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OSCILLATORY BEHAVIOR OF SECOND ORDER NONLINEAR DAMPED NEUTRAL DIFFERENCE EQUATION

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Abstract. This paper deals with oscillatory properties of solutions of a class of second order nonlinear damped neutral difference equation of the form:

$$\Delta(a_n(\Delta z_n)^\alpha) + p_n(\Delta z_n)^\alpha + q_n f(x_{n-l}) = 0,\ n \geq n_0. \tag{E}$$

where $z_n = x_n - b_n x_{n-k}$. Some new sufficient conditions are obtained which ensure that all solutions of equation (E) are oscillatory. The results presented in this paper extend and improve some of the related results reported in the literature. Examples are provided to illustrate the importance of the main results.

1. Introduction

Consider the second order neutral difference equation with damping term of the form:

$$\Delta(a_n(\Delta z_n)^\alpha) + p_n(\Delta z_n)^\alpha + q_n f(x_{n-l}) = 0,\ n \geq n_0, \tag{1.1}$$

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where n_0 is a nonnegative integer, α is a ratio of odd positive integers, and $z_n = x_n - b_n x_{n-k}$.

Throughout this paper and without further mention, we assume that:

- (H_1) $\{a_n\}$ is a positive real sequence, and $\{b_n\}$ is a real sequence with $0 \leq b_n \leq b < 1$ for all $n \geq n_0$;
- (H_2) $\{p_n\}$ is a real sequence, and $\{q_n\}$ is a positive real sequence for all $n \geq n_0$;
- (H_3) l and k are positive integers;
- (H_4) $f : \mathbb{R} \to \mathbb{R}$ is a continuous function with $uf(u) > 0$ for $u \neq 0$, and there exists a positive constant M such that $\frac{f(u)}{u^\beta} \geq M$ for all $u \neq 0$, where β is the ratio of odd positive integers.

Let $\theta = \max\{k, l\}$. By a solution of equation (1.1), we mean a real sequence $\{x_n\}$ defined for all $n \geq n_0 - \theta$, and satisfying equation (1.1) for all $n \geq n_0$.

A nontrivial solution of equation (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, and it is nonoscillatory otherwise.

From the review of literature, it is known that there are many results available on the oscillatory behavior of solutions of equation (1.1) when $p_n \equiv 0$ and $b_n \leq 0$ for $n \geq n_0$, see for example [1, 2, 3, 14, 15] and the references cited therein. However few results available on the oscillatory behavior of equation (1.1) when $p_n \equiv 0$ and $b_n \geq 0$ for $n \geq n_0$, see for example [1, 5, 6, 7, 12, 15].

In [7], the authors considered equation (1.1) with $p_n \equiv 0$ and $\alpha = \beta$, and established some new conditions which ensure that any solution $\{x_n\}$ of equation (1.1) is either oscillatory or converges to zero. In [5, 6], the authors considered equation (1.1) with $p_n \equiv 0$ and $\alpha = 1$, and established some new conditions which ensure that every solution of equation (1.1) is oscillatory.

If $p_n \equiv 0$, $b_n \equiv 0$ and $\alpha = 1$, then equation (1.1) considered in [4, 8, 9, 10, 11] and established sufficient conditions for the oscillation of all solutions of equation (1.1).

Motivated by the work in [5], [6] and [7], and the papers mentioned above, in the present paper, by employing Riccati type transformation and summation averaging technique, we establish some new sufficient conditions which ensure that every solution of equation (1.1) is oscillatory. Therefore the results obtained in this paper improve and complement to the results reported in [4, 5, 6, 7, 8, 9, 10, 11].

2. Oscillation results:

In this section we present sufficient conditions for the oscillation of all solutions of equation (1.1) when

$$\sum_{n=n_0}^{\infty} \frac{1}{(a_n E_n)^{\frac{1}{\alpha}}} = \infty, \tag{2.1}$$

where

$$E_n = \prod_{s=n_0}^{n-1} \left(\frac{a_s}{a_s - p_s} \right) \text{ and } a_n - p_n > 0 \text{ for all } n \geq n_0. \tag{2.2}$$

Note that using (2.2) we can write equation (1.1) in the equivalent form:

$$\Delta(E_n a_n (\Delta z_n)^\alpha) + E_{n+1} q_n f(x_{n-l}) = 0,\ n \geq n_0, \tag{2.3}$$

and it is easy to see that from (2.2) that every solution of equation (1.1) is oscillatory if and only if every solution of equation (2.3) is oscillatory.

First we study the oscillatory behavior of equation (1.1) when condition (2.1) and (2.2) hold.

