established the Banach contraction mapping principle in such spaces. Moreover, they defined the convergence through interior points of the cone. Such an approach allows the investigation of the case that the cone is not necessarily normal. Afterwards, several authors have studied fixed point problems in cone metric spaces. Some of these works are noted in [1, 2, 5, 23, 24].

In this paper, we analyze the existence and uniqueness of common fixed points of a family of self mappings under implicit relation in cone metric spaces. As an application, we prove well-posedness of a common fixed point problem.

2. Preliminaries

In this section, we recall the definition of cone metric space and some of their properties. The following notions will be used in order to prove the main results.

Definition 2.1. Let E be a real Banach space. A subset P of E is called a cone if the following conditions are satisfied:

(i) P is closed, nonempty and $P \neq \{0\}$;
(ii) $a, b \in \mathbb{R}, a, b \geq 0$ and $x, y \in P$ imply that $ax + by \in P$.
(iii) $P \cap (-P) = \{0\}$.

Given a cone P of E, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in intP$.

A cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$,

$$0 \leq x \leq y \text{ implies } \|x\| \leq K\|y\|.$$

The least positive number satisfying the above inequality is called the normal constant of P.

Definition 2.2. Let X be a nonempty set and $d : X \times X \to E$ be a mapping such that the following conditions hold:

(i) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Example 2.3. Let $X = \mathbb{R}$, $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x, y \geq 0\} \subset \mathbb{R}^2$ and $d : X \times X \to E$ such that $d(x, y) = (|x-y|, \delta|x-y|)$, where $\delta \geq 0$ is a constant. Then (X,d) is a cone metric space.

Example 2.4. Let $E = C^1_{\mathbb{R}}([0, 1])$ with norm $\|f\| = \|f\|_\infty + \|f^{'}\|_\infty$. The cone $P = \{f \in E : f \geq 0\}$ is a non-normal cone.

Definition 2.5. Let (X, d) be a cone metric space. We say that $\{x_n\}$ is

(i) a Cauchy sequence if for every $c \in E$ with $0 \ll c$, there is N such that for all $m, n > N, d(x_n, x_m) \ll c$;

(ii) a convergent sequence if for every $c \in E$ with $0 \ll c$, there is N such that for all $n > N, d(x_n, x) \ll c$, for some $x \in X$. We denote it by $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$.

A cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X. The limit of a convergent sequence is unique provided P is a normal cone with normal constant K (see [8]).

Lemma 2.6. *Let (X, d) be a cone metric space and P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to 0$ as $m, n \to \infty$.*

Definition 2.7. (Implicit Relation) Let Φ be the class of real valued continuous functions $\phi : \mathbb{R}^3_+ \to \mathbb{R}_+$ which are non-decreasing in the first argument and satisfying the following condition: for $x, y > 0$,

(i) $x \leq \phi\left(y, \dfrac{x+y}{2}, \dfrac{x+y}{2}\right)$ or

(ii) $x \leq \phi(x, 0, x)$,

there exists a real number $0 < h < 1$ such that $x \leq hy$.

Example 2.8. Let $\phi(r, s, t) = r - \alpha \min(s, t) + (2 + \alpha)t$, where $\alpha > 0$.

Example 2.9. Let $\phi(r, s, t) = r^2 - ar \max(s, t) - bs$, where $a > 0$, $b > 0$.

Example 2.10. Let $\phi(r, s, t) = r + c \max(s, t)$, where $c \geq 0$.

Definition 2.11. A sequence $\{x_n\}$ in a cone metric space X is said to be asymptotically T-regular if $\lim\limits_{n\to\infty} d(x_n, Tx_n) = 0$.

3. Main results

We shall now prove our main results.

Theorem 3.1. *Let (X,d) be a complete cone metric space, S and T be two continuous self-mappings of X such that for all $x,y \in X$ satisfying the condition*

$$d(Sx,Ty) \leq \phi\left(d(x,y), \frac{d(x,Sx)+d(y,Ty)}{2}, \frac{d(x,Ty)+d(y,Sx)}{2}\right). \tag{3.1}$$

Then S and T have a unique common fixed point in X.

