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SEQUENCE OF MULTI-VALUED PEROV TYPE CONTRACTION MAPPINGS

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Abstract. In this papere, we have extended the result of Perov [5] for a sequence of multi-valued mappings in two different ways. First, we discuss the result for a sequence of admissible multi-valued self mappings, secondly our discussion is for sequence of nonself type multi-valued mappings. As consequence of our results, we have obtained some new fixed point theorems.

1. Introduction and preliminaries

Banach contraction principle is the basis of metric fixed point theory. This result has been generalized by several authors in different ways. One of them is the result of Semat *et al*. [7] for the admissible mappings. The results of Perov [5] are considered from the earlier and worthwhile generalizations of Banach contraction principle. Ali *et al*. [1] generalized the results of [5] by extending the concept of admissible mappings. Here, we extend the result of Perov [5] for a sequence of multi-valued mappings in two different settings. The consequences of our results contain many new results. Some consequences can be concluded as an extension of the result of Ali *et al*. [1].

Now, we introduce some basic notions and results: Let X be a nonempty set and $\mathbb{R}_m$ be the set of all $m \times 1$ real matrices. If $\alpha, \beta \in \mathbb{R}_m$, $\alpha =$

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$(\alpha_1, \alpha_2, \dots, \alpha_m)^T$, $\beta = (\beta_1, \beta_2, \dots, \beta_m)^T$ and $c \in \mathbb{R}$, then by $\alpha \leq \beta$ (resp., $\alpha < \beta$) we mean $\alpha_i \leq \beta_i$ (resp., $\alpha_i < \beta_i$) for each $i \in \{1, 2, \dots, m\}$ and by $\alpha \geq c$ we mean that $\alpha_i \geq c$ for each $i \in \{1, 2, \dots, m\}$. A mapping $d \colon X \times X \to \mathbb{R}_m$ is called a vector-valued metric on X if the following properties are satisfied:

(d_1) $d(x, y) \geq 0$ for all $x, y \in X$; if $d(x, y) = 0$ then $x = y$, and viceversa;
(d_2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(d_3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

A set X endowed with a vector-valued metric d is called a generalized metric space. It is denoted by (X, d). The convergence sequence and Cauchy sequence in a generalized metric space are defined in a similar manner as in a metric space.

Throughout this paper, we denote the set of all nonempty closed subsets of X by $CL(X)$, the set of all $m \times m$ matrices with non-negative elements by $M_{m,m}(\mathbb{R}_+)$, the zero $m \times m$ matrix by $\bar{0}$ and the identity $m \times m$ matrix by I. Also note that $A^0 = I$. A matrix A is said to be convergent to zero if and only if $A^n \to \bar{0}$ as $n \to \infty$ (see [9]).

Theorem 1.1. [3] *Let $A \in M_{m,m}(\mathbb{R}_+)$. Then the following conditions are equivalent.*

(1) *A is convergent to zero;*
(2) *The eigenvalues of A are in the open unit disc, that is, $|\lambda| < 1$, for every $\lambda \in \mathbb{C}$ with $det(A - \lambda I) = 0$;*
(3) *The matrix $I - A$ is nonsingular and*

$$(I - A)^{-1} = I + A + \cdots + A^n + \cdots . \tag{1.1}$$

Example 1.2. The following matrices are convergent to zero.

(1) $A := \begin{pmatrix} a & a \\ b & b \end{pmatrix}$, where $a, b \in \mathbb{R}_+$ and $a + b < 1$;
(2) $B := \begin{pmatrix} a & b \\ a & b \end{pmatrix}$, where $a, b \in \mathbb{R}_+$ and $a + b < 1$;
(3) $C := \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$, where $a, b, c \in \mathbb{R}_+$ and $\max\{a, c\} < 1$.

Perov [5] extended the Banach contraction principle by proving the following fixed point theorem.

Theorem 1.3. [5] *Let (X, d) be a complete generalized metric space and $f \colon X \to X$ be a mapping for which there exists a matrix $A \in M_{m,m}(R_+)$ such that $d(f(x), f(y)) \leq Ad(x, y)$ for all $x, y \in X$. If A is a convergent matric to zero, then the sequence of successive approximations $\{x_n\}$ with $x_n = f^n(x_0)$, is convergent to $x^* \in Fix(f)$, for all $x_0 \in X$.*

You can find some more contributions to this topic in [2, 3, 4, 6, 8]. Recently Ali *et al*. [1] extended the result of Perov [5] in the following way:

Theorem 1.4. [1] *Let (X, d) be a complete generalized metric space and $\Lambda : X \times X \to M_{m,m}(\mathbb{R}_+)$. Let $f : X \to X$ be a mapping such that for each $x, y \in X$ we have*

$$\begin{aligned} \Lambda(x,y)d(fx,fy) &\leq A_1 d(x,y) + A_2 d(x,fx) + A_3 d(y,fy) \\ &\quad + A_4 d(x,fy) + Bd(y,fx), \end{aligned} \tag{1.2}$$

where $\Lambda, A_1, A_2, A_3, A_4, B \in M_{m,m}(\mathbb{R}_+)$ and $(I - A_3 - A_4)^{-1}$ exists. Further, assume that the following conditions hold:

