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A HIGH ORDER ITERATIVE SCHEME ASSOCIATED WITH A DIRICHLET-ROBIN PROBLEM FOR A NONLINEAR CARRIER EQUATION IN THE ANNULAR MEMBRANE

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Abstract. In this paper, we consider an initial and boundary value problem for a nonlinear Carrier wave equation, with a source term containing four variables

$$f = f(x, t, u, \int_{\rho}^{1} xu^2 (x, t)\, dx).$$

Motivated by recent results for Carrier equation, we establish here a high order iterative scheme to obtain a convergent sequence at a rate of order N.

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1. Introduction

In this paper, we consider the following nonlinear Carrier wave equation in the annular membrane

$$u_{tt} - \mu(\|u(t)\|_0^2)(u_{xx} + \frac{1}{x}u_x) = f(x, t, u, \|u(t)\|_0^2),\ \rho < x < 1,\ 0 < t < T, \quad (1.1)$$

associated with Robin-Dirichlet conditions

$$u(\rho, t) = u_x(1, t) + \zeta u(1, t) = 0, \quad (1.2)$$

and initial conditions

$$u(x, 0) = \tilde{u}_0(x),\ u_t(x, 0) = \tilde{u}_1(x), \quad (1.3)$$

where μ, f, $\tilde{u}_0$, $\tilde{u}_1$ are given functions, and ρ, ζ are given constants, with $0 < \rho < 1$. In equation (1.1), the nonlinear terms $\mu(\|u(t)\|_0^2)$ and $f(x, t, u, \|u(t)\|_0^2)$ depend on the integral $\|u(t)\|_0^2 = \int_\rho^1 xu^2(x, t)\, dx$. Equation (1.1) herein is the bidimensional nonlinear wave equation describing nonlinear vibrations of the annular membrane $\Omega_1 = \{(x, y) : \rho^2 < x^2 + y^2 < 1\}$. In the vibration processing, the area of the annular membrane and the tension at various points change in time. The condition on the boundary $\Gamma_1 = \{(x, y) : x^2 + y^2 = 1\}$, that is $u_x(1, t) + \zeta u(1, t) = 0$, describes elastic constraints where ζ the constant has a mechanical signification. Here, the boundary conditions on $\Gamma_\rho = \{(x, y) : x^2 + y^2 = \rho^2\}$ requiring $u(\rho, t) = 0$, it means that the annular membrane is fixed.

In [1], Carrier established the equation which models vibrations of an elastic string when changes in tension are not small

$$\rho u_{tt} - \left(1 + \frac{EA}{LT_0}\int_0^L u^2(y, t)dy\right) u_{xx} = 0, \quad (1.4)$$

where $u(x, t)$ is the x - derivative of the deformation, T_0 is the tension in the rest position, E is the Young modulus, A is the cross - section of a string, L is the length of a string and ρ is the density of a material.

In [10], a high order iterative scheme was established in order to get a convergent sequence at a rate of order N $(N \geq 1)$ to a local unique weak solution of a nonlinear Carrier wave equation as follows

$$u_{tt} - \mu(\|u(t)\|_0^2)(u_{xx} + \frac{1}{x}u_x) = f(x, t, u),\ \rho < x < 1,\ 0 < t < T,$$

associated with the Robin-Dirichlet conditions (1.2), where $\mu \in C^1(\mathbb{R}_+)$ and there exist constants $p > 1$, $\mu_* > 0$, $\mu_1 > 0$, $\mu_2 > 0$ such that

$$\begin{cases} 0 < \mu_* \leq \mu(z) \leq \mu_1(1 + z^p), \text{ for all } z \geq 0, \\ |\mu'(z)| \leq \mu_2(1 + z^{p-1}), \text{ for all } z \geq 0. \end{cases} \quad (1.5)$$

Motivated by results for nonlinear wave equations in [3] and [4], where recurrent sequences converge at a rate of order 1 or 2, we will construct a high order iterative scheme to obtain a convergent sequence at a rate of order N to a local weak solution of problems (1.1)-(1.3). This scheme is established based on a high order method for solving operator equation $F(x) = 0$, it also has been applied in [5]-[10], [14] and some other works. It is well known that Newton's method and its variants are used to solve nonlinear operator equations, see [11] and references therein.

In this paper, we associate with equation (1.1) a recurrent sequence $\{u_m\}$ defined by

$$\begin{aligned} &\frac{\partial^2 u_m}{\partial t^2} - \mu(\|u_m(t)\|_0^2)\left(\frac{\partial^2 u_m}{\partial x^2} + \frac{1}{x}\frac{\partial u_m}{\partial x}\right) \qquad (1.6) \\ = &\sum_{0\le i+j\le N-1} A_{ij}f[u_{m-1}](u_m - u_{m-1})^i\left(\|u_m(t)\|_0^2 - \|u_{m-1}(t)\|_0^2\right)^j, \end{aligned}$$

$\rho < x < 1$, $0 < t < T$, where $A_{ij}f[u_{m-1}] = \frac{1}{i!j!}D_3^iD_4^jf(x,t,u_{m-1},\|u_{m-1}(t)\|_0^2)$, with u_m satisfying (1.2), (1.3) and $u_0 \equiv 0$. If $f \in C^N([\rho,1]\times\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}_+)$, then we prove that the sequence $\{u_m\}$ converges at a rate of order N to a local weak solution of problems (1.1)-(1.3). In our proofs, the Faedo - Galerkin approximation method associated with a priori estimates, the weak convergence, compactness techniques and a known fixed point theorem are used. We would like to emphasize here that the assumptions for the function μ and its derivatives are bounded by a polynomials as in [4]-[6], [14] will be ignored in the process of proving. Our results can be regarded as an extension and improvement of the corresponding results of [3]-[10], [13] and [14].

2. PRELIMINARIES

Put $\Omega = (\rho, 1)$, $Q_T = \Omega \times (0,T)$, $T > 0$. We will omit the definitions of the usual function spaces and denote them by the notations $L^p = L^p(\Omega)$, $H^m = H^m(\Omega)$. Let $\langle\cdot,\cdot\rangle$ be either the scalar product in L^2 or the dual pairing of a continuous linear functional and an element of a function space. The notation $\|\cdot\|$ stands for the norm in L^2 and we denote by $\|\cdot\|_X$ the norm in the Banach space X. We call X' the dual space of X. We denote $L^p(0,T;X)$, $1 \le p \le \infty$ the Banach space of real measurable functions $u : (0,T) \to X$, such that $\|u\|_{L^p(0,T;X)} < +\infty$, with

$$\|u\|_{L^p(0,T;X)} = \begin{cases} \left(\int_0^T \|u(t)\|_X^p\,dt\right)^{1/p}, & \text{if} \quad 1 \le p < \infty, \\ ess\sup\limits_{0<t<T} \|u(t)\|_X, & \text{if} \quad p = \infty. \end{cases}$$

For $f \in C^k([\rho,1] \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+)$, $f = f(x,t,y_1,y_2)$, we put $D_1 f = \dfrac{\partial f}{\partial x}$, $D_2 f = \dfrac{\partial f}{\partial t}$, $D_{2+i} f = \dfrac{\partial f}{\partial y_i}$ with $i = 1,2$, and $D^\alpha f = D_1^{\alpha_1} ... D_4^{\alpha_4} f$, $\alpha = (\alpha_1, ..., \alpha_4) \in \mathbb{Z}_+^4$, $|\alpha| = \alpha_1 + ... + \alpha_4 = k$, $D^{(0,...,0)} f = f$;

On $H^1 \equiv H^1(\Omega)$, $H^2 \equiv H^2(\Omega)$, we shall use the following norms

$$\|v\|_{H^1} = \left(\|v\|^2 + \|v_x\|^2 \right)^{\frac{1}{2}} \tag{2.1}$$

and

$$\|v\|_{H^2} = \left(\|v\|^2 + \|v_x\|^2 + \|v_{xx}\|^2 \right)^{\frac{1}{2}}, \tag{2.2}$$

respectively.

Note that L^2, H^1, H^2 are also the Hilbert spaces with respect to the corresponding scalar products

$$\langle u, v \rangle = \int_\rho^1 x u(x) v(x) \, dx, \ \langle u, v \rangle + \langle u_x, v_x \rangle, \ \langle u, v \rangle + \langle u_x, v_x \rangle + \langle u_{xx}, v_{xx} \rangle, \tag{2.3}$$

respectively. The norms in L^2 and H^1 induced by the corresponding scalar products (2.3) are denoted by $\|\cdot\|_0$, $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively.

We put

$$V_1 = \left\{ v \in H^1 : v(\rho) = 0 \right\}. \tag{2.4}$$

Then, V_1 is a closed subspace of H^1 and two norms $\|v\|_{H^1}$ and $\|v_x\|$ are equivalent norms on V_1, and V_1 is continuously and densely embedded in L^2. Identifying L^2 with $(L^2)'$ (the dual of L^2), we have $V_1 \hookrightarrow L^2 \hookrightarrow V_1'$; On the other hand, the notation $\langle \cdot, \cdot \rangle$ is used for the pairing between V_1 and V_1'.

We then have the following lemmas.

Lemma 2.1. *The following inequalities are fulfilled:*

(i) $\sqrt{\rho} \|v\| \leq \|v\|_0 \leq \|v\|$ *for all* $v \in L^2$,

(ii) $\sqrt{\rho} \|v\|_{H^1} \leq \|v\|_1 \leq \|v\|_{H^1}$ *for all* $v \in H^1$.