If $\{x_n\}$ is a solution of equation (1.1), then $y_n = -x_n$ is a solution of the equation.

$$\Delta(a_n(\Delta w_n)^\alpha) + p_n(\Delta w_n)^\alpha + q_n f^*(y_{n-l}) = 0,\ n \geq n_0,$$

where $w_n = y_n - b_n y_{n-k}$ and $f^*(y_{n-l}) = -f(-y_{n-l})$, and $uf^*(u) > 0$ for $u \neq 0$. Thus concerning nonoscillatory solution of equation (1.1) we can restrict our attention only to solutions which are positive for all large n.

Define

$$R_n = \sum_{s=N}^{n-1} \frac{1}{(a_s E_s)^{\frac{1}{\alpha}}},\ N \geq n_0.$$

We begin with the following lemma.

Lemma 2.1. *Assume conditions* (2.1) *and* (2.2) *hold. Let* $\{x_n\}$ *be an eventually positive solution of equation* (1.1)*. Then one of the following two cases holds for all sufficiently large* n :

(1) $z_n > 0,\ \Delta z_n > 0,\ \Delta(E_n a_n(\Delta z_n)^\alpha) \leq 0;$
(2) $z_n < 0,\ \Delta z_n > 0,\ \Delta(E_n a_n(\Delta z_n)^\alpha) \leq 0.$

Proof. Assume that $x_n > 0$, $x_{n-k} > 0$ and $x_{n-l} > 0$ for all $n \geq n_1 \geq n_0$. Multiplying equation (1.1) by E_{n+1} and simplifying we have equation (2.3).

From (H_2), (2.2) and (2.3) we obtain

$$\Delta(a_n E_n (\Delta z_n)^\alpha) = -E_{n+1} q_n f(x_{n-l}) \leq 0,\ n \geq n_1, \tag{2.4}$$

so $\{a_n E_n (\Delta z_n)^\alpha\}$ is eventually decreasing for all $n \geq n_2 \geq n_1$. We claim that $\Delta z_n > 0$ for all $n \geq n_2$. If this is not the case, there exists an integer $n_3 \geq n_2$ such that $\Delta z_{n_3} \leq 0$. In view of (2.4) there is an integer $n_4 \geq n_3$ such that

$$E_n a_n (\Delta z_n)^\alpha \leq E_{n_4} a_{n_4} (\Delta z_{n_4}) = c < 0,\ \text{for } n \geq n_4.$$

Hence

$$\Delta z_n \leq \frac{c^{\frac{1}{\alpha}}}{(a_n E_n)^{\frac{1}{\alpha}}}$$

from which it follows that

$$z_n \le z_{n_4} + c^{\frac{1}{\alpha}} \sum_{s=n_4}^{n-1} \frac{1}{(a_s E_s)^{\frac{1}{\alpha}}}.$$

In view of condition (2.1), the last inequality implies that

$$\lim_{n\to\infty} z_n = -\infty. \tag{2.5}$$

Therefore, there are two cases to consider:

Case (1): If $\{x_n\}$ is unbounded, then there exists a sequence of integers $\{n_j\}$ such that $\lim_{j\to\infty} n_j = \infty$ and $\lim_{j\to\infty} x_{n_j} = \infty$, where $x_{n_j} = \max\{x_s : n_0 \le s \le n_j\}$. Since $n-k < n$, we have

$$\begin{aligned} x_{n_j-k} &= \max\{x_s : n_0 \le s \le n_j - k\} \\ &\le \max\{x_s : n_0 \le s \le n_j\} = x_{n_j}. \end{aligned} \tag{2.6}$$

So from the definition of z_n, we see that

$$z_{n_j} = x_{n_j} - b_{n_j} x_{n_j-k} \ge (1-b_{n_j}) x_{n_j} \ge (1-b) x_{n_j} > 0$$

which is a contradiction with (2.5).

Case (2): If $\{x_n\}$ is bounded, then from the definition of z_n and (H_1), we see that $\{z_n\}$ is bounded which again contradicts (2.5). Thus we conclude that $\Delta z_n > 0$ for all $n \ge n_2$, and this completes the proof. □

