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THE CONVERGENCE THEOREMS OF FIXED POINTS FOR GENERALIZED φ-WEAKLY CONTRACTIVE MAPPINGS

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Abstract. In this paper, we generalize results of Corollary 2.2 of Zhang and Song[Fixed point theory for generalized φ-weak contractions, Appl. Math. Lett. 22(2009)75-78].

1. Introduction

Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be weak contraction if there exists a continuous and nondecreasing function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$ such that

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)) \tag{1.1}$$

for all $x, y \in X$.

The above concept was introduced by Alber and Guerre-Delabriere [1] in 1997, and they showed the existence of fixed points for the weak contractive mappings on Hilbert spaces. In fact, Banach contraction principle is a special case of weak contraction by taking $\varphi(t) = (1-k)t$ for $0 \leq k < 1$. Therefore, it is very meaningful to study the existence of fixed points for the weak contraction in the general spaces. To this end, Rhoades [2], in 2001, obtained the most generalized interesting fixed point theorem.

Theorem 1.1. [2] *Let (X, d) be a complete metric space and let T be a weak contraction on X. Then T has a unique fixed point.*

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Indeed, Alber and Guerre-Delabriere [1] assumed an additional condition on φ which is $\lim_{t\to\infty}\varphi(t)=\infty$. But we noted the result of Rhoades [2] without using this assumption. Furthermore, according to the definition of weak contraction, it is not difficult to find that if T is a weak contraction, then T satisfies the following inequality

$$d(Tx,Ty)\le d(x,y)-\varphi(d(Tx,Ty)) \tag{1.2}$$

for all $x,y\in X$. The converse is not true in general. See example as follows.

Example 1.2. Let $X=A\cup B$ with $A=\{0,\frac{1}{2},\frac{1}{3},\cdots,\frac{1}{n},\cdots\}$ and $B=\{2,3,4,\cdots\}$. Define a metric by

$$d(x,y)=\begin{cases}\max\{x,y\}, & x\neq y\\ 0, & x=y.\end{cases}$$

Let $T:X\to X$ be a mapping by

$$Tx=\begin{cases}x^3, & x\in A\\ \frac{1}{x}, & x\in B.\end{cases}$$

Assumed that $\varphi(t):[0,+\infty)\to[0,+\infty)$ is defined by $\varphi(t)=\frac{6t}{7}$. Then $Fix(T)=\{0\}$ and T satisfies (1.2). However the mapping T does not satisfy (1.1).

In order to prove that these facts hold, we consider the following possible cases(without loss of generality, let $x<y$).

Case 1. Let $x,y\in A$ with $x<y$. Then we have

$$d(Tx,Ty)=d(x^3,y^3)=y^3<y-\frac{6y^3}{7}=d(x,y)-\varphi(d(Tx,Ty)).$$

Case 2-1. Let $x\in A$, $y\in B$ with $Tx<Ty$. Then we have

$$d(Tx,Ty)=d(x^3,\frac{1}{y})=\frac{1}{y}\le y-\frac{6}{7y}=d(x,y)-\varphi(d(Tx,Ty)).$$

Case 2-2. Let $x\in A$, $y\in B$ with $Tx\ge Ty$. Then we have

$$d(Tx,Ty)=d(x^3,\frac{1}{y})=x^3\le y-\frac{6x^3}{7}=d(x,y)-\varphi(d(Tx,Ty)).$$

Case 3. Let $x,y\in B$ with $Tx\ge Ty$. Then we have

$$d(Tx,Ty)=d(\frac{1}{x},\frac{1}{y})=\frac{1}{x}\le y-\frac{6}{7x}=d(x,y)-\varphi(d(Tx,Ty)).$$

On the other hand, we choose $x=2$ and $y=3$, then

$$d(T2,T3)=d(\frac{1}{2},\frac{1}{3})=\frac{1}{2}>3-\frac{18}{7}=d(2,3)-\varphi(d(2,3)).$$

Thus it explains that T is not a weak contraction.

For the sake of convenience, we define that the new mapping (1.2) is called φ-weak contraction [3]. From above example, we easily obtain the following conclusions:

$$\{contractions\} \subset \{weak\ contractions\} \subset \{\varphi - weak\ contractions\}.$$

Recently, weak contractions have been widely generalized by many authors(see [2],[4-5]). Meanwhile, the result of Zhang and Song [4] is much attention.

Theorem 1.3. [4, *Corollary* 2.2] *Let (X,d) be a complete metric space and $T : X \to X$ be a self-map such that for all $x, y \in X$,*

$$d(Tx,Ty) \le M(x,y) - \varphi(M(x,y)), \tag{1.3}$$

where

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(Ty,y), \frac{1}{2}[d(x,Ty) + d(y,Tx)]\},$$

$\varphi : [0,+\infty) \to [0,+\infty)$ is a lower semi-continuous function with $\varphi(0) = 0$ and $\varphi(t) > 0$ for $t > 0$. Then there exists a unique fixed point of T in X.