Lemma 2.2. *The embedding* $V_1 \hookrightarrow C^0(\overline{\Omega})$ *is compact and for all* $v \in V_1$, *we have*

(i) $\|v\|_{C^0(\overline{\Omega})} \leq \sqrt{1-\rho} \|v_x\|$,

(ii) $\|v\| \leq \frac{1-\rho}{\sqrt{2}} \|v_x\|$,

(iii) $\|v\|_0 \leq \frac{1-\rho}{\sqrt{2\rho}} \|v_x\|_0$,

(iv) $\|v_x\|_0^2 + v^2(1) \geq \|v\|_0^2$,

(v) $|v(1)| \leq \sqrt{3}\,\|v\|_1$.

Remark 2.3. Since two norms $v \mapsto \|v\|$ and $v \mapsto \|v\|_0$ are equivalent on L^2, two norms $v \mapsto \|v\|_{H^1}$ and $v \mapsto \|v\|_1$ are equivalent on H^1, and five norms $v \mapsto \|v\|_{H^1}$, $v \mapsto \|v\|_1$, $v \mapsto \|v_x\|$, $v \mapsto \|v_x\|_0$ and $v \mapsto \sqrt{\|v_x\|_0^2 + v^2(1)}$ are equivalent on V_1.

Now, we define the bilinear form

$$a(u,v) = \zeta u(1)\, v(1) + \int_\rho^1 x u_x(x)\, v_x(x)\, dx, \text{ for all } u,\, v \in V_1, \tag{2.5}$$

where $\zeta \geq 0$ is a constant. We then have the following lemma.

Lemma 2.4. *The symmetric bilinear form $a(\cdot,\cdot)$ defined by (2.5) is continuous on $V_1 \times V_1$ and coercive on V_1, i.e.,*

(i) $|a(u,v)| \leq C_1 \|u\|_1 \|v\|_1$,

(ii) $a(v,v) \geq C_0 \|v\|_1^2$,

for all $u,\, v \in V_1$, with $C_0 = \frac{1}{2}\min\{1, \frac{2\rho}{(1-\rho)^2}\}$ and $C_1 = 1 + 3\zeta$.

Remark 2.5. By Lemma 2.4, the norms $v \mapsto \|v\|_{H^1}$, $v \mapsto \|v\|_1$, $v \mapsto \|v_x\|$, $v \mapsto \|v_x\|_0$, $v \mapsto \sqrt{\|v_x\|_0^2 + v^2(1)}$ and $v \mapsto \|v\|_a = \sqrt{a(v,v)} = \sqrt{\|v_x\|_0^2 + \zeta v^2(1)}$ are equivalent on V_1.

Lemma 2.6. *There exists the Hilbert orthonormal base $\{w_j\}$ of the space L^2 consisting of eigenfunctions w_j corresponding to eigenvalues λ_j such that*

(i) $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \lambda_{j+1} \leq \cdots$, $\lim_{j\to+\infty} \lambda_j = +\infty$,

(ii) $a(w_j, v) = \lambda_j \langle w_j, v\rangle$ *for all* $v \in V_1$, $j = 1, 2, \ldots$.

Furthermore, the sequence $\{w_j/\sqrt{\lambda_j}\}$ is also the Hilbert orthonormal base of V_1 with respect to the scalar product $a(\cdot,\cdot)$.

On the other hand, w_j satisfies the following boundary value problem

$$\begin{cases} Aw_j \equiv -w_{jxx} - \dfrac{1}{x} w_{jx} = -\dfrac{1}{x}\dfrac{\partial}{\partial x}(x w_{jx}) = \lambda_j w_j, \ in\ \Omega, \\ w_j(\rho) = w_{jx}(1) + \zeta w_j(1) = 0, \ w_j \in C^\infty([\rho,1]). \end{cases} \tag{2.6}$$

The proof of Lemma 2.6 can be found in [[12], p.87, Theorem 7.7], with $H = L^2$ and $V = V_1$, and $a(\cdot,\cdot)$ as defined by (2.5).

We also note that the operator $A : V_1 \longrightarrow V_1'$ in (2.6) is uniquely defined by the Lax-Milgram's lemma, i.e.,

$$a(u, v) = \langle Au, v\rangle \text{ for all } u, v \in V_1. \tag{2.7}$$

Lemma 2.7. *Three norms*

$$v \mapsto \|v\|_{H^2},$$

$$v \mapsto \|v\|_2 = \sqrt{\|v\|_0^2 + \|v_x\|_0^2 + \|v_{xx}\|_0^2}$$

and

$$v \mapsto \sqrt{\|v_x\|_0^2 + \|Av\|_0^2}$$

are equivalent on $V_1 \cap H^2$.

Finally, we need the following lemma.

Lemma 2.8. *Let* $g \in C(\mathbb{R}_+;\mathbb{R})$. *if the function* Φ_g *is defined as follows:*

$$\Phi_g(r) = \begin{cases} \sup\limits_{0\le u\le r} |g(u)|, & r > 0, \\ |g(0)|, & r = 0, \end{cases} \tag{2.8}$$

then $\Phi_g \in C(\mathbb{R}_+;\mathbb{R}_+)$ *and* Φ_g *s nondecreasing such that*

$$g(x) \le \Phi_g(x), \text{ for all } x \in \mathbb{R}_+. \tag{2.9}$$

The proof of Lemma 2.8 can be found in [[15], Appendix 1].

3. A high order iterative scheme

First, we say that u is a weak solution of problems (1.1)-(1.3) if

$$u \in L^\infty(0,T;V_1 \cap H^2),\ u_t \in L^\infty(0,T;V_1),\ u_{tt} \in L^\infty(0,T;L^2)\}, \tag{3.1}$$

and u satisfies the following variational equation

$$\langle u_{tt}(t), v\rangle + \mu\left(\|u(t)\|_0^2\right) a(u(t), v) = \langle f(x,t,u,\|u(t)\|_0^2), v\rangle, \tag{3.2}$$

for all $v \in V_1$, and a.e., $t \in (0,T)$, together with the initial conditions

$$u(0) = \tilde{u}_0,\ u_t(0) = \tilde{u}_1, \tag{3.3}$$

where $a(\cdot,\cdot)$ is the symmetric bilinear form on V_1 defined by (2.5).

Now, we make the following assumptions:

(H_1) $\tilde{u}_0 \in V_1 \cap H^2$, $\tilde{u}_1 \in V_1$;

(H_2) $\mu \in C^1(\mathbb{R}_+)$, and there exists a constant $\mu_* > 0$ such that $\mu(z) \geq \mu_* > 0$, for all $z \geq 0$;

(H_3) $f \in C^0([\rho,1] \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+)$ such that $f(\rho,t,0,z) = 0$, $\forall t,\ z \geq 0$ and
  (i) $D_3^i D_4^j f \in C^0([\rho,1] \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+)$, $1 \leq i+j \leq N$,
  (ii) $D_1 D_3^i D_4^j f \in C^0([\rho,1] \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+)$, $0 \leq i+j \leq N-1$.

Remark 3.1. The following assumptions for the function μ made in [10] are not needed here: $\mu \in C^1(\mathbb{R}_+)$ and there exist constants $p > 1$, $\mu_* > 0$, $\mu_1 > 0$, $\mu_2 > 0$ such that

(i) $0 < \mu_* \leq \mu(z) \leq \mu_1(1 + z^p)$, for all $z \geq 0$,
(ii) $|\mu'(z)| \leq \mu_2(1 + z^{p-1})$, for all $z \geq 0$.

Fix $T^* > 0$. For each $M > 0$ given, we set the constants $\tilde{K}_M(\mu)$, $\bar{K}_M(f)$ as follows:

$$\tilde{K}_M(\mu) = \sup_{0 \leq z \leq M^2} \left(\mu(z) + \left|\mu'(z)\right|\right)$$

and

$$\bar{K}_M(f) = \max_{0<i+j \leq N-1} \left\| D_3^i D_4^j f \right\|_{C^0(A_*(M))},$$

where $\|f\|_{C^0(A_*(M))} = \sup\{|f(x,t,y,z)| : (x,t,y,z) \in A_*(M)\}$, and

$$A_*(M) = [\rho,1] \times [0,T^*] \times \left[-\sqrt{\frac{1-\rho}{\rho}}M, \sqrt{\frac{1-\rho}{\rho}}M\right] \times \left[-M^2, M^2\right].$$

For each $M > 0$ and $T \in (0,T^*]$, we put

$$\begin{aligned} W(M,T) = \{&u \in L^\infty\left(0,T;V_1 \cap H^2\right) : u_t \in L^\infty(0,T;V_1),\ u_{tt} \in L^2(Q_T), \\ &\|u\|_{L^\infty(0,T;V_1\cap H^2)} \leq M,\ \|u_t\|_{L^\infty(0,T;V_1)} \leq M,\ \|u_{tt}\|_{L^2(Q_T)} \leq M\}, \\ W_1(M,T) = \{&u \in W(M,T) : u_{tt} \in L^\infty\left(0,T;L^2\right)\}. \end{aligned}$$

Now, we establish the following recurrent sequence $\{u_m\}$. The first term is chosen as $u_0 \equiv 0$, suppose that

$$u_{m-1} \in W_1(M,T), \tag{3.4}$$

we find $u_m \in W_1(M,T)$ $(m \geq 1)$ satisfying the linear variational problem

$$\begin{cases} \langle u_m''(t), v\rangle + \mu_m(t)\, a(u_m(t), v) = \langle F_m(t), v\rangle,\ \forall v \in V_1, \\ u_m(0) = \tilde{u}_0,\ u_m'(0) = \tilde{u}_1, \end{cases} \tag{3.5}$$

in which

$$\begin{cases} \mu_m(t) = \mu\left(\|u_m(t)\|_0^2\right), \\ F_m(x,t) = \sum\limits_{0\le i+j\le N-1} A_{ij}f[u_{m-1}](u_m - u_{m-1})^i\left(\|u_m(t)\|_0^2 - \|u_{m-1}(t)\|_0^2\right)^j, \\ A_{ij}f[u_{m-1}] = \frac{1}{i!j!}D_3^iD_4^jf(x,t,u_{m-1},\|u_{m-1}(t)\|_0^2), \\ \qquad\qquad i,j\in\mathbb{Z}_+,\ 0\le i+j\le N-1. \end{cases} \tag{3.6}$$

Now, we have the following theorem.

