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GENERIC LIGHTLIKE SUBMANIFOLDS OF AN INDEFINITE TRANS-SASAKIAN MANIFOLD WITH A SEMI-SYMMETRIC METRIC CONNECTION

Dae Ho Jin

Department of Mathematics, Dongguk University
Gyeongju 780-714, Republic of Korea
e-mail: jindh@dongguk.ac.kr

Abstract. We study the geometry of generic lightlike submanifolds M of an indefinite trans-Sasakian manifold $\bar{M}$ with a semi-symmetric metric connection subject such that the characteristic vector field ζ of $\bar{M}$ is identical with structure vector field of $\bar{M}$ and ζ is tangent to M. Under the same conditions, we also characterize the geometry of generic lightlike submanifolds of an indefinite generalized Sasakian space form $\bar{M}(f_1, f_2, f_2)$.

1. Introduction

A linear connection $\bar{\nabla}$ on a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is said to be a *semi-symmetric connection* if its torsion tensor $\bar{T}$ satisfies

$$\bar{T}(\bar{X}, \bar{Y}) = \theta(\bar{Y})\bar{X} - \theta(\bar{X})\bar{Y}, \tag{1.1}$$

where θ is a 1-form associated with a smooth unit vector field ζ, which is called the *characteristic vector field*, by $\theta(\bar{X}) = \bar{g}(\bar{X}, \zeta)$. Moreover, if this connection $\bar{\nabla}$ is a metric connection, *i.e.*, it saisfies $\bar{\nabla}\bar{g} = 0$, then $\bar{\nabla}$ is called a *semi-symmetric metric connection*. The notion of semi-symmetric metric connection on a Riemannian manifold was introduced by Yano [14]. In the followings, we denote by $\bar{X}, \bar{Y}$ and $\bar{Z}$ the smooth vector fields on $\bar{M}$.

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Let $\widetilde{\nabla}$ be the Levi-Civita connection of the semi-Riemannian manifold $(\bar{M}, \bar{g})$ with respect to the metric $\bar{g}$. It is known that *a linear connection* $\bar{\nabla}$ *on* $\bar{M}$ *is a semi-symmetric metric connection if and only if it satisfies*

$$\bar{\nabla}_{\bar{X}}\bar{Y} = \widetilde{\nabla}_{\bar{X}}\bar{Y} + \theta(\bar{Y})\bar{X} - \bar{g}(\bar{X}, \bar{Y})\zeta. \tag{1.2}$$

A lightlike submanifold M of an indefinite almost contact manifold $\bar{M}$ is called *generic* if there exists a screen distribution $S(TM)$ of M such that

$$J(S(TM)^{\perp}) \subset S(TM), \tag{1.3}$$

where $S(TM)^{\perp}$ is the orthogonal complement of $S(TM)$ in the tangent bundle $T\bar{M}$ of $\bar{M}$, that is, $T\bar{M} = S(TM) \oplus_{orth} S(TM)^{\perp}$. The notion of generic lightlike submanifolds was introduced by Jin-Lee [9] and later, studied by Duggal-Jin [5], Jin [6, 7] and Jin-Lee [10]. The geometry of generic lightlike submanifolds is an extension of that of lightlike hypersurface and half lightlike submanifold of codimension 2. Much of its theory will be immediately generalized in a formal way to general lightlike submanifolds.

The notion of a trans-Sasakian manifold of type (α, β) was introduced by Oubina [13]. Sasakian, Kenmotsu and cosymplectic manifolds are important kinds of the trans-Sasakian manifold such that α and β satisfy

$$\alpha = \epsilon, \;\; \beta = 0; \quad \alpha = 0, \;\; \beta = \epsilon; \quad \alpha = \beta = 0,$$

respectively, where $\epsilon = \pm 1$. If a trans-Sasakian manifold is a semi-Riemannian manifold, then it is called an *indefinite trans-Sasakian manifold*.

In this paper, we study the geometry of generic lightlike submanifolds of an indefinite trans-Sasakian manifold $(\bar{M}, J, \zeta, \theta, \bar{g})$ with a semi-symmetric metric connection $\bar{\nabla}$ in which the characteristic vector field ζ of $\bar{M}$ is identical with the structure vector field ζ of $(\bar{M}, J, \zeta, \theta, \bar{g})$ and ζ is tangent to M. Under the same conditions, we also characterize generic lightlike submanifolds of an indefinite generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$.

2. Semi-symmetric metric connections

An odd-dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$ is called an *indefinite almost contact metric manifold* if there exists a set $\{J, \zeta, \theta, \bar{g}\}$, where J is a (1, 1)-type tensor field, ζ is a vector field and θ is a 1-form such that

$$J^2\bar{X} = -\bar{X} + \theta(\bar{X})\zeta, \;\; \bar{g}(J\bar{X}, J\bar{Y}) = \bar{g}(\bar{X}, \bar{Y}) - \epsilon\theta(\bar{X})\theta(\bar{Y}), \;\; \theta(\zeta) = 1, \tag{2.1}$$

where $\epsilon = 1$ or -1 according as ζ is spacelike or timelike, respectively. The set $\{J, \zeta, \theta, \bar{g}\}$ is called an *indefinite almost contact metric structure*.

From (2.1), we show that

$$J\zeta = 0, \quad \theta \circ J = 0, \quad \theta(\bar{X}) = \epsilon\bar{g}(\bar{X}, \zeta), \quad \bar{g}(J\bar{X}, \bar{Y}) = -\bar{g}(\bar{X}, J\bar{Y}).$$

In the entire discussion of this article, we shall assume that the structure vector field ζ is a spacelike one, *i.e.*, $\epsilon = 1$, without loss of generality.

Definition 2.1. An indefinite almost contact metric manifold $(\bar{M}, \bar{g})$ is said to be an *indefinite trans-Sasakian manifold* [13] if, for the Levi-Civita connection $\widetilde{\nabla}$, there exist two smooth functions α and β such that

$$(\widetilde{\nabla}_{\bar{X}}J)\bar{Y} = \alpha\{\bar{g}(\bar{X},\bar{Y})\zeta - \theta(\bar{Y})\bar{X}\} + \beta\{\bar{g}(J\bar{X},\bar{Y})\zeta - \theta(\bar{Y})J\bar{X}\}.$$

$\{J, \zeta, \theta, \bar{g}\}$ is called an *indefinite trans-Sasakian structure, of type* (α, β).

Let $\bar{\nabla}$ be a semi-symmetric metric connection on $\bar{M} = (\bar{M},\, J,\, \zeta,\, \theta,\, \bar{g})$. By using (1.2), (2.1) and the facts that $J\zeta = 0$ and $\theta \circ J = 0$, we see that

$$\begin{aligned}(\bar{\nabla}_{\bar{X}}J)\bar{Y} \;=\; & \alpha\{\bar{g}(\bar{X},\bar{Y})\zeta - \theta(\bar{Y})\bar{X}\} \\ & +\; (\beta+1)\{\bar{g}(J\bar{X},\bar{Y})\zeta - \theta(\bar{Y})J\bar{X}\}.\end{aligned} \tag{2.2}$$

Replacing $\bar{Y}$ by ζ to (2.2) and using $J\zeta = 0$ and $\theta(\bar{\nabla}_{\bar{X}}\zeta) = 0$, we obtain

$$\bar{\nabla}_{\bar{X}}\zeta \;= -\,\alpha J\bar{X} + (\beta+1)\{\bar{X} - \theta(\bar{X})\zeta\}. \tag{2.3}$$

Let (M, g) be an m-dimensional lightlike submanifold of an indefinite trans-Sasakian manifold $\bar{M}$ of dimension $(m+n)$. Then the radical distribution $Rad(TM) = TM \cap TM^{\perp}$ of M is a subbundle of the tangent bundle TM and the normal bundle $TM^{\perp}$, of rank $r\,(1 \le r \le \min\{m,\, n\})$. We say that M is *r-lightlike submanifold* [4] if $1 \le r < \min\{m,\, n\}$. In the sequel, by saying that M is a lightlike submanifold we shall mean that it is an r-lightlike submanifold. For an r-lightlike submanifold M, there exist two complementary non-degenerate distributions $S(TM)$ and $S(TM^{\perp})$ of $Rad(TM)$ in TM and $TM^{\perp}$, respectively, which are called the *screen distribution* and the *co-screen distribution* of M, such that

$$TM = Rad(TM) \oplus_{orth} S(TM),\;\; TM^{\perp} = Rad(TM) \oplus_{orth} S(TM^{\perp}),$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle E over M. Also denote by $(2.1)_i$ the i-th equation of (2.1). We use the same notations for any others. Let X, Y, Z and W be the vector fields on M, unless otherwise specified. We use the following range of indices:

$$i,\, j,\, k,\, ... \in \{1,\, ...\, ,\, r\}, \qquad a,\, b,\, c,\, ... \in \{r\,+\,1,\, ...\, ,\, n\}.$$

