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DIFFERENTIABILITY PROPERTIES OF FUNCTIONS OF GENERALIZED QUATERNIONIC VARIABLES

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Abstract. This paper gives general base i_α and j_β, which are similar with two bases i and j in quaternions. From properties of i_α and j_β, we present an expression of a generalized quaternion. Based on the analytic properties of real and complex analysis, we propose a regularity which is related to a differentiability of functions of generalized quaternionic variables.

1. Introduction

The quaternions are represented by the form $z = \sum_{r=0}^{3} x_r e_r$, where each e_r is non-commutative base for four dimensional real field $\mathbb{R}^4$ and $x_r \in \mathbb{R}$ $(r = 0, 1, 2, 3)$. By the algebraic and analytic properties of quaternions, many theories of functions of quaternionic variables have been studied. Fueter [3] gave a definition of hyperholomorphic functions of quaternionic variables in $\mathbb{R}^4$. Deavours [2] and Sudbery [11] developed theories of quaternionic analysis and refined differential operators and integral formulae in complex analysis to use in quaternionic analysis. Naser [10] and Koriyama et al. [8] investigated the notions and arithmetic operations of hyperholomorphic functions and generalized holomorphic functions with values in quaternions. Kajiwara et al. [5]

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showed that for any complex-valued harmonic function f_1 in a quaternion domain D, there exists a function f_2 in D such that the function $f_1 + f_2 j$ is hyperholomorphic in D. Kim et al. [6] obtained some results for the regularity of functions on the reduced quaternion field in Clifford analysis. Also, Kim and Shon [7] researched corresponding Cauchy-Riemann systems and properties of a regularity of functions with values in dual split quaternions.

Generalized quaternions have been introduced for studying the quadratic form theory. This construction is a natural generalization of quaternions. Actually, there are studies on the algebraic aspect of generalized quaternions (see [1, 4, 9, 12]). We investigate what features are available in generalized quaternions, by comparing with differentiability of functions of quaternionic variables. From these results, if the differentiability and regularity of functions of generalized quaternionic variables is well-defined, we can induce analytic properties of generalized quaternion-valued functions without the process of a generalization of quaternionic analysis.

In this paper, we consider generalized quaternions, represented by base i_α and j_β. We study representations and some characteristics of differentiability, and regularity of generalized quaternionic functions.

2. Notations and Preliminaries

For non-zero scalar numbers α and β, the algebra of the elements of the following form

$$z = x_0 \cdot 1 + x_1 i_\alpha + x_2 j_\beta + x_3 k, \quad x_r \in \mathbb{R} \ (r = 0, 1, 2, 3),$$

is said to be a generalized quaternion which is a four-dimensional noncommutative and associative real field and is composed of bases i_α, j_β and k, where

$$i_\alpha^2 = \alpha, \ j_\beta^2 = \beta, \ k = i_\alpha j_\beta = -j_\beta i_\alpha.$$

Especially, if $\alpha = \beta = -1$, then z is said to be a quaternion. Also, if $\alpha = -1$ and $\beta = 1$, then z is said to be a split quaternion. If $\alpha = -1$ and $\beta = 0$, then z is said to be a dual quaternion. We consider the set of generalized quaternions as follows:

$$\mathbb{G}_\mathbb{H} = \{z \mid z = x_0 \cdot 1 + x_1 i_\alpha + x_2 j_\beta + x_3 k\},$$

where the element 1 is the identity of $\mathbb{G}_\mathbb{H}$ and $\mathbb{G}_\mathbb{H}$ is isomorphic to $\mathbb{R}^4$.

The conjugate number z^* of z in $\mathbb{G}_\mathbb{H}$ is given by the form:

$$z^* = x_0 - x_1 i_\alpha - x_2 j_\beta - x_3 k.$$

Also, the norm $|z|$ of z and the inverse z^{-1} of z are given by the forms:

$$|z|^2 = z(z^*) = z^*(z) = x_0^2 - x_1^2\alpha - x_2^2\beta + x_3^2\alpha\beta$$

and

$$z^{-1} = \frac{z^*}{|z|^2},$$

respectively. The existence of z^{-1} is guaranteed except two cases:

$$[x_0 = \pm x_1\sqrt{\alpha},\ x_2 = \pm x_3\sqrt{\alpha}] \quad \text{or} \quad [x_0 = \pm x_1\sqrt{\alpha},\ \beta = 0].$$

From the form of z, the scalar part of z, denoted by $Sc(z)$, is $Sc(z) = x_0$ and the pure generalized quaternion of z, denoted by $Pu(z)$, is $Pu(z) = x_1 i_\alpha + x_2 j_\beta + x_3 k$.

