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CONVERGENCE THEOREMS FOR GENERALIZED ASYMPTOTICALLY NONEXPANSIVE MAPPINGS AND ASYMPTOTICALLY NONEXPANSIVE NON-SELF MAPPINGS IN UNIFORMLY CONVEX BANACH SPACES

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Abstract. In this paper, we proposed and study a new two-step iteration scheme of hybrid mixed type for two generalized asymptotically nonexpansive self mappings and two asymptotically nonexpansive non-self mappings and establish some strong and weak convergence theorems for mentioned scheme and mappings in uniformly convex Banach spaces. Our results extend and generalize the corresponding results given in the current existing literature.

1. Introduction and preliminaries

Let K be a nonempty subset of a real Banach space E and $T\colon K \to K$ be a nonlinear mapping. $F(T)$ denotes the set of fixed points of T, that is, $F(T) = \{x \in K : Tx = x\}$ and

$$F := F(S_1) \cap F(S_2) \cap F(T_1) \cap F(T_2)$$

denotes the set of common fixed points of the mappings S_1, S_2, T_1 and T_2. Recall the following definitions.

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T is said to be nonexpansive if

$$\|T(x) - T(y)\| \leq \|x - y\|, \ \forall\, x, y \in K.$$

T is said to be asymptotically nonexpansive [6] if there exists a positive sequence $\{k_n\}$ in $[1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n(x) - T^n(y)\| \quad \leq \quad k_n\|x - y\|, \ \ \forall\, x, y \in K,\, n \geq 1.$$

It is easy to see that every nonexpansive mapping is asymptotically nonexpansive with the constant sequence $\{1\}$. The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [6] as a generalization of the class of nonxpansive mappings. They proved that if K is a nonempty closed convex subset of a real uniformly convex Banach space and T is an asymptotically nonexpansive mapping on K, then T has a fixed point.

T is said to be asymptotically nonexpansive in the intermediate sense [2] if it is continuous and the following inequality holds:

$$\limsup_{n\to\infty} \sup_{x,y\in K} \Big(\|T^n(x) - T^n(y)\| - \|x - y\|\Big) \quad \leq \quad 0. \tag{1.1}$$

Putting $c_n = \max\big\{0, \sup_{x,y\in K}(\|T^n(x) - T^n(y)\| - \|x - y\|)\big\}$, we see that $c_n \to 0$ as $n \to \infty$. Then (1.1) is reduced to the following:

$$\|T^n(x) - T^n(y)\| \leq \|x - y\| + c_n, \quad \forall\, x, y \in K,\ n \geq 1.$$

The class of asymptotically nonexpansive mappings in the intermediate sense was introduced by Bruck *et al.* [2] as a generalization of the class of asymptotically nonxpansive mappings. It is known that if K is a nonempty closed convex and bounded subset of a real Hilbert space, then every asymptotically nonexpansive self mapping in the intermediate sense has a fixed point; see [18] for more details.

In 2011, Agarwal *et al.* [1] introduced the notion of generalized asymptotically nonexpansive mapping as follows:

Definition 1.1. A mapping $T\colon K \to K$ is said to be generalized asymptotically nonexpansive if it is continuous and there exists a positive sequence $\{h_n\}$ in $[1, \infty)$ with $\lim_{n\to\infty} h_n = 1$ such that

$$\limsup_{n\to\infty} \sup_{x,y\in K} \Big(\|T^n(x) - T^n(y)\| - h_n\|x - y\|\Big) \quad \leq \quad 0. \tag{1.2}$$

Putting $d_n = \max\big\{0, \sup_{x,y\in K}(\|T^n(x) - T^n(y)\| - h_n\|x - y\|)\big\}$, we see that $d_n \to 0$ as $n \to \infty$. Then (1.2) is reduced to the following:

$$\|T^n(x) - T^n(y)\| \leq h_n\|x - y\| + d_n, \quad \forall\, x, y \in K,\ n \geq 1.$$

We remark that if $h_n = 1$, then the class of generalized asymptotically nonexpansive mappings is reduced to the class of asymptotically nonexpansive mappings in the intermediate sense.

Definition 1.2. A subset K of a Banach space E is said to be a retract of E if there exists a continuous mapping $P\colon E \to K$ (called a retraction) such that $P(x) = x$ for all $x \in K$. If, in addition P is nonexpansive, then P is said to be a nonexpansive retract of E.

If $P\colon E \to K$ is a retraction, then $P^2 = P$. A retract of a Hausdorff space must be a closed subset. Every closed convex subset of a uniformly convex Banach space is a retract.

Chidume *et al.* [3] introduced the concept of non-self asymptotically nonexpansive mappings as follows.

Definition 1.3. Let K be a nonempty subset of a real Banach space E and let $P\colon E \to K$ be a nonexpansive retraction of E onto K. A non-self mapping $T\colon K \to E$ is said to be asymptotically nonexpansive if there exists a positive sequence $\{k_n\}$ in $[1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T(PT)^{n-1}(x) - T(PT)^{n-1}(y)\| \quad \leq \quad k_n\|x - y\|, \ \forall\, x, y \in K,\ n \geq 1. \tag{1.3}$$

Example 1.4. Let $E = \mathbb{R}$ be a normed linear space, $K = [0, 1]$ and P be the identity mapping. For each $x \in K$, we define

$$T(x) = \begin{cases} \lambda\, x, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}$$

where $0 < \lambda < 1$. Then

$$|T^n x - T^n y| = \lambda^n |x - y| \leq |x - y|$$

for all $x, y \in K$ and $n \geq 1$. Thus T is an asymptotically nonexpansive mapping with constant sequence $\{k_n\} = \{1\}$ for all $n \geq 1$ and uniformly L-Lipschtzian mappings with $L = \sup_{n\geq 1}\{k_n\}$.

In 2006, Wang [15] studied the iteration process as follows:

$$\begin{cases} x_1 = x \in K, \\ x_{n+1} = P((1 - \alpha_n)x_n + \alpha_n T_1(PT_1)^{n-1} y_n), \\ y_n = P((1 - \beta_n)x_n + \beta_n T_2(PT_2)^{n-1} x_n), \ \ n \geq 1, \end{cases} \tag{1.4}$$

where T_1, $T_2\colon K \to E$ are two asymptotically nonexpansive non-self mappings and $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences in $[0, 1)$, and proved some strong and weak convergence theorems for asymptotically nonexpansive non-self mappings .

In 2012, Guo *et al.* [8] generalized the iteration process (1.4) as follows:

$$\begin{cases} x_1 = x \in K, \\ x_{n+1} = P((1-\alpha_n)S_1^n x_n + \alpha_n T_1(PT_1)^{n-1} y_n), \\ y_n = P((1-\beta_n)S_2^n x_n + \beta_n T_2(PT_2)^{n-1} x_n), \ n \geq 1, \end{cases} \tag{1.5}$$

where S_1, $S_2\colon K \to K$ are two asymptotically nonexpansive self mappings and T_1, $T_2\colon K \to E$ are two asymptotically nonexpansive non-self mappings and $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences in $[0,1)$, and proved some strong and weak convergence theorems for mixed type asymptotically nonexpansive mappings.

Recently, Wei and Guo [17] studied the following: Let E be a real Banach space, K be a nonempty closed convex subset of E and $P\colon E \to K$ be a nonexpansive retraction of E onto K. Let S_1, $S_2\colon K \to K$ be two asymptotically nonexpansive self mappings and T_1, $T_2\colon K \to E$ be two asymptotically nonexpansive non-self mappings. Then Wei and Guo [17] defined the new iteration scheme of mixed type with mean errors as follows:

$$\begin{cases} x_1 = x \in K, \\ x_{n+1} = P(\alpha_n S_1^n x_n + \beta_n T_1(PT_1)^{n-1} y_n + \gamma_n u_n), \\ y_n = P(\alpha'_n S_2^n x_n + \beta'_n T_2(PT_2)^{n-1} x_n + \gamma'_n u'_n), \ n \geq 1, \end{cases} \tag{1.6}$$

where $\{u_n\}$, $\{u'_n\}$ are bounded sequences in E, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$ are real sequences in $[0,1)$ satisfying $\alpha_n + \beta_n + \gamma_n = 1 = \alpha'_n + \beta'_n + \gamma'_n$ for all $n \geq 1$, and proved some weak convergence theorems in the setting of real uniformly convex Banach spaces.