Let $tr(TM)$ and $ltr(TM)$ be complementary vector bundles to TM in $T\bar{M}_{|M}$ and $TM^{\perp}$ in $S(TM)^{\perp}$, respectively, and let $\{N_1,\, \cdots,\, N_r\}$ be a null basis of $ltr(TM)_{|\mathcal{U}}$, where $\mathcal{U}$ is a coordinate neighborhood of M, such that

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \qquad \bar{g}(N_i, N_j) = 0,$$

where $\{\xi_1, \cdots, \xi_r\}$ is a null basis of $Rad(TM)_{|\mathcal{U}}$. Then we have

$$\begin{aligned} T\bar{M} &= TM \oplus tr(TM) = \{Rad(TM) \oplus tr(TM)\} \oplus_{orth} S(TM) \\ &= \{Rad(TM) \oplus ltr(TM)\} \oplus_{orth} S(TM) \oplus_{orth} S(TM^{\perp}). \end{aligned}$$

We call $tr(TM)$, $ltr(TM)$ and N_i the *transversal vector bundle*, the *lightlike transversal vector bundle* and the *null transversal vector fields* of M with respect to the screen distribution $S(TM)$, respectively. Hence the local quasi-orthonormal field of frames on $\bar{M}$ along M is given by

$$\{\xi_1, \cdots, \xi_r, N_1, \cdots, N_r, F_{r+1}, \cdots, F_m, E_{r+1}, \cdots, E_n\},$$

where $\{F_{r+1}, \cdots, F_m\}$ and $\{E_{r+1}, \cdots, E_n\}$ are orthonormal bases of $S(TM)$ and $S(TM^{\perp})$, respectively. Denote $\epsilon_a = \bar{g}(E_a, E_a)$. Then $\epsilon_a \delta_{ab} = \bar{g}(E_a, E_b)$.

Let P be the projection morphism of TM on $S(TM)$. Then the local Gauss-Weingarten formulae of M and $S(TM)$ are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + \sum_{i=1}^{r} h_i^{\ell}(X,Y)N_i + \sum_{a=r+1}^{n} h_a^s(X,Y)E_a, \tag{2.4}$$

$$\bar{\nabla}_X N_i = -A_{N_i} X + \sum_{j=1}^{r} \tau_{ij}(X)N_j + \sum_{a=r+1}^{n} \rho_{ia}(X)E_a, \tag{2.5}$$

$$\bar{\nabla}_X E_a = -A_{E_a} X + \sum_{i=1}^{r} \phi_{ai}(X)N_i + \sum_{b=r+1}^{n} \sigma_{ab}(X)E_b, \tag{2.6}$$

$$\nabla_X PY = \nabla_X^* PY + \sum_{i=1}^{r} h_i^*(X,PY)\xi_i, \tag{2.7}$$

$$\nabla_X \xi_i = -A_{\xi_i}^* X - \sum_{j=1}^{r} \tau_{ji}(X)\xi_j, \tag{2.8}$$

where ∇ and ∇^* are induced linear connections on TM and $S(TM)$, respectively, h_i^{ℓ} and h_a^s are called the *local second fundamental forms* on TM, h_i^* are called the *local second fundamental forms* on $S(TM)$. A_{N_i}, A_{E_a} and $A_{\xi_i}^*$ are called the *shape operators*, and τ_{ij}, ρ_{ia}, ϕ_{ai} and σ_{ab} are 1-forms on TM.

The connection ∇ is a semi-symmetric non-metric connection and satisfy

$$(\nabla_X g)(Y,Z) = \sum_{i=1}^{r} \{h_i^{\ell}(X,Y)\eta_i(Z) + h_i^{\ell}(X,Z)\eta_i(Y)\}, \tag{2.9}$$

$$T(X,Y) = \theta(Y)X - \theta(X)Y, \tag{2.10}$$

and the results: h_i^{ℓ} and h_a^s are symmetric, where η_i's are 1-forms such that

$$\eta_i(X) = \bar{g}(X, N_i).$$

From the facts that $h_i^\ell(X,Y) = \bar{g}(\bar{\nabla}_X Y, \xi_i)$ and $\epsilon_a h_a^s(X,Y) = \bar{g}(\bar{\nabla}_X Y, E_a)$, we know that h_i^ℓ and h_a^s are independent of the choice of $S(TM)$. The local second fundamental forms are related to their shape operators by

$$h_i^\ell(X,Y) = g(A^*_{\xi_i}X, Y) - \sum_{k=1}^{r} h_k^\ell(X,\xi_i)\eta_k(Y), \tag{2.11}$$

$$\epsilon_a h_a^s(X,Y) = g(A_{E_a}X, Y) - \sum_{k=1}^{r} \phi_{ak}(X)\eta_k(Y), \tag{2.12}$$

$$h_i^*(X, PY) = g(A_{N_i}X, PY). \tag{2.13}$$

Applying $\bar{\nabla}_X$ to $g(\xi_i,\xi_j) = 0$, $\bar{g}(\xi_i, E_a) = 0$, $\bar{g}(N_i, N_j) = 0$, $\bar{g}(N_i, E_a) = 0$ and $\bar{g}(E_a, E_b) = \epsilon\delta_{ab}$ by turns and using (2.4) ∼ (2.6), we obtain

$$\begin{aligned} &h_i^\ell(X,\xi_j) + h_j^\ell(X,\xi_i) = 0, \qquad h_a^s(X,\xi_i) = -\epsilon_a\phi_{ai}(X),\\ &\eta_j(A_{N_i}X) + \eta_i(A_{N_j}X) = 0, \quad\ \ \eta_i(A_{E_a}X) = \epsilon_a\rho_{ia}(X),\\ &\epsilon_b\sigma_{ab} + \epsilon_a\sigma_{ba} = 0; \quad h_i^\ell(X,\xi_i) = 0,\ h_i^\ell(\xi_j,\xi_k) = 0,\ A^*_{\xi_i}\xi_i = 0. \end{aligned} \tag{2.14}$$

Definition 2.2. We say that a lightlike submanifold M of $\bar{M}$ is

(1) *irrotational* [12] if $\bar{\nabla}_X\xi_i \in \Gamma(TM)$ for all $i \in \{1, \cdots, r\}$,
(2) *solenoidal* [11] if A_{E_a} and A_{N_i} are $S(TM)$-valued for all α and i.

Remark 2.3. From (2.4) and $(2.14)_2$, the item (1) is equivalent to

$$h_j^\ell(X,\xi_i) = 0, \quad h_a^s(X,\xi_i) = \phi_{ai}(X) = 0. \tag{2.15}$$

By using $(2.14)_4$, the item (2) is equivalent to

$$\eta_j(A_{N_i}X) = 0, \quad \rho_{ia}(X) = \eta_i(A_{E_a}X) = 0. \tag{2.16}$$

3. Generic lightlike submanifolds

Let M be a generic lightlike submanifold of $\bar{M}$. From (1.3) we show that $J(Rad(TM))$, $J(ltr(TM))$ and $J(S(TM^\perp))$ are subbundles of $S(TM)$. Now we shall assume that ζ is tangent to M. Călin [2] proved that *if ζ is tangent to M, then it belongs to $S(TM)$* which we assume in this paper. Then there exist two non-degenerate almost complex distributions H_o and H with respect to J, that is, $J(H_o) = H_o$ and $J(H) = H$, such that

$$S(TM) = \{J(Rad(TM)) \oplus J(ltr(TM))\} \oplus_{orth} J(S(TM^\perp)) \oplus_{orth} H_o,$$
$$H = Rad(TM) \oplus_{orth} J(Rad(TM)) \oplus_{orth} H_o.$$

In this case, the tangent bundle TM of M is decomposed as follow:

$$TM = H \oplus J(ltr(TM)) \oplus_{orth} J(S(TM^\perp)). \tag{3.1}$$

Consider $2r$ local null vector fields U_i and V_i, $(n-r)$ local non-null unit vector fields W_a on $S(TM)$ and their 1-forms u_i, v_i and w_a defined by

$$U_i = -JN_i, \quad V_i = -J\xi_i, \quad W_a = -JE_a, \tag{3.2}$$
$$u_i(X) = g(X, V_i), \quad v_i(X) = g(X, U_i), \quad w_a(X) = \epsilon_a g(X, W_a). \tag{3.3}$$

Denote by S the projection morphism of TM on H and by F the tensor field of type $(1,1)$ globally defined on M by $F = J \circ S$. Then JX is expressed as

$$JX = FX + \sum_{i=1}^{r} u_i(X)N_i + \sum_{a=r+1}^{n} w_a(X)E_a. \tag{3.4}$$

Applying J to (3.4) and using $(2.1)_1$ and (3.2), we have

$$F^2X = -X + \theta(X)\zeta + \sum_{i=1}^{r} u_i(X)U_i + \sum_{a=r+1}^{n} w_a(X)W_a. \tag{3.5}$$

We say that the tensor field F is the *structure tensor field* of M and the vector fields U_i and W_a are the *structure vector fields* of M.