We give the addition and multiplication of two generalized quaternions $z = x_0 + x_1 i_\alpha + x_2 j_\beta + x_3 k$ and $w = y_0 + y_1 i_\alpha + y_2 j_\beta + y_3 k$, where $y_m \in \mathbb{R}$ ($m = 0, 1, 2, 3$), as follows:

$$z + w = (x_0 + y_0) + (x_1 + y_1)i_\alpha + (x_2 + y_2)j_\beta + (x_3 + y_3)k$$

and

$$\begin{aligned} zw = x_0y_0 \cdot 1 + x_1y_1\alpha + x_2y_2\beta - x_3y_3\alpha\beta + (x_1y_0 + x_0y_1 + x_3y_2\beta - x_2y_3\beta)i_\alpha \\ + (x_2y_0 - x_3y_1\alpha + x_0y_2 + x_1y_3\alpha)j_\beta + (x_3y_0 - x_2y_1 + x_1y_2 + x_0y_3)k, \end{aligned}$$

respectively. We have a vector space $\mathbb{G}_{\mathbb{H}} = \mathbb{R} \bigoplus \mathcal{V}$, where $\mathcal{V}$ is a three-dimensional Euclidean vector space, which has the product of two elements $Pu(z)$ and $Pu(w)$ of $\mathcal{P}$ as follows:

$$Pu(z)Pu(w) = Pu(z) \cdot_G Pu(w) + Pu(z) \times_G Pu(w),$$

where

$$Pu(z) \cdot_G Pu(w) = x_1y_1\alpha + x_2y_2\beta - x_3y_3\alpha\beta$$

and

$$Pu(z) \times_G Pu(w) = (x_3y_2 - x_2y_3)\beta i_\alpha - (x_3y_1 - x_1y_3)\alpha j_\beta + (x_1y_2 - x_2y_1)k.$$

We consider differential operators as follows:

$$D := \frac{1}{2}\Big(\frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_1}\alpha^{-1}i_\alpha + \frac{\partial}{\partial x_2}\beta^{-1}j_\beta - \frac{\partial}{\partial x_3}\alpha^{-1}\beta^{-1}k\Big)$$

and

$$D^* := \frac{1}{2}\Big(\frac{\partial}{\partial x_0} - \frac{\partial}{\partial x_1}\alpha^{-1}i_\alpha - \frac{\partial}{\partial x_2}\beta^{-1}j_\beta + \frac{\partial}{\partial x_3}\alpha^{-1}\beta^{-1}k\Big),$$

where α^{-1} satisfies $\alpha\alpha^{-1} = 1$ and β^{-1} satisfies $\beta\beta^{-1} = 1$. When $\alpha,\ \beta = 0$, for example, if z is a dual quaternion, the differential operators are defined by another ways. Then by the express of differential operators, the Laplacian operator in generalized quaternions is

$$DD^* = D^*D = \frac{\partial^2}{\partial x_0^2} - \frac{\partial^2}{\partial x_1^2}\alpha^{-1} - \frac{\partial^2}{\partial x_2^2}\beta^{-1} + \frac{\partial^2}{\partial x_3^2}\alpha^{-1}\beta^{-1}.$$

3. Differentiability of generalized quaternion-valued functions

Let Ω be an open subset of $\mathbb{R}^4$. For z in $\mathbb{G}_{\mathbb{H}}$, consider a function $f:\Omega\to\mathbb{G}_{\mathbb{H}}$ such that

$$f(z):=f(x_0,x_1,x_2,x_3)=u_0+u_1i_\alpha+u_2j_\beta+u_3k,$$

where $u_r=u_r(x_0,x_1,x_2,x_3)$ $(r=0,1,2,3)$ are real-valued functions.