Proof. For $x_0 \in X$, we define a sequence $\{x_n\}$ as follows:

$$x_{2n+1} = Sx_{2n}, \quad x_{2n+2} = Tx_{2n+1}, \quad n = 0,1,2,\cdots$$

Now, for all $u \in X$, from (3.1), we have

$$\begin{aligned}
d(x_{2n+1},x_{2n}) &= d(Sx_{2n},Tx_{2n-1}) \\
&\leq \phi\Bigg(d(x_{2n},x_{2n-1}), \frac{d(x_{2n},Sx_{2n})+d(x_{2n-1},Tx_{2n-1})}{2}, \\
&\qquad \frac{d(x_{2n},Tx_{2n-1})+d(x_{2n-1},Sx_{2n})}{2}\Bigg) \\
&\leq \phi\Bigg(d(x_{2n},x_{2n-1}), \frac{d(x_{2n},x_{2n+1})+d(x_{2n-1},x_{2n})}{2}, \\
&\qquad \frac{d(x_{2n},x_{2n})+d(x_{2n-1},x_{2n+1})}{2}\Bigg) \\
&\leq \phi\Bigg(d(x_{2n},x_{2n-1}), \frac{d(x_{2n}x_{2n+1})+d(x_{2n-1},x_{2n})}{2}, \\
&\qquad \frac{d(x_{2n-1},x_{2n})+d(x_{2n},x_{2n+1})}{2}\Bigg).
\end{aligned}$$

Hence by Definition 2.7 (i), we have

$$d(x_{2n+1},x_{2n}) \leq hd(x_{2n},x_{2n-1}), \text{ for } 0 < h < 1.$$

Proceeding in the similar way, we obtain

$$d(x_{2n+1},x_{2n}) \leq h^{2n}d(x_1,x_0), \quad n = 1,2,3,\cdots.$$

Also for $n > m$, we have

$$\begin{aligned}
d(x_n,x_m) &\leq d(x_n,x_{n-1}) + d(x_{n-1},x_{n-2}) + \cdots + d(x_{m+1},x_m) \\
&\leq (h^{n-1} + h^{n-2} + \cdots + h^m)d(x_1,x_0) \\
&\leq \frac{h^m}{1-h}d(x_1,x_0).
\end{aligned}$$

Note that $\dfrac{h^m}{1-h} \to 0$ as $n \to \infty$, since $0 < h < 1$. Thus $d(x_n, x_m) \to 0$ as $m, n \to \infty$, which shows that $\{x_n\}$ is a Cauchy sequence in X. Hence there exists a point $z \in X$ such that $x_n \to z$ as $n \to \infty$. By the continuity of S and T, it is clear that $Sz = Tz = z$. Therefore, z is a common fixed point of S and T.

In order to prove the uniqueness, let take another common fixed point of S and T, say v with $v \neq z$. Then

$$\begin{aligned} d(v,z) &= d(Sv,Tz) \\ &\leq \phi\left(d(v,z), \frac{d(v,Sv)+d(z,Tz)}{2}, \frac{d(v,Tz)+d(z,Sv)}{2}\right) \\ &\leq \phi\Big(d(v,z), 0, d(v,z)\Big). \end{aligned}$$

Now, by Definition 2.7 (ii), we get

$$d(v,z) \leq hd(v,z), \text{ for } 0 < h < 1.$$

This means that z is a unique common fixed point of S and T . □

Remark 3.2. Theorem 3.1 extends the result of Pitchaimani and Ramesh [12] to cone metric spaces.

Theorem 3.3. *Let (X, d) be a complete cone metric space, S and T be two continuous self-mappings of X such that*

$$d(S^p x, T^q y) \leq \phi\left(d(x,y), \frac{d(x,S^p x)+d(y,T^q y)}{2} + \frac{d(x,T^q y)+d(y,S^p x)}{2}\right)$$

for all $x, y \in X$, where p and q are some positive integers. Then S and T have a unique common fixed point.