Theorem 3.2. *Let* (H_1), (H_2), (H_3) *hold. Then there exist a constant* $M>0$ *depending on* $\tilde{u}_0$, $\tilde{u}_1$, μ, ζ, ρ *and* $T>0$ *depending on* $\tilde{u}_0$, $\tilde{u}_1$, μ, f, ζ,ρ *such that, for* $u_0\equiv 0$, *there exists a recurrent sequence* $\{u_m\}\subset W_1(M,T)$ *defined by* (3.5) *and* (3.6).

*Proof. Step 1***.** *Approximating solutions.* Consider the basis $\{w_j\}$ for V_1 as in Lemma 2.6. Put

$$u_m^{(k)}(t) = \sum\nolimits_{j=1}^k c_{mj}^{(k)}(t)w_j, \tag{3.7}$$

where the coefficients $c_{mj}^{(k)}$ satisfy the system of nonlinear differential equations

$$\begin{cases} \langle \ddot{u}_m^{(k)}(t), w_j\rangle + \mu_m^{(k)}(t)\,a(u_m^{(k)}(t),w_j) = \langle F_m^{(k)}(t), w_j\rangle,\ j=1,...,k, \\ u_m^{(k)}(0) = \tilde{u}_{0k},\ \dot{u}_m^{(k)}(0) = \tilde{u}_{1k}, \end{cases} \tag{3.8}$$

in which

$$\begin{cases} \tilde{u}_{0k} = \sum_{j=1}^k \alpha_j^{(k)}w_j \longrightarrow \tilde{u}_0 \text{ strongly in } V_1\cap H^2, \\ \tilde{u}_{1k} = \sum_{j=1}^k \beta_j^{(k)}w_j \longrightarrow \tilde{u}_1 \text{ strongly in } V_1, \end{cases} \tag{3.9}$$

and

$$\begin{cases} \mu_m^{(k)}(t) = \mu\left(\left\|u_m^{(k)}(t)\right\|_0^2\right), \\ F_m^{(k)}(x,t) = \sum\limits_{0\le i+j\le N-1} A_{ij}f[u_{m-1}](u_m^{(k)} - u_{m-1})^i\left(\left\|u_m^{(k)}(t)\right\|_0^2 - \|u_{m-1}(t)\|_0^2\right)^j \end{cases} \tag{3.10}$$

with

$$A_{ij}f[u_{m-1}] = \frac{1}{i!j!}D_3^iD_4^jf(x,t,u_{m-1},\|u_{m-1}(t)\|_0^2),\ i,\ j\in\mathbb{Z}_+,\ 0\le i+j\le N-1. \tag{3.11}$$

The system (3.8), (3.9) can be written in the form

$$\begin{cases} \ddot{c}_{mj}^{(k)}(t) + \lambda_j\mu_m^{(k)}(t)c_{mj}^{(k)}(t) = \langle F_m^{(k)}(t), w_j\rangle, \quad 1\le j\le k, \\ c_{mj}^{(k)}(0) = \alpha_j^{(k)}, \quad \dot{c}_{mj}^{(k)}(0) = \beta_j^{(k)}. \end{cases} \tag{3.12}$$

It can see that, system (3.12) is equivalent to system of integral equations

$$\begin{aligned} c_{mj}^{(k)}(t) &= \alpha_j^{(k)} + t\beta_j^{(k)} - \lambda_j \int_0^t (t-s)\mu[c_m^{(k)}](s)c_{mj}^{(k)}(s)ds \\ &\quad + \int_0^t (t-s)F_{mj}[c_m^{(k)}]\,(s)\,ds, \end{aligned} \tag{3.13}$$

$1 \leq j \leq k$, where

$$\begin{aligned} \mu[c_m^{(k)}](t) &= \mu\left(\left\|u_m^{(k)}(t)\right\|_0^2\right), \\ F_{mj}[c_m^{(k)}]\,(t) &= \langle F_m^{(k)}\,(t)\,, w_j\rangle,\ 1 \leq j \leq k, \\ c_m^{(k)} &= (c_{m1}^{(k)}, ..., c_{mk}^{(k)}). \end{aligned}$$

Note that by (3.4), it is not difficult to prove that the system (3.13) has a unique solution $c_{mj}^{(k)}(t)$, $1 \leq j \leq k$ on interval $[0, T_m^{(k)}] \subset [0, T]$, so let us omit the details.

The following estimates allow one to take $T_m^{(k)} = T$ independent of m and k.

Step 2. *A priori estimates.* Put

$$S_m^{(k)}\,(t) = X_m^{(k)}\,(t) + Y_m^{(k)}\,(t) + \int_0^t \left\|\ddot{u}_m^{(k)}(s)\right\|_0^2 ds, \tag{3.14}$$

where

$$\begin{cases} X_m^{(k)}\,(t) = \left\|\dot{u}_m^{(k)}(t)\right\|_0^2 + \mu_m^{(k)}\,(t)\left\|u_m^{(k)}(t)\right\|_a^2, \\ Y_m^{(k)}\,(t) = \left\|\dot{u}_m^{(k)}(t)\right\|_a^2 + \mu_m^{(k)}\,(t)\left\|Au_m^{(k)}(t)\right\|_0^2. \end{cases} \tag{3.15}$$

Then, it follows from (3.8), (3.14) and (3.15) that

$$\begin{aligned} S_m^{(k)}\,(t) &= S_m^{(k)}\,(0) + \int_0^t \dot{\mu}_m^{(k)}\,(s)\left[\left\|u_m^{(k)}(s)\right\|_a^2 + \left\|Au_m^{(k)}(s)\right\|_0^2\right] ds \\ &\quad +2\int_0^t \langle F_m^{(k)}\,(s)\,, \dot{u}_m^{(k)}(s)\rangle ds + 2\int_0^t a\left(F_m^{(k)}\,(s)\,, \dot{u}_m^{(k)}(s)\right) ds \\ &\quad + \int_0^t \left\|\ddot{u}_m^{(k)}(s)\right\|_0^2 ds \\ &= S_m^{(k)}\,(0) + \sum\nolimits_{j=1}^4 I_j. \end{aligned} \tag{3.16}$$

We shall estimate all the terms I_j on the right-hand side of (3.16).

The term I_1. By (3.10), we have

$$\dot{\mu}_m^{(k)}\,(t) = \frac{\partial}{\partial t}\mu\left(\left\|u_m^{(k)}(t)\right\|_0^2\right) = 2\mu'\left(\left\|u_m^{(k)}(t)\right\|_0^2\right)\langle u_m^{(k)}\,(t)\,, \dot{u}_m^{(k)}(t)\rangle. \tag{3.17}$$

By using assumption (H_2) and the following inequalities:

$$\left\|u_m^{(k)}(t)\right\|_0 \leq \left\|u_m^{(k)}(t)\right\|_1 \leq \frac{1}{\sqrt{C_0}}\left\|u_m^{(k)}(t)\right\|_a \leq \sqrt{\frac{1}{C_0\mu_*}}\sqrt{S_m^{(k)}(t)}, \tag{3.18}$$

$$\left\|\dot{u}_m^{(k)}(t)\right\|_0 \leq \sqrt{S_m^{(k)}(t)}, \tag{3.19}$$

we deduce from (2.9) and (3.17) that

$$\begin{aligned}\left|\dot{\mu}_m^{(k)}(t)\right| &\leq 2\left|\mu'\left(\left\|u_m^{(k)}(t)\right\|_0^2\right)\right|\left\|u_m^{(k)}(t)\right\|_0\left\|\dot{u}_m^{(k)}(t)\right\|_0 \\ &\leq 2\Phi_{\mu'}\left(\left\|u_m^{(k)}(t)\right\|_0^2\right)\sqrt{\frac{1}{C_0\mu_*}}S_m^{(k)}(t) \\ &\leq 2\Phi_{\mu'}\left(\frac{1}{C_0\mu_*}S_m^{(k)}(t)\right)\sqrt{\frac{1}{C_0\mu_*}}S_m^{(k)}(t).\end{aligned} \tag{3.20}$$

Using the following inequality

$$\begin{aligned}S_m^{(k)}(t) &\geq \mu_m^{(k)}(t)\left[\left\|u_m^{(k)}(t)\right\|_a^2 + \left\|Au_m^{(k)}(t)\right\|_0^2\right] \\ &\geq \mu_*\left[\left\|u_m^{(k)}(t)\right\|_a^2 + \left\|Au_m^{(k)}(t)\right\|_0^2\right],\end{aligned} \tag{3.21}$$

from (3.20), it leads to

$$\begin{aligned}I_1 &= \int_0^t \dot{\mu}_m^{(k)}(s)\left[\left\|u_m^{(k)}(s)\right\|_a^2 + \left\|Au_m^{(k)}(s)\right\|_0^2\right]ds \\ &\leq 2\int_0^t \Phi_{\mu'}\left(\frac{1}{C_0\mu_*}S_m^{(k)}(s)\right)\sqrt{\frac{1}{C_0\mu_*}}S_m^{(k)}(s)\,\frac{1}{\mu_*}S_m^{(k)}(s)\,ds \\ &= \int_0^t \Phi_1\left(S_m^{(k)}(s)\right)ds,\end{aligned} \tag{3.22}$$

where $\Phi_1(z) = \frac{2}{\mu_*}\sqrt{\frac{1}{C_0\mu_*}}z^2\Phi_{\mu'}\left(\frac{1}{C_0\mu_*}z\right)$.