Replacing Y by ζ to (2.4) and using (2.3) and (3.4), we have

$$\nabla_X\zeta = -\alpha FX + (\beta+1)\{X - \theta(X)\zeta\}, \tag{3.6}$$
$$h_i^\ell(X,\zeta) = -\alpha u_i(X), \quad h_a^s(X,\zeta) = -\alpha w_a(X). \tag{3.7}$$

Applying $\bar\nabla_X$ to $\bar g(\zeta, N_i) = 0$ and using (2.3), (2.5) and (2.13), we get

$$h_i^*(X,\zeta) = -\alpha v_i(X) + (\beta+1)\eta_i(X). \tag{3.8}$$

Applying $\bar\nabla_X$ to (3.2), (3.3) and (3.4) by turns and using (2.2), (2.4) $\sim$ (2.8), (2.11) $\sim$ (2.13) and (3.2) $\sim$ (3.4), we have

$$\begin{aligned} &h_j^\ell(X,U_i) = h_i^*(X,V_j), \quad \epsilon_a h_i^*(X,W_a) = h_a^s(X,U_i),\\ &h_j^\ell(X,V_i) = h_i^\ell(X,V_j), \quad \epsilon_a h_i^\ell(X,W_a) = h_a^s(X,V_i),\\ &\epsilon_b h_b^s(X,W_a) = \epsilon_a h_a^s(X,W_b), \end{aligned} \tag{3.9}$$

$$\begin{aligned} \nabla_X U_i = {} & F(A_{N_i}X) + \sum_{j=1}^{r}\tau_{ij}(X)U_j + \sum_{a=r+1}^{n}\rho_{ia}(X)W_a \\ & - \{\alpha\eta_i(X) + (\beta+1)v_i(X)\}\zeta, \end{aligned} \tag{3.10}$$

$$\begin{aligned} \nabla_X V_i = {} & F(A^*_{\xi_i}X) - \sum_{j=1}^{r}\tau_{ji}(X)V_j + \sum_{j=1}^{r}h_j^\ell(X,\xi_i)U_j \\ & - \sum_{a=r+1}^{n}\epsilon_a\phi_{ai}(X)W_a - (\beta+1)u_i(X)\zeta, \end{aligned} \tag{3.11}$$

$$\begin{aligned}\nabla_X W_a \;=\; & F(A_{E_a} X) + \sum_{i=1}^{r} \phi_{ai}(X) U_i + \sum_{b=r+1}^{n} \sigma_{ab}(X) W_b \\ & - \epsilon_a (\beta + 1) w_a(X) \zeta,\end{aligned} \tag{3.12}$$

$$\begin{aligned}(\nabla_X F)(Y) \;=\; & \sum_{i=1}^{r} u_i(Y) A_{N_i} X + \sum_{a=r+1}^{n} w_a(Y) A_{E_a} X \\ & - \sum_{i=1}^{r} h_i^{\ell}(X, Y) U_i - \sum_{a=r+1}^{n} h_a^s(X, Y) W_a \\ & + \alpha\{g(X, Y)\zeta - \theta(Y) X\} \\ & + (\beta + 1)\{\bar{g}(JX, Y)\zeta - \theta(Y) FX\},\end{aligned} \tag{3.13}$$

$$\begin{aligned}(\nabla_X u_i)(Y) = & - \sum_{j=1}^{r} u_j(Y) \tau_{ji}(X) - \sum_{a=r+1}^{n} w_a(Y) \phi_{ai}(X) \\ & - h_i^{\ell}(X, FY) - (\beta + 1)\theta(Y) u_i(X),\end{aligned} \tag{3.14}$$

$$\begin{aligned}(\nabla_X v_i)(Y) = & \sum_{j=1}^{r} v_j(Y) \tau_{ij}(X) + \sum_{a=r+1}^{n} \epsilon_a w_a(Y) \rho_{ia}(X) \\ & - \sum_{j=1}^{r} u_j(Y) \eta_j(A_{N_i} X) - g(A_{N_i} X, FY) \\ & - \{\alpha \eta_i(X) + (\beta + 1) v_i(X)\}\theta(Y).\end{aligned} \tag{3.15}$$

Theorem 3.1. *Let M be a generic lightlike submanifold of an indefinite trans-Sasakian manifold $\bar{M}$ with a semi-symmetric metric connection. If U_is are parallel with respect to the connection ∇, then $\tau_{ij} = 0$, M is solenoidal and $\bar{M}$ is an indefinite Kenmotsu manifold such that $\alpha = 0$ and $\beta = -1$.*

Proof. Assume that U_is are parallel with respect to ∇. Taking the scalar product with ζ, V_j, U_j, W_a and N_j to (3.10) by turns, we get

$$\alpha = 0, \quad \beta = -1; \quad \tau_{ij} = 0, \quad \eta_j(A_{N_i} X) = 0, \quad \rho_{ia} = 0, \tag{3.16}$$

$$h_i^*(X, U_j) = 0, \tag{3.17}$$

respectively. As $\alpha = 0$ and $\beta = -1$, $\bar{M}$ is an indefinite Kenmotsu manifold. As $\eta_j(A_{N_i} X) = 0$ and $\rho_{ia} = 0$, M is solenoidal. □

Theorem 3.2. *Let M be a generic lightlike submanifold of an indefinite trans-Sasakian manifold $\bar{M}$ with a semi-symmetric metric connection. If V_is are parallel with respect to the connection ∇, then $\tau_{ij} = 0$, M is irrotational and $\bar{M}$ is an indefinite Kenmotsu manifold such that $\alpha = 0$ and $\beta = -1$.*

Proof. Assume that V_is are parallel with respect to ∇. Taking the scalar product with V_j, W_a, U_j, ζ and N_j to (3.11) by turns, we obtain

$$h^\ell_j(X,\xi_i)=0, \quad \phi_{ai}=0, \quad \tau_{ij}=0, \quad \beta=-1, \\ h^\ell_i(X,U_k)=0, \tag{3.18}$$

respectively. As $h^\ell_j(X,\xi_i)=0$ and $\phi_{ai}=0$, M is irrotational. Replacing X by ζ to (3.18) and using $(3.7)_1$, we have $\alpha=0$. Thus

$$\alpha=0, \quad \beta=-1, \quad \tau_{ij}=0, \quad h^\ell_j(X,\xi_i)=0, \quad \phi_{ai}=0. \tag{3.19}$$

As $\alpha=0$ and $\beta=-1$, $\bar{M}$ is an indefinite Kenmotsu manifold. □

4. Recurrent and Lie recurrent submanifolds

Definition 4.1. ([8]) The structure tensor field F of M is said to be *recurrent* if there exists a 1-form ϖ on M such that

$$(\nabla_X F)Y=\varpi(X)FY.$$

A lightlike submanifold M of an indefinite trans-Sasakian manifold $\bar{M}$ is called *recurrent* if it admits a recurrent structure tensor field F.

Theorem 4.2. *Let M be a recurrent generic lightlike submanifold of an indefinite trans-Sasakian manifold $\bar{M}$ with a semi-symmetric metric connection. Then we have the following results:*

1. *$\bar{M}$ is an indefinite Kenmotsu manifold, i.e., $\alpha=0$ and $\beta=-1$,*
2. *F is parallel with respect to the induced connection ∇ on M,*
3. *M is irrotational and solenoidal,*
4. *H, $J(ltr(TM))$ and $J(S(TM^\perp))$ are parallel distributions on M,*
5. *M is locally a product manifold $M_r\times M_{n-r}\times M^\sharp$, where M_r, M_{n-r} and $M^\sharp$ are leaves of $J(ltr(TM))$, $J(S(TM^\perp))$ and H, respectively.*

Proof. From the above definition and (3.13), we obtain

$$\begin{aligned}\varpi(X)FY \;=\; &\sum_{i=1}^r u_i(Y)A_{N_i}X+\sum_{a=r+1}^n w_a(Y)A_{E_a}X \\ &-\sum_{i=1}^r h^\ell_i(X,Y)U_i-\sum_{a=r+1}^n h^s_a(X,Y)W_a \\ &+\alpha\{g(X,Y)\zeta-\theta(Y)X\} \\ &+(\beta+1)\{\bar{g}(JX,Y)\zeta-\theta(Y)FX\}.\end{aligned} \tag{4.1}$$

Replacing Y by ζ to (4.1) and using (2.1), (3.5) and (3.7), we get

$$\alpha F^2X-(\beta+1)FX=0.$$

Taking $X = \xi_i$ to this and using the fact that $F\xi_i = -V_i$, we have

$$-\alpha\xi_i + (\beta + 1)V_i = 0.$$

Taking the scalar product with N_i and U_i to this by turns, we obtain

$$\alpha = 0, \quad \beta = -1. \tag{4.2}$$

Therefore, $\bar{M}$ is an indefinite Kenmotsu manifold.

(2) Replacing Y by ξ_i to (4.1) and using (4.2), we have

$$\varpi(X)V_i = \sum_{j=1}^{r} h_j^\ell(X, \xi_i)U_j + \sum_{a=r+1}^{n} h_a^s(X, \xi_i)W_a.$$

Taking the scalar product with U_i, V_k and W_b to this by turns, we get

$$\varpi = 0, \quad h_k^\ell(X, \xi_i) = 0, \quad h_b^s(X, \xi_i) = 0. \tag{4.3}$$

As $\varpi = 0$, F is parallel with respect to the induced connection ∇.