If $\gamma_n = \gamma'_n = 0$, for all $n \geq 1$, then the iteration scheme (1.6) reduces to the scheme (1.5).

Inspired and motivated by [8, 17] and many others, we proposed the following hybrid mixed type iteration scheme:

Let S_1, $S_2\colon K \to K$ be two generalized asymptotically nonexpansive self mappings and T_1, $T_2\colon K \to E$ are two asymptotically nonexpansive non-self mappings, then we define the hybrid mixed type iteration scheme as follows:

$$\begin{cases} x_1 = x \in K, \\ x_{n+1} = P((1-a_n-c_n)S_1^n x_n + a_n T_1(PT_1)^{n-1} y_n + c_n u_n), \\ y_n = P((1-b_n-d_n)S_2^n x_n + b_n T_2(PT_2)^{n-1} x_n + d_n v_n), \ n \geq 1, \end{cases} \tag{1.7}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$ are four real sequences in $[0,1]$ satisfying $a_n + c_n \leq 1$, $b_n + d_n \leq 1$ and $\{u_n\}$, $\{v_n\}$ are bounded sequences in K.

The aim of this paper is to study and establish some strong and weak convergence theorems of iteration scheme (1.7) for above mentioned mappings in the setting of uniformly convex Banach spaces. Our results extend and generalize several results from the current existing literature.

For the sake of convenience theorems, we restate the following notion and results.

Let E be a Banach space with its dimension greater than or equal to 2. The modulus of convexity of E is the function $\delta_E(\varepsilon)\colon (0,2] \to [0,1]$ defined by

$$\delta_E(\varepsilon) = \inf\Big\{1 - \|\frac{1}{2}(x+y)\| : \|x\| = 1,\ \|y\| = 1,\ \varepsilon = \|x-y\|\Big\}.$$

A Banach space E is uniformly convex if and only if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$.

Definition 1.5. Let $\mathcal{S} = \{x \in E : \|x\| = 1\}$ and let E^* be the dual of E, that is, the space of all continuous linear functionals f on E. The space E has:

(i) G*â*teaux differentiable norm if

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists for each x and y in $\mathcal{S}$.

(ii) Fr*é*chet differentiable norm [13] if for each x in $\mathcal{S}$, the above limit exists and is attained uniformly for y in $\mathcal{S}$ and in this case, it is also well known that

$$\begin{aligned}\langle h,\ J(x)\rangle + \frac{1}{2}\|x\|^2 &\le \frac{1}{2}\|x+h\|^2 \\ &\le \langle h,\ J(x)\rangle + \frac{1}{2}\|x\|^2 + b(\|x\|)\end{aligned} \tag{1.8}$$

for all $x, h \in E$, where J is the Fr*é*chet derivative of the functional $\frac{1}{2}\|\cdot\|^2$ at $x \in E$, $\langle .\,, .\rangle$ is the pairing between E and E^*, and b is an increasing function defined on $[0,\infty)$ such that $\lim_{t\to 0}\frac{b(t)}{t} = 0$.

(iii) Opial condition [9] if for any sequence $\{x_n\}$ in E, x_n converges to x weakly it follows that $\limsup_{n\to\infty}\|x_n - x\| < \limsup_{n\to\infty}\|x_n - y\|$ for all $y \in E$ with $y \neq x$. Examples of Banach spaces satisfying Opial condition are Hilbert spaces and all spaces $l^p (1 < p < \infty)$. On the other hand, $L^p[0,2\pi]$ with $1 < p \neq 2$ fail to satisfy Opial condition.

Definition 1.6. A mapping $T\colon K \to K$ is said to be demiclosed at zero, if for any sequence $\{x_n\}$ in K, the condition $\{x_n\}$ converges weakly to $x \in K$ and $\{Tx_n\}$ converges strongly to 0 imply $Tx = 0$.

Definition 1.7. ([12]) A Banach space E has the Kadec-Klee property, if for every sequence $\{x_n\}$ in E, $x_n \to x$ weakly and $\|x_n\| \to \|x\|$ it follows that $\|x_n - x\| \to 0$.

Next we state the following useful lemmas to prove our main results.

Lemma 1.8. ([14]) *Let* $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ *and* $\{r_n\}_{n=1}^{\infty}$ *be sequences of non-negative numbers satisfying the inequality*

$$\alpha_{n+1} \leq (1+\beta_n)\alpha_n + r_n, \ \forall\, n \geq 1.$$

If $\sum_{n=1}^{\infty} \beta_n < \infty$ *and* $\sum_{n=1}^{\infty} r_n < \infty$, *then* $\lim_{n\to\infty} \alpha_n$ *exists.*

In particular, if $\{\alpha_n\}_{n=1}^{\infty}$ *has a subsequence which converges strongly to zero, then* $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 1.9. ([11]) *Let* E *be a uniformly convex Banach space and* $0 < \alpha \leq t_n \leq \beta < 1$ *for all* $n \in \mathbb{N}$. *Suppose that* $\{x_n\}$ *and* $\{y_n\}$ *are sequences of* E *such that* $\limsup_{n\to\infty} \|x_n\| \leq a, \limsup_{n\to\infty} \|y_n\| \leq a$ *and*

$$\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = a$$

hold for some $a \geq 0$. *Then* $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 1.10. ([12]) *Let* E *be a real reflexive Banach space with its dual* E^* *has the Kadec-Klee property. Let* $\{x_n\}$ *be a bounded sequence in* E *and* $p, q \in W_w(x_n)$ *(where* $W_w(x_n)$ *denotes the set of all weak subsequential limits of* $\{x_n\}$*). Suppose* $\lim_{n\to\infty} \|tx_n + (1-t)p - q\|$ *exists for all* $t \in [0,1]$. *Then* $p = q$.

Lemma 1.11. ([12]) *Let* K *be a nonempty convex subset of a uniformly convex Banach space* E. *Then there exists a strictly increasing continuous convex function* $\phi\colon [0,\infty) \to [0,\infty)$ *with* $\phi(0) = 0$ *such that for each Lipschitzian mapping* $T\colon K \to K$ *with the Lipschitz constant* L,

$$\|tTx + (1-t)Ty - T(tx + (1-t)y)\| \leq L\phi^{-1}\Big(\|x-y\| - \frac{1}{L}\|Tx - Ty\|\Big)$$

for all $x, y \in K$ *and all* $t \in [0,1]$.

2. Strong convergence theorems

In this section, we prove some strong convergence theorems of iteration scheme (1.7) for two generalized asymptotically nonexpansive self mappings and two asymptotically nonexpansive non-self mappings in real uniformly convex Banach spaces. First, we shall need the following lemmas.

Lemma 2.1. *Let E be a real Banach space, K be a nonempty closed convex subset of E. Let S_1, $S_2\colon K \to K$ be two generalized asymptotically nonexpansive self mappings with sequence $\{k_n\} \subset [1,\infty)$ and T_1, $T_2\colon K \to E$ be two asymptotically nonexpansive non-self mappings with sequence $\{h_n\} \subset [1,\infty)$. Put*

$$G_n = \max\Big\{0, \sup_{x,\, y\in K,\, n\geq 1} \Big(\|S_1^n x - S_1^n y\| - k_n\|x-y\|\Big), \sup_{x,\, y\in K,\, n\geq 1} \Big(\|S_2^n x - S_2^n y\| - k_n\|x-y\|\Big)\Big\} \tag{2.1}$$

such that $\sum_{n=1}^{\infty} G_n < \infty$. Suppose that

$$F := F(S_1)\bigcap F(S_2)\bigcap F(T_1)\bigcap F(T_2) \neq \emptyset.$$

Let

$$\mu = \sup_n a_n,\ \ \omega_1 = \sup_n k_n \geq 1,\ \ \omega_2 = \sup_n h_n \geq 1$$

and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$ be four real sequences in $[0,1]$ which satisfy the following conditions:

(i) $\sum_{n=1}^{\infty}(k_n h_n - 1) < \infty$;
(ii) $\sum_{n=1}^{\infty} c_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$;
(iii) $\mu\omega_1\omega_2 < 1$ *or* $\mu < \frac{1}{\omega_1^2\omega_2^2}$.