Lemma 2.2. *Assume conditions* (2.1) *and* (2.2) *hold. If* $\{x_n\}$ *is an eventually positive solution of equation* (1.1) *such that case (1) of Lemma 2.1 holds, then*

$$z_n \ge R_n E_n^{1/\alpha} a_n^{1/\alpha} \Delta z_n, \; n \ge N \ge n_0 \tag{2.7}$$

and $\{\frac{z_n}{R_n}\}$ *is eventually decreasing.*

Proof. The proof is similar to that of Lemma 2 of [12] and is omitted. □

Theorem 2.3. *Let conditions* (2.1), (2.2), $\alpha = \beta$, *and* $l > k$ *be hold. If there exists a positive nondecreasing sequence* $\{\rho_n\}$ *such that*

$$\lim_{n\to\infty} \sup \sum_{s=N}^{n} \left[M \rho_s E_{s+1} q_s \left(\frac{R_{s-l}}{R_{s+1}} \right)^{\alpha} - \frac{\Delta \rho_s}{R_{s+1}^{\alpha}} \right] = \infty \tag{2.8}$$

and

$$\lim_{n\to\infty} \sup \sum_{s=n-l+k}^{n-1} \left(\frac{1}{a_s E_s} \sum_{t=s}^{n-1} E_{t+1} q_t \right)^{\frac{1}{\alpha}} > \frac{b}{M^{\frac{1}{\alpha}}}, \tag{2.9}$$

then every solution of equation (1.1) *is oscillatory.*

Proof. Assume that there exists a nonoscillatory solution x_n of equation (1.1), say $x_n > 0,\ x_{n-k} > 0$ and $x_{n-l} > 0$ for all $n \ge N \ge n_0$, where N is chosen so that two cases of Lemma 2.1 hold for all $n \ge N$.

Case (1): From the definition of z_n and (H_1), we have $x_n \ge z_n$, and set

$$w_n = \rho_n \frac{E_n a_n (\Delta z_n)^\alpha}{z_n^\alpha},\ n \ge N. \tag{2.10}$$

Then $w_n > 0$ for $n \ge N$, and from equation (1.1), (2.2), (H_4) and (2.10), we obtain

$$\begin{aligned}
&\Delta w_n \\
= \ & \Delta\rho_n \frac{E_{n+1}a_{n+1}(\Delta z_{n+1})^\alpha}{z_{n+1}^\alpha} + \rho_n \frac{\Delta(E_n a_n(\Delta z_n)^\alpha)}{z_{n+1}^\alpha} - \rho_n \frac{E_n a_n (\Delta z_n)^\alpha}{z_n^\alpha z_{n+1}^\alpha}\Delta z_n^\alpha \\
\le \ & \Delta\rho_n \frac{E_{n+1}a_{n+1}(\Delta z_{n+1})^\alpha}{z_{n+1}^\alpha} - M\rho_n E_{n+1}\frac{q_n z_{n-l}^\alpha}{z_{n+1}^\alpha} - \frac{\rho_n E_n a_n(\Delta z_n)^\alpha}{z_n^\alpha z_{n+1}^\alpha}\Delta z_n^\alpha.
\end{aligned} \tag{2.11}$$

By the mean value theorem and the fact that $\{z_n\}$ is nondecreasing, we have

$$\Delta z_n^\alpha \ge \begin{cases} \alpha z_n^{\alpha-1}\Delta z_n, & \text{if } \alpha \ge 1; \\ \alpha z_{n+1}^{\alpha-1}\Delta z_n, & \text{if } \alpha < 1. \end{cases} \tag{2.12}$$

Using (2.12) in (2.11), we obtain

$$\begin{aligned}
\Delta w_n \ \le \ & -M\rho_n E_{n+1} q_n \left(\frac{z_{n-l}}{z_{n+1}}\right)^\alpha + \frac{\Delta\rho_n E_{n+1} a_{n+1}(\Delta z_{n+1})^\alpha}{z_{n+1}^\alpha} \\
& - \frac{\alpha\rho_{n+1}E_n a_n (\Delta z_n)^{\alpha+1}}{z_{n+1}^{\alpha+1}},\ n \ge N.
\end{aligned} \tag{2.13}$$

In view of the fact that $\Delta z_n > 0$ for $n \ge N$, it follows from (2.13) that

$$\Delta w_n \le -M\rho_n E_{n+1} q_n \left(\frac{z_{n-l}}{z_{n+1}}\right) + \Delta\rho_n E_{n+1} a_{n+1} \left(\frac{\Delta z_{n+1}}{z_{n+1}}\right)^\alpha. \tag{2.14}$$

Using (2.7) and the decreasing nature of $\{\frac{z_n}{R_n}\}$, (2.14) implies

$$\Delta w_n \le -M\rho_n E_{n+1} q_n \left(\frac{R_{n-l}}{R_{n+1}}\right)^\alpha + \frac{\Delta\rho_n}{R_{n+1}^\alpha},\ n \ge N. \tag{2.15}$$

Summing the inequality (2.15) from N to $n-1$, we have

$$\sum_{s=N}^{n-1}\left(M\rho_s E_{s+1} q_s \left(\frac{R_{s-l}}{R_{s+1}}\right)^\alpha - \frac{\Delta\rho_s}{R_{s+1}^\alpha}\right) \le w_N < \infty.$$

Taking limit sup as $n \to \infty$ in the above inequality, we obtain a contradiction with (2.8).