Definition 3.1. Let Ω be an open subset of $\mathbb{R}^4$. A function $f:\Omega\to\mathbb{G}_{\mathbb{H}}$ is said to be generalized quaternionic differentiable at z on the left if the following limit

$$\frac{df}{dz}:=\lim_{h\to0}h^{-1}(f(z+h)-f(z))$$

exists, where $h\to0$ is set such that the norm $|h|$ of h is approximated to 0 and

$$|h|^2=h_0^2-h_1^2\alpha-h_2^2\beta+h_3^2\alpha\beta\neq0.$$

Similarly, if the limit

$$\lim_{h\to0}(f(z+h)-f(z))h^{-1}$$

exists, a function f is said to be generalized quaternionic differentiable at z on the right.

Remark 3.2. By properties of the calculation of the differential operators D and D^*, we have the following results:

$$\begin{aligned}Df&=\frac{\partial u_0}{\partial x_0}+\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}-\frac{\partial u_3}{\partial x_3}+\Big(\frac{\partial u_0}{\partial x_1}\alpha^{-1}+\frac{\partial u_1}{\partial x_0}-\frac{\partial u_2}{\partial x_3}\alpha^{-1}-\frac{\partial u_3}{\partial x_2}\Big)i_\alpha\\&\quad+\Big(\frac{\partial u_0}{\partial x_2}\beta^{-1}+\frac{\partial u_1}{\partial x_3}\beta^{-1}+\frac{\partial u_2}{\partial x_0}+\frac{\partial u_3}{\partial x_1}\Big)j_\beta\\&\quad+\Big(-\frac{\partial u_0}{\partial x_3}\alpha^{-1}\beta^{-1}-\frac{\partial u_1}{\partial x_2}\beta^{-1}+\frac{\partial u_2}{\partial x_1}\alpha^{-1}+\frac{\partial u_3}{\partial x_0}\Big)k\end{aligned}$$

and

$$\begin{aligned}D^*f&=\frac{\partial u_0}{\partial x_0}-\frac{\partial u_1}{\partial x_1}-\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}+\Big(\frac{\partial u_1}{\partial x_0}-\frac{\partial u_0}{\partial x_1}\alpha^{-1}+\frac{\partial u_2}{\partial x_3}\alpha^{-1}+\frac{\partial u_3}{\partial x_2}\Big)i_\alpha\\&\quad+\Big(\frac{\partial u_2}{\partial x_0}-\frac{\partial u_0}{\partial x_2}\beta^{-1}-\frac{\partial u_1}{\partial x_3}\beta^{-1}-\frac{\partial u_3}{\partial x_1}\Big)j_\beta\\&\quad+\Big(\frac{\partial u_3}{\partial x_0}-\frac{\partial u_2}{\partial x_1}\alpha^{-1}+\frac{\partial u_0}{\partial x_3}\alpha^{-1}\beta^{-1}+\frac{\partial u_1}{\partial x_2}\beta^{-1}\Big)k.\end{aligned}$$

Definition 3.3. Let Ω be an open subset of $\mathbb{R}^4$. A function $f\in\mathcal{C}^1(\Omega)$, where $\mathcal{C}^1(\Omega)$ is a class which is composed of continuously differentiable functions, is said to be regular if f satisfies the equation $D^*f=0$.

Theorem 3.4. *Let Ω be an open subset of $\mathbb{R}^4$. Suppose that a function f is defined on Ω. Then f is differentiable if and only if it is regular on Ω.*

Proof. From Definition 3.1 and referring [11], if f is differentiable, then we have the following equation:

$$h\frac{df}{dz} = df_z(h),$$

that is,

$$dz\frac{df}{dz} = df_z.$$

By equaling coefficients of dx_0, dx_1, dx_2 and dx_3, we have

$$\frac{\partial f}{\partial x_0} = \alpha^{-1}\frac{\partial f}{\partial x_1}\ i_\alpha = \beta^{-1}\frac{\partial f}{\partial x_2}\ j_\beta = -\alpha^{-1}\beta^{-1}\frac{\partial f}{\partial x_3}\ k. \tag{3.1}$$