Let $\{x_n\}$ be the sequence defined by (1.7)*. Then $\lim_{n\to\infty}\|x_n - q\|$ and $\lim_{n\to\infty} D(x_n, F)$ both exist for each $q \in F$, where $D(x_n, F) = \inf_{q\in F}\|x_n - q\|$.*

Proof. Let $q \in F$. From (1.7) and (2.1), we have

$$\begin{aligned}
&\|y_n - q\| \\
&= \|P((1-b_n-d_n)S_2^n x_n + b_n T_2(PT_2)^{n-1}x_n + d_n v_n) - P(q)\| \\
&\leq \|(1-b_n-d_n)S_2^n x_n + b_n T_2(PT_2)^{n-1}x_n + d_n v_n - q\| \\
&= \|(1-b_n-d_n)(S_2^n x_n - q) + b_n(T_2(PT_2)^{n-1}x_n - q) + d_n(v_n - q)\| \\
&\leq (1-b_n-d_n)\|S_2^n x_n - q\| + b_n\|T_2(PT_2)^{n-1}x_n - q\| + d_n\|v_n - q\| \\
&\leq (1-b_n-d_n)[k_n\|x_n - q\| + G_n] + b_n h_n\|x_n - q\| + d_n\|v_n - q\| \\
&\leq (1-b_n)k_n\|x_n - q\| + b_n h_n\|x_n - q\| + d_n\|v_n - q\| + G_n \\
&\leq (1-b_n)k_n h_n\|x_n - q\| + b_n k_n h_n\|x_n - q\| + d_n\|v_n - q\| + G_n \\
&= k_n h_n\|x_n - q\| + d_n\|v_n - q\| + G_n.
\end{aligned} \tag{2.2}$$

Again using (1.7) and (2.1), we have

$$\begin{aligned}
&\|x_{n+1}-q\| \\
&= \|P((1-a_n-c_n)S_1^n x_n + a_n T_1(PT_1)^{n-1}y_n + c_n u_n) - P(q)\| \\
&\le \|(1-a_n-c_n)S_1^n x_n + a_n T_1(PT_1)^{n-1}y_n + c_n u_n - q\| \\
&= \|(1-a_n-c_n)(S_1^n x_n - q) + a_n(T_1(PT_1)^{n-1}y_n - q) + c_n(u_n-q)\| \\
&\le (1-a_n-c_n)\|S_1^n x_n - q\| + a_n\|T_1(PT_1)^{n-1}y_n - q\| + c_n\|u_n-q\| \\
&\le (1-a_n-c_n)[k_n\|x_n-q\| + G_n] + a_n h_n\|y_n-q\| + c_n\|u_n-q\| \\
&\le (1-a_n)k_n\|x_n-q\| + a_n h_n\|y_n-q\| + c_n\|u_n-q\| + G_n \\
&\le (1-a_n)k_n h_n\|x_n-q\| + a_n k_n h_n\|y_n-q\| + c_n\|u_n-q\| + G_n. \qquad (2.3)
\end{aligned}$$

Using equation (2.2) in (2.3), we obtain

$$\begin{aligned}
\|x_{n+1}-q\| &\le (1-a_n)k_n h_n\|x_n-q\| + a_n k_n h_n[k_n h_n\|x_n-q\| \\
&\quad + d_n\|v_n-q\| + G_n] + c_n\|u_n-q\| + G_n \\
&\le (1-a_n)k_n^2 h_n^2\|x_n-q\| + a_n k_n^2 h_n^2\|x_n-q\| \\
&\quad + a_n k_n h_n d_n\|v_n-q\| + c_n\|u_n-q\| + a_n k_n h_n G_n + G_n \\
&= k_n^2 h_n^2\|x_n-q\| + a_n k_n h_n d_n\|v_n-q\| + c_n\|u_n-q\| \\
&\quad + (a_n k_n h_n + 1)G_n \\
&= [1 + (k_n^2 h_n^2 - 1)]\|x_n-q\| + a_n k_n h_n d_n\|v_n-q\| \\
&\quad + c_n\|u_n-q\| + (a_n k_n h_n + 1)G_n \\
&= [1+m_n]\|x_n-q\| + f_n, \qquad (2.4)
\end{aligned}$$

where

$$m_n = (k_n^2 h_n^2 - 1)$$

and

$$f_n = a_n k_n h_n d_n\|v_n-q\| + c_n\|u_n-q\| + (a_n k_n h_n + 1)G_n.$$

By assumption of the lemma, we get

$$\begin{aligned}
\sum_{n=1}^{\infty} m_n &= \sum_{n=1}^{\infty}(k_n^2 h_n^2 - 1) = \sum_{n=1}^{\infty}(k_n h_n + 1)(k_n h_n - 1) \\
&\le (\omega_1\omega_2 + 1)\sum_{n=1}^{\infty}(k_n h_n - 1) < \infty
\end{aligned}$$

and boundedness of the sequences $\{\|u_n-q\|\}$, $\{\|v_n-q\|\}$ with condition (ii) of the lemma

$$\begin{aligned}\sum_{n=1}^{\infty} f_n &= \sum_{n=1}^{\infty}[a_nk_nh_nd_n\|v_n - q\| + c_n\|u_n - q\| + (a_nk_nh_n + 1)G_n]\\ &= a_nk_nh_n\sum_{n=1}^{\infty} d_n\|v_n - q\| + \sum_{n=1}^{\infty} c_n\|u_n - q\| + (a_nk_nh_n + 1)\sum_{n=1}^{\infty} G_n\\ &\le \mu\omega_1\omega_2\sum_{n=1}^{\infty} d_n\|v_n - q\| + \sum_{n=1}^{\infty} c_n\|u_n - q\|(\mu\omega_1\omega_2 + 1)\sum_{n=1}^{\infty} G_n\\ &< \infty.\end{aligned}$$

Now taking $\rho_n = \|x_n - q\|$ in (2.4), we obtain

$$\rho_{n+1} \le (1 + m_n)\rho_n + f_n.$$

Hence by Lemma 1.8, we have $\lim_{n\to\infty}\|x_n - q\|$ exists.

Now, taking the infimum over all $q \in F$ in (2.4), we have

$$D(x_{n+1}, F) \quad\le\quad [1 + m_n]D(x_n, F) + f_n \tag{2.5}$$

for all $n \ge 1$. It follows from $\sum_{n=1}^{\infty} m_n < \infty$, $\sum_{n=1}^{\infty} f_n < \infty$ and Lemma 1.8 that $\lim_{n\to\infty} D(x_n, F)$ exists. This completes the proof. □

Lemma 2.2. *Let E be a real uniformly convex Banach space, K be a nonempty closed convex subset of E. Let S_1, $S_2\colon K \to K$ be two generalized asymptotically nonexpansive self mappings with sequence $\{k_n\} \subset [1,\infty)$ and T_1, $T_2\colon K \to E$ be two asymptotically nonexpansive non-self mappings with sequence $\{h_n\} \subset [1,\infty)$ and G_n be taken as in Lemma 2.1. Suppose that*

$$F := F(S_1)\bigcap F(S_2)\bigcap F(T_1)\bigcap F(T_2) \ne \emptyset,$$

Let

$$\mu = \sup_n a_n,\;\; \omega_1 = \sup_n k_n \ge 1,\;\; \omega_2 = \sup_n h_n \ge 1$$

and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$ be four real sequences in $[0,1]$ which satisfy the following conditions:

(i) $\sum_{n=1}^{\infty}(k_nh_n - 1) < \infty$;
(ii) $\sum_{n=1}^{\infty} c_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$;
(iii) $\mu\omega_1\omega_2 < 1$ *or* $\mu < \frac{1}{\omega_1^2\omega_2^2}$;
(iv)

$$\|x - T_1(PT_1)^{n-1}y\| \le \|S_1^nx - T_1(PT_1)^{n-1}y\|$$

and

$$\|x - T_i(PT_i)^{n-1}x\| \le \|S_i^nx - T_i(PT_i)^{n-1}x\|$$

for all $x, y \in K$ and for $i = 1, 2$.