Case (2): From the definition of z_n and (H_1), we have

$$x_{n-k} > \left(\frac{-z_n}{b}\right). \tag{2.16}$$

From equation (1.1), (2.2), (H_4) and (2.16), we obtain

$$\Delta(a_nE_n(\Delta z_n)^\alpha) - \frac{M}{b^\alpha}E_{n+1}q_nz_{n-l+k}^\alpha \leq 0,\ n \geq N.$$

Summing the last inequality from s to $n-1$ for $n > s+1$, we have

$$a_nE_n(\Delta z_n)^\alpha - a_sE_s(\Delta z_s)^\alpha - \frac{M}{b^\alpha}\sum_{t=s}^{n-1}E_{t+1}q_tz_{t-l+k}^\alpha \leq 0,\ n \geq N,$$

it implies that

$$-\Delta z_s \leq \frac{M^{\frac{1}{\alpha}}}{b}\left(\frac{1}{a_sE_s}\sum_{t=s}^{n-1}E_{t+1}q_tz_{t-l+k}^\alpha\right)^{\frac{1}{\alpha}},\ n \geq N.$$

Again summing the last inequality from $n-l+k$ to $n-1$ for s, we obtain

$$z_{n-l+k} - z_n \leq \frac{M^{\frac{1}{\alpha}}}{b}z_{n-l+k}\sum_{s=n-l+k}^{n-1}\left(\frac{1}{a_sE_s}\sum_{t=s}^{n-1}E_{t+1}q_t\right)^{\frac{1}{\alpha}},$$

it implies that

$$\frac{b}{M^{\frac{1}{\alpha}}} \geq \sum_{s=n-l+k}^{n-1}\left(\frac{1}{a_sE_s}\sum_{t=s}^{n-1}E_{t+1}q_t\right)^{\frac{1}{\alpha}}.$$

Taking limit sup as $n \to \infty$, we obtain a contradiction to (2.9). This completes the proof. □

Theorem 2.4. *Let conditions* (2.1), (2.2), $\alpha > \beta$, *and* $l > k$ *be hold. If there exists a positive nondecreasing sequence* $\{\rho_n\}$ *such that, for all sufficiently large* N_4 *and for* $N > N_4$,

$$\lim_{n\to\infty}\sup\sum_{s=N}^{n-1}\left[M\rho_sE_{s+1}q_s\left(\frac{R_{s-l}}{R_{s+1}}\right)^\beta - \left(\frac{\alpha}{c_1\beta}\right)^\alpha\frac{a_sE_s(\Delta\rho_s)^{\alpha+1}}{(\alpha+1)^{\alpha+1}p_s^\alpha R_{s+1}^{\beta-\alpha}}\right] = \infty \tag{2.17}$$

with $c_1 > 0$ *and*

$$\lim_{n\to\infty}\sup\sum_{s=n-l+k}^{n-1}\left(\frac{1}{a_sE_s}\sum_{t=s}^{n-1}E_{t+1}q_t\right)^{\frac{1}{\alpha}} = \infty, \tag{2.18}$$

then every solution of equation (1.1) *is oscillatory.*

Proof. Assume that there exists a nonoscillatory solution x_n of equation (1.1), say $x_n > 0,\ x_{n-k} > 0$ and $x_{n-l} > 0$ for all $n \geq N \geq n_0$, where N is chosen so that two cases Lemma 2.1 hold for all $n \geq N$.

Case(1): From the definition of z_n and (H_1), we have $x_n \geq z_n$, and set

$$w_n = \rho_n \frac{E_n a_n (\Delta z_n)^\alpha}{z_n^\beta},\ n \geq N. \tag{2.19}$$

Then proceeding as in the proof of Theorem 2.3, we obtain

$$\Delta w_n \quad \leq \quad -M\rho_n E_{n+1} q_n \left(\frac{z_{n-l}}{z_{n+1}}\right)^\beta + \frac{\Delta \rho_n}{\rho_{n+1}} w_{n+1} - \frac{\beta \rho_n w_{n+1}^{1+\frac{1}{\alpha}}}{\rho_{n+1}^{1+\frac{1}{\alpha}} (a_n E_n)^{\frac{1}{\alpha}}} z_{n+1}^{\frac{\beta}{\alpha}-1}. \tag{2.20}$$