The term I_2. Using the inequalities $(a+b)^p \leq 2^{p-1}(a^p+b^p)$, for all a, $b \geq 0$, $p \geq 1$; $s^q \leq 1 + s^p$, for all $s \geq 0$, $lq \in (0,p]$, we get from (3.10) that

$$\begin{aligned}&\left|F_m^{(k)}(x,t)\right| \\ \leq\ & \overline{K}_M(f)\sum_{0\leq i+j\leq N-1}\frac{1}{i!j!}\left|u_m^{(k)}(t) - u_{m-1}(t)\right|^i \\ &\times\left|\left\|u_m^{(k)}(t)\right\|_0^2 - \|u_{m-1}(t)\|_0^2\right|^j\end{aligned}$$

$$
\begin{aligned}
&\leq \overline{K}_M(f) \sum_{0\leq i+j\leq N-1} \frac{1}{i!j!} \left(\left| u_m^{(k)}(t) \right| + |u_{m-1}(t)| \right)^i \\
&\quad \times \left(\left\| u_m^{(k)}(t) \right\|_0 + \|u_{m-1}(t)\|_0 \right)^{2j} \\
&\leq \overline{K}_M(f) \sum_{0\leq i+j\leq N-1} \frac{1}{i!j!} \left(\left\| u_m^{(k)}(t) \right\|_{C^0([\rho,1])} + \|u_{m-1}(t)\|_{C^0([\rho,1])} \right)^i \\
&\quad \times \left(\left\| u_m^{(k)}(t) \right\|_0 + \|u_{m-1}(t)\|_0 \right)^{2j} \\
&\leq \overline{K}_M(f) \sum_{0\leq i+j\leq N-1} \frac{1}{i!j!} \left(\sqrt{1-\rho} \right)^i \left(\left\| \nabla u_m^{(k)}(t) \right\| + \|\nabla u_{m-1}(t)\| \right)^i \\
&\quad \times \left(\left\| u_m^{(k)}(t) \right\|_1 + \|u_{m-1}(t)\|_1 \right)^{2j} \\
&\leq \overline{K}_M(f) \sum_{0\leq i+j\leq N-1} \frac{1}{i!j!} \left(\sqrt{\frac{1-\rho}{\rho}} \right)^i \left(\left\| \nabla u_m^{(k)}(t) \right\|_0 + \|\nabla u_{m-1}(t)\|_0 \right)^i \\
&\quad \times \left(\left\| u_m^{(k)}(t) \right\|_1 + \|u_{m-1}(t)\|_1 \right)^{2j} \\
&\leq \overline{K}_M(f) \sum_{0\leq i+j\leq N-1} \frac{1}{i!j!} \left(\sqrt{\frac{1-\rho}{\rho}} \right)^i \left(\left\| u_m^{(k)}(t) \right\|_1 + \|u_{m-1}(t)\|_1 \right)^i \\
&\quad \times \left(\left\| u_m^{(k)}(t) \right\|_1 + \|u_{m-1}(t)\|_1 \right)^{2j} \\
&= \overline{K}_M(f) \sum_{0\leq i+j\leq N-1} \frac{1}{i!j!} \left(\sqrt{\frac{1-\rho}{\rho}} \right)^i \left(\left\| u_m^{(k)}(t) \right\|_1 + \|u_{m-1}(t)\|_1 \right)^{i+2j} \\
&\leq \overline{K}_M(f) \sum_{0\leq i+j\leq N-1} \frac{1}{i!j!} \left(\sqrt{\frac{1-\rho}{\rho}} \right)^i \left(\sqrt{\frac{S_m^{(k)}(t)}{C_0\mu_*}} + M \right)^{i+2j} \\
&\leq \overline{K}_M(f) \sum_{0\leq i+j\leq N-1} \frac{1}{i!j!} \left(\sqrt{\frac{1-\rho}{\rho}} \right)^i \qquad (3.23) \\
&\quad \times 2^{i+2j-1} \left[\left(\sqrt{\frac{S_m^{(k)}(t)}{C_0\mu_*}} \right)^{i+2j} + M^{i+2j} \right] \\
&\leq \overline{K}_M(f) \sum_{0\leq i+j\leq N-1} \frac{1}{i!j!} \left(\sqrt{\frac{1-\rho}{\rho}} \right)^i
\end{aligned}
$$

$$
\begin{aligned}
& \times 2^{i+2j-1}\left[1+\left(\frac{S_m^{(k)}(t)}{C_0\mu_*}\right)^{N-1}+1+M^{2N-2}\right] \\
\leq\ & \overline{K}_M(f)\left(1+M^{2N-2}\right)\sum_{0\leq i+j\leq N-1}\frac{1}{i!j!}\left(\sqrt{\frac{1-\rho}{\rho}}\right)^i \\
& \times 2^{i+2j}\left[1+\left(\frac{S_m^{(k)}(t)}{C_0\mu_*}\right)^{N-1}\right] \\
\leq\ & \overline{K}_M(f)\left(1+M^{2N-2}\right)\left[1+\left(\frac{1}{C_0\mu_*}\right)^{N-1}\right]\sum_{0\leq i+j\leq N-1}\frac{1}{i!j!}\left(\sqrt{\frac{1-\rho}{\rho}}\right)^i \\
& \times 2^{i+2j}\left[1+\left(S_m^{(k)}(t)\right)^{N-1}\right].
\end{aligned}
$$

Hence

$$
\begin{aligned}
\left\|F_m^{(k)}(t)\right\|_0 \leq\ & \overline{K}_M(f)\left(1+M^{2N-2}\right)\left[1+\left(\frac{1}{C_0\mu_*}\right)^{N-1}\right]\sqrt{\frac{1-\rho^2}{2}} \\
& \times\sum_{0\leq i+j\leq N-1}\frac{1}{i!j!}\left(\sqrt{\frac{1-\rho}{\rho}}\right)^i 2^{i+2j}\left[1+\left(S_m^{(k)}(t)\right)^{N-1}\right] \\
=\ & \xi_2(M,\rho)\left[1+\left(S_m^{(k)}(t)\right)^{N-1}\right],
\end{aligned} \tag{3.24}
$$

where

$$
\begin{aligned}
\xi_2(M,\rho) =\ & \overline{K}_M(f)\left(1+M^{2N-2}\right)\left[1+\left(\frac{1}{C_0\mu_*}\right)^{N-1}\right]\sqrt{\frac{1-\rho^2}{2}} \\
& \times\sum_{0\leq i+j\leq N-1}\frac{1}{i!j!}\left(\sqrt{\frac{1-\rho}{\rho}}\right)^i 2^{i+2j}.
\end{aligned}
$$

It implies from (3.15) and (3.24) that

$$
\begin{aligned}
I_2 =\ & 2\int_0^t\left\langle F_m^{(k)}(s),\dot{u}_m^{(k)}(s)\right\rangle ds \\
\leq\ & 2\xi_2(M,\rho)\int_0^t\left[1+\left(S_m^{(k)}(s)\right)^{N-1}\right]\sqrt{S_m^{(k)}(s)}ds \\
=\ & \int_0^t\Phi_2\left(S_m^{(k)}(s)\right)ds,
\end{aligned} \tag{3.25}
$$

where $\Phi_2(z) = 2\xi_2(M,\rho)\left(1+z^{N-1}\right)\sqrt{z}$. *The term* I_3. By (3.10), we have

$$\begin{aligned}
&\nabla F_m^{(k)}(x,t) \\
= \ & D_1 f[u_{m-1}] + D_3 f[u_{m-1}]\nabla u_{m-1} \\
&+ \sum_{1\le i+j\le N-1} \left(D_1 A_{ij} f[u_{m-1}] + D_3 A_{ij} f[u_{m-1}]\nabla u_{m-1}\right)(u_m^{(k)} - u_{m-1})^i \\
&\times \left(\left\|u_m^{(k)}(t)\right\|_0^2 - \|u_{m-1}(t)\|_0^2\right)^j \qquad (3.26)\\
&+ \sum_{1\le i+j\le N-1} A_{ij} f[u_{m-1}] i (u_m^{(k)} - u_{m-1})^{i-1}(\nabla u_m^{(k)} - \nabla u_{m-1}) \\
&\times \left(\left\|u_m^{(k)}(t)\right\|_0^2 - \|u_{m-1}(t)\|_0^2\right)^j,
\end{aligned}$$