Let $\{x_n\}$ *be the sequence defined by* (1.7). *Then* $\lim_{n\to\infty}\|x_n - S_i x_n\| = 0$ *and* $\lim_{n\to\infty}\|x_n - T_i x_n\| = 0$ *for each* $i = 1, 2$.

Proof. By Lemma 2.1, $\lim_{n\to\infty}\|x_n - q\|$ exists for all $q \in F$ and therefore $\{x_n\}$ is bounded. Let $\lim_{n\to\infty}\|x_n - q\| = r_1$. Then $r_1 > 0$ otherwise there is nothing to prove.

Now (1.7) implies that

$$\limsup_{n\to\infty}\|y_n - q\| \quad \leq \quad r_1. \tag{2.6}$$

Also, we have

$$\|S_2^n x_n - q\| \quad \leq \quad k_n\|x_n - q\| + G_n, \;\; \forall\, n \geq 1,$$

$$\|T_2(PT_2)^{n-1}x_n - q\| \quad \leq \quad h_n\|x_n - q\|, \;\; \forall\, n \geq 1$$

and

$$\|S_1^n x_n - q\| \quad \leq \quad k_n\|x_n - q\| + G_n, \;\; \forall\, n \geq 1.$$

Hence

$$\limsup_{n\to\infty}\|S_2^n x_n - q\| \quad \leq \quad r_1, \tag{2.7}$$

$$\limsup_{n\to\infty}\|T_2(PT_2)^{n-1}x_n - q\| \quad \leq \quad r_1 \tag{2.8}$$

and

$$\limsup_{n\to\infty}\|S_1^n x_n - q\| \quad \leq \quad r_1. \tag{2.9}$$

Next,

$$\|T_1(PT_1)^{n-1}y_n - q\| \quad \leq \quad h_n\|y_n - q\|$$

gives by virtue of (2.6) that

$$\limsup_{n\to\infty}\|T_1(PT_1)^{n-1}y_n - q\| \quad \leq \quad r_1. \tag{2.10}$$

Also, it follows from

$$\begin{aligned} r_1 &= \lim_{n\to\infty}\|x_{n+1} - q\| \\ &= \lim_{n\to\infty}\|(1 - a_n - c_n)S_1^n x_n + a_n T_1(PT_1)^{n-1}y_n + c_n u_n - q\| \\ &= \lim_{n\to\infty}\|(1 - a_n)[(S_1^n x_n - q) + c_n(u_n - S_1^n x_n)] \\ &\quad + a_n[(T_1(PT_1)^{n-1}y_n - q) + c_n(u_n - S_1^n x_n)]\| \end{aligned}$$

and Lemma 1.9 that

$$\lim_{n\to\infty}\|S_1^n x_n - T_1(PT_1)^{n-1}y_n\| = 0. \tag{2.11}$$

By condition (iv), it follows that

$$\|x_n - T_1(PT_1)^{n-1}y_n\| \quad \leq \quad \|S_1^n x_n - T_1(PT_1)^{n-1}y_n\|$$

and so, from (2.11), we have

$$\lim_{n\to\infty} \|x_n - T_1(PT_1)^{n-1}y_n\| = 0. \tag{2.12}$$

Since

$$\begin{aligned} \|x_n - q\| &\leq \|x_n - T_1(PT_1)^{n-1}y_n\| + \|T_1(PT_1)^{n-1}y_n - q\| \\ &\leq \|x_n - T_1(PT_1)^{n-1}y_n\| + h_n\|y_n - q\|, \end{aligned}$$

taking limit infimum on both sides in the above inequality, we have

$$\liminf_{n\to\infty} \|y_n - q\| \quad \geq \quad r_1$$

by (2.12) and so

$$\lim_{n\to\infty} \|y_n - q\| = r_1. \tag{2.13}$$

Now, note that

$$\begin{aligned} r_1 &= \lim_{n\to\infty} \|y_n - q\| \\ &= \lim_{n\to\infty} \|(1 - b_n - d_n)S_2^n x_n + b_n T_2(PT_2)^{n-1}x_n + d_n u_n - q\| \\ &= \lim_{n\to\infty} \|(1 - b_n)[(S_2^n x_n - q) + d_n(v_n - S_2^n x_n)] \\ &\quad + b_n[(T_2(PT_2)^{n-1}x_n - q) + d_n(v_n - S_2^n x_n)]\| \end{aligned}$$

and Lemma 1.9 that

$$\lim_{n\to\infty} \|S_2^n x_n - T_2(PT_2)^{n-1}x_n\| = 0. \tag{2.14}$$

By condition (iv), it follows that

$$\|x_n - T_2(PT_2)^{n-1}x_n\| \quad \leq \quad \|S_2^n x_n - T_2(PT_2)^{n-1}x_n\|$$

and so, from (2.14), we have

$$\lim_{n\to\infty} \|x_n - T_2(PT_2)^{n-1}x_n\| = 0. \tag{2.15}$$

Since $S_2^n x_n = P(S_2^n x_n)$ and $P\colon E \to K$ is a nonexpansive retraction of E onto K, we have

$$\begin{aligned} \|y_n - S_2^n x_n\| &= \|(1 - b_n - d_n)S_2^n x_n + b_n T_2(PT_2)^{n-1}x_n + d_n v_n - S_2^n x_n\| \\ &\leq b_n\|S_2^n x_n - T_2(PT_2)^{n-1}x_n\| + d_n\|v_n - S_2^n x_n\| \\ &\leq b_n\|S_2^n x_n - T_2(PT_2)^{n-1}x_n\| + d_n\|v_n - S_2^n x_n\| \end{aligned}$$

and so

$$\lim_{n\to\infty} \|y_n - S_2^n x_n\| = 0. \tag{2.16}$$

Again, we have

$$\begin{aligned}\|y_n - x_n\| &\leq \|y_n - S_2^n x_n\| + \|S_2^n x_n - T_2(PT_2)^{n-1}x_n\| \\ &\quad + \|T_2(PT_2)^{n-1}x_n - x_n\|.\end{aligned}$$

Thus, it follows from (2.14), (2.15) and (2.16) that

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{2.17}$$

Since $\|x_n - T_1(PT_1)^{n-1}y_n\| \leq \|S_1^n x_n - T_1(PT_1)^{n-1}y_n\|$ by condition (iv) and

$$\begin{aligned}\|S_1^n x_n - T_1(PT_1)^{n-1}x_n\| &\leq \|S_1^n x_n - T_1(PT_1)^{n-1}y_n\| \\ &\quad + \|T_1(PT_1)^{n-1}y_n - T_1(PT_1)^{n-1}x_n\| \\ &\leq \|S_1^n x_n - T_1(PT_1)^{n-1}y_n\| + h_n\|y_n - x_n\|,\end{aligned}$$

using (2.11), (2.17) and $h_n \to 1$ as $n \to \infty$, we have

$$\lim_{n\to\infty} \|S_1^n x_n - T_1(PT_1)^{n-1}x_n\| = 0 \tag{2.18}$$

and by condition (iv), we have

$$\lim_{n\to\infty} \|x_n - T_1(PT_1)^{n-1}x_n\| = 0. \tag{2.19}$$

It follows from

$$\begin{aligned}&\|x_{n+1} - S_1^n x_n\| \\ &= \|P((1 - a_n - c_n)S_1^n x_n + a_n T_1(PT_1)^{n-1}y_n + c_n u_n) - P(S_1^n x_n)\| \\ &\leq \|(1 - a_n - c_n)S_1^n x_n + a_n T_1(PT_1)^{n-1}y_n + c_n u_n - S_1^n x_n\| \\ &\leq a_n\|S_1^n x_n - T_1(PT_1)^{n-1}y_n\| + c_n\|u_n - S_1^n x_n\| \\ &\leq \mu\|S_1^n x_n - T_1(PT_1)^{n-1}y_n\| + c_n\|u_n - S_1^n x_n\|,\end{aligned}$$