Let $z = z_1 + z_2 j_\beta$, where $z_1 = x_0 + x_1 i_\alpha$ and $z_2 = x_2 + x_3 i_\alpha$, and let $f(z) = f_1(z_1, z_2) + f_2(z_1, z_2) j_\beta$, where f_1 and f_2 are complex-valued functions of complex variables z_1 and z_2. Then we have

$$\frac{\partial f}{\partial x_0} = \frac{\partial f_1}{\partial x_0} + \frac{\partial f_2}{\partial x_0}\ j_\beta,$$

$$\alpha^{-1}\frac{\partial f}{\partial x_1}\ i_\alpha = \alpha^{-1}\Big(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_1}\ j_\beta\Big)\ i_\alpha = \alpha^{-1}\Big(\frac{\partial f_1}{\partial x_1}\ i_\alpha + \frac{\partial f_2}{\partial x_1}\ i_\alpha\ j_\beta\Big),$$

$$\beta^{-1}\frac{\partial f}{\partial x_2}\ j_\beta = \beta^{-1}\Big(\frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x_2}\ j_\beta\Big)\ j_\beta = \beta^{-1}\Big(\frac{\partial f_1}{\partial x_2}\ j_\beta + \frac{\partial f_2}{\partial x_2}\beta\Big)$$

and

$$\begin{aligned} -\alpha^{-1}\beta^{-1}\frac{\partial f}{\partial x_3}\ k \quad &= \quad -\alpha^{-1}\beta^{-1}\Big(\frac{\partial f_1}{\partial x_3} + \frac{\partial f_2}{\partial x_3}\ j_\beta\Big)\ k \\ &\qquad -\alpha^{-1}\beta^{-1}\Big(\frac{\partial f_1}{\partial x_3}\ i_\alpha\ j_\beta - \frac{\partial f_2}{\partial x_3}\ \beta\ i_\alpha\Big). \end{aligned}$$

By equation (3.1), we obtain the following equations:

$$\frac{\partial f_1}{\partial x_0} = \alpha^{-1}\frac{\partial f_1}{\partial x_1}\ i_\alpha = \frac{\partial f_2}{\partial x_2} = \alpha^{-1}\frac{\partial f_2}{\partial x_3}\ i_\alpha$$

and

$$\frac{\partial f_2}{\partial x_0} = \alpha^{-1}\frac{\partial f_2}{\partial x_1}\ i_\alpha = \beta^{-1}\frac{\partial f_1}{\partial x_2} = -\alpha^{-1}\beta^{-1}\frac{\partial f_1}{\partial x_3}\ i_\alpha.$$

Conversely, from Remark 3.2, we have

$$\begin{aligned} D^*f &= (\frac{\partial}{\partial x_0} - \frac{\partial}{\partial x_1}\alpha^{-1}i_\alpha - \frac{\partial}{\partial x_2}\beta^{-1}j_\beta + \frac{\partial}{\partial x_3}\alpha^{-1}\beta^{-1}k) \\ &\quad \times (u_0 + u_1 i_\alpha + u_2 j_\beta + u_3 k) \\ &= \frac{\partial u_0}{\partial x_0} - \frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} + (-\frac{\partial u_0}{\partial x_1}\alpha^{-1} + \frac{\partial u_1}{\partial x_0} + \frac{\partial u_2}{\partial x_3}\alpha^{-1} + \frac{\partial u_3}{\partial x_2})i_\alpha \\ &\quad +(-\frac{\partial u_0}{\partial x_2}\beta^{-1} - \frac{\partial u_1}{\partial x_3}\beta^{-1} + \frac{\partial u_2}{\partial x_0} - \frac{\partial u_3}{\partial x_1})j_\beta \\ &\quad +(\frac{\partial u_0}{\partial x_3}\alpha^{-1}\beta^{-1} + \frac{\partial u_1}{\partial x_2}\beta^{-1} - \frac{\partial u_2}{\partial x_1}\alpha^{-1} + \frac{\partial u_3}{\partial x_0})k. \end{aligned}$$

Hence, the equation $D^*f = 0$ is equivalent to the following corresponding Cauchy-Riemann system in $\mathbb{G}_{\mathbb{H}}$:

$$\begin{cases} \frac{\partial u_0}{\partial x_0} - \frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = 0, \\ \frac{\partial u_0}{\partial x_1}\alpha^{-1} + \frac{\partial u_1}{\partial x_0} + \frac{\partial u_2}{\partial x_3}\alpha^{-1} + \frac{\partial u_3}{\partial x_2} = 0, \\ -\frac{\partial u_0}{\partial x_2}\beta^{-1} - \frac{\partial u_1}{\partial x_3}\beta^{-1} + \frac{\partial u_2}{\partial x_0} - \frac{\partial u_3}{\partial x_1} = 0, \\ \frac{\partial u_0}{\partial x_3}\alpha^{-1}\beta^{-1} + \frac{\partial u_1}{\partial x_2}\beta^{-1} - \frac{\partial u_2}{\partial x_1}\alpha^{-1} + \frac{\partial u_3}{\partial x_0} = 0. \end{cases}$$