so

$$\begin{aligned}
&\left|\nabla F_m^{(k)}(x,t)\right| \\
\le \ & |D_1 f[u_{m-1}] + D_3 f[u_{m-1}]\nabla u_{m-1}| \\
&+ \sum_{1\le i+j\le N-1} \left|\left(D_1 A_{ij} f[u_{m-1}] + D_3 A_{ij} f[u_{m-1}]\nabla u_{m-1}\right)\left(u_m^{(k)} - u_{m-1}\right)^i\right. \\
&\left.\times \left(\left\|u_m^{(k)}(t)\right\|_0^2 - \|u_{m-1}(t)\|_0^2\right)^j\right| \\
&+ \sum_{1\le i+j\le N-1} \left|A_{ij} f[u_{m-1}] i (u_m^{(k)} - u_{m-1})^{i-1}(\nabla u_m^{(k)} - \nabla u_{m-1})\right. \\
&\left.\times \left(\left\|u_m^{(k)}(t)\right\|_0^2 - \|u_{m-1}(t)\|_0^2\right)^j\right| \qquad (3.27)\\
\le \ & \overline{K}_M(f)\left(1 + |\nabla u_{m-1}|\right) \\
&+ \overline{K}_M(f) \sum_{1\le i+j\le N-1} \frac{1}{i!j!}\left(1 + |\nabla u_{m-1}|\right)\left|u_m^{(k)} - u_{m-1}\right|^i \\
&\times \left|\left\|u_m^{(k)}(t)\right\|_0^2 - \|u_{m-1}(t)\|_0^2\right|^j \\
&+ \overline{K}_M(f) \sum_{1\le i+j\le N-1} \frac{i}{i!j!}\left|u_m^{(k)} - u_{m-1}\right|^{i-1}\left|\nabla u_m^{(k)} - \nabla u_{m-1}\right| \\
&\times \left|\left\|u_m^{(k)}(t)\right\|_0^2 - \|u_{m-1}(t)\|_0^2\right|^j \\
\le \ & \overline{K}_M(f)\left(1 + |\nabla u_{m-1}|\right)
\end{aligned}$$

$$
\begin{aligned}
&+\overline{K}_M(f)\sum_{1\le i+j\le N-1}\frac{1}{i!j!}\left(1+|\nabla u_{m-1}|\right)\left(\sqrt{\frac{1-\rho}{\rho}}\right)^i \\
&\times\left(\left\|\nabla u_m^{(k)}(t)\right\|_0+\|\nabla u_{m-1}(t)\|_0\right)^i\left(\left\|u_m^{(k)}(t)\right\|_1+\|u_{m-1}(t)\|_1\right)^{2j} \\
&+\overline{K}_M(f)\sum_{1\le i+j\le N-1}\frac{i}{i!j!}\left|\nabla u_m^{(k)}-\nabla u_{m-1}\right|\left(\sqrt{\frac{1-\rho}{\rho}}\right)^i \\
&\times\left(\left\|\nabla u_m^{(k)}(t)\right\|_0+\|\nabla u_{m-1}(t)\|_0\right)^{i-1}\left(\left\|u_m^{(k)}(t)\right\|_1+\|u_{m-1}(t)\|_1\right)^{2j}.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
&\left\|\nabla F_m^{(k)}(t)\right\|_0 \qquad\qquad (3.28)\\
\le\ &\overline{K}_M(f)\left(\sqrt{\frac{1-\rho^2}{2}}+\|\nabla u_{m-1}\|_0\right) \\
&+\overline{K}_M(f)\sum_{1\le i+j\le N-1}\frac{1}{i!j!}\left(\sqrt{\frac{1-\rho^2}{2}}+\|\nabla u_{m-1}\|_0\right)\left(\sqrt{\frac{1-\rho}{\rho}}\right)^i \\
&\times\left(\left\|u_m^{(k)}(t)\right\|_1+\|u_{m-1}(t)\|_1\right)^i\left(\left\|u_m^{(k)}(t)\right\|_1+\|u_{m-1}(t)\|_1\right)^{2j} \\
&+\overline{K}_M(f)\sum_{1\le i+j\le N-1}\frac{i}{i!j!}\left\|\nabla u_m^{(k)}(t)-\nabla u_{m-1}(t)\right\|_0\left(\sqrt{\frac{1-\rho}{\rho}}\right)^{i-1} \\
&\times\left(\left\|u_m^{(k)}(t)\right\|_1+\|u_{m-1}(t)\|_1\right)^{i-1}\left(\left\|u_m^{(k)}(t)\right\|_1+\|u_{m-1}(t)\|_1\right)^{2j} \\
\le\ &\overline{K}_M(f)\left(\sqrt{\frac{1-\rho^2}{2}}+M\right)+\overline{K}_M(f)\sum_{1\le i+j\le N-1}\frac{1}{i!j!}\left(\sqrt{\frac{1-\rho^2}{2}}+M\right) \\
&\times\left(\sqrt{\frac{1-\rho}{\rho}}\right)^i\left(\sqrt{\frac{S_m^{(k)}(t)}{C_0\mu_*}}+M\right)^{i+2j} \\
&+\overline{K}_M(f)\sum_{1\le i+j\le N-1}\frac{i}{i!j!}\left(\sqrt{\frac{1-\rho}{\rho}}\right)^{i-1}\left(\sqrt{\frac{S_m^{(k)}(t)}{C_0\mu_*}}+M\right)^{i+2j} \\
\le\ &\overline{K}_M(f)(1+M) \\
&+\overline{K}_M(f)(1+M)\sum_{1\le i+j\le N-1}\frac{1}{i!j!}\left(\sqrt{\frac{1}{\rho}}\right)^i\left(\sqrt{\frac{S_m^{(k)}(t)}{C_0\mu_*}}+M\right)^{i+2j}
\end{aligned}
$$

$$
\begin{aligned}
& +\overline{K}_M(f) \sum_{1 \le i+j \le N-1} \frac{i}{i!j!} \left(\sqrt{\frac{1}{\rho}}\right)^{i-1} \left(\sqrt{\frac{S_m^{(k)}(t)}{C_0\mu_*}} + M\right)^{i+2j} \\
\le\ & \overline{K}_M(f)(1+M) \\
& +2N\overline{K}_M(f)(1+M) \sum_{1 \le i+j \le N-1} \frac{1}{i!j!} \left(\sqrt{\frac{1}{\rho}}\right)^{i} \left(\sqrt{\frac{S_m^{(k)}(t)}{C_0\mu_*}} + M\right)^{i+2j} \\
\le\ & 2N\overline{K}_M(f)(1+M) \sum_{0 \le i+j \le N-1} \frac{1}{i!j!} \left(\sqrt{\frac{1}{\rho}}\right)^{i} \left(\sqrt{\frac{S_m^{(k)}(t)}{C_0\mu_*}} + M\right)^{i+2j} \\
\le\ & 2N\overline{K}_M(f)(1+M) \sum_{0 \le i+j \le N-1} \frac{1}{i!j!} \left(\sqrt{\frac{1}{\rho}}\right)^{i} 2^{i+2j-1} \\
& \times \left[\left(\sqrt{\frac{S_m^{(k)}(t)}{C_0\mu_*}}\right)^{i+2j} + M^{i+2j}\right] \\
\le\ & 2N\overline{K}_M(f)(1+M) \\
& \times \sum_{0 \le i+j \le N-1} \frac{1}{i!j!} \left(\sqrt{\frac{1}{\rho}}\right)^{i} 2^{i+2j-1} \left[1 + \left(\frac{S_m^{(k)}(t)}{C_0\mu_*}\right)^{N-1} + 1 + M^{2N-2}\right] \\
\le\ & 2N\overline{K}_M(f)(1+M)\left(1+M^{2N-2}\right) \\
& \times \sum_{0 \le i+j \le N-1} \frac{1}{i!j!} \left(\sqrt{\frac{1}{\rho}}\right)^{i} 2^{i+2j} \left[1 + \left(\frac{S_m^{(k)}(t)}{C_0\mu_*}\right)^{N-1}\right] \\
\le\ & 2N\overline{K}_M(f)(1+M)\left(1+M^{2N-2}\right) \left[1 + \left(\frac{1}{C_0\mu_*}\right)^{N-1}\right] \\
& \times \sum_{0 \le i+j \le N-1} \frac{1}{i!j!} \left(\sqrt{\frac{1}{\rho}}\right)^{i} 2^{i+2j} \left[1 + \left(S_m^{(k)}(t)\right)^{N-1}\right] \\
=\ & \xi_3(M,\rho)\left[1 + \left(S_m^{(k)}(t)\right)^{N-1}\right],
\end{aligned}
$$

where

$$
\begin{aligned}
\xi_3(M,\rho) =\ & 2N\overline{K}_M(f)(1+M)\left(1+M^{2N-2}\right) \\
& \times \left[1 + \left(\frac{1}{C_0\mu_*}\right)^{N-1}\right] \sum_{0 \le i+j \le N-1} \frac{1}{i!j!} \left(\sqrt{\frac{1}{\rho}}\right)^{i} 2^{i+2j}.
\end{aligned}
$$

On the other hand, (3.24) and (3.28) give

$$\begin{aligned}
\left\|F_m^{(k)}(t)\right\|_1 &= \sqrt{\left\|F_m^{(k)}(t)\right\|_0^2 + \left\|\nabla F_m^{(k)}(t)\right\|_0^2} \\
&\leq \left\|F_m^{(k)}(t)\right\|_0 + \left\|\nabla F_m^{(k)}(t)\right\|_0 \\
&\leq \left[\xi_2(M,\rho) + \xi_3(M,\rho)\right]\left[1 + \left(S_m^{(k)}(t)\right)^{N-1}\right].
\end{aligned} \tag{3.29}$$

Hence, it follows from (3.15) and (3.29) that

$$\begin{aligned}
I_3 &= 2\int_0^t a\left(F_m^{(k)}(s), \dot{u}_m^{(k)}(s)\right) ds \\
&\leq \frac{2C_1}{\sqrt{C_0}}\int_0^t \left\|F_m^{(k)}(s)\right\|_1 \sqrt{S_m^{(k)}(s)}ds \\
&\leq \frac{2C_1}{\sqrt{C_0}}\left[\xi_2(M,\rho) + \xi_3(M,\rho)\right]\int_0^t \left[1 + \left(S_m^{(k)}(s)\right)^{N-1}\right]\sqrt{S_m^{(k)}(s)}ds \\
&= \int_0^t \Phi_3\left(S_m^{(k)}(s)\right) ds,
\end{aligned} \tag{3.30}$$

where $\Phi_3(z) = \frac{2C_1}{\sqrt{C_0}}\left[\xi_2(M,\rho) + \xi_3(M,\rho)\right]\left(1 + z^{N-1}\right)\sqrt{z}$.