(2.11) and $c_n \to 0$ as $n \to \infty$ that

$$\lim_{n\to\infty} \|x_{n+1} - S_1^n x_n\| = 0. \tag{2.20}$$

In addition, we have

$$\|x_{n+1} - T_1(PT_1)^{n-1}y_n\| \leq \|x_{n+1} - S_1^n x_n\| + \|S_1^n x_n - T_1(PT_1)^{n-1}y_n\|.$$

Using (2.11) and (2.20), we have

$$\lim_{n\to\infty} \|x_{n+1} - T_1(PT_1)^{n-1}y_n\| = 0. \tag{2.21}$$

Now, using (2.18), (2.19) and the inequality

$$\|S_1^n x_n - x_n\| \leq \|S_1^n x_n - T_1(PT_1)^{n-1}x_n\| + \|T_1(PT_1)^{n-1}x_n - x_n\|,$$

we have $\lim_{n\to\infty} \|S_1^n x_n - x_n\| = 0$. It follows from (2.15) and the inequality

$$\|S_1^n x_n - T_2(PT_2)^{n-1}x_n\| \leq \|S_1^n x_n - x_n\| + \|x_n - T_2(PT_2)^{n-1}x_n\|$$

that

$$\lim_{n\to\infty} \|S_1^n x_n - T_2(PT_2)^{n-1} x_n\| = 0. \tag{2.22}$$

Since

$$\|x_{n+1} - T_2(PT_2)^{n-1} y_n\| \le \|x_{n+1} - S_1^n x_n\| + \|S_1^n x_n - T_2(PT_2)^{n-1} x_n\| + h_n\|x_n - y_n\|,$$

from (2.17), (2.20), (2.22) and $h_n \to 1$ as $n \to \infty$, it follows that

$$\lim_{n\to\infty} \|x_{n+1} - T_2(PT_2)^{n-1} y_n\| = 0. \tag{2.23}$$

Since T_i for $i = 1, 2$ is uniformly continuous, P is nonexpansive retraction, it follows from (2.23) that

$$\begin{aligned}\|T_i(PT_i)^{n-1} y_{n-1} - T_i x_n\| &= \|T_i[(PT_i)(PT)^{n-2})y_{n-1}] - T_i(Px_n)\| \\ &\to\ 0\ \text{ as }\ n \to \infty,\end{aligned} \tag{2.24}$$

for $i = 1, 2$. Moreover, we have

$$\begin{aligned}\|x_{n+1} - y_n\| \le\ & \|x_{n+1} - T_1(PT_1)^{n-1} y_n\| \\ &+ \|T_1(PT_1)^{n-1} y_n - x_n\| + \|x_n - y_n\|.\end{aligned}$$

Using (2.12), (2.17) and (2.21), we have

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = 0. \tag{2.25}$$

In addition, we have

$$\begin{aligned}\|x_n - T_1 x_n\| &\le \|x_n - T_1(PT_1)^{n-1} x_n\| + \|T_1(PT_1)^{n-1} x_n - T_1(PT_1)^{n-1} y_{n-1}\| \\ &\quad + \|T_1(PT_1)^{n-1} y_{n-1} - T_1 x_n\| \\ &\le \|x_n - T_1(PT_1)^{n-1} x_n\| + h_n\|x_n - y_{n-1}\| \\ &\quad + \|T_1(PT_1)^{n-1} y_{n-1} - T_1 x_n\|.\end{aligned}$$

Thus, it follows from (2.19), (2.24), (2.25) and $h_n \to 1$ as $n \to \infty$, that

$$\lim_{n\to\infty} \|x_n - T_1 x_n\| = 0. \tag{2.26}$$

Similarly, we can prove that

$$\lim_{n\to\infty} \|x_n - T_2 x_n\| = 0. \tag{2.27}$$

Finally, by condition (iv), we have

$$\begin{aligned}\|x_n - S_1 x_n\| &\le \|x_n - T_1(PT_1)^{n-1} x_n\| + \|S_1 x_n - T_1(PT_1)^{n-1} x_n\| \\ &\le \|x_n - T_1(PT_1)^{n-1} x_n\| + \|S_1^n x_n - T_1(PT_1)^{n-1} x_n\|.\end{aligned}$$

Thus, it follows from (2.18) and (2.19) that

$$\lim_{n\to\infty} \|x_n - S_1 x_n\| = 0. \tag{2.28}$$

Similarly, we can prove that

$$\lim_{n\to\infty} \|x_n - S_2 x_n\| = 0. \tag{2.29}$$

This completes the proof. □

Theorem 2.3. *Let E be a real Banach space, K be a nonempty closed convex subset of E. Let S_1, $S_2\colon K \to K$ be two generalized asymptotically nonexpansive self mappings with sequence $\{k_n\} \subset [1,\infty)$ and T_1, $T_2\colon K \to E$ be two asymptotically nonexpansive non-self mappings with sequence $\{h_n\} \subset [1,\infty)$ and G_n be taken as in Lemma 2.1. Suppose that*

$$F := F(S_1)\bigcap F(S_2)\bigcap F(T_1)\bigcap F(T_2) \neq \emptyset,$$

and is closed. Let

$$\mu = \sup_n a_n,\ \ \omega_1 = \sup_n k_n \geq 1,\ \ \omega_2 = \sup_n h_n \geq 1$$

and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$ be four real sequences in $[0,1]$ which satisfy the following conditions:

(i) $\sum_{n=1}^{\infty}(k_n h_n - 1) < \infty$;
(ii) $\sum_{n=1}^{\infty} c_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$;
(iii) $\mu\omega_1\omega_2 < 1$ *or* $\mu < \frac{1}{\omega_1^2\omega_2^2}$.

Let $\{x_n\}$ be the sequence defined by (1.7). Then $\{x_n\}$ converges strongly to a common fixed point of S_1, S_2, T_1 and T_2 if and only if

$$\liminf_{n\to\infty} D(x_n, F) = 0,$$

where $D(x,F) = \inf\{\|x-p\| : p \in F\}$.

Proof. The necessity is obvious. Indeed, if $x_n \to q \in F$ as $n \to \infty$, then

$$D(x_n, F) = \inf_{q\in F} \|x_n - q\| \leq \|x_n - q\| \to 0 \quad (n \to \infty).$$

Thus $\liminf_{n\to\infty} D(x_n, F) = 0$.

Conversely, suppose that $\liminf_{n\to\infty} D(x_n, F) = 0$. By Lemma 2.1, we have that $\lim_{n\to\infty} D(x_n, F)$ exists. Further, by assumption $\liminf_{n\to\infty} D(x_n, F) = 0$, from (2.3) and Lemma 1.8, we conclude that $\lim_{n\to\infty} D(x_n, F) = 0$. Now, we show that $\{x_n\}$ is a Cauchy sequence in E. Indeed, from (2.4), we have

$$\|x_{n+1} - q\| \quad \leq \quad [1 + m_n]\|x_n - q\| + f_n$$

for each $n \geq 1$, where

$$m_n = (k_n^2 h_n^2 - 1)$$

and

$$f_n = a_n k_n h_n d_n \|v_n - q\| + c_n \|u_n - q\| + (a_n k_n h_n + 1)G_n$$

with $\sum_{n=1}^{\infty} m_n < \infty$, $\sum_{n=1}^{\infty} f_n < \infty$ and $q \in F$. For any m, n with $m > n \geq 1$, we have

$$\begin{aligned}
\|x_m - q\| &\leq [1 + m_{m-1}]\|x_{m-1} - q\| + f_{m-1} \\
&\leq e^{m_{m-1}}\|x_{m-1} - q\| + f_{m-1} \\
&\vdots \\
&\leq \left(e^{\sum_{i=n}^{m-1} m_i}\right)\|x_n - q\| + \left(e^{\sum_{i=n+1}^{m-1} m_i}\right)\sum_{i=n}^{m-1} f_i \\
&\leq \mathcal{Z}\|x_n - q\| + \mathcal{Z}\sum_{i=n}^{m-1} f_i
\end{aligned}$$

where $\mathcal{Z} = e^{\sum_{i=n}^{\infty} m_i}$. Thus for any $q \in F$, we have

$$\begin{aligned}
\|x_n - x_m\| &\leq \|x_n - q\| + \|x_m - q\| \\
&\leq \|x_n - q\| + \mathcal{Z}\|x_n - q\| + \mathcal{Z}\sum_{i=n}^{m-1} f_i \\
&\leq (\mathcal{Z} + 1)\|x_n - q\| + \mathcal{Z}\sum_{i=n}^{\infty} f_i.
\end{aligned}$$