Since $\{\frac{z_n}{R_n}\}$ is decreasing, we have $z_n \leq cR_n$ for $c > 0$ and $n \geq N$, from (2.20) we obtain

$$\Delta w_n \quad \leq \quad -M\rho_n E_{n+1} q_n \left(\frac{R_{n-l}}{R_{n+1}}\right)^\beta + \frac{\Delta \rho_n}{\rho_{n+1}} w_{n+1} - \frac{\beta c_1 R_{n+1}^{\frac{\beta}{\alpha}-1} \rho_n w_{n+1}^{1+\frac{1}{\alpha}}}{\rho_{n+1}^{1+\frac{1}{\alpha}} (a_n E_n)^{\frac{1}{\alpha}}}, \tag{2.21}$$

where we have used $c_1 = c_1^{1-\frac{\beta}{\alpha}}$ and $\{\frac{z_n}{R_n}\}$ is decreasing. Now using the inequality $Au - Bu^{1+\frac{1}{\alpha}} \leq \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{A^{\alpha+1}}{B^\alpha}$ for $B > 0$ with $A = \frac{\Delta\rho_n}{\rho_{n+1}}$, $B = \frac{\beta c_1 \rho_n R_{n+1}^{\frac{\beta}{\alpha}-1}}{\rho_{n+1}^{1+\frac{1}{\alpha}} (a_n E_n)^{\frac{1}{\alpha}}}$ in (2.21), we obtain

$$\Delta w_n \quad \leq \quad -M\rho_n E_{n+1} q_n \left(\frac{R_{n-l}}{R_{n+1}}\right)^\beta + \left(\frac{\alpha}{c_1\beta}\right)^\alpha \frac{(\Delta\rho_n)^{\alpha+1} a_n E_n}{(\alpha+1)^{\alpha+1} \rho_n^\alpha R_{n+1}^{\beta-\alpha}}, n \geq N.$$

Summing the last inequality from N to $n - l$, we have

$$\sum_{s=N}^{n-1} \left[M\rho_s E_{s+1} q_s \left(\frac{R_{s-l}}{R_{s+1}}\right)^\beta - \left(\frac{\alpha}{c_1\beta}\right)^\alpha \frac{(\Delta\rho_s)^{\alpha+1} a_s E_s}{(\alpha+1)^{\alpha+1} \rho_s^\alpha R_{s+1}^{\beta-\alpha}} \right] \leq w_N < \infty.$$

Taking limit sup as $n \to \infty$ in the last inequality, we obtain a contradiction to (2.17).

Case(2): Proceeding as in Case (2) of Theorem 2.3, we have

$$a_n E_n (\Delta z_n)^\alpha - a_s E_s (\Delta z_s)^\alpha - \frac{M}{b^\beta} \sum_{t=s}^{n-1} E_{t+1} q_t z_{t-l+k}^\beta \leq 0,\ n \geq N \tag{2.22}$$

Let $\lim_{n\to\infty} z_n = c = 0$. Summing (2.22) from $n - l + k$ to $n - 1$ for s, we have

$$z_{n-l+k} - z_n \leq \frac{M^{\frac{1}{\alpha}}}{b^{\frac{\beta}{\alpha}}} \sum_{s=n-l+k}^{n-1} \left(\frac{1}{a_s E_s} \sum_{t=s}^{n-1} E_{t+1} q_t \right)^{\frac{1}{\alpha}} z_{n-l+k}^{\frac{\beta}{\alpha}},$$

it implies that

$$\frac{z_{n-l+k}}{z_{n-l+k}^{\frac{\beta}{\alpha}}} \geq \frac{M^{\frac{1}{\alpha}}}{b^{\frac{\beta}{\alpha}}} \sum_{s=n-l+k}^{n-1} \left(\frac{1}{a_s E_s} \sum_{t=s}^{n-1} E_{t+1} q_t \right)^{\frac{1}{\alpha}}. \tag{2.23}$$

Since $\frac{z_{n-l+k}}{z_{n-l+k}^{\frac{\beta}{\alpha}}} = |z_{n-l+k}|^{1-\frac{\beta}{\alpha}}$ and $1 - \frac{\beta}{\alpha} > 0$, we have from (2.23) that

$$\lim_{n\to\infty} \sup \sum_{s=n-l+k}^{n-1} \left(\frac{1}{a_s E_s} \sum_{t=s}^{n-1} E_{t+1} q_t \right)^{\frac{1}{\alpha}} \leq 0,$$

which contradicts (2.18). Next assume that $\lim_{n\to\infty} z_n = c < 0$. From (2.22), we have

$$\Delta z_s + \frac{M^{\frac{1}{\alpha}}}{b^{\frac{\beta}{\alpha}}} z_n^{\frac{\beta}{\alpha}} \left(\frac{1}{a_s E_s} \sum_{t=s}^{n-1} E_{t+1} q_t \right)^{\frac{1}{\alpha}}.$$