The term I_4. Equation (3.8) can be rewritten as follows:

$$\langle \ddot{u}_m^{(k)}(t), w_j\rangle + \mu_m^{(k)}(t)\langle Au_m^{(k)}(t), w_j\rangle = \langle F_m^{(k)}(t), w_j\rangle,\ j = 1, ..., k. \tag{3.31}$$

Thus, it follows after replacing w_j with $\ddot{u}_m^{(k)}(t)$ and integrating that

$$\begin{aligned}
I_4 &= \int_0^t \left\|\ddot{u}_m^{(k)}(s)\right\|_0^2 ds \\
&\leq \int_0^t \left[\mu_m^{(k)}(s)\left\|Au_m^{(k)}(s)\right\|_0 + \left\|F_m^{(k)}(s)\right\|_0\right]^2 ds \\
&\leq 2\int_0^t \left(\mu_m^{(k)}(s)\right)^2 \left\|Au_m^{(k)}(s)\right\|_0^2 ds + 2\int_0^t \left\|F_m^{(k)}(s)\right\|_0^2 ds \\
&= I_4^{(1)} + I_4^{(2)}.
\end{aligned} \tag{3.32}$$

We shall estimate step by step two integrals $I_4^{(1)}$, $I_4^{(2)}$.

Estimate $I_4^{(1)}$. By using (H_2) and (2.9), we deduce from (3.10) and (3.18), that

$$\left|\mu_m^{(k)}(t)\right| = \mu\left(\left\|u_m^{(k)}(t)\right\|_0^2\right) \leq \Phi_\mu\left(\left\|u_m^{(k)}(t)\right\|_0^2\right) \leq \Phi_\mu\left(\frac{S_m^{(k)}(t)}{C_0\mu_*}\right). \tag{3.33}$$

Therefore, we obtain from (3.21) and (3.33) that

$$\begin{aligned} I_4^{(1)} &= 2\int_0^t \left(\mu_m^{(k)}(s)\right)^2 \left\|Au_m^{(k)}(s)\right\|_0^2 ds \\ &\leq 2\int_0^t \Phi_\mu\left(\frac{S_m^{(k)}(s)}{C_0\mu_*}\right) S_m^{(k)}(s)\,ds. \end{aligned} \tag{3.34}$$

Estimate $I_4^{(2)}$. We again use the inequality (3.24), we have

$$\begin{aligned} I_4^{(2)} &= 2\int_0^t \left\|F_m^{(k)}(s)\right\|_0^2 ds \\ &\leq 2\xi_2^2(M,\rho)\int_0^t \left[1+\left(S_m^{(k)}(s)\right)^{N-1}\right]^2 ds \\ &\leq 4\xi_2^2(M,\rho)\int_0^t \left[1+\left(S_m^{(k)}(s)\right)^{2(N-1)}\right] ds. \end{aligned} \tag{3.35}$$

It follows from (3.32), (3.34) and (3.35) that

$$\begin{aligned} I_4 &\leq 2\int_0^t \Phi_\mu\left(\frac{S_m^{(k)}(s)}{C_0\mu_*}\right) S_m^{(k)}(s)\,ds \\ &\quad +4\xi_2^2(M,\rho)\int_0^t \left[1+\left(S_m^{(k)}(s)\right)^{2(N-1)}\right] ds \\ &= \int_0^t \Phi_4\left(S_m^{(k)}(s)\right) ds, \end{aligned} \tag{3.36}$$

where $\Phi_4(z) = 2z\Phi_\mu\left(\frac{z}{C_0\mu_*}\right) + 4\xi_2^2(M,\rho)\left(1+z^{2(N-1)}\right)$.

Now, we need an estimate on the term $S_m^{(k)}(0)$. We have

$$S_m^{(k)}(0) = \|\tilde{u}_{1k}\|_0^2 + \|\tilde{u}_{1k}\|_a^2 + \mu\left(\|\tilde{u}_{0k}\|_0^2\right)\left[\|\tilde{u}_{0k}\|_a^2 + \|A\tilde{u}_{0k}\|_0^2\right]. \tag{3.37}$$

By means of the convergences (3.9), we can deduce the existence of a constant $M > 0$ independent of k and m such that

$$S_m^{(k)}(0) \leq \frac{M^2}{4}. \tag{3.38}$$

Combining (3.16), (3.22), (3.25), (3.30), (3.36) and (3.38), it leads to

$$S_m^{(k)}(t) \leq \frac{M^2}{4} + \int_0^t \Psi\left(S_m^{(k)}(s)\right) ds, \text{ for } 0 \leq t \leq T_m^{(k)} \leq T, \tag{3.39}$$

where

$$\Psi(z) = \sum\nolimits_{i=1}^4 \Phi_i(z),\ \Psi \in C(\mathbb{R}_+;\mathbb{R}_+) \text{ and } \Psi \text{ is nondecreasing.} \tag{3.40}$$

Then we can show that there exists a constant $T>0$ independent of k and m such that

$$S_m^{(k)}(t) \le M^2 \ \forall t\in[0,T], \ \textit{for all } k \textit{ and } m\in\mathbb{N}. \tag{3.41}$$

Indeed, put $y(t)=\frac{M^2}{4}+\int_0^t \Psi\left(S_m^{(k)}(s)\right)ds$ and $G(w)=\int_0^w \dfrac{dz}{\Psi(z)}$, we have

$$\begin{cases} S_m^{(k)}(t)\le y(t), \\ y(0)=\frac{M^2}{4}=M_1, \\ y'(t)=\Psi\left(S_m^{(k)}(t)\right)\le \Psi(y(t)). \end{cases}$$

Hence

$$G(y(t))-G(M_1)=\int_{M_1}^{y(t)}\frac{dz}{\Psi(z)}=\int_0^t \frac{y'(s))ds}{\Psi(y(s))}\le t. \tag{3.42}$$

By (3.40), we can deduce the existence of a constant $C_*=C_*(M)>0$ such that

$$\Psi(z)\ge C_*(M)(1+z^2),$$

for all $z\ge 0$. So $G_\infty=\int_0^\infty \dfrac{dz}{\Psi(z)}<\infty$ and $w\longmapsto G(w)=\int_0^w \dfrac{dz}{\Psi(z)}$ is a continuous and nondecreasing function on $\mathbb{R}_+$, it leads to the function $G^{-1}: [0,G_\infty)\to\mathbb{R}_+$ is defined and continuous, nondecreasing on $[0,G_\infty)$.

Choose $T\in(0,T^*]$ such that $T\le \int_{M_1}^{M^2}\dfrac{dz}{\Psi(z)}=G(M^2)-G(M_1)$. Then we obtain

$$G(y(t))\le t+G(M_1)\le T+G(M_1)\le G(M^2)<G_\infty, \ \forall t\in[0,T],$$

it means that

$$S_m^{(k)}(t)\le y(t)\le G^{-1}G(M^2)=M^2.$$

Hence we can take constant $T_m^{(k)}=T$ for all k and m. Therefore, we have

$$u_m^{(k)}\in W(M,T), \text{ for all } k \text{ and } m\in\mathbb{N}. \tag{3.43}$$

Step 3. *Convergence*. From (3.43), we can extract from $\{u_m^{(k)}\}$ a subsequence $\{u_m^{(k_i)}\}$ such that

$$\begin{cases} u_m^{(k_i)}\to u_m \text{ in } L^\infty\left(0,T;V_1\cap H^2\right) \text{ weak*}, \\ \dot u_m^{(k_i)}\to u_m' \text{ in } L^\infty(0,T;V_1) \text{ weak*}, \\ \ddot u_m^{(k_i)}\to u_m'' \text{ in } L^2(Q_T) \text{ weak}, \\ u_m\in W(M,T). \end{cases} \tag{3.44}$$

By the compactness lemma of Lions ([2], p. 57) and applying the theorem of Fischer-Riesz, from (3.44), one has a subsequence of $\{u_m^{(k)}\}$, denoted by the same symbol satisfying

$$\begin{cases} u_m^{(k)} \to u_m \text{ strongly in } L^2(0,T;V_1) \text{ and a.e. in } Q_T, \\ \dot{u}_m^{(k)} \to u_m' \text{ strongly in } L^2(Q_T) \text{ and a.e. in } Q_T. \end{cases} \tag{3.45}$$