Taking the infimum over all $q \in F$, we obtain

$$\|x_n - x_m\| \leq (\mathcal{Z} + 1)D(x_n, F) + \mathcal{Z}\sum_{i=n}^{\infty} f_i.$$

Thus it follows from $\lim_{n\to\infty} D(x_n, F) = 0$ and $f_n \to 0$ as $n \to \infty$ that $\{x_n\}$ is a Cauchy sequence in K. Since K is closed subset of E, the sequence $\{x_n\}$ converges strongly to some $y^* \in K$. Next, we show that $y^* \in F$. Now, $\lim_{n\to\infty} D(x_n, F) = 0$ gives that $D(y^*, F) = 0$. Since F is closed, $y^* \in F$. Thus y^* is a common fixed point of S_1, S_2, T_1 and T_2. This completes the proof. □

Theorem 2.4. *Let E be a real Banach space, K be a nonempty closed convex subset of E. Let S_1, $S_2 \colon K \to K$ be two generalized asymptotically nonexpansive self mappings with sequence $\{k_n\} \subset [1, \infty)$ and T_1, $T_2 \colon K \to E$ be two asymptotically nonexpansive non-self mappings with sequence $\{h_n\} \subset [1, \infty)$ and G_n be taken as in Lemma 2.1. Suppose that*

$$F := F(S_1) \bigcap F(S_2) \bigcap F(T_1) \bigcap F(T_2) \neq \emptyset.$$

Let

$$\mu = \sup_n a_n, \;\; \omega_1 = \sup_n k_n \geq 1, \;\; \omega_2 = \sup_n h_n \geq 1$$

and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$ *be four real sequences in* $[0,1]$ *which satisfy the following conditions:*

(i) $\sum_{n=1}^{\infty}(k_n h_n - 1) < \infty$;
(ii) $\sum_{n=1}^{\infty} c_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$;
(iii) $\mu\omega_1\omega_2 < 1$ *or* $\mu < \frac{1}{\omega_1^2\omega_2^2}$;
(iv) $\|x - T_1(PT_1)^{n-1}y\| \leq \|S_1^n x - T_1(PT_1)^{n-1}y\|$ *and* $\|x - T_i(PT_i)^{n-1}x\| \leq \|S_i^n x - T_i(PT_i)^{n-1}x\|$ *for all* $x, y \in K$ *and for* $i = 1, 2$.

Let $\{x_n\}$ *be the sequence defined by* (1.7). *If one of* S_1, S_2, T_1 *and* T_2 *is semi-compact, then* $\{x_n\}$ *converges strongly to a common fixed point of* S_1, S_2, T_1 *and* T_2.

Proof. By Lemma 2.2, we have that

$$\lim_{n\to\infty} \|x_n - S_i x_n\| = \lim_{n\to\infty} \|x_n - T_i x_n\| = 0$$

for $i = 1, 2$ and one of S_1, S_2, T_1 and T_2 is semi-compact, there exists a subsequence $\{x_{n_r}\}$ of $\{x_n\}$ such that $\{x_{n_r}\}$ converges strongly to some $f^* \in K$. Moreover, by the continuity of S_1, S_2, T_1 and T_2, we have

$$\|f^* - S_i f^*\| = \lim_{r\to\infty} \|x_{n_r} - S_i x_{n_r}\| = 0$$

and

$$\|f^* - T_i f^*\| = \lim_{r\to\infty} \|x_{n_r} - T_i x_{n_r}\| = 0$$

for $i = 1, 2$. Thus it follows that $f^* \in F = F(S_1) \bigcap F(S_2) \bigcap F(T_1) \bigcap F(T_2)$. Since $\lim_{n\to\infty} \|x_n - f^*\|$ exists by Lemma 2.1, we have $\lim_{n\to\infty} \|x_n - f^*\| = 0$. This shows that the sequence $\{x_n\}$ converges strongly to a common fixed point of S_1, S_2, T_1 and T_2. This completes the proof. □

Theorem 2.5. *Let* E *be a real Banach space,* K *be a nonempty closed convex subset of* E. *Let* $S_1, S_2 \colon K \to K$ *be two generalized asymptotically nonexpansive self mappings with sequence* $\{k_n\} \subset [1, \infty)$ *and* $T_1, T_2 \colon K \to E$ *be two asymptotically nonexpansive non-self mappings with sequences* $\{h_n\} \subset [1, \infty)$ *and* G_n *be taken as in Lemma* 2.1. *Suppose that*

$$F := F(S_1) \bigcap F(S_2) \bigcap F(T_1) \bigcap F(T_2) \neq \emptyset.$$

Let

$$\mu = \sup_n a_n, \;\; \omega_1 = \sup_n k_n \geq 1, \;\; \omega_2 = \sup_n h_n \geq 1$$

and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$ *be four real sequences in* $[0,1]$ *which satisfy the following conditions:*

(i) $\sum_{n=1}^{\infty}(k_n h_n - 1) < \infty$;
(ii) $\sum_{n=1}^{\infty} c_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$;
(iii) $\mu\omega_1\omega_2 < 1$ *or* $\mu < \frac{1}{\omega_1^2\omega_2^2}$;
(iv)

$$\|x - T_1(PT_1)^{n-1}y\| \leq \|S_1^n x - T_1(PT_1)^{n-1}y\|$$

and

$$\|x - T_i(PT_i)^{n-1}x\| \leq \|S_i^n x - T_i(PT_i)^{n-1}x\|$$

for all $x, y \in K$ *and for* $i = 1, 2$.

Let $\{x_n\}$ *be the sequence defined by* (1.7). *If* S_1, S_2, T_1 *and* T_2 *satisfy the following conditions:*

(C_1) $\lim_{n\to\infty} \|x_n - S_i x_n\| = \lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ *for* $i = 1, 2$;
(C_2) *there exists a continuous function* $\varphi\colon [0,\infty) \to [0,\infty)$ *with* $\varphi(0) = 0$ *and* $\varphi(t) > 0$ *for all* $t \in (0,\infty)$ *such that*

$$\begin{aligned}\varphi(D(x,F)) \quad &\leq \quad a_1\|x - S_1x\| + a_2\|x - S_2x\| \\ &\quad\; + a_3\|x - T_1x\| + a_4\|x - T_2x\|\end{aligned}$$

for all $x \in K$, *and* a_1, a_2, a_3, a_4 *are nonnegative real numbers such that* $a_1 + a_2 + a_3 + a_4 = 1$.

Then the sequence $\{x_n\}$ *converges strongly to a common fixed point of* S_1, S_2, T_1 *and* T_2.

Proof. It follows from the hypothesis that

$$\begin{aligned}\lim_{n\to\infty} \varphi(D(x_n, F)) &\leq a_1.\|x_n - S_1x_n\| + a_2.\|x_n - S_2x_n\| \\ &\quad + a_3.\|x_n - T_1x_n\| + a_4.\|x_n - T_2x_n\| \\ &= 0,\end{aligned}$$

that is,

$$\lim_{n\to\infty} \varphi(D(x_n, F)) = 0.$$

Since $\varphi\colon [0,\infty) \to [0,\infty)$ is a continuous function and $\varphi(0) = 0$, therefore we have

$$\lim_{n\to\infty} D(x_n, F) = 0.$$

Therefore, Theorem 2.3 implies that $\{x_n\}$ must converges strongly to a common fixed point of S_1, S_2, T_1 and T_2. This completes the proof. $\square$

3. Weak convergence theorems

In this section, we prove some weak convergence theorems of iteration scheme (1.7) for two generalized asymptotically nonexpansive self mappings and two asymptotically nonexpansive non-self mappings in real uniformly convex Banach spaces. First, we shall need the following lemmas.