Summing the last inequality from N to $n-1$, we obtain

$$z_N - z_n \leq \frac{M^{\frac{1}{\alpha}}}{b^{\frac{\beta}{\alpha}}} z_n^{\frac{\beta}{\alpha}} \sum_{s=N}^{n-1} \left(\frac{1}{a_s E_s} \sum_{t=s}^{n-1} E_{t+1} q_t \right)^{\frac{1}{\alpha}},$$

it implies that

$$\frac{b^{\frac{\beta}{\alpha}} z_N}{M^{\frac{1}{\alpha}} z_n^{\frac{\beta}{\alpha}}} \geq \sum_{s=N}^{n-1} \left(\frac{1}{a_s E_s} \sum_{t=s}^{n-1} E_{t+1} q_t \right)^{\frac{1}{\alpha}}.$$

In view of $c < 0$, the term $\{\frac{b^{\frac{\beta}{\alpha}} z_N}{M^{\frac{1}{\alpha}} z_n^{\frac{\beta}{\alpha}}}\}$ has an upper bound, so

$$\lim_{n\to\infty} \sum_{s=N}^{n-1} \left(\frac{1}{a_s E_s} \sum_{t=s}^{n-1} E_{t+1} q_t \right)^{\frac{1}{\alpha}} < \infty,$$

which again contradicts (2.18). This completes the proof. □

Theorem 2.5. *Let conditions* (2.1), (2.2), $\alpha \geq \beta \geq 1$, *and* $l > k$ *be hold. If there exists a positive nondecreasing sequence* $\{\rho_n\}$ *such that, for all sufficiently large integer* N_* *and for* $N > N_*$,

$$\lim_{n\to\infty} \sup \sum_{s=N}^{n} \left[M \rho_s E_{s+1} q_s \left(\frac{R_{s-l}}{R_{s+1}} \right)^{\beta} - \frac{c_3 (\Delta \rho_s)^{\alpha} (a_s E_s)^{\frac{1}{\alpha}}}{4\beta R_{s+1}^{\beta-1} \rho_s} \right] = \infty \tag{2.24}$$

with $c_3 > 0$ *and condition* (2.18) *holds, then every solution of equation* (1.1) *is oscillatory.*

Proof. Assume that there exists a nonoscillatory solution x_n of equation (1.1), say $x_n > 0,\ x_{n-k} > 0$ and $x_{n-l} > 0$ for all $n \geq N \geq n_0$, where N is chosen so that two cases of Lemma 2.1 hold for all $n \geq N$.

Case(1): Proceeding exactly as in the proof of Theorem 2.4 (Case(1)), we obtain (2.21) which can be rewritten as

$$\Delta w_n \leq -M\rho_n E_{n+1} q_n \left(\frac{R_{n-l}}{R_{n+1}}\right)^\beta + \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} - \frac{\beta c_1 \rho_n R_{n+1}^{\frac{\beta}{\alpha}-1}}{\rho_{n+1}^{1+\frac{1}{\alpha}} (a_n E_n)^{\frac{1}{\alpha}}} w_{n+1}^2 w_{n+1}^{\frac{1}{\alpha}-1}, \tag{2.25}$$

for $n \geq N$. From (2.19), we get

$$\begin{aligned} w_{n+1}^{\frac{1}{\alpha}-1} &= (\rho_{n+1} E_{n+1} a_{n+1})^{\frac{1}{\alpha}-1} \left(\frac{(\Delta z_{n+1})^\alpha}{z_{n+1}^\beta}\right)^{\frac{1}{\alpha}-1} \\ &= (\rho_{n+1} E_{n+1} a_{n+1})^{\frac{1}{\alpha}-1} \left(\frac{z_{n+1}}{\Delta z_{n+1}}\right)^{\alpha-1} z_{n+1}^{(\alpha-1)(\frac{\beta}{\alpha}-1)}. \end{aligned} \tag{2.26}$$

From (2.8), we have

$$\left(\frac{z_{n+1}}{\Delta z_{n+1}}\right)^{\alpha-1} \geq (E_{n+1} a_{n+1})^{\frac{\alpha-1}{\alpha}} R_{n+1}^{\alpha-1},\ n \geq N. \tag{2.27}$$