(3) From $(4.3)_{2,3}$, we see that M is irrotational.

Taking thescalar product with N_j to (4.1), we obtain

$$\sum_{j=1}^{r} u_j(Y)\bar{g}(A_{N_j}X, N_i) + \sum_{a=r+1}^{n} w_a(Y)\bar{g}(A_{E_a}X, N_i) = 0.$$

Taking $Y = U_k$ and $Y = W_b$ to this equation by turns, we have

$$\bar{g}(A_{N_k}X, N_i) = 0, \quad \bar{g}(A_{E_b}X, N_i) = 0. \tag{4.4}$$

Thus, by Remark 2.3, we see that M is solenoidal.

(4) Taking the scalar product with V_i and W_a to (4.1) by turns, we obtain

$$h_i^\ell(X, Y) = \sum_{k=1}^{r} u_k(Y)u_i(A_{N_k}X) + \sum_{a=r+1}^{n} w_a(Y)u_i(A_{E_a}X),$$

$$h_a^s(X, Y) = \sum_{i=1}^{r} u_i(Y)w_a(A_{N_i}X) + \sum_{b=r+1}^{n} w_b(Y)w_a(A_{E_b}X).$$

Taking $Y = V$ and $Y = FZ_o$, $Z_o \in \Gamma(H_o)$ to these equations by turns and using the results: $u_i(FZ_o) = w_a(FZ_o) = 0$ as $FZ_o = JZ_o \in \Gamma(H_o)$, we have

$$h_i^\ell(X, V_j) = 0, \; h_i^\ell(X, FZ_o) = 0, \; h_a^s(X, V_j) = 0, \; h_a^s(X, FZ_o) = 0. \tag{4.5}$$

In general, by using (2.1), (2.8), (2.11), (3.4), (3.11) and (3.12), we derive

$$g(\nabla_X\xi_i, V_j) = -h_i^\ell(X, V_j), \quad g(\nabla_X\xi_i, W_a) = -h_i^\ell(X, W_a),$$
$$g(\nabla_X V_i, V_j) = h_j^\ell(X, \xi_i), \quad g(\nabla_X V_i, W_a) = -\phi_{ai}(X),$$
$$g(\nabla_X Z_o, V_i) = b_i^\ell(X, FZ_o), \quad g(\nabla_X Z_o, W_a) = b_a^s(X, FZ_o).$$

From these equations and $(3.9)_4$, (4.3) and (4.5), we see that

$$\nabla_X Y \in \Gamma(H), \quad \forall\, X \in \Gamma(TM), \quad \forall\, Y \in \Gamma(H).$$

It follows that H is a parallel distribution on M.

Taking $Y = U_i$ and $Y = W_a$ to (4.1) by turns and using (4.2), we have

$$A_{N_i} X = \sum_{j=1}^{r} h_j^{\ell}(X, U_i) U_j + \sum_{a=r+1}^{n} h_a^s(X, U_i) W_a. \tag{4.6}$$

$$A_{E_a} X = \sum_{i=1}^{r} h_j^{\ell}(X, W_a) U_i + \sum_{b=r+1}^{n} h_b^s(X, W_a) W_b. \tag{4.7}$$

Applying F to (4.6) and (4.7) by turns and using $FU_i = FW_a = 0$, we get

$$F(A_{N_i} X) = 0, \quad F(A_{E_a} X) = 0.$$

Using this result and (4.2)$\sim$(4.4), Eq.s (3.10) and (3.12) are reduced to

$$\nabla_X U_i = \sum_{j=1}^{r} \tau_{ij}(X) U_j, \quad \nabla_X W_a = \sum_{b=r+1}^{n} \sigma_{ab}(X) W_b. \tag{4.8}$$

Thus $J(ltr(TM))$ and $J(S(TM^{\perp}))$ are also parallel distributions on M.

(5) As H, $J(ltr(TM))$ and $J(S(TM^{\perp}))$ are parallel distributions and satisfy the decomposition form (3.1), by the de Rham's decomposition theorem [3], M is locally a product manifold $M_r \times M_{n-r} \times M^{\sharp}$, where M_r, M_{n-r} and $M^{\sharp}$ are leaves of $J(ltr(TM))$, $J(S(TM^{\perp}))$ and H, respectively. □

Definition 4.3. ([8]) The structure tensor field F of M is said to be *Lie recurrent* if there exists a 1-form ϑ on M such that

$$(\mathcal{L}_X F)Y = \vartheta(X) FY,$$

where $\mathcal{L}_X$ denotes the Lie derivative on M with respect to X. In case $\vartheta = 0$, we say that F is *Lie parallel*. A lightlike submanifold M is called *Lie recurrent* if it admits a Lie recurrent structure tensor field F.

Theorem 4.4. *Let M be a Lie recurrent generic lightlike submanifold of an indefinite trans-Sasakian manifold $\bar{M}$ with a semi-symmetric metric connection. Then we have the following results:*

(1) $\alpha = 0$ *and $\bar{M}$ is an indefinite β-Kenmotsu manifold,*
(2) F *is Lie parallel,*
(3) τ_{ij} *and ρ_{ia} are satisfied $\tau_{ij} \circ F = 0$ and $\rho_{ia} \circ F = 0$. Moreover,*

$$\tau_{ij}(X) = \sum_{k=1}^{r} u_k(X) g(A_{N_k} V_j, N_i) - \beta \delta_{ij} \theta(X).$$

Proof. (1) Using (2.10), (3.13) and the fact that $\theta \circ F = 0$, we get

$$\begin{aligned} \vartheta(X)FY &= -\nabla_{FY}X + F\nabla_Y X \\ &+ \sum_{i=1}^{r} u_i(Y)A_{N_i}X + \sum_{a=r+1}^{n} w_a(Y)A_{E_a}X \\ &- \sum_{i=1}^{r} h_i^{\ell}(X,Y)U_i - \sum_{a=r+1}^{n} h_a^{s}(X,Y)W_a \\ &+ \alpha\{g(X,Y)\zeta - \theta(Y)X\} \\ &+ (\beta+1)\bar{g}(JX,Y)\zeta - \beta\theta(Y)FX. \end{aligned} \tag{4.9}$$

Taking $Y = \xi_j$ and $Y = V_j$ to (4.9) by turns, we have

$$\begin{aligned} -\vartheta(X)V_j &= \nabla_{V_j}X + F\nabla_{\xi_j}X + (\beta+1)u_j(X)\zeta \\ &- \sum_{i=1}^{r} h_i^{\ell}(X,\xi_j)U_i - \sum_{a=r+1}^{n} h_a^{s}(X,\xi_j)W_a, \end{aligned} \tag{4.10}$$

$$\begin{aligned} \vartheta(X)\xi_j &= -\nabla_{\xi_j}X + F\nabla_{V_j}X + \alpha u_j(X)\zeta \\ &- \sum_{i=1}^{r} h_i^{\ell}(X,V_j)U_i - \sum_{a=r+1}^{n} h_a^{s}(X,V_j)W_a. \end{aligned} \tag{4.11}$$

Taking the scalar product with ζ to (4.11) such that $X = U_j$ and using (3.10), we obtain $\alpha = 0$. Thus $\bar{M}$ is an indefinite β-Kenmotsu manifold.

(2) Taking the product with U_i to (4.10) and N_i to (4.11), we obtain

$$\begin{aligned} -\delta_{ij}\vartheta(X) &= g(\nabla_{V_j}X, U_i) - \bar{g}(\nabla_{\xi_j}X, N_i), \\ \delta_{ij}\vartheta(X) &= g(\nabla_{V_j}X, U_i) - \bar{g}(\nabla_{\xi_j}X, N_i), \end{aligned} \tag{4.12}$$

respectively. From these equations, we get $\vartheta = 0$. Thus F is Lie parallel.

(3) Taking the scalar product with N_i to (4.10) such that $X = W_a$ and using (2.12), $(2.14)_4$ and (3.12), we get $h_a^s(U_i, V_j) = \rho_{ia}(\xi_j)$. Also, taking the scalar product with W_a to (4.11) such that $X = U_i$ and using (3.10), we have $h_a^s(U_i, V_j) = -\rho_{ia}(\xi_j)$. Thus $\rho_{ia}(\xi_j) = 0$ and $h_a^s(U_i, V_j) = 0$.

Taking the scalar product with U_i to (4.10) such that $X = W_a$ and using $(2.14)_{2,4}$ and (3.12), we get $\epsilon_a\rho_{ia}(V_j) = \phi_{aj}(U_i)$. Also, taking the scalar product with W_a to (4.10) such that $X = U_i$ and using $(2.14)_2$ and (3.10), we get $\epsilon_a\rho_{ia}(V_j) = -\phi_{aj}(U_i)$. Thus $\rho_{ia}(V_j) = 0$ and $\phi_{aj}(U_i) = 0$.