Proof. Since S^p and T^q satisfy the conditions of Theorem 3.1, S^p and T^q have a unique common fixed point. Let v be the common fixed point. Now

$$\begin{aligned} S^p v = v &\Rightarrow S(S^p v) = Sv, \\ S^p(Sv) &= Sv. \end{aligned}$$

If $Sv = x_0$ then $S^p(x_0) = x_0$. So, Sv is a fixed point of S^p. Similarly, $T^q(Tv) = Tv$. Now, we have

$$\begin{aligned}
&d(v, Tv)\\
&= d(S^p v, T^q(Tv))\\
&\leq \phi\left(d(v, Tv), \frac{d(v, S^p v) + d(Tv, T^q(Tv))}{2}, \frac{d(v, T^q(Tv)) + d(Tv, S^p v)}{2}\right)\\
&= \phi\Big(d(v, Tv), 0, d(v, Tv)\Big).
\end{aligned}$$

Hence, by Definition 2.7 (ii), we obtain

$$d(v, Tv) \leq 0.$$

Thus $v = Tv$. Similarly, $v = Sv$.

For uniqueness of v, let $w \neq v$ be another common fixed point of S and T. Then clearly w is also a common fixed point of S^p and T^q which implies $w = v$. Hence S and T have a unique common fixed point. □

Hence we have proved that if x_0 is a unique common fixed point of S^p and T^q for some positive integers p and q then x_0 is a unique common fixed point of S and T. Next we generalize Theorem 3.1 to the case of family of mappings satisfying the condition (3.1).

Theorem 3.4. *Let (X, d) be a complete cone metric space and $\{F_\alpha\}$ be a family of continuous self-mappings on X satisfying*

$$\begin{aligned}
&d(F_\alpha x, F_\beta y)\\
&\leq \phi\left(d(x, y), \frac{d(x, F_\alpha x) + d(y, F_\beta y)}{2}, \frac{d(x, F_\beta y) + d(y, F_\alpha x)}{2}\right)
\end{aligned} \tag{3.2}$$

for $\alpha, \beta \in \Lambda$ with $\alpha \neq \beta$ and $x, y \in X$, where Λ is an index set. Then there exists a unique $v \in X$ satisfying $F_\alpha v = v$ for all $\alpha \in \Lambda$.

Proof. As the proof is similar to that of Theorem 3.1, we omit the proof here. □

Theorem 3.5. *Let (X, d) be a complete metric space and $\{F_n\}$ be a sequence of self-mappings on X such that $\{F_n\}$ converging pointwise to a self-mapping F and*

$$d(F_n x, F_n y) \leq \phi\left(d(x, y), \frac{d(x, F_n x) + d(y, F_n y)}{2}, \frac{d(x, F_n y) + d(y, F_n x)}{2}\right),$$

for all $x, y \in X$. If $\{F_n\}$ has a fixed point v_n and F has a fixed point v. Then the sequence $\{v_n\}$ converges to v.

Proof. Note that $F_n v_n = v_n$ and $Fv = v$. Now consider

$$\begin{aligned} d(v, v_n) &= d(Fv, F_n v_n) \\ &\leq d(Fv, F_n v) + d(F_n v, F_n v_n) \\ &\leq d(Fv, F_n v) + \phi\Bigg(d(v, v_n), \frac{d(v, F_n v) + d(v_n, F_n v_n)}{2}, \\ &\qquad \frac{d(v, F_n v_n) + d(v_n, F_n v)}{2}\Bigg). \end{aligned}$$

By the fact that $F_n v \to Fv$ as $n \to \infty$, we get

$$d(v, v_n) \leq \phi\Bigg(d(v, v_n), 0, d(v, v_n)\Bigg).$$

Hence, by Implicit Relation 2.7 (ii), we obtain

$$d(v, v_n) \leq 0$$

which implies that $v_n \to v$ as $n \to \infty$. □

Theorem 3.6. *Let (X, d) be a complete cone metric space and $T : X \to X$ be a self-mapping such that*

$$d(Tx, Ty) \leq \phi\Bigg(d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}, \frac{d(x, Ty) + d(y, Tx)}{2}\Bigg).$$

Then T has a unique fixed point $v \in X$ and $\{z_n\}$ is asymptotically T-regular if and only if T is continuous at $v \in X$.