Since $\{\frac{z_n}{R^n}\}$ is decreasing and $(\alpha-1)(\frac{\beta}{\alpha}-1) \leq 0$, we obtain

$$z_{n+1}^{(\alpha-1)(\frac{\beta}{\alpha}-1)} \geq c_2^{(\alpha-1)(\frac{\beta}{\alpha}-1)} R_{n+1}^{(\alpha-1)(\frac{\beta}{\alpha}-1)},\ n \geq N. \tag{2.28}$$

Substituting (2.27), and (2.28) into (2.26) gives

$$w_{n+1}^{\frac{1}{\alpha}-1} \geq \rho_{n+1}^{\frac{1}{\alpha}-1} c_2^{(\alpha-1)(\frac{\beta}{\alpha}-1)} R_{n+1}^{\beta-\frac{\beta}{\alpha}}. \tag{2.29}$$

Using (2.29) in (2.25), we obtain

$$\begin{aligned} \Delta w_n \leq\ & -M\rho_n E_{n+1} q_n \left(\frac{R_{n-l}}{R_{n+1}}\right)^\beta + \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} \\ & - \frac{\beta R_{n+1}^{\beta-1}\rho_n}{c_3 \rho_{n+1}^2 (a_n E_n)^{\frac{1}{\alpha}}} w_{n+1}^2. \end{aligned} \tag{2.30}$$

Completing square with w_{n+1}, it follows from (2.30) that

$$\Delta w_n \leq -M\rho_n E_{n+1} q_n \left(\frac{R_{n-l}}{R_{n+1}}\right)^\beta + \frac{1}{4}\frac{c_3 (\Delta\rho_n)^2 (a_n E_n)^{\frac{1}{\alpha}}}{\beta R_{n+1}^{\beta-1}\rho_n}. \tag{2.31}$$

Summing the last inequality from N to n leads to

$$\sum_{s=N}^{n}\left[M\rho_s E_{s+1}q_s\left(\frac{R_{s-l}}{R_{s+1}}\right)^{\beta}-\frac{c_3(\Delta\rho_s)^2(a_sE_s)^{\frac{1}{\alpha}}}{4\beta R_{s+1}^{\beta-1}\rho_s}\right]\le w_N<\infty$$

which contradicts condition (2.24).

Case(2): The proof is similar to that of Case (2) of Theorem 2.5 and the proof of the theorem is completed. □

Theorem 2.6. *Let conditions* (2.1)*,* (2.2) *and* $\alpha\le\beta$ *be hold. If there exists a positive nondecreasing sequence* $\{\rho_n\}$ *such that, for all sufficiently large* N_* *and for* $N>N_*$

$$\lim_{n\to\infty}\sup\sum_{s=N}^{n}\left[MG_{s+1}E_{s+1}q_s\left(\frac{R_{s-l}}{R_{s+1}}\right)^{\beta}-\frac{c^{\alpha-\beta}}{R_{s+1}^{\alpha}E_{s+1}a_{s+1}}\right]=\infty \tag{2.32}$$

with $c>0$ *and* $G_n=\sum_{s=n_0}^{n-1}\frac{1}{a_sE_s}$*, then every solution of equation* (1.1) *is either oscillatory or tends to zero as* $n\to\infty$*.*

Proof. Proceeding as in the proof of Theorem 2.4, we see that the two cases of Lemma 2.1 hold for all $n\ge N\ge n_0$.

Case(1): Proceeding as in the proof of Theorem 2.4 (Case(1)), we have

$$\Delta(E_na_n(\Delta z_n)^{\alpha})+ME_{n+1}q_nz_{n-l}^{\beta}\le 0,\ n\ge N. \tag{2.33}$$

Define

$$w_n=G_n\frac{E_na_n(\Delta z_n)^{\alpha}}{z_n^{\beta}},\ n\ge N. \tag{2.34}$$

Then $w_n>0$ for $n\ge N$, and from (2.33), (2.34), we obtain

$$\begin{aligned}\Delta w_n &\le -MG_{n+1}E_{n+1}q_n\left(\frac{z_{n-l}}{z_{n+1}}\right)^{\beta}+\frac{(\Delta z_{n+1})^{\alpha}}{z_{n+1}^{\beta}}\\ &\le -MG_{n+1}E_{n+1}q_n\left(\frac{R_{n-l}}{R_{n+1}}\right)^{\beta}+\left(\frac{\Delta z_{n+1}}{z_{n+1}}\right)^{\alpha}z_{n+1}^{\alpha-\beta}.\end{aligned} \tag{2.35}$$