Taking the scalar product with V_i to (4.10) such that $X = W_a$ and using $(2.14)_2$, $(3.9)_4$ and (3.12), we obtain $\phi_{ai}(V_j) = -\phi_{aj}(V_i)$. Also, taking the scalar product with W_a to (4.10) such that $X = V_i$ and using $(2.14)_2$ and (3.11), we have $\phi_{ai}(V_j) = \phi_{aj}(V_i)$. Thus we obtain $\phi_{ai}(V_j) = 0$.

Taking the scalar product with W_a to (4.10) such that $X = \xi_i$ and using (2.8), (2.11) and $(2.14)_2$, we get $h_i^\ell(V_j, W_a) = \phi_{ai}(\xi_j)$. Also, taking the scalar product with V_i to (4.11) such that $X = W_a$ and using (3.12), we have $h_i^\ell(V_j, W_a) = -\phi_{ai}(\xi_j)$. Thus $\phi_{ai}(\xi_j) = 0$ and $h_i^\ell(V_j, W_a) = 0$.

Summarizing the above results, we obtain

$$\begin{aligned}\rho_{ia}(\xi_j) = 0,\ \rho_{ia}(V_j) &= 0,\ \phi_{ai}(U_j) = 0,\ \phi_{ai}(V_j) = 0,\ \phi_{ai}(\xi_j) = 0, \\ h_a^s(U_i, V_j) &= h_j^\ell(U_i, W_a) = 0, \quad h_i^\ell(V_j, W_a) = h_a^s(V_j, V_i) = 0.\end{aligned} \tag{4.13}$$

Taking the scalar product with N_i to (4.9) and using $(2.14)_4$, we have

$$\begin{aligned}&-\bar{g}(\nabla_{FY}X, N_i) + g(\nabla_Y X, U_i) - \beta\theta(Y)v_i(X) \\ &+\sum_{k=1}^{r} u_k(Y)\bar{g}(A_{N_k}X, N_i) + \sum_{a=r+1}^{n} \epsilon_a w_a(Y)\rho_{ia}(X) = 0.\end{aligned} \tag{4.14}$$

Taking $X = V_j$ and $X = \xi_j$ by turns and using (2.8) and (3.11), we get

$$h_j^\ell(FX, U_i) + \tau_{ij}(X) + \beta\delta_{ij}\theta(X) = \sum_{k=1}^{r} u_k(X)\bar{g}(A_{N_k}V_j, N_i), \tag{4.15}$$

$$h_j^\ell(X, U_i) = \sum_{k=1}^{r} u_k(X)\bar{g}(A_{N_k}\xi_j, N_i) + \tau_{ij}(FX), \tag{4.16}$$

due to $(4.13)_{1,2}$. Taking $X = U_k$ to (4.16), we have

$$h_i^*(U_k, V_j) = h_j^\ell(U_k, U_i) = \bar{g}(A_{N_k}\xi_j, N_i). \tag{4.17}$$

Replacing X by U_i to (4.9) and using (2.13), (3.3), (3.5), (3.8), $(3.9)_{1,2}$, (3.10) and the fact that $\alpha = 0$, we obtain

$$\begin{aligned}&\sum_{k=1}^{r} u_k(Y)A_{N_k}U_i + \sum_{a=r+1}^{n} w_a(Y)A_{E_a}U_i - A_{N_i}Y + (\beta+1)\eta_i(Y)\zeta \\ &- F(A_{N_i}FY) - \sum_{j=1}^{r}\tau_{ij}(FY)U_j - \sum_{a=r+1}^{n}\rho_{ia}(FY)W_a = 0.\end{aligned} \tag{4.18}$$

Taking the scalar product with V_j to (4.18) and using (2.12), (2.13), $(2.14)_3$, $(3.9)_1$ and (4.17), we get

$$h_j^\ell(X, U_i) = -\sum_{k=1}^{r} u_k(X)\bar{g}(A_{N_k}\xi_j, N_i) - \tau_{ij}(FX).$$

Comparing this equation with (4.16), we obtain

$$\tau_{ij}(FX) + \sum_{k=1}^{r} u_k(X)\bar{g}(A_{N_k}\xi_j, N_i) = 0.$$

Replacing X by U_h to this equation, we have $\bar{g}(A_{N_k}\xi_j, N_i) = 0$. Thus

$$\tau_{ij}(FX) = 0, \qquad h_j^\ell(X, U_i) = 0. \tag{4.19}$$

Taking $X = FY$ to $(4.19)_2$, we get $h_j^\ell(FX, U_i) = 0$. Thus (4.15) reduces

$$\tau_{ij}(X) = \sum_{k=1}^{r} u_k(X)\bar{g}(A_{N_k}V_j, N_i) - \beta\delta_{ij}\theta(X). \tag{4.20}$$

Replacing Y by W_a to (4.18), we obtain $A_{N_i}W_a = A_{E_a}U_i$. Taking the scalar product with U_j to this and using (2.12), (2.13) and $(3.9)_2$, we have

$$h_i^*(W_a, U_j) = \epsilon_a h_a^s(U_i, U_j) = \epsilon_a h_a^s(U_j, U_i) = h_i^*(U_j, W_a). \tag{4.21}$$

Taking the scalar product with W_a to (4.18) and using (2.12), we have

$$\epsilon_a\rho_{ia}(FY) = -h_i^*(Y, W_a) + \sum_{k=1}^{r} u_k(Y)h_k^*(U_i, W_a) + \sum_{b=r+1}^{n} \epsilon_b w_b(Y)h_b^s(U_i, W_a).$$

Taking the scalar product with U_i to (4.9) and then, taking $X = W_a$ and using (2.12), (2.13), $(2.14)_4$, $(3.9)_2$, (3.12) and (4.21), we obtain

$$\epsilon_a\rho_{ia}(FY) = h_i^*(Y, W_a) - \sum_{k=1}^{r} u_k(Y)h_k^*(U_i, W_a) - \sum_{b=r+1}^{n} \epsilon_b w_b(Y)h_b^s(U_i, W_a).$$

Comparing the last two equations, we obtain $\rho_{ia}(FY) = 0$. □

5. Indefinite generalized Sasakian space forms

Definition 5.1. An indefinite trans-Sasakian manifold $\bar{M}$ is called *indefinite generalized Sasakian space form* and denoted by $\bar{M}(f_1, f_2, f_3)$ if there exist three smooth functions f_1, f_2 and f_3 on $\bar{M}$ such that

$$\begin{aligned}\widetilde{R}(\bar{X}, \bar{Y})\bar{Z} \;=\; & f_1\{\bar{g}(\bar{Y}, \bar{Z})\bar{X} - \bar{g}(\bar{X}, \bar{Z})\bar{Y}\} \\ & + f_2\{\bar{g}(\bar{X}, J\bar{Z})J\bar{Y} - \bar{g}(\bar{Y}, J\bar{Z})J\bar{X} + 2\bar{g}(\bar{X}, J\bar{Y})J\bar{Z}\} \\ & + f_3\{\theta(\bar{X})\theta(\bar{Z})\bar{Y} - \theta(\bar{Y})\theta(\bar{Z})\bar{X} \\ & + \bar{g}(\bar{X}, \bar{Z})\theta(\bar{Y})\zeta - \bar{g}(\bar{Y}, \bar{Z})\theta(\bar{X})\zeta\},\end{aligned} \tag{5.1}$$

where $\widetilde{R}$ denote the curvature tensor of the Levi-Civita connection $\widetilde{\nabla}$ on $\bar{M}$.

Generalized Sasakian space form was introduced by Alegre *et. al.* [1]. Sasakian space form, Kenmotsu space form and cosymplectic space form are important kinds of generalized Sasakian space forms such that

$$f_1 = \tfrac{c+3}{4}, f_2 = f_3 = \tfrac{c-1}{4}; \quad f_1 = \tfrac{c-3}{4}, f_2 = f_3 = \tfrac{c+1}{4}; \quad f_1 = f_2 = f_3 = \tfrac{c}{4}$$

respectively, where c is a constant J-sectional curvature of each space forms.

By directed calculations from (1.1) and (1.2), we see that

$$\begin{aligned}\bar{R}(\bar{X},\bar{Y})\bar{Z} = \widetilde{R}(\bar{X},\bar{Y})\bar{Z} + \bar{g}(\bar{X},\bar{Z})\bar{\nabla}_{\bar{Y}}\zeta - \bar{g}(\bar{Y},\bar{Z})\bar{\nabla}_{\bar{X}}\zeta \\ + \{(\bar{\nabla}_{\bar{X}}\theta)(\bar{Z}) - \bar{g}(\bar{X},\bar{Z})\}\bar{Y} - \{(\bar{\nabla}_{\bar{Y}}\theta)(\bar{Z}) - \bar{g}(\bar{Y},\bar{Z})\}\bar{X},\end{aligned} \tag{5.2}$$

where $\bar{R}$ is the curvature tensor of the semi-symmetric metric connection $\bar{\nabla}$.