Therefore, these equations are equivalent to

$$\frac{\partial f_1}{\partial x_0} = \alpha^{-1}\frac{\partial f_1}{\partial x_1}\ i_\alpha = \frac{\partial f_2}{\partial x_2} = \alpha^{-1}\frac{\partial f_2}{\partial x_3}\ i_\alpha$$

and

$$\frac{\partial f_2}{\partial x_0} = \alpha^{-1}\frac{\partial f_2}{\partial x_1}\ i_\alpha = \beta^{-1}\frac{\partial f_1}{\partial x_2} = -\alpha^{-1}\beta^{-1}\frac{\partial f_1}{\partial x_3}\ i_\alpha.$$

This completes the proof. □

Example 3.5. Let Ω be an open subset of $\mathbb{R}^4$. Suppose that a function $f(z) = z$ is defined on Ω. Since

$$\frac{df}{dz} := \lim_{h\to 0} h^{-1}(f(z+h) - f(z)) = \lim_{h\to 0} \frac{h^*}{|h|^2}((z+h) - z) = 1,$$

the function f is differentiable on Ω. Also, from Remark 3.2, we have

$$\begin{aligned} D^*f &= \frac{\partial x_0}{\partial x_0} - \frac{\partial x_1}{\partial x_1} - \frac{\partial x_2}{\partial x_2} + \frac{\partial x_3}{\partial x_3} + \Big(\frac{\partial x_1}{\partial x_0} - \frac{\partial x_0}{\partial x_1}\alpha^{-1} + \frac{\partial x_2}{\partial x_3}\alpha^{-1} + \frac{\partial x_3}{\partial x_2}\Big) i_\alpha \\ &\quad + \Big(\frac{\partial x_2}{\partial x_0} - \frac{\partial x_0}{\partial x_2}\beta^{-1} - \frac{\partial x_1}{\partial x_3}\beta^{-1} - \frac{\partial x_3}{\partial x_1}\Big) j_\beta \\ &\quad + \Big(\frac{\partial x_3}{\partial x_0} - \frac{\partial x_2}{\partial x_1}\alpha^{-1} + \frac{\partial x_0}{\partial x_3}\alpha^{-1}\beta^{-1} + \frac{\partial x_1}{\partial x_2}\beta^{-1}\Big) k \\ &= 1 - 1 - 1 + 1 = 0. \end{aligned}$$

So, the function f is regular on Ω.

Example 3.6. Let Ω be an open subset of $\mathbb{R}^4$. Suppose that a function $f(z) = z^*$ is defined on Ω. Since the limit

$$\frac{df}{dz} := \lim_{h\to 0} h^{-1}(f(z+h) - f(z)) = \lim_{h\to 0} \frac{h^*}{|h|^2}((z+h)^* - z^*) = \lim_{h\to 0} \frac{(h^*)^2}{|h|^2}$$

goes to infinity, the function f is not differentiable on Ω. Also, from Remark 3.2, we have

$$D^*f = \frac{\partial x_0}{\partial x_0} - \frac{\partial(-x_1)}{\partial x_1} - \frac{\partial(-x_2)}{\partial x_2} + \frac{\partial(-x_3)}{\partial x_3} = 2 \neq 0.$$

So, the function f is not regular on Ω.

Theorem 3.7. *Let Ω be an open subset of $\mathbb{R}^4$. If a function f is regular on Ω, then it satisfies the following equations:*

$$Df = Sc(D)f = Pu(D)f,$$

where $Sc(D)f = \frac{\partial f}{\partial x_0}$ and

$$Pu(D)f = i_\alpha \frac{\partial f}{\partial x_1}\alpha^{-1} + i_\beta \frac{\partial f}{\partial x_2}\beta^{-1} - k\frac{\partial f}{\partial x_3}\alpha^{-1}\beta^{-1}.$$