Since $\{z_n\}$ is nondecreasing, we have $z_n\ge c>0$, and from (2.8) we have

$$\left(\frac{\Delta z_{n+1}}{z_{n+1}}\right)^{\alpha}\le\frac{1}{R_{n+1}^{\alpha}E_{n+1}a_{n+1}},\ n\ge N. \tag{2.36}$$

Using (2.36) in (2.35), we obtain

$$\Delta w_n\le -MG_{n+1}E_{n+1}q_n\left(\frac{R_{n-l}}{R_{n+1}}\right)^{\beta}+\frac{c^{\alpha-\beta}}{R_{n+1}^{\alpha}E_{n+1}a_{n+1}},\ n\ge N.$$

Summing the last inequality from N to n, we have

$$\sum_{s=N}^{n}\left[MG_{s+1}E_{s+1}q_s\left(\frac{R_{s-l}}{R_{s+1}}\right)^{\beta}-\frac{c^{\alpha-\beta}}{R_{s+1}^{\alpha}E_{s+1}a_{s+1}}\right]\leq w_N<\infty$$

which contradicts (2.32).

Case(2): In this case we have $z_n<0$ and $\Delta z_n>0$ for all $n\geq N$. Then by Lemma 1 of [7], one can see that $\lim_{n\to\infty}x_n=0$. This completes the proof. □

Remark 2.7. Theorems 2.3 and 2.4 improve and extend the results established in [7] in the sense that our criteria ensure the oscillation of all solutions of equation (1.1). Further the results established in the paper extend and generalize that of in [4, 5, 6, 8, 9, 10, 11].

3. Examples

In this section, we provide two examples to illustrate the main results.

Example 3.1. Consider the second order neutral difference equation:

$$\Delta((\Delta(x_n-\frac{1}{2}x_{n-2}))^3)+\frac{1}{n+1}(\Delta(x_n-\frac{1}{2}x_{n-2}))^3+\frac{n}{n+1}x_{n-3}^3=0,\ n\geq 1. \quad (3.1)$$

Here $a_n=1,\ p_n=\frac{1}{n+1},\ q_n=\frac{n}{n+1},\ b_n=\frac{1}{2},\ l=3,\ k=2$ and $\alpha=\beta=3$. So $E_n=n$ and $R_n=\sum_{s=1}^{n-1}\frac{1}{s^{1/3}}$ and $M=1$. Further $R_n\to\infty$ as $n\to\infty$ and $n-1\geq R_n\geq(n-1)^{2/3}$. By taking $\rho_n=1$, we see that

$$\lim_{n\to\infty}\sup\sum_{s=1}^{n}(s+1)\frac{s}{s+1}\left(\frac{R_{s-3}}{R_{s+1}}\right)^3\geq\lim_{n\to\infty}\sup\sum_{s=1}^{n}\frac{(s-4)^2}{s^2}=\infty$$

and

$$\lim_{n\to\infty}\sup\sum_{s=n-1}^{n-1}\left(\frac{1}{s}\sum_{t=s}^{n-1}(t+1)\frac{t}{t+1}\right)=\lim_{n\to\infty}\sup\left(\frac{1}{n-1}\right)(n-1)=1>\frac{1}{2}.$$

Thus, all conditions of Theorem 2.3 are satisfied and hence every solution of equation (3.1) is oscillatory. In fact $\{x_n\}=\{(-1)^n\}$ is one such oscillatory of equation (3.1).

Example 3.2. Consider the second order neutral difference equation:

$$\Delta(n(\Delta z_n)^3)+(n-1)(\Delta z_n)^3+n^2x_{n-2}=0,\ n\geq 2. \quad (3.2)$$

Where $z_n=x_n-\frac{1}{3}x_{n-1}$. Here $a_n=n,\ p_n=n-1,\ q_n=n^2,\ b_n=\frac{1}{3},\ l=2,\ k=1,\ \alpha=3$, and $\beta=1$. So $E_n=\frac{n^2-n-2}{2}$, $(n-1)^{1/3}\geq R_n\geq(\frac{n-2}{n-1})$ and $M=1$. Further $R_n\to\infty$ as $n\to\infty$ and by taking $\rho_n=1$, we see that all conditions of Theorem 2.4 are satisfied. Hence every solution of equation (3.2) is oscillatory.

4. Conclusion

In this paper we have established some new oscillation results which can be easily extended to more general neutral type difference equations and neutral dynamic equations on any time scales. Further note that the results in the present paper will contribute to the studies on oscillatory and asymptotic behavior of solutions of neutral type difference equations with damping term. It is interesting to extend the results of this paper when the condition (2.1) fails to hold or $a_n - p_n \leq 0$. This would be left to further research.

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