Denote by R and R^* the curvature tensors of the induced linear connection ∇ and ∇^* on M and $S(TM)$ respectively. Using the Gauss-Weingarten formulae, we obtain Gauss equations for M and $S(TM)$, respectively:

$$\begin{aligned}\bar{R}(X,Y)Z &= R(X,Y)Z + \sum_{i=1}^{r}\{h_i^\ell(X,Z)A_{N_i}Y - h_i^\ell(Y,Z)A_{N_i}X\} \\ &+ \sum_{a=r+1}^{n}\{h_a^s(X,Z)A_{E_a}Y - h_a^s(Y,Z)A_{E_a}X\} \\ &+ \sum_{i=1}^{r}\{(\nabla_X h_i^\ell)(Y,Z) - (\nabla_Y h_i^\ell)(X,Z) \\ &+ \sum_{j=1}^{r}[\tau_{ji}(X)h_j^\ell(Y,Z) - \tau_{ji}(Y)h_j^\ell(X,Z)] \\ &+ \sum_{a=r+1}^{n}[\phi_{ai}(X)h_a^s(Y,Z) - \phi_{ai}(Y)h_a^s(X,Z)] \\ &- \theta(X)h_i^\ell(Y,Z) + \theta(Y)h_i^\ell(X,Z)\}N_i \qquad (5.3) \\ &+ \sum_{a=r+1}^{n}\{(\nabla_X h_a^s)(Y,Z) - (\nabla_Y h_a^s)(X,Z) \\ &+ \sum_{i=1}^{r}[\rho_{ia}(X)h_i^\ell(Y,Z) - \rho_{ia}(Y)h_i^\ell(X,Z)] \\ &+ \sum_{b=r+1}^{n}[\sigma_{ba}(X)h_b^s(Y,Z) - \sigma_{ba}(Y)h_b^s(X,Z)] \\ &- \theta(X)h_a^s(Y,Z) + \theta(Y)h_a^s(X,Z)\}E_a\end{aligned}$$

and

$$R(X,Y)PZ = R^*(X,Y)PZ + \sum_{i=1}^{r}\{h_i^*(X,PZ)A^*_{\xi_i}Y - h_i^*(Y,PZ)A_{\xi_i}X\}$$

$$
\begin{aligned}
&+\sum_{i=1}^{r}\{(\nabla_X h_i^*)(Y,PZ)-(\nabla_Y h_i^*)(X,PZ)\\
&+\sum_{k=1}^{r}[\tau_{ik}(Y)h_k^*(X,PZ)-\tau_{ik}(X)h_k^*(Y,PZ)] \qquad (5.4)\\
&-\theta(X)h_i^*(Y,PZ)+\theta(Y)h_i^*(X,PZ)\}\xi_i.
\end{aligned}
$$

Taking the scalar product with ξ_i and N_i to (5.2) by turns and then, substituting (5.3) and (5.1) and using (2.3), $(2.14)_4$ and (5.4), we get

$$
\begin{aligned}
&(\nabla_X h_i^\ell)(Y,Z)-(\nabla_Y h_i^\ell)(X,Z)\\
&+\sum_{j=1}^{r}\{\tau_{ji}(X)h_j^\ell(Y,Z)-\tau_{ji}(Y)h_j^\ell(X,Z)\}\\
&+\sum_{a=r+1}^{n}\{\phi_{ai}(X)h_a^s(Y,Z)-\phi_{ai}(Y)h_a^s(X,Z)\} \qquad (5.5)\\
&-\theta(X)h_i^\ell(Y,Z)+\theta(Y)h_i^\ell(X,Z)\\
&+\alpha\{u_i(Y)g(X,Z)-u_i(X)g(Y,Z)\}\\
=\ & f_2\{u_i(Y)\bar{g}(X,JZ)-u_i(X)\bar{g}(Y,JZ)+2u_i(Z)\bar{g}(X,JY)\}
\end{aligned}
$$

and

$$
\begin{aligned}
&(\nabla_X h_i^*)(Y,\,PZ)-(\nabla_Y h_i^*)(X,\,PZ)\\
&+\sum_{j=1}^{r}\{\tau_{ij}(Y)h_j^*(X,\,PZ)-\tau_{ij}(X)h_j^*(Y,\,PZ)\}\\
&+\sum_{j=1}^{r}\{h_j^\ell(X,\,PZ)\eta_i(A_{N_j}Y)-h_j^\ell(Y,\,PZ)\eta_i(A_{N_j}X)\}\\
&+\sum_{a=r+1}^{n}\epsilon_a\{\rho_{ia}(Y)h_a^s(X,\,PZ)-\rho_{ia}(X)h_a^s(Y,\,PZ)\}\\
&-\theta(X)h_i^*(Y,PZ)+\theta(Y)h_i^*(X,PZ)\\
&-\{(\bar{\nabla}_X\theta)(PZ)+\beta g(X,PZ)\}\eta_i(Y) \qquad (5.6)\\
&+\{(\bar{\nabla}_Y\theta)(PZ)+\beta g(Y,PZ)\}\eta_i(X)\\
&+\alpha\{v_i(Y)g(X,PZ)-v_i(X)g(Y,PZ)\}\\
=\ & f_1\{g(Y,PZ)\eta_i(X)-g(X,PZ)\eta_i(Y)\}\\
&+f_2\{v_i(Y)\bar{g}(X,JPZ)-v_i(X)\bar{g}(Y,JPZ)+2v_i(PZ)\bar{g}(X,JY)\}\\
&+f_3\{\theta(X)\eta_i(Y)-\theta(Y)\eta_i(X)\}\theta(PZ).
\end{aligned}
$$

Theorem 5.2. *Let M be a generic lightlike submanifold of an indefinite generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ with a semi-symmetric metric connection. Then the functions α, β, f_1, f_2 and f_3 satisfy*

(1) *α is a constant on M,*
(2) *$\alpha\beta = 0$,*
(3) *$f_1 - f_2 = \alpha^2 - \beta^2$ and $f_1 - f_3 = \alpha^2 - \beta^2 - \zeta\beta$.*

Proof. Applying ∇_X to $h_j^\ell(Y, U_i) = h_i^*(Y, V_j)$ and using (2.1), (2.11), (2.13), (3.2), (3.3), (3.4), $(3.7)_1$, (3.8), $(3.9)_{1,2,4}$, (3.10) and (3.11), we have

$$\begin{aligned}
(\nabla_X h_j^\ell)(Y, U_i) \;=\; & (\nabla_X h_i^*)(Y, V_j) - \sum_{k=1}^{r}\{\tau_{kj}(X)h_k^\ell(Y, U_i) + \tau_{ik}(X)h_k^*(Y, V_j)\} \\
& - \sum_{a=r+1}^{n}\{\phi_{aj}(X)h_a^s(Y, U_i) + \epsilon_a\rho_{ia}(X)h_a^s(Y, V_j)\} \\
& + \sum_{k=1}^{r}\{h_i^*(Y, U_k)h_k^\ell(X, \xi_j) + h_i^*(X, U_k)h_k^\ell(Y, \xi_j)\} \\
& - g(A^*_{\xi_j}X, F(A_{N_i}Y)) - g(A^*_{\xi_j}Y, F(A_{N_i}X)) \\
& - \sum_{k=1}^{r} h_j^\ell(X, V_k)\eta_k(A_{N_i}Y) \\
& + \alpha(\beta+1)\{u_j(X)v_i(Y) - u_j(Y)v_i(X)\} \\
& - \alpha^2 u_j(Y)\eta_i(X) + (\beta+1)^2 u_j(X)\eta_i(Y).
\end{aligned}$$

Substituting this into (5.5) such that replace i by j and Z by U_i, we have

$$\begin{aligned}
& (\nabla_X h_i^*)(Y, V_j) - (\nabla_Y h_i^*)(X, V_j) \\
& - \sum_{k=1}^{r}\{\tau_{ik}(X)h_k^*(Y, V_j) - \tau_{ik}(Y)h_k^*(X, V_j)\} \\
& - \sum_{a=r+1}^{n}\epsilon_a\{h_a^s(Y, V_j)\rho_{ia}(X) - h_a^s(X, V_j)\rho_{ia}(Y)\} \\
& - \sum_{k=1}^{r}\{h_k^\ell(Y, V_j)\eta_i(A_{N_k}X) - h_k^\ell(X, V_j)\eta_i(A_{N_k}Y)\} \\
& - \theta(X)h_i^*(Y, V_j) + \theta(Y)h_i^*(X, V_j)\} \\
& - \alpha(2\beta+1)\{u_j(Y)v_i(X) - u_j(X)v_i(Y)\} \\
& - \{\alpha^2 + (\beta+1)^2\}\{u_j(X)\eta_i(Y) - u_j(Y)\eta_i(X)\} \\
& = f_2\{u_j(Y)\eta_i(X) - u_j(X)\eta_i(Y) + 2\delta_{ij}\bar{g}(X, JY)\}.
\end{aligned} \tag{5.7}$$

Applying $\bar{\nabla}_X$ to $\theta(V_i) = 0$ and using (2.4) and (3.11), we obtain

$$(\bar{\nabla}_X \theta)(V_i) = (\beta + 1)u_i(X). \tag{5.8}$$

Comparing (5.7) and (5.6) with $Z = V_j$ and using $(2.14)_3$ and $(3.9)_3$, we get

$$\begin{aligned}&\{f_1 - f_2 - \alpha^2 + \beta^2\}[u_j(Y)\eta_i(X) - u_j(X)\eta_i(Y)] \\ &= 2\alpha\beta\{u_j(Y)v_i(X) - u_j(X)v_i(Y)\}.\end{aligned}$$