Proof. Since f satisfies the equation $D^*f = 0$, we have

$$Df = \frac{\partial u_0}{\partial x_0} + \frac{\partial u_1}{\partial x_0} i_\alpha + \frac{\partial u_2}{\partial x_0} j_\beta + \frac{\partial u_3}{\partial x_0} k = \frac{\partial f}{\partial x_0}$$

and

$$\begin{aligned} Df &= \frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}+\Big(\frac{\partial u_0}{\partial x_1}\alpha^{-1}-\frac{\partial u_2}{\partial x_3}\alpha^{-1}-\frac{\partial u_3}{\partial x_2}\Big)i_\alpha \\ &\quad +\Big(\frac{\partial u_0}{\partial x_2}\beta^{-1}+\frac{\partial u_1}{\partial x_3}\beta^{-1}+\frac{\partial u_3}{\partial x_1}\Big)j_\beta \\ &\quad +\Big(\frac{\partial u_2}{\partial x_1}\alpha^{-1}-\frac{\partial u_0}{\partial x_3}\alpha^{-1}\beta^{-1}-\frac{\partial u_1}{\partial x_2}\beta^{-1}\Big)k \\ &= i_\alpha\frac{\partial f}{\partial x_1}\alpha^{-1}+j_\beta\frac{\partial f}{\partial x_2}\beta^{-1}-k\frac{\partial f}{\partial x_3}\alpha^{-1}\beta^{-1}. \end{aligned}$$

Therefore, we obtain the following desired result:

$$Df = Sc(D)f = Pu(D)f.$$

□

Example 3.8. Let Ω be an open subset of $\mathbb{R}^4$. For a function $f(z)=z$, from Example 3.5, f is regular on Ω. By the definitions of D, $Sc(D)$ and $Pu(D)$, we have

$$Df = 1\ ,\ Sc(D)f=\frac{\partial f}{\partial x_0}=1$$

and

$$\begin{aligned} Pu(D)f &= i_\alpha\frac{\partial f}{\partial x_1}\alpha^{-1}+j_\beta\frac{\partial f}{\partial x_2}\beta^{-1}-k\frac{\partial f}{\partial x_3}\alpha^{-1}\beta^{-1} \\ &= i_\alpha i_\alpha\alpha^{-1}+j_\beta j_\beta\beta^{-1}-kk\alpha^{-1}\beta^{-1}=1. \end{aligned}$$

Thus, f satisfies

$$Df = Sc(D)f = Pu(D)f.$$

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References

[1] A.A. Albert, *Integral domains of rational generalized quaternion algebras,* Bull. Amer. Math. Soc., **40**(2) (1934), 164–176.

[2] C.A. Deavours, *The quaternion calculus,* Amer. Math. Monthly, **80** (1973), 995–1008.

[3] R. Fueter, *Die Funktionentheorie der Differentialgleichungen* $\Delta u = 0$ *and* $\Delta\Delta u = 0$ *mit vier reellen Variablen,* Comm. Math. Helv., **7**(1) (1935), 307–330.

[4] T. Hayami and K. Sanada, *Cohomology ring of the generalized quaternion group with coefficients in an order,* J. Comm. Alg., **30**(8) (2002), 3611–3628.

[5] J. Kajiwara, X.D. Li and K.H. Shon, *Regeneration in complex, quaternion and Clifford analysis,* In Finite or Infinite Dimensional Complex Analysis and Applications(ICFIDCAA), pp. 287–298, Springer US (2004).

[6] J.E. Kim, S.J. Lim and K.H. Shon, *Regularity of functions on the reduced quaternion field in Clifford analysis,* Abst. Appl. Anal., **2014**(ID.654798) (2014), 8 pages.
[7] J.E. Kim and K.H. Shon, *The Regularity of functions on Dual split quaternions in Clifford analysis,* Abst. Appl. Anal., **2014**(ID.369430) (2014), 8 pages.
[8] H. Koriyama, H. Mae and K. Nôno, *Hyperholomorphic Fucntions and Holomorphic Functions in Quaternionic Analysis,* Bull. Fukuoka Univ. Ed., **60**(III) (2011), 1–9.
[9] J.J. Malone, *Generalized quaternion groups and distributively generated near-rings,* Proc. Edinburgh Math. Soc., **18** (1973), 235–238.
[10] M. Naser, *Hyperholomorphic functions,* Sib. Math. J., **12** (1971), 959–968.
[11] A. Sudbery, *Quaternionic analysis,* Math. Proc. Camb. Phil. Soc., **85**(2) (1979), 199–224.
[12] T. Unger and N. Markin, *Quadratic forms and space-time block codes from generalized quaternion and biquaternion algebras,* IEEE Trans. Inform. Theory, **57**(9) (2011), 6148–6156.