(i) *the matrix $A = (I - A_3 - A_4)^{-1}(A_1 + A_2 + A_4)$ converges to zero;*
(ii) *there exists $x_0 \in X$ such that $\Lambda(x_0, fx_0) \geq I$;*
(iii) *f is Λ_*-admissible, that is, $\Lambda(x,y) \geq I \implies \Lambda(fx, fy) \geq I$ for each $x, y \in X$;*
(iv) *for each sequence $\{x_n\} \subseteq X$ such that $x_n \to x$ and $\Lambda(x_n, x_{n+1}) \geq I$ for all $n \in \mathbb{N}$, we have $\Lambda(x_n, x) \geq I$.*

Then f has a fixed point.

2. Main Results

We begin this section with the following definition.

Definition 2.1. Let X be a nonempty set, $\Lambda : X \times X \to M_{m,m}(\mathbb{R}_+)$ and $\{T_i : X \to CL(X) : i \in \mathbb{N}\}$ be a sequence of mappings. The sequence $\{T_i\}$ is said to be Λ_*-admissible if for $x \in X$ and $y \in T_i x$ for some $i \in \mathbb{N}$ satisfying $\Lambda(x, y) \geq I$, then we have $\Lambda(y, z) \geq I$ for each $z \in T_{i+1}y$, where I is an $m \times m$ identity matrix and the inequality between matrices means entrywise inequality.

Now, we are in a position to introduce the main theorem.

Theorem 2.2. *Let (X, d) be a complete generalized metric space and $\Lambda : X \times X \to M_{m,m}(\mathbb{R}_+)$. Let $\{T_i : X \to CL(X)\}$ be a sequence of mappings such that for each $x, y \in X$ with $\Lambda(x, y) \geq I$ and $u \in T_i x$, we have $v \in T_{i+1}y$ satisfying the following inequality:*

$$\begin{aligned} d(u,v) &\leq A_1 d(x,y) + A_2 d(x,u) + A_3 d(y,v) \\ &\quad + A_4 d(x,v) + Bd(y,u), \end{aligned} \tag{2.1}$$

where $A_1, A_2, A_3, A_4, B \in M_{m,m}(\mathbb{R}_+)$ and $(I - A_3 - A_4)^{-1}$ exists. Further, assume that the following conditions hold:

(i) *the matrix $A = (I - A_3 - A_4)^{-1}(A_1 + A_2 + A_4)$ converges to zero;*
(ii) *there exist $x_0 \in X$ and $x_1 \in T_1 x_0$ such that $\Lambda(x_0, x_1) \geq I$;*
(iii) *$\{T_i\}$ is Λ_*-admissible;*

(iv) *for each sequence* $\{x_n\} \subseteq X$ *such that* $x_n \to x$ *and* $\Lambda(x_n, x_{n+1}) \geq I$ *for all* $n \in \mathbb{N}$, *we have* $\Lambda(x_n, x) \geq I$.

Then there exists an $\omega \in X$ *such that* $\omega \in T_i\omega$ *for sufficiently large* i.

Proof. Using hypothesis (ii), we get $x_0 \in X$ and $x_1 \in T_1x_0$ such that $\Lambda(x_0, x_1) \geq I$. From (2.1), for $x_0, x_1 \in X$ with $\Lambda(x_0, x_1) \geq I$ and $x_1 \in T_1x_0$, we have $x_2 \in T_2x_1$ which satisfies the following inequality

$$\begin{aligned} d(x_1, x_2) &\leq A_1d(x_0, x_1) + A_2d(x_0, x_1) + A_3d(x_1, x_2) \\ &\quad +A_4d(x_0, x_2) + Bd(x_1, x_1) \\ &\leq A_1d(x_0, x_1) + A_2d(x_0, x_1) + A_3d(x_1, x_2) \\ &\quad +A_4[d(x_0, x_1) + d(x_1, x_2)] + B0. \end{aligned}$$

After simplifying this inequality, we get

$$\begin{aligned} d(x_1, x_2) &\leq (I - A_3 - A_4)^{-1}(A_1 + A_2 + A_4)d(x_0, x_1) \\ &= Ad(x_0, x_1). \end{aligned} \tag{2.2}$$