Taking $X = \xi_i, Y = U_j$ and $X = V_i$, $Y = U_j$ to this by turns, we obtain

$$f_1 - f_2 = \alpha^2 - \beta^2, \qquad \alpha\beta = 0. \tag{5.9}$$

Applying $\bar{\nabla}_X$ to $\theta(\zeta) = 1$ and using (2.3), we obtain

$$(\bar{\nabla}_X \theta)(\zeta) = 0. \tag{5.10}$$

Applying $\bar{\nabla}_X$ to $\eta_i(Y) = \bar{g}(Y, N_i)$ and using (2.5), we have

$$(\nabla_X \eta_i)Y = -g(A_{N_i} X, Y) + \sum_{j=1}^{r} \tau_{ij}(X)\eta_j(Y).$$

Applying ∇_X to (3.8) and using (2.13) and (3.6) and (3.8), (3.16), we have

$$\begin{aligned}(\nabla_X h_i^*)(Y, \zeta) \;=\; & -(X\alpha)v_i(Y) + (X\beta)\eta_i(Y) \\ & - \alpha\{\sum_{j=1}^{r} v_j(Y)\tau_{ij}(X) + \sum_{a=r+1}^{n} w_a(Y)\rho_{ia}(X) \\ & - \sum_{j=1}^{r} u_j(Y)\eta_j(A_{N_i} X) - g(A_{N_i} X, FY) - g(A_{N_i} Y, FX) \\ & - \alpha\theta(Y)\eta_i(X) + \theta(X)v_i(Y) - \theta(Y)v_i(X)\} \\ & + (\beta + 1)\{\sum_{j=1}^{r} \tau_{ij}(X)\eta_j(Y) + (\beta + 1)\theta(X)\eta_i(Y) \\ & - g(A_{N_i} X, Y) - g(A_{N_i} Y, X)\}.\end{aligned}$$

Substituting this and (3.7) into (5.5) with $PZ = \zeta$ and using (5.9), we get

$$\begin{aligned}& -(X\alpha)v_i(Y) + (Y\alpha)v_i(X) + (X\beta)\eta_i(Y) - (Y\beta)\eta_i(X) \\ = \;& (f_1 - f_3 - \alpha^2 + \beta^2)\{\theta(Y)\eta_i(X) - \theta(X)\eta_i(Y)\}.\end{aligned}$$

Taking $Y = \zeta$, $X = \xi_i$ and $Y = U_j$, $X = V_i$ to this by turns, we obtain

$$f_1 - f_3 = \alpha^2 - \beta^2 - \zeta\beta, \quad U_i\alpha = 0.$$

Applying ∇_Y to $(3.7)_1$ and using (3.6), $(3.7)_1$ and (3.14), we have

$$\begin{aligned}(\nabla_X h_i^\ell)(Y,\zeta) \;&=\; -(X\alpha)u_i(Y) + \alpha\{h_i^\ell(X,FY) + h_i^\ell(Y,FX) \\ &\quad + \sum_{j=1}^{r} u_j(Y)\tau_{ji}(X) + \sum_{a=r+1}^{n} w_a(Y)\phi_{ai}(X) \\ &\quad + \theta(Y)u_i(X) - \theta(X)u_i(Y)\} - (\beta+1)h_i^\ell(X,Y).\end{aligned}$$

Substituting this and $(3.7)_1$ into (5.5) with $Z=\zeta$ and using (3.7), we get

$$(X\alpha)u_i(Y) = (Y\alpha)u_i(X).$$

Taking $Y = U_i$ to this, we get $X\alpha = 0$. Thus α is a constant on M. □

Definition 5.3. (1) A screen distribution $S(TM)$ is called *totally umbilical* [5] if there exist smooth functions γ_i on a neighborhood $\mathcal{U}$ such that

$$h_i^*(X,PY) = \gamma_i g(X,PY).$$

In case $\gamma_i = 0$, we say that $S(TM)$ is *totally geodesic* in M.

(2) A generic lightlike submanifold M is said to be *screen conformal* [5] if there exist non-vanishing smooth functions φ_i on $\mathcal{U}$ such that

$$h_i^*(X,PY) = \varphi_i h_i^\ell(X,PY). \tag{5.11}$$

Theorem 5.4. *Let M be a generic lightlike submanifold of an indefinite generalized Sasakian space form $\bar{M}(f_1,f_2,f_3)$ with a semi-symmetric metric connection. If one of the following five conditions is satisfied,*

(1) *M is recurrent,*
(2) *$S(TM)$ is totally umbilical,*
(3) *M is screen conformal,*
(4) *U_is is parallel with respect to the induced connection ∇,*
(5) *V_is is parallel with respect to the induced connection ∇,*

then $\bar{M}(f_1,f_2,f_3)$ is an indefinite Kenmotsu space form such that

$$\alpha = 0, \quad \beta = -1; \quad f_1 = -1, \quad f_2 = f_3 = 0. \tag{5.12}$$

Proof. (1) As M is recurrent, by Theorem 4.2, we obtain $\alpha = 0$, $\beta = -1$ and the fact that M is irrotational and solenoidal, *i.e.*, (2.15) and (2.16) are satisfied. By directed calculation from $(4.8)_1$, we obtain

$$R(X,Y)U_i = \sum_{j=1}^{r} 2d\tau_{ij}(X,Y)U_j. \tag{5.13}$$

On the other hand, since $\alpha = 0$ and $\beta = -1$, we have $\bar{\nabla}_X\zeta = 0$ by (2.3) and $f_1 + 1 = f_2 = f_3$ by Theorem 5.2. Comparing the tangential components of the right and left terms of (5.2) and using (5.1) and (5.3), we obtain

$$\begin{aligned} R(X,Y)Z &= \sum_{i=1}^{r}\{h_i^\ell(Y,Z)A_{N_i}X - h_i^\ell(X,Z)A_{N_i}Y\} \\ &+ \sum_{a=r+1}^{n}\{h_a^s(Y,Z)A_{E_a}X - h_a^s(X,Z)A_{E_a}Y\} \\ &+ (\bar\nabla_X\theta)(Z)Y - (\bar\nabla_Y\theta)(Z)X \\ &+ (f_1+1)\{g(Y,Z)X - g(X,Z)Y\} \\ &+ f_2\{\bar g(X,JZ)FY - \bar g(Y,JZ)FX + 2\bar g(X,JY)FZ\} \\ &+ f_3\{\theta(X)\theta(Z)Y - \theta(Y)\theta(Z)X \\ &+ \bar g(X,Z)\theta(Y)\zeta - \bar g(Y,Z)\theta(X)\zeta\}. \end{aligned} \tag{5.14}$$

Applying $\bar\nabla_X$ to $\theta(U_i)=0$ and using (2.4) and (3.10), we obtain

$$(\bar\nabla_X\theta)(U_i) = \alpha\eta_i(X) + (\beta+1)v_i(X). \tag{5.15}$$

Replacing Z by U_i to (5.14) and using (5.13) and (5.15), we get

$$\begin{aligned} \sum_{j=1}^{r} 2d\tau_{ij}(X,Y)U_j &= \sum_{j=1}^{r}\{h_j^\ell(Y,U_i)A_{N_j}X - h_j^\ell(X,U_i)A_{N_j}Y\} \\ &+ \sum_{a=r+1}^{n}\{h_a^s(Y,U_i)A_{E_a}X - h_a^s(X,U_i)A_{E_a}Y\} \\ &+ (f_1+1)\{v_i(Y)X - v_i(X)Y\} \\ &+ f_2\{\eta_i(X)FY - \eta_i(Y)FX\} \\ &+ f_3\{v_i(X)\theta(Y) - v_i(Y)\theta(X)\}\zeta. \end{aligned}$$

Taking the scalar product with N_j and using (2.15) and (2.16), we get

$$f_2\{v_i(Y)\eta_j(X) - v_i(X)\eta_j(Y)\} + f_2\{v_j(Y)\eta_i(X) - v(X)\eta_i(Y)\} = 0.$$

Taking $Y=V_i$ and $X=\xi_j$, we get $f_2=0$. Thus $f_1=-1$ and $f_2=f_3=0$.