In this paper our aim is to extend Theorem 1.3 to the other class of φ-weakly contractive mappings.

2. Main Results

Theorem 2.1. *Let (X,d) be a complete metric space and $T : X \to X$ be a self-map such that*

$$d(Tx,Ty) \le N(x,y) - \varphi(N(x,y)) \tag{2.1}$$

for all $x, y \in X$, where $\varphi : [0,+\infty) \to [0,+\infty)$ is a lower semi-continuous function with $\varphi(0) = 0$ and $\varphi(t) > 0$ for $t > 0$,

$$N(x,y) = ad(x,y) + bd(x,Tx) + cd(Ty,y) + e[d(x,Ty) + d(y,Tx)]$$

with $a, b, c, e \ge 0$ and $a + b + c + 2e \le 1$. Then there exists a unique fixed point of T.

Proof. For given $x_0 \in X$, let $\{x_n\}$ be defined by $x_{n+1} = Tx_n$ with $x_{n+1} \ne x_n$ for any n. Then we have

$$\begin{aligned} d(x_{n+1},x_n) &= d(Tx_n,Tx_{n-1}) \\ &\le N(x_n,x_{n-1}) - \varphi(N(x_n,x_{n-1})) \\ &\le N(x_n,x_{n-1}) \\ &= ad(x_n,x_{n-1}) + bd(x_n,x_{n+1}) + cd(x_n,x_{n-1}) \\ &\quad + e[d(x_n,x_n) + d(x_{n+1},x_{n-1})] \\ &\le (a+c+e)d(x_n,x_{n-1}) + (b+e)d(x_n,x_{n+1}). \end{aligned} \tag{2.2}$$

If $b+e=1$, then $e=0, b=1$, that is, $a=c=e=0$. From (2.1), we get

$$\begin{aligned} d(x_{n+1},x_n) &= d(Tx_n,Tx_{n-1}) \\ &\le d(x_n,x_{n+1})-\varphi(d(x_n,x_{n+1})), \end{aligned} \tag{2.3}$$

which is a contradiction, so $b+e<1$. Thus (2.2) implies that

$$d(x_{n+1},x_n) \le \tfrac{a+c+e}{1-b-e} d(x_n,x_{n-1}) \le d(x_n,x_{n-1}) \tag{2.4}$$

and

$$d(x_{n+1},x_n) \le N(x_n,x_{n-1}) \le d(x_n,x_{n-1}). \tag{2.5}$$

Then the sequence $\{d(x_{n+1},x_n)\}$ is monotone decreasing and bounded below, this implies that there exists $r \ge 0$ such that

$$\lim_{n\to\infty} d(x_{n+1},x_n) = \lim_{n\to\infty} N(x_n,x_{n-1}) = r. \tag{2.6}$$

Since φ is lower semi-continuous, we have

$$\varphi(r) \le \liminf_{n\to\infty} \varphi(N(x_n,x_{n-1})). \tag{2.7}$$

It follows from (2.1) that

$$\liminf_{n\to\infty} \varphi(N(x_n,x_{n-1})) + \lim_{n\to\infty} d(x_{n+1},x_n) \le \lim_{n\to\infty} N(x_n,x_{n-1}).$$

This means that $\varphi(r) \le 0$, i.e., $r=0$. Hence $\lim_{n\to\infty} d(x_{n+1},x_n)=0$.

Now we prove that $\{x_n\}$ is a Cauchy sequence. Suppose that it is not true, there exists $\epsilon>0$ for any k we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ such that $n(k)$ is the smallest index for which $n(k)>m(k)>k$ and $d(x_{m(k)},x_{n(k)}) \ge \epsilon$. This implies that $d(x_{m(k)},x_{n(k)-1})<\epsilon$ for all $k \ge 1$. Using the triangle inequality, we have

$$\begin{aligned} \epsilon &\le d(x_{m(k)},x_{n(k)}) \\ &\le d(x_{m(k)},x_{n(k)-1}) + d(x_{n(k)-1},x_{n(k)}) \\ &\le \epsilon + d(x_{n(k)-1},x_{n(k)}). \end{aligned} \tag{2.8}$$