Proof. Let $v \in X$ and $z_n \to v$ as $n \to \infty$. Now

$$d(Tz_n, Tv) \quad \leq \quad \phi\Bigg(d(z_n, v), \frac{d(z_n, Tz_n) + d(v, Tv)}{2}, \frac{d(z_n, Tv) + d(v, Tz_n)}{2}\Bigg).$$

Since T has a fixed point and $\{z_n\}$ is asymptotically T-regular, we get

$$d(Tz_n, Tv) \quad \leq \quad \phi\Bigg(d(Tz_n, Tv), 0, d(Tz_n, Tv)\Bigg).$$

By Implicit Relation 2.7 (ii), there exists $0 < h < 1$ such that

$$d(Tz_n, Tv) \leq hd(Tz_n, Tv),$$

this implies that

$$Tz_n \to Tv \quad \text{as} \quad n \to \infty.$$

Hence T is continuous at $v \in X$. Conversely, assume that T is continuous at $v \in X$. Note that

$$z_n \to v \ \Rightarrow \ Tz_n \to Tv \quad \text{as} \quad n \to \infty,$$

which implies that

$$d(z_n, Tz_n) \quad \to \quad d(v, Tv) = 0,$$

since T has a fixed point. This completes the proof. $\square$

Remark 3.7. Theorem 3.4 extends the study of Saluja [22] to a family of continuous self-mappings using implicit relation in the setting of cone metric spaces. In Theorem 3.5, convergence of sequence of self-mappings to another self-mapping implies convergence of corresponding sequence of fixed points. One can notice that the continuity of mappings is not essential in Theorem 3.5.

4. Applications

The concept of well-posedness of a fixed point problem has generated much interest to several mathematicians, for example [3, 4, 9, 17, 20]. Here, we study well-posedness of a common fixed point problem of mappings in Theorem 3.1.

Definition 4.1. Let (X, d) be a complete cone metric space and f be a self-mapping. Then the fixed point problem of f is said to be well-posed if

(i) f has a unique fixed point $x_0 \in X$,
(ii) for any sequence $\{x_n\} \subset X$ and $\lim_{n\to\infty} d(x_n, fx_n) = 0$, we have

$$\lim_{n\to\infty} d(x_n, x_0) = 0.$$

Let $CFP(T, f, X)$ denote a common fixed point problem of self-mappings T and f on X and $CF(T, f)$ denote the set of all common fixed points of T and f.

Definition 4.2. $CFP(T, f, X)$ is called well-posed if $CF(T, f)$ is singleton and for any sequence $\{x_n\}$ in X with

$$\tilde{x} \in CF(T, f) \text{ and } \lim_{n\to\infty} d(x_n, fx_n) = \lim_{n\to\infty} d(x_n, Tx_n) = 0$$

implies $\tilde{x} = \lim_{n\to\infty} x_n$.

Theorem 4.3. *Let (X, d) be a complete cone metric space and T, f be self-mappings on X as in* Theorem 3.1*. Then the common fixed point problem of f and T is well-posed.*

Proof. From Theorem 3.1, the mappings f and T have a unique common fixed point, say $v \in X$. Let $\{x_n\}$ be a sequence in X and $\lim_{n\to\infty} d(fx_n, x_n) = \lim_{n\to\infty} d(Tx_n, x_n) = 0$. Without loss of generality, assume that $v \neq x_n$ for any non-negative integer n. Using (3.1) and $fv = Tv = v$, we get

$$\begin{aligned} d(v, x_n) &\leq d(Tv, Tx_n) + d(Tx_n, x_n) \\ &= d(fv, Tx_n) + d(Tx_n, x_n) \\ &\leq d(Tx_n, x_n) + \phi\bigg(d(v, x_n), \frac{d(v, fv) + d(x_n, Tx_n)}{2}, \\ &\quad \frac{d(v, Tx_n) + d(x_n, fv)}{2}\bigg) \\ &= \phi(d(v, x_n), 0, d(v, x_n)). \end{aligned}$$

Hence by Implicit Relation 2.7 (ii), we obtain $d(v, x_n) \to 0$ as $n \to \infty$. This completes the proof. □

Corollary 4.4. *Let (X, d) be a complete cone metric space and T be a self-mapping on X such that*

$$d(Tx, Ty) \leq \phi\bigg(d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}, \frac{d(x, Ty) + d(y, Tx)}{2}\bigg),$$

for all $x, y \in X$. Then the fixed point problem of T is well-posed.

References

[1] M. Abbas and G. Jungck, *Common fixed point results for non-commuting mappings without continuity in cone metric spaces*, J. Math. Anal. Appl., **341** (2008), 416–420.

[2] M. Abbas and B.E. Rhoades, *Fixed and periodic point results in cone metric spaces*, Appl. Math. Lett., **22** (2009), 511–515.