(2) Assume that $S(TM)$ is totally umbilical. Then (3.8) is reduced to

$$\gamma_i\theta(X) = -\alpha v_i(X) + (\beta+1)\eta_i(X).$$

Taking $X=\zeta$, $X=V_i$ and $X=\xi_i$ by turns, we have $\gamma_i=0$, $\alpha=0$ and $\beta=-1$. As $\gamma_i=0$, $S(TM)$ is totally geodesic and, from $(3.9)_{1,2}$. we have

$$h_j^\ell(X,U_i)=0, \qquad h_a^s(X,U_i)=0. \tag{5.16}$$

As $\alpha=0$ and $\beta=-1$, $\bar M$ is an indefinite Kenmotsu manifold and $f_1+1=f_2=f_3$ by Theorem 5.2. Taking $PZ=U_j$ to (5.6) such that $h_i^*=0$ and using

(5.15), (5.16) and the fact that $f_1 + 1 = f_2$, we get

$$f_2\{[v_j(Y)\eta_i(X) - v_j(X)\eta_i(Y)] + [v_i(Y)\eta_j(X) - v_i(X)\eta_j(Y)]\} = 0.$$

Taking $X = \xi_i$ and $Y = V_j$ to this equation, we get $f_2 = 0$. Thus $\bar{M}(f_1, f_2, f_3)$ is an indefinite Kenmotsu space form satisfying (5.12).

(3) Taking $PY = \zeta$ to (5.11) and using $(3.7)_1$ and (3.8), we get

$$\alpha v_i(X) - (\beta + 1)\eta_i(X) = \alpha\varphi u_i(X).$$

Taking $X = V_i$ and $X = \xi_i$ by turns, we have $\alpha = 0$ and $\beta = -1$ respectively. Thus $\bar{M}$ is an indefinite Kenmotsu manifold such that $f_1 + 1 = f_2 = f_3$.

Denote by μ_i, $i \in \{1, \cdots, r\}$ the r-th vector fields on $S(TM)$ such that $\mu_i = U_i - \varphi_i V_i$. Then $J\mu_i = N_i - \varphi_i\xi_i$. Using $(3.9)_{1,2,3,4}$, we get

$$h^\ell_j(X, \mu_i) = 0, \quad h^s_a(X, \mu_i) = 0. \tag{5.17}$$

Applying ∇_Y to (5.11), we have

$$(\nabla_X h^*_i)(Y, PZ) = (X\varphi_i)h^\ell_i(Y, PZ) + \varphi_i(\nabla_X h^\ell_i)(Y, PZ).$$

Substituting this equation and (5.11) into (5.6) and using (5.5), we have

$$\begin{aligned}
&\sum_{j=1}^{r}\{(X\varphi_i)\delta_{ij} - \varphi_i\tau_{ji}(X) - \varphi_j\tau_{ij}(X) - \eta_i(A_{N_j}X)\}h^\ell_j(Y, PZ)\\
&- \sum_{j=1}^{r}\{(Y\varphi_i)\delta_{ij} - \varphi_i\tau_{ji}(Y) - \varphi_j\tau_{ij}(Y) - \eta_i(A_{N_j}Y)\}h^\ell_j(X, PZ)\\
&- \sum_{a=r+1}^{n}\{\epsilon_a\rho_{ia}(X) + \varphi_i\phi_{ai}(X)\}h^s_a(Y, PZ)\\
&+ \sum_{a=r+1}^{n}\{\epsilon_a\rho_{ia}(Y) + \varphi_i\phi_{ai}(Y)\}h^s_a(X, PZ)\\
&- \{(\bar{\nabla}_X\theta)(PZ) - g(X, PZ)\}\eta_i(Y)\\
&+ \{(\bar{\nabla}_Y\theta)(PZ) - g(Y, PZ)\}\eta_i(X)\\
= \;& f_1\{g(Y, PZ)\eta_i(X) - g(X, PZ)\eta_i(Y)\}\\
&+ f_2\{g(Y, \mu_i)\bar{g}(X, JPZ) - g(X, \mu_i)\bar{g}(Y, JPZ)\\
&+ 2g(PZ, \mu_i)\bar{g}(X, JY)\}\\
&+ f_3\{\theta(X)\eta_i(Y) - \theta(Y)\eta_i(X)\}\theta(PZ).
\end{aligned}$$

Replacing PZ by μ_i to this and using (5.8), (5.15) and (5.17), we obtain

$$\begin{aligned} &f_2\{[v_k(Y)\eta_i(X) - v_k(X)\eta_i(Y) + v_i(Y)\eta_k(X) - v_i(X)\eta_k(Y)] \\ &\quad - \varphi_k[u_k(Y)\eta_i(X) - u_k(X)\eta_i(Y)] + \varphi_i[u_i(Y)\eta_k(X) - u_i(X)\eta_k(Y)] \\ &\quad + 2[\varphi_k\delta_{ki} - \varphi_i\delta_{ki}]\bar{g}(X, JY)\} = 0. \end{aligned}$$

Taking $X = \xi_i$ and $Y = V_k$, we get $f_2 = 0$. Thus $\bar{M}(f_1, f_2, f_3)$ is an indefinite Kenmotsu space form satisfying (5.12).

(4) If U_is is parallel with respect to ∇, then we have (3.16) and (3.17). As $\alpha = 0$ and $\beta = 0$, we get $f_1 + 1 = f_2 = f_3$ by Theorem 5.2.

Applying ∇_Y to (3.17) and using the fact that $\nabla_X U_i = 0$, we obtain

$$(\nabla_X h_i^*)(Y, U_j) = 0.$$

Substituting this equation and (3.17 into (5.6) such that $PZ = U_j$ and using (5.15), (3.16) and the fact that $f_1 + 1 = f_2$, we have

$$f_2\{[v_j(Y)\eta_i(X) - v_j(X)\eta_i(Y)] + [v_i(Y)\eta_j(X) - v_i(X)\eta_j(Y)]\} = 0.$$

Taking $X = \xi_i$ and $Y = V_j$ to this equation, we get $f_2 = 0$. Thus $\bar{M}(f_1, f_2, f_3)$ is an indefinite Kenmotsu space form satisfying (5.12).

(5) If V_is is parallel with respect to the connection ∇, then we have (3.18) and (3.19). As $\alpha = 0$ and $\beta = -1$, we get $f_1 + 1 = f_2 = f_3$ by Theorem 5.2. From $(3.9)_1$ and (3.18), we have

$$h_i^*(Y, V_j) = 0.$$

Applying ∇_X to this equation and using the fact that $\nabla_X V_j = 0$, we have

$$(\nabla_X h_i^*)(Y, V_j) = 0.$$

Substituting these two equations into (5.6) such that $PZ = V_j$, we obtain

$$\begin{aligned} &\sum_{k=1}^{r} \{h_k^\ell(X, V_j)\eta_i(A_{N_k}Y) - h_k^\ell(Y, V_j)\eta_i(A_{N_k}X)\} \\ &+ \sum_{a=r+1}^{n} \epsilon_a\{\rho_{ia}(Y)h_a^s(X, V_j) - \rho_{ia}(X)h_a^s(Y, V_j)\} \\ &= f_2\{u_j(Y)\eta_i(X) - u_j(X)\eta_i(Y) + 2\delta_{ij}\bar{g}(X, JY)\}. \end{aligned}$$

Taking $X = \xi_i$ and $Y = U_j$ to this and using (3.18), (3.19) and the fact that $h_a^s(U_j, V_j) = \epsilon_a h_i^\ell(U_j, W_a) = 0$ due to $(3.9)_4$ and (3.18), we get $f_2 = 0$. Thus $\bar{M}(f_1, f_2, f_3)$ is an indefinite Kenmotsu space form satisfying (5.12). □

Theorem 5.5. *Let M be a Lie recurrent generic lightlike submanifold of an indefinite generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ with a semi-symmetric*

metric connection. Then $\bar{M}(f_1, f_2, f_3)$ *is a space form with an indefinite* β-*Kenmotsu structure such that*

$$f_1 = -\beta^2, \quad f_2 = 0, \quad f_3 = \zeta\beta.$$

Proof. If M is Lie recurrent, then, by Theorem 4.4 we get $\alpha = 0$ and

$$h_i^\ell(X, U_j) = 0. \tag{5.18}$$

Applying ∇_Y to (5.18) and using $(3.7)_1$ and (3.10), we have

$$(\nabla_X h_i^\ell)(Y, U_j) = -h_i^\ell(Y, F(A_{N_j} X)) - \sum_{a=r+1}^{n} \rho_{ja}(X) h_i^\ell(Y, W_a).$$

Substituting the last two equations into (5.5) with $Z = U_j$, we have

$$\begin{aligned} & h_i^\ell(X, F(A_{N_j} Y)) - h_i^\ell(Y, F(A_{N_j} X)) \\ & + \sum_{a=r+1}^{n} \{\rho_{ja}(Y) h_i^\ell(X, W_a) - \rho_{ja}(X) h_i^\ell(Y, W_a)\} \\ & + \sum_{a=r+1}^{n} \{\phi_{ai}(X) h_a^s(Y, U_j) - \phi_{ai}(Y) h_a^s(X, U_j)\} \\ = \; & f_2\{u_i(Y)\eta_j(X) - u_i(X)\eta_j(Y) + 2\delta_{ij}\bar{g}(X, JY)\}. \end{aligned}$$

Taking $Y = U_i$ and $X = \xi_j$ to this and using $(2.14)_2$, $(3.9)_4$, $(4.13)_{1,3,4,5}$ and (5.18), we have $f_2 = 0$. As $f_2 = 0$, we have $f_1 = -\beta^2$ and $f_3 = \zeta\beta$. □

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