Letting $k\to\infty$ in (2.8), we obtain

$$\lim_{k\to\infty} d(x_{m(k)},x_{n(k)}) = \epsilon.$$

Since

$$\begin{aligned} d(x_{m(k)},x_{n(k)}) - d(x_{n(k)-1},x_{n(k)}) &\le d(x_{m(k)},x_{n(k)-1}) \\ &\le d(x_{m(k)},x_{n(k)}) + d(x_{n(k)},x_{n(k)-1}). \end{aligned} \tag{2.9}$$

we have

$$\lim_{k\to\infty} d(x_{m(k)},x_{n(k)-1}) = \epsilon.$$

Similarly, we can show that

$$\begin{aligned}\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)}) &= \lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)-1})\\ &= \lim_{k\to\infty} d(x_{m(k)+1}, x_{n(k)})\\ &= \lim_{k\to\infty} d(x_{m(k)}, x_{n(k)+1})\\ &= \epsilon.\end{aligned}$$

Then we get

$$\lim_{k\to\infty} N(x_{m(k)-1}, x_{n(k)-1}) = \epsilon.$$

In fact

$$\begin{aligned}\epsilon \ &\leq d(x_{m(k)}, x_{n(k)})\\ &\leq d(Tx_{m(k)-1}, Tx_{n(k)-1})\\ &\leq N(x_{m(k)-1}, x_{n(k)-1}) - \varphi(N(x_{m(k)-1}, x_{n(k)-1}))\\ &\leq N(x_{m(k)-1}, x_{n(k)-1})\\ &= ad(x_{m(k)-1}, x_{n(k)-1}) + bd(x_{m(k)}, x_{m(k)-1}) + cd(x_{n(k)}, x_{n(k)-1})\\ &\quad +e[d(x_{n(k)-1}, x_{m(k)}) + d(x_{m(k)-1}, x_{n(k)})]\\ &\to (a+b+c+2e)\epsilon = \epsilon\end{aligned} \tag{2.10}$$

as $k \to \infty$. It leads to

$$\lim_{k\to\infty} N(x_{m(k)-1}, x_{n(k)-1}) = \epsilon.$$

On the other hand, it follows from (2.10) that

$$\varphi(N(x_{m(k)-1}, x_{n(k)-1})) + d(x_{m(k)}, x_{n(k)}) \leq N(x_{m(k)-1}, x_{n(k)-1}). \tag{2.11}$$

Taking lower limits as $k \to \infty$ on (2.10)

$$\varphi(\epsilon) + \epsilon \leq \epsilon,$$

which is a contradiction. This shows that $\{x_n\}$ is a Cauchy sequence and hence it is convergent in X. Let $\lim_{n\to\infty} x_n = q$.

Now let us show that q is the fixed point of T. If $q \neq Tq$, then for $d(q, Tq) > 0$, there exists N such that

$$d(x_n, q) < \frac{1}{2}d(q, Tq),\ d(x_{n+1}, x_n) < \frac{1}{2}d(q, Tq),$$

for any $n > N$. Then we have

$$\begin{aligned}
d(Tq,q) - d(x_{n+1},q) &\le d(Tq,x_{n+1}) \\
&= d(Tq,Tx_n) \\
&\le N(q,x_n) - \varphi(N(q,x_n)) \\
&\le N(q,x_n) \\
&= ad(q,x_n) + bd(q,Tq) + cd(x_{n+1},x_n) \\
&\quad +e[d(q,x_{n+1}) + d(x_n,Tq)] \\
&\le \tfrac{1}{2}ad(q,Tq) + bd(q,Tq) + \tfrac{1}{2}cd(q,Tq) \\
&\quad +e[\tfrac{1}{2}d(q,Tq) + \tfrac{1}{2}d(q,Tq) + d(q,Tq)]\} \\
&\le (a+b+c+2e)d(q,Tq) \\
&\le d(q,Tq).
\end{aligned} \tag{2.12}$$

It implies that

$$\lim_{n\to\infty} N(q,x_n) = d(Tq,q).$$

Again using above inequality (2.12)

$$d(Tq,x_{n+1}) + \varphi(N(q,x_n)) \le N(q,x_n). \tag{2.13}$$

Taking lower limit in (2.12), we have

$$d(Tq,q) + \varphi(d(Tq,q)) \le d(Tq,q),$$

which is a contradiction and so $q = Tq$.

For the uniqueness of fixed point. If it is false, there exists $p \neq q$ such that $p = Tp$. Then

$$\begin{aligned}
d(p,q) &= d(Tp,Tq) \\
&\le N(p,q) - \varphi(N(p,q)) \\
&\le ad(p,q) + bd(p,Tp) + cd(q,Tq) + e[d(p,Tq) + d(q,Tp)] \\
&\le (a+2e)d(p,q) \\
&\le d(p,q),
\end{aligned} \tag{2.14}$$

it implies that $N(p,q) = d(p,q)$. It follows from (2.14) that

$$\begin{aligned}
d(p,q) &= d(Tp,Tq) \\
&\le N(p,q) - \varphi(N(p,q)) \\
&\le d(p,q) - \varphi(d(p,q)),
\end{aligned} \tag{2.15}$$

which is a contradiction. Hence $q = p$. The proof is completed. □

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