Lemma 3.1. *Under the assumptions of Lemma* 2.1*, for all* $q_1, q_2 \in F = F(S_1) \cap F(S_2) \cap F(T_1) \cap F(T_2)$*, the limit*

$$\lim_{n\to\infty} \|tx_n + (1-t)q_1 - q_2\|$$

exists for all $t \in [0,1]$*, where* $\{x_n\}$ *is the sequence defined by* (1.7).

Proof. By Lemma 2.1, $\lim_{n\to\infty} \|x_n - z\|$ exists for all $z \in F$ and therefore $\{x_n\}$ is bounded. Letting

$$a_n(t) = \|tx_n + (1-t)q_1 - q_2\|$$

for all $t \in [0,1]$. Then $\lim_{n\to\infty} a_n(0) = \|q_1 - q_2\|$ and $\lim_{n\to\infty} a_n(1) = \lim \|x_n - q_2\|$ exists by Lemma 2.1. Now it remains to prove the Lemma 3.1 for $t \in (0,1)$. For all $x \in K$, we define the mapping $\mathcal{R}_n \colon K \to K$ by:

$$\mathcal{A}_n(x) \quad = \quad P((1-b_n-d_n)S_2^n x + b_n T_2(PT_2)^{n-1}x + d_n v_n)$$

and

$$\mathcal{R}_n(x) \quad = \quad P((1-a_n-c_n)S_1^n x + a_n T_1(PT_1)^{n-1}\mathcal{A}_n(x) + c_n u_n).$$

Then it follows that $x_{n+1} = \mathcal{R}_n x_n$, $\mathcal{R}_n p = p$ for all $p \in F$. Now from (2.2) and (2.4) of Lemma 2.1, we see that

$$\|\mathcal{A}_n(x) - \mathcal{A}_n(y)\| \quad \le \quad k_n h_n \|x-y\|$$

and

$$\begin{aligned} \|\mathcal{R}_n(x) - \mathcal{R}_n(y)\| &\le [1+m_n]\|x-y\| + f_n \\ &= \mathcal{J}_n\|x-y\| + f_n, \end{aligned} \tag{3.1}$$

where

$$m_n = (k_n^2 h_n^2 - 1)$$

and

$$f_n = a_n k_n h_n d_n \|v_n - q\| + c_n\|u_n - q\| + (a_n k_n h_n + 1)G_n$$

with $\sum_{n=1}^{\infty} m_n < \infty$, $\sum_{n=1}^{\infty} f_n < \infty$, $\mathcal{J}_n = 1 + m_n$ and $\mathcal{J}_n \to 1$ as $n \to \infty$. Setting

$$S_{n,m} = \mathcal{R}_{n+m-1}\mathcal{R}_{n+m-2}\cdots\mathcal{R}_n, \ m \ge 1 \tag{3.2}$$

and

$$b_{n,m} \quad = \quad \|S_{n,m}(tx_n + (1-t)q_1) - (tS_{n,m}x_n + (1-t)S_{n,m}q_2)\|.$$

From (3.1) and (3.2), we have

$$\begin{aligned}
&\|S_{n,m}(x) - S_{n,m}(y)\| \\
&= \|\mathcal{R}_{n+m-1}\mathcal{R}_{n+m-2}\dots\mathcal{R}_n(x) - \mathcal{R}_{n+m-1}\mathcal{R}_{n+m-2}\dots\mathcal{R}_n(y)\| \\
&\le \mathcal{J}_{n+m-1}\|\mathcal{R}_{n+m-2}\dots\mathcal{R}_n(x) - \mathcal{R}_{n+m-2}\dots\mathcal{R}_n(y)\| + f_{n+m-1} \\
&\le \mathcal{J}_{n+m-1}\mathcal{J}_{n+m-2}\|\mathcal{R}_{n+m-3}\dots\mathcal{R}_n(x) - \mathcal{R}_{n+m-3}\dots\mathcal{R}_n(y)\| \\
&\quad + f_{n+m-1} + f_{n+m-2} \\
&\quad \vdots \\
&\le \Big(\prod_{i=n}^{n+m-1}\mathcal{J}_i\Big)\|x-y\| + \sum_{i=n}^{n+m-1} f_i \\
&= \mathcal{H}_n\|x-y\| + \sum_{i=n}^{n+m-1} f_i \qquad (3.3)
\end{aligned}$$

for all $x, y \in K$, where $\mathcal{H}_n = \prod_{i=n}^{n+m-1}\mathcal{J}_i$ and $S_{n,m}x_n = x_{n+m}$ and $S_{n,m}p = p$ for all $p \in F$. Thus

$$\begin{aligned}
a_{n+m}(t) &= \|tx_{n+m} + (1-t)q_1 - q_2\| \\
&\le b_{n,m} + \|S_{n,m}(tx_n + (1-t)q_1) - q_2\| \\
&\le b_{n,m} + \mathcal{H}_n a_n(t) + \sum_{i=n}^{n+m-1} f_i. \qquad (3.4)
\end{aligned}$$

By using Theorem 2.3 in [5], we have

$$\begin{aligned}
b_{n,m} &\le \phi^{-1}(\|x_n - u\| - \|S_{n,m}x_n - S_{n,m}u\|) \\
&\le \phi^{-1}(\|x_n - u\| - \|x_{n+m} - u + u - S_{n,m}u\|) \\
&\le \phi^{-1}(\|x_n - u\| - (\|x_{n+m} - u\| - \|S_{n,m}u - u\|))
\end{aligned}$$

and so the sequence $\{b_{n,m}\}$ converges uniformly to 0, *i.e.*, $b_{n,m} \to 0$ as $n \to \infty$. Since $\lim_{n\to\infty}\mathcal{H}_n = 1$ and $\lim_{n\to\infty} f_n = 0$, from (3.4), we have

$$\limsup_{n\to\infty} a_n(t) \le \lim_{n,m\to\infty} b_{n,m} + \liminf_{n\to\infty} a_n(t) = \liminf_{n\to\infty} a_n(t).$$

This shows that $\lim_{n\to\infty} a_n(t)$ exists, that is,

$$\lim_{n\to\infty}\|tx_n + (1-t)q_1 - q_2\|$$

exists for all $t \in [0,1]$. This completes the proof. □

Lemma 3.2. *Under the assumptions of Lemma* 2.1, *if* E *has a Frěchet differentiable norm, then for all* $q_1,\, q_2 \in F = F(S_1) \cap F(S_2) \cap F(T_1) \cap F(T_2)$, *the limit*

$$\lim_{n\to\infty} \langle x_n, J(q_1 - q_2)\rangle$$

exists, where $\{x_n\}$ *is the sequence defined by* (1.7). *If* $W_w(\{x_n\})$ *denotes the set of all weak subsequential limits of* $\{x_n\}$, *then* $\langle u_1 - u_2, J(q_1 - q_2)\rangle = 0$ *for all* $q_1,\, q_2 \in F$ *and* $u_1,\, u_2 \in W_w(\{x_n\})$.