From (3.45) and dominated convergence theorem, we obtain

$$F_m^{(k)} \to F_m \text{ strongly in } L^2(Q_T). \tag{3.46}$$

On the other hand, we have

$$\begin{aligned} \left|\mu_m^{(k)}(t) - \mu_m(t)\right| &= \left|\mu\left(\left\|u_m^{(k)}(t)\right\|_0^2\right) - \mu\left(\|u_m(t)\|_0^2\right)\right| \\ &\leq 2M\tilde{K}_M(\mu)\left\|u_m^{(k)}(t) - u_m(t)\right\|_0. \end{aligned} \tag{3.47}$$

Hence, from (3.44) and (3.47), we get

$$\mu_m^{(k)} \to \mu_m \text{ strongly in } L^2(0,T). \tag{3.48}$$

Passing to limit in (3.8), (3.9), we have u_m satisfying (3.5), (3.6) in $L^2(0,T)$.
On the other hand, it follows from (3.5) and (3.44) that

$$u_m'' = -\mu_m(t)\,Au_m + F_m \in L^\infty(0,T;L^2). \tag{3.49}$$

Therefore, $u_m \in W_1(M,T)$ and Theorem 3.2 is proved. □

Next, to obtain the main result, we put

$$W_1(T) = \{v \in L^\infty(0,T;V_1) : v' \in L^\infty(0,T;L^2)\},$$

then $W_1(T)$ is a Banach space with respect to the norm

$$\|v\|_{W_1(T)} = \|v\|_{L^\infty(0,T;V_1)} + \|v'\|_{L^\infty(0,T;L^2)}.$$

Theorem 3.3. *Let (H_1)-(H_3) hold. Then, there exist constants $M > 0$ and $T > 0$ such that*

(i) *Problem (1.1)-(1.3) has a unique weak solution $u \in W_1(M,T)$.*

(ii) *The recurrent sequence $\{u_m\}$, defined by (3.5) and (3.6) converges at a rate of order N to the solution u strongly in the space $W_1(T)$ in the snse*

$$\|u_m - u\|_{W_1(T)} \leq C\,\|u_{m-1} - u\|_{W_1(T)}^N, \tag{3.50}$$

for all $m \geq 1$, where C is a suitable constant. On the other hand, the estimate is fulfilled

$$\|u_m - u\|_{W_1(T)} \leq C_T\beta_T^{N^m}, \text{ for all } m \in \mathbb{N}, \tag{3.51}$$

where C_T and $0 < \beta_T < 1$ are the constants depending only on T.

Proof. Existence. We can prove that $\{u_m\}$ is a Cauchy sequence in $W_1(T)$.

Indeed, let $w_m = u_{m+1} - u_m$. Then w_m satisfies the variational problem

$$\begin{cases} \langle w''_m(t), v\rangle + \mu_{m+1}(t)\, a(w_m(t), v) + [\mu_{m+1}(t) - \mu_m(t)]\, \langle Au_m(t), v\rangle \\ \qquad = \langle F_{m+1}(t) - F_m(t), v\rangle, \ \forall v \in V_1, \\ w_m(0) = w'_m(0) = 0. \end{cases} \tag{3.52}$$

Taking $v = w'_m(t)$ in $(3.52)_1$, after integrating in t, we get

$$\begin{aligned} Z_m(t) &= \int_0^t \mu'_{m+1}(s)\, \|w_m(s)\|_a^2\, ds \\ &\quad -2\int_0^t [\mu_{m+1}(s) - \mu_m(s)]\, \langle Au_m(s), w'_m(s)\rangle ds \\ &\quad +2\int_0^t \left\langle F_{m+1}(s) - F_m(s), w'_m(s)\right\rangle ds \\ &:= J_1 + J_2 + J_3, \end{aligned} \tag{3.53}$$

where

$$\begin{aligned} Z_m(t) &= \left\|w'_m(t)\right\|_0^2 + \mu_{m+1}(t)\, \|w_m(t)\|_a^2 \\ &\geq \left\|w'_m(t)\right\|_0^2 + \mu_*\, \|w_m(t)\|_a^2 \\ &\geq \left\|w'_m(t)\right\|_0^2 + \mu_* C_0\, \|w_m(t)\|_1^2 \\ &\geq 2\sqrt{\mu_* C_0}\, \left\|w'_m(t)\right\|_0 \|w_m(t)\|_1, \end{aligned} \tag{3.54}$$

and all integrals on the right - hand side of (3.53) are estimated as follows.

Estimating J_1. It follows from (3.44) that

$$\begin{aligned} \left|\mu'_m(t)\right| &= 2\left|\mu'\left(\|u_m(t)\|_0^2\right)\right| \left|\langle u_m(t), u'_m(t)\rangle\right| \\ &\leq 2\tilde{K}_M(\mu)\, \|u_m(t)\|_0 \left\|u'_m(t)\right\|_0 \\ &\leq 2\tilde{K}_M(\mu)\, \|u_m(t)\|_1 \left\|u'_m(t)\right\|_0 \\ &\leq 2M^2\tilde{K}_M(\mu), \end{aligned} \tag{3.55}$$

this implies that

$$J_1 = \int_0^t \bar{\mu}'_{m+1}(s)\, \|w_m(s)\|_a^2\, ds \leq \frac{2}{\mu_*} M^2 \tilde{K}_M(\mu) \int_0^t Z_m(s) ds. \tag{3.56}$$

Estimating J_2.

$$\begin{aligned} |\mu_{m+1}(t) - \mu_m(t)| &= \left|\mu\left(\|u_{m+1}(t)\|_0^2\right) - \mu\left(\|u_m(t)\|_0^2\right)\right| \\ &\leq \tilde{K}_M(\mu) \left|\|u_{m+1}(t)\|_0^2 - \|u_m(t)\|_0^2\right| \\ &\leq 2M\tilde{K}_M(\mu)\, \|w_m(t)\|_0 . \end{aligned} \tag{3.57}$$

Thus

$$\begin{aligned} J_2 &= -2\int_0^t [\mu_{m+1}(s) - \mu_m(s)] \langle Au_m(s), w_m'(s)\rangle ds \qquad (3.58) \\ &\leq 4M\tilde{K}_M(\mu)\int_0^t \|w_m(s)\|_0 \|Au_m(s)\|_0 \left\|w_m'(s)\right\|_0 ds \\ &\leq 4M^2\tilde{K}_M(\mu)\int_0^t \|w_m(s)\|_1 \left\|w_m'(s)\right\|_0 ds \\ &\leq \frac{2}{\sqrt{\mu_* C_0}} M^2 \tilde{K}_M(\mu) \int_0^t Z_m(s) ds. \end{aligned}$$

Estimating J_3. By using Taylor's expansion for the function $f\left(x,t,u_m,\|u_m\|_0^2\right)$ around the point $\left(x,t,u_{m-1},\|u_{m-1}\|_0^2\right)$ up to order N, we obtain

$$\begin{aligned} & f[u_m] - f[u_{m-1}] \\ := \; & f\left(x,t,u_m,\|u_m\|_0^2\right) - f\left(x,t,u_{m-1},\|u_{m-1}\|_0^2\right) \qquad (3.59) \\ = \; & \sum_{1\leq i+j\leq N-1} A_{ij} f[u_{m-1}]\, w_{m-1}^i(t) \left(\|u_m(t)\|_0^2 - \|u_{m-1}(t)\|_0^2\right)^j \\ & + \sum_{i+j=N} A_{ij} f[\tau_m]\, w_{m-1}^i(t) \left(\|u_m(t)\|_0^2 - \|u_{m-1}(t)\|_0^2\right)^j, \end{aligned}$$

where

$$\tau_m = \left(x,t,u_{m-1}+\theta w_{m-1}, \theta\|u_m\|_0^2 + (1-\theta)\|u_{m-1}\|_0^2\right), \;\; 0<\theta<1.$$

Hence, it follows from (3.6), (3.59) that

$$\begin{aligned} & F_{m+1}(t) - F_m(t) \qquad (3.60) \\ = \; & \sum_{1\leq i+j\leq N-1} A_{ij} f[u_m]\, w_m^i(t) \left(\|u_{m+1}(t)\|_0^2 - \|u_m(t)\|_0^2\right)^j \\ & + \sum_{i+j=N} A_{ij} f[\tau_m]\, w_{m-1}^i(t) \left(\|u_m(t)\|_0^2 - \|u_{m-1}(t)\|_0^2\right)^j. \end{aligned}$$