[3] M. Akkouchi, *Well posedness of the fixed point problem for certain asymptotically regular mappings*, Annalas Mathematicae Silesianae, **23** (2009), 43–52.

[4] M. Akkouchi and V. Popa, *Well posedness of the fixed point problem for mappings satisfying an implicit relations*, Demonstratio Mathematica, **23**(4) (2010), 923–929.

[5] A. Azam, M. Arshad and I. Beg, *Common fixed points of two maps in cone metric spaces*, Rendiconti del Circolo Matematico di Palermo, **57** (2008), 433–441.

[6] Lj.B. Ćirić, *A generalization of Banach's contraction principle*, Proc. Amer. Math. Soc., **45** (1974), 267–273.

[7] Lj.B. Ćirić, *Generalized contractions and fixed point theorems*, Publ. Inst. Math. (Beograd), **12**(26) (1971), 19–26.

[8] LG. Huang and X. Zhang, *Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl., **332**(2) (2007), 1468–1476.

[9] B.K. Lahiri, *Well-Posedness and certain classes of operators*, Demonstratio Mathematica, **38** (2005), 170–176.

[10] J.J. Nieto, R.L. Pouso and R. Rodríguez-López, *Fixed point theorems in ordered abstract spaces*, Proc. Am. Math. Soc., **135** (2007), 2505–2517.

[11] H.K. Pathak and N. Hussain, *Common fixed points for Banach operator pairs with applications*, Nonlinear Anal., **69** (2008), 2788–2802.

[12] M. Pitchaimani and D. Ramesh Kumar, *Some common fixed point theorems using implicit relation in 2-Banach spaces*, Surveys in Mathematics and its Applications, **10** (2015), 159–168.

[13] M. Pitchaimani and D. Ramesh Kumar, *Common and coincidence fixed point theorems for asymptotically regular mappings in 2-Banach Space*, Nonlinear Func. Anal. Appl., **21**(1) (2016), 131–144.

[14] M. Pitchaimani and D. Ramesh Kumar, *On construction of fixed point theory under implicit relation in Hilbert spaces*, Nonlinear Func. Anal. Appl., **21**(3) (2016), 513–522.

[15] M. Pitchaimani and D. Ramesh Kumar, *On Nadler type results in ultrametric spaces with application to well-posedness*, Asian-European Journal of Mathematics, **10**(4) (2017), 1750073(1-15) DOI: 10.1142/S1793557117500735.

[16] M. Pitchaimani and D. Ramesh Kumar, *Generalized Nadler type results in ultrametric spaces with application to well-posedness*, Afr. Mat., (2017) DOI: 10.1007/s13370-017-0496-6.

[17] V. Popa, *Well- posedness of fixed problem in compact metric space*, Bull. Univ. Petrol-Gaze, Ploicsti, sec. Mat Inform. Fiz., **60**(1) (2008), 1–4.

[18] D. Ramesh Kumar and M. Pitchaimani, *Set-valued contraction mappings of Prešić-Reich type in ultrametric spaces*, Asian-European Journal of Mathematics, **10**(4) (2017), 1750065(1-15) DOI: 10.1142/S1793557117500656.

[19] D. Ramesh Kumar and M. Pitchaimani, *A generalization of set-valued Prešić-Reich type contractions in ultrametric spaces with applications*, J. Fixed Point Theory Appl., (2016), DOI 10.1007/s11784-016-0338-4.

[20] S. Reich and A.T. Zaslawski, *Well-posedness of fixed point problems*, Far East J. Math. Sci, Special volume, part **III** (2011), 393–401.

[21] A.M. Saddeek, *Coincidence points by generalized Mann iterates with applications in Hilbert spaces*, Nonlinear Anal., **72** (2010), 2262–2270.

[22] G.S. Saluja, *On common fixed point theorems under implicit relation in 2-Banach spaces*, Nonlinear Funct. Anal. Appl., **20**(2) (2015), 215–221.

[23] G. Song, X. Sun, Y. Zhao and G. Wang, *New common fixed point theorems for maps on cone metric spaces*, Appl. Math. Lett., **23** (2010), 1033–1037.

[24] S. Wang and B. Guo, *Distance in cone metric spaces and common fixed point theorems*, Appl. Math. Lett., **24** (2011), 1735–1739.