Proof. Suppose that $x = q_1 - q_2$ with $q_1 \neq q_2$ and $h = t(x_n - q_1)$ in inequality (1.8). Then, we get

$$\begin{aligned} & t\,\langle x_n - q_1,\, J(q_1 - q_2)\rangle + \frac{1}{2}\|q_1 - q_2\|^2 \\ & \leq \frac{1}{2}\|tx_n + (1-t)q_1 - q_2\|^2 \\ & \leq t\,\langle x_n - q_1,\, J(q_1 - q_2)\rangle + \frac{1}{2}\|q_1 - q_2\|^2 + b(t\|x_n - q_1\|). \end{aligned}$$

Since $\sup_{n\geq 1}\|x_n - q_1\| \leq \mathcal{M}$ for some $\mathcal{M} > 0$, we have

$$\begin{aligned} & t\,\limsup_{n\to\infty}\langle x_n - q_1,\, J(q_1 - q_2)\rangle + \frac{1}{2}\|q_1 - q_2\|^2 \\ & \leq \frac{1}{2}\lim_{n\to\infty}\|tx_n + (1-t)q_1 - q_2\|^2 \\ & \leq t\,\liminf_{n\to\infty}\langle x_n - q_1,\, J(q_1 - q_2)\rangle + \frac{1}{2}\|q_1 - q_2\|^2 + b(t\mathcal{M}), \end{aligned}$$

that is,

$$\limsup_{n\to\infty}\langle x_n - q_1,\, J(q_1 - q_2)\rangle \leq \liminf_{n\to\infty}\langle x_n - q_1,\, J(q_1 - q_2)\rangle + \frac{b(t\mathcal{M})}{t\mathcal{M}}\mathcal{M}.$$

If $t \to 0$, then $\lim_{n\to\infty}\langle x_n - q_1,\, J(q_1 - q_2)\rangle$ exists for all $q_1,\, q_2 \in F$; in particular, we have $\langle u_1 - u_2,\, J(q_1 - q_2)\rangle = 0$ for all $u_1,\, u_2 \in W_w(\{x_n\})$. This completes the proof. $\square$

Theorem 3.3. *Under the assumptions of Lemma* 2.2, *if* E *has Frěchet differentiable norm, then the sequence* $\{x_n\}$ *defined by* (1.7) *converges weakly to a common fixed point of* S_1, S_2, T_1 *and* T_2.

Proof. By Lemma 3.2, $\langle u_1 - u_2,\, J(q_1 - q_2)\rangle = 0$ for all $u_1,\, u_2 \in W_w(\{x_n\})$. Therefore $\|t^* - s^*\|^2 = \langle t^* - s^*,\, J(t^* - s^*)\rangle = 0$ implies $t^* = s^*$. Consequently, $\{x_n\}$ converges weakly to a common fixed point in $F = F(S_1) \cap F(S_2) \cap F(T_1) \cap F(T_2)$. This completes the proof. $\square$

Theorem 3.4. *Under the assumptions of Lemma* 2.2*, if the dual space* E^* *of* E *has the Kadec-Klee (KK) property and the mappings* $I - S_i$ *and* $I - T_i$ *for* $i = 1, 2$*, where* I *denotes the identity mapping, are demiclosed at zero, then the sequence* $\{x_n\}$ *defined by* (1.7) *converges weakly to a common fixed point of* S_1*,* S_2*,* T_1 *and* T_2*.*

Proof. By Lemma 2.1, $\{x_n\}$ is bounded and since E is reflexive, there exists a subsequence $\{x_{n_r}\}$ of $\{x_n\}$ which converges weakly to some $u^* \in K$. By Lemma 2.2, we have

$$\lim_{r\to\infty} \|x_{n_r} - S_i x_{n_r}\| = 0 \quad \text{and} \quad \lim_{r\to\infty} \|x_{n_r} - T_i x_{n_r}\| = 0$$

for $i = 1, 2$. Since by hypothesis the mappings $I - S_i$ and $I - T_i$ for $i = 1, 2$ are demiclosed at zero, therefore $S_i u^* = u^*$ and $T_i u^* = u^*$ for $i = 1, 2$, which means $u^* \in F = F(S_1) \cap F(S_2) \cap F(T_1) \cap F(T_2)$. Now, we show that $\{x_n\}$ converges weakly to u^*. Suppose $\{x_{n_s}\}$ is another subsequence of $\{x_n\}$ converges weakly to some $v^* \in K$. By the same method as above, we have $v^* \in F$ and u^*, $v^* \in W_w(\{x_n\})$. By Lemma 3.1, the limit

$$\lim_{n\to\infty} \|t x_n + (1-t)u^* - v^*\|$$

exists for all $t \in [0, 1]$ and so $u^* = v^*$ by Lemma 1.10. Thus, the sequence $\{x_n\}$ converges weakly to $u^* \in F$. This completes the proof. □

Theorem 3.5. *Under the assumptions of Lemma* 2.2*, if* E *satisfies Opial's condition and the mappings* $I - S_i$ *and* $I - T_i$ *for* $i = 1, 2$*, where* I *denotes the identity mapping, are demiclosed at zero, then the sequence* $\{x_n\}$ *defined by* (1.7) *converges weakly to a common fixed point of* S_1*,* S_2*,* T_1 *and* T_2*.*

Proof. Let $b_* \in F$, from Lemma 2.1 the sequence $\{\|x_n - b_*\|\}$ is convergent and hence bounded. Since E is uniformly convex, every bounded subset of E is weakly compact. Thus there exists a subsequence $\{x_{n_r}\} \subset \{x_n\}$ such that $\{x_{n_r}\}$ converges weakly to $y^* \in K$. From Lemma 2.2, we have

$$\lim_{r\to\infty} \|x_{n_r} - S_i x_{n_r}\| = 0 \quad \text{and} \quad \lim_{r\to\infty} \|x_{n_r} - T_i x_{n_r}\| = 0$$

for $i = 1, 2$. Since the mappings $I - S_i$ and $I - T_i$ for $i = 1, 2$ are demiclosed at zero, $S_i y^* = y^*$ and $T_i y^* = y^*$ for $i = 1, 2$, which means $y^* \in F$. Finally, let us prove that $\{x_n\}$ converges weakly to y^*. Suppose on contrary that there is a subsequence $\{x_{n_s}\} \subset \{x_n\}$ such that $\{x_{n_s}\}$ converges weakly to $z^* \in K$ and $y^* \neq z^*$. Then by the same method as given above, we can also prove that $z^* \in F$. From Lemma 2.1 the limits $\lim_{n\to\infty} \|x_n - y^*\|$ and $\lim_{n\to\infty} \|x_n - z^*\|$

exist. By virtue of the Opial's condition of E, we obtain

$$\begin{aligned} \lim_{n\to\infty} \|x_n - y^*\| &= \lim_{n_r\to\infty} \|x_{n_r} - y^*\| \\ &< \lim_{n_r\to\infty} \|x_{n_r} - z^*\| = \lim_{n\to\infty} \|x_n - z^*\| = \lim_{n_s\to\infty} \|x_{n_s} - z^*\| \\ &< \lim_{n_s\to\infty} \|x_{n_s} - y^*\| \\ &= \lim_{n\to\infty} \|x_n - y^*\|, \end{aligned}$$

which is a contradiction, so $y^* = z^*$. Thus $\{x_n\}$ converges weakly to a common fixed point of S_1, S_2, T_1 and T_2. This completes the proof. □

Example 3.6. Let $\mathbb{R}$ be the real line with the usual norm $|.|$ and let $K = [-1, 1]$. Define two mappings $S, T\colon K \to K$ by

$$T(x) = \begin{cases} -2\sin\frac{x}{2}, & \text{if } x \in [0, 1], \\ 2\sin\frac{x}{2}, & \text{if } x \in [-1, 0). \end{cases}$$

and

$$S(x) = \begin{cases} x, & \text{if } x \in [0, 1], \\ -x, & \text{if } x \in [-1, 0). \end{cases}$$

Then both S and T are asymptotically nonexpansive mappings with constant sequence $\{k_n\} = \{1\}$ for all $n \geq 1$ and uniformly L-Lipschtzian mappings with $L = \sup_{n\geq 1}\{k_n\}$. Also the unique common fixed point of S and T, that is, $F = F(S) \cap F(T) = \{0\}$.

4. Conclusion

In this paper, we study hybrid mixed type iteration scheme for two generalized asymptotically nonexpansive self mappings and two asymptotically nonexpansive non-self mappings and establish some strong convergence theorems using semi-compactness condition and the condition which is weaker than that of compactness and some weak convergence theorems using the following conditions: (c_1) the space E has a Frĕchet differentiable norm (c_2) dual space E^* of E has the Kadec-Klee (KK) property (c_3) the space E satisfies Opial's condition. Our results extend and generalize the corresponding results of [3, 4, 7, 8, 10, 11], [14]-[17] and many others from the existing literature to the case of more general class of mappings and newly proposed hybrid mixed type iteration scheme considered in this paper.

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