From (3.60), it yields

$$\begin{aligned} & \|F_{m+1}(t) - F_m(t)\|_0 \qquad (3.61) \\ \leq \; & \bar{K}_M(f) \sum_{1\leq i+j\leq N-1} \frac{1}{i!j!} \left(\sqrt{\frac{1-\rho}{\rho}}\|w_{mx}(t)\|_0\right)^i \left(\|u_{m+1}(t)\|_0 + \|u_m(t)\|_0\right)^j \\ & \times \left(\|u_{m+1}(t)\|_0 - \|u_m(t)\|_0\right)^j \end{aligned}$$

$$
\begin{aligned}
& +\bar{K}_M(f) \sum_{i+j=N} \frac{1}{i!j!} \left(\sqrt{\frac{1-\rho}{\rho}} \left\| w_{m-1} \right\|_{W_1(T)} \right)^i \left(\left\| u_m(t) \right\|_0 + \left\| u_{m-1}(t) \right\|_0 \right)^j \\
& \times \left(\left\| u_m(t) \right\|_0 - \left\| u_{m-1}(t) \right\|_0 \right)^j \\
\leq \; & \bar{K}_M(f) \sum_{1 \leq i+j \leq N-1} \frac{1}{i!j!} \left(\sqrt{\frac{1-\rho}{\rho}} \right)^i \left\| w_{mx}(t) \right\|_0^i (2M)^j \left\| w_m(t) \right\|_0^j \\
& +\bar{K}_M(f) \sum_{i+j=N} \frac{1}{i!j!} \left(\sqrt{\frac{1-\rho}{\rho}} \right)^i \left\| w_{m-1} \right\|_{W_1(T)}^i (2M)^j \left\| w_{m-1}(t) \right\|_0^j \\
\leq \; & \bar{K}_M(f) \sum_{1 \leq i+j \leq N-1} \frac{1}{i!j!} \left(\sqrt{\frac{1-\rho}{\rho}} \right)^i (2M)^j \left\| w_m(t) \right\|_1^{i+j-1} \left\| w_m(t) \right\|_1 \\
& +\bar{K}_M(f) \sum_{i+j=N} \frac{1}{i!j!} \left(\sqrt{\frac{1-\rho}{\rho}} \right)^i \left\| w_{m-1} \right\|_{W_1(T)}^{i+j} (2M)^j \\
\leq \; & \bar{K}_M(f) \sum_{1 \leq i+j \leq N-1} \frac{1}{i!j!} \left(\sqrt{\frac{1-\rho}{\rho}} \right)^i 2^j M^{i+2j-1} \left\| w_m(t) \right\|_1 \\
& +\bar{K}_M(f) \sum_{i+j=N} \frac{1}{i!j!} \left(\sqrt{\frac{1-\rho}{\rho}} \right)^i (2M)^j \left\| w_{m-1} \right\|_{W_1(T)}^N \\
= \; & \alpha_T \left\| w_m(t) \right\|_1 + \bar{\alpha}_T \left\| w_{m-1} \right\|_{W_1(T)}^N ,
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha_T &= \bar{K}_M(f) \sum_{1 \leq i+j \leq N-1} \frac{1}{i!j!} \left(\sqrt{\frac{1-\rho}{\rho}} \right)^i 2^j M^{i+2j-1}, \\
\bar{\alpha}_T &= \bar{K}_M(f) \sum_{i+j=N} \frac{1}{i!j!} \left(\sqrt{\frac{1-\rho}{\rho}} \right)^i (2M)^j .
\end{aligned}
$$

It leads to

$$
\begin{aligned}
J_3 &= 2 \int_0^t \left\langle F_{m+1}(s) - F_m(s), w_m'(s) \right\rangle ds \qquad (3.62) \\
&\leq 2 \int_0^t \left\| F_{m+1}(s) - F_m(s) \right\|_0 \left\| w_m'(s) \right\|_0 ds \\
&\leq 2 \int_0^t \left(\alpha_T \left\| w_m(s) \right\|_1 + \bar{\alpha}_T \left\| w_{m-1} \right\|_{W_1(T)}^N \right) \left\| w_m'(s) \right\|_0 ds \\
&\leq 2\alpha_T \int_0^t \left\| w_m(s) \right\|_1 \left\| w_m'(s) \right\|_0 ds + 2\bar{\alpha}_T \int_0^t \left\| w_{m-1} \right\|_{W_1(T)}^N \left\| w_m'(s) \right\|_0 ds
\end{aligned}
$$

$$\leq \; T\bar{\alpha}_T \left\| w_{m-1} \right\|_{W_1(T)}^{2N} + \left(\frac{\alpha_T}{\sqrt{\mu_* C_0}} + \bar{\alpha}_T \right) \int_0^t Z_m(s) ds.$$

Then, we deduce, from (3.53), (3.56), (3.58) and (3.62) that

$$Z_m(t) \leq T\bar{\alpha}_T \left\| w_{m-1} \right\|_{W_1(T)}^{2N} + \bar{\beta}_T \int_0^t Z_m(s) ds, \tag{3.63}$$

where $\bar{\beta}_T = \frac{2}{\mu_*} M^2 \tilde{K}_M(\mu) + \frac{2}{\sqrt{\mu_* C_0}} M^2 \tilde{K}_M(\mu) + \left(\frac{\alpha_T}{\sqrt{\mu_* C_0}} + \bar{\alpha}_T \right)$.

By using Gronwall's lemma, we obtain from (3.63) that

$$\left\| w_m \right\|_{W_1(T)} \leq \mu_T \left\| w_{m-1} \right\|_{W_1(T)}^{N}, \tag{3.64}$$

with $\mu_T = \left(1 + \frac{1}{\sqrt{\mu_* C_0}} \right) \sqrt{T\bar{\alpha}_T \exp(T\bar{\beta}_T)}$. Then, it follows from (3.64) that

$$||u_m - u_{m+p}||_{W_1(T)} \leq (1 - \beta_T)^{-1} (\mu_T)^{\frac{-1}{N-1}} \beta_T^{N^m}, \tag{3.65}$$

for all m and p.

Taking $T > 0$ small enough, such that $\beta_T = (\mu_T)^{\frac{1}{N-1}} M < 1$. It follows that $\{u_m\}$ is the Cauchy sequence in $W_1(T)$. Then there exists $u \in W_1(T)$ such that

$$u_m \longrightarrow u \text{ strongly in } W_1(T). \tag{3.66}$$

Uniqueness. Applying a similar argument used in the proof of Theorem 3.2, $u \in W_1(M, T)$ is a unique local weak solution of problem (1.1)-(1.3). Taking the limit in (3.65) as $p \to +\infty$ for fixed m, we get (3.51). Also with a similar argument, (3.50) follows. Theorem 3.3 is proved. □

Remark 3.4. In order to construct a N- order iterative scheme, we need the condition (H_3). Then, we get a convergent sequence at a rate of order N to a unique local weak solution of the problem. This condition of f can be relaxed if we only consider the existence of solutions (see [3], [14]).

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References

[1] G. F. Carrier, *On the nonlinear vibrations problem of elastic string*, Quart. J. Appl. Math. **3** (1945), 157-165.

[2] J. L. Lions, *Quelques méthodes de résolution des problèmes aux limites nonlinéaires*, Dunod; Gauthier-Villars, Paris, 1969.

[3] N. T. Long, A. P. N. Dinh and T. N. Diem, *Linear recursive schemes and asymptotic expansion associated with the Kirchhoff-Carrier operator*, J. Math. Anal. Appl. **267**(1) (2002), 116-134.

[4] N. T. Long and L. T. P. Ngoc, *On a nonlinear Kirchhoff - Carrier wave equation in the unit membrane: The quadratic convergence and asymptotic expansion of solutions*, Demonstratio Math. **40**(2) (2007), 365-392.

[5] L. T. P. Ngoc, L. X. Truong and N. T. Long, *An N-order iterative scheme for a nonlinear Kirchhoff-Carrier wave equation associated with mixed homogeneous conditions*, Acta Mathematica Vietnamica, **35**(2) (2010), 207-227.

[6] L. T. P. Ngoc, L. X. Truong and N. T. Long, *High-order iterative methods for a nonlinear Kirchhoff wave equation*, Demonstratio Math. **43**(3) (2010), 605-634.

[7] L. T. P. Ngoc, N. A. Triet, N. T. Duy and N. T. Long, *An N-order iterative scheme for a nonlinear Love equation*, Vietnam Journal of Mathematics, **44**(4) (2016), 801-816.

[8] L. T. P. Ngoc, B. M. Tri and N. T. Long, *An N-order iterative scheme for a nonlinear wave equation containing a nonlocal term*, FILOMAT, **31**(6) (2017), 1755-1767.

[9] L. T. P. Ngoc, L. H. K. Son, T. M. Thuyet and N. T. Long, *Linear approximation and asymptotic expansion of solutions for a nonlinear Carrier wave equation in an annular membrane with Robin-Dirichlet conditions*, Mathematical Problems in Engineering, Vol. **2016** (2016), Article ID 8031638, 18 pages.

[10] L. T. P. Ngoc, L. H. K. Son, T. M. Thuyet and N. T. Long, *An N-order iterative scheme for a nonlinear Carrier wave equation in the annular with Robin-Dirichlet conditions,* Nonlinear Functional Analysis and Applications, **22**(1) (2017), 147-169.

[11] P. K. Parida and D. K. Gupta, *Recurrence relations for a Newton-like method in Banach spaces*, J. Comput. Appl. Math. **206** (2007), 873 -887.

[12] R. E. Showater, *Hilbert space methods for partial differential equations*, Electronic J. Differential Equations, Monograph 01, 1994.

[13] N. A. Triet, L. T. P. Ngoc and N. T. Long, *A mixed Dirichlet-Robin problem for a nonlinear Kirchhoff-Carrier wave equation*, Nonlinear Anal. RWA. **13**(2) (2012), 817-839.

[14] L. X. Truong, L. T. P. Ngoc and N. T. Long, *The N-order iterative schemes for a nonlinear Kirchhoff-Carrier wave equation associated with the mixed inhomogeneous conditions*, Applied Mathematics and Computation, **215**(5) (2009), 1908-1925.

[15] L. T. P. Ngoc, N. V. Y, T. M. Thuyet and N. T. Long, *On a nonlinear heat equation with viscoelastic term associate with Robin conditions*, Applicable Analysis, **96**(16) (2017), 2717-2736.