Since $\{T_i\}$ is Λ_*-admissible, we have $\Lambda(x_1, x_2) \geq I$. Again by using (2.1), for $x_1, x_2 \in X$ with $\Lambda(x_1, x_2) \geq I$ and $x_2 \in T_2x_1$, we have $x_3 \in T_3x_2$ such that

$$\begin{aligned} d(x_2, x_3) &\leq A_1d(x_1, x_2) + A_2d(x_1, x_2) + A_3d(x_2, x_3) \\ &\quad +A_4d(x_1, x_3) + Bd(x_2, x_2) \\ &\leq A_1d(x_1, x_2) + A_2d(x_1, x_2) + A_3d(x_2, x_3) \\ &\quad +A_4[d(x_1, x_2) + d(x_2, x_3)] + B0. \end{aligned}$$

This implies that

$$\begin{aligned} d(x_2, x_3) &\leq (I - A_3 - A_4)^{-1}(A_1 + A_2 + A_4)d(x_1, x_2) \\ &= Ad(x_1, x_2). \end{aligned} \tag{2.3}$$

From (2.2) and (2.3), we have

$$d(x_2, x_3) \leq A^2d(x_0, x_1).$$

Continuing in this process, we get a sequence $\{x_n\} \subseteq X$ such that

$$x_n \in T_nx_{n-1},\ \Lambda(x_{n-1}, x_n) \geq I$$

and

$$d(x_n, x_{n+1}) \leq A^nd(x_0, x_1),\ \forall\ n \in \mathbb{N}.$$

Next we will prove that $\{x_n\}$ is a Cauchy sequence. Let n, m be arbitrary natural numbers with $m > n$, by using the triangle inequality, we get

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \\ &\leq \sum_{i=n}^{m-1} A^i d(x_0, x_1) \\ &\leq A^n \left(\sum_{i=0}^{\infty} A^i \right) d(x_0, x_1) \\ &= A^n (I - A)^{-1} d(x_0, x_1). \end{aligned}$$

Letting $n \to \infty$ to the above inequality, since the matrix A converges to zero, we get $\lim_{n\to\infty} d(x_n, x_{n+m}) = 0$. Hence $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists an element x^* in X such that $x_n \to x^*$. By using the hypothesis (iii), we conclude that $\Lambda(x_n, x^*) \geq 1$ for all $n \in \mathbb{N}$. From (2.1), for $x_n, x^* \in X$ with $\Lambda(x_n, x^*) \geq 1$ and $x_{n+1} \in T_{n+1}x_n$, we have $v_{n+2} \in T_{n+2}x^*$ such that

$$\begin{aligned} d(x_{n+1}, v_{n+2}) \leq\ & A_1 d(x_n, x^*) + A_2 d(x_n, x_{n+1}) + A_3 d(x^*, v_{n+2}) \\ & + A_4 d(x_n, v_{n+2}) + B d(x^*, x_{n+1}). \end{aligned}$$

By using the triangle inequality and the above inequality, we have

$$\begin{aligned} d(x^*, v_{n+2}) &\leq d(x^*, x_{n+1}) + d(x_{n+1}, v_{n+2}) \\ &\leq d(x^*, x_{n+1}) + A_1 d(x_n, x^*) + A_2 d(x_n, x_{n+1}) \\ &\quad + A_3 d(x^*, v_{n+2}) + A_4 [d(x_n, x^*) + d(x^*, v_{n+2})] \\ &\quad + B d(x^*, x_{n+1}). \end{aligned}$$

This implies that

$$(I - (A_3 + A_4)) d(x^*, v_{n+2}) \leq 0 \text{ as } n \to \infty.$$

Since the inverse of the matrix $I - (A_3 + A_4)$ exists, we have $d(x^*, v_{n+2}) \to 0$ as $n \to \infty$. Hence $v_{n+2} \to x^*$. This means that $x^* \in T_n x^*$ for sufficiently large n. □

Example 2.3. Let $X = [0, \infty) \times [0, \infty)$ be equipped with a generalized metric defined by

$$d(x, y) = \begin{cases} \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}, & \text{if } x \neq y \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & \text{if } x = y \end{cases} \qquad \text{for each } x = (x_1, x_2), y = (y_1, y_2) \in X.$$

Define the mappings

$$T\colon X \to CL(X), \quad T(x_1,x_2) = \Big\{(1,1), \Big(\frac{x_1}{3}, \frac{x_2}{3}\Big)\Big\}$$

and

$$S\colon X \to CL(X), \quad S(x_1,x_2) = \begin{cases} \big\{(1,1), \big(\frac{x_1}{4}, \frac{x_2}{4}\big)\big\}, & \text{if } x_1, x_2 \le 4 \\ \big\{(0,0), \big(x_1^2, x_2^2\big)\big\}, & \text{otherwise.} \end{cases}$$

Define $\Lambda : X \times X \to M_{2,2}(\mathbb{R}_+)$ by

$$\Lambda((x_1,x_2),(y_1,y_2)) = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \text{if } x_1, x_2, y_1, y_2 \le 4 \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & \text{otherwise.} \end{cases}$$

Then, it is easy to see that the sequence of mappings $\{T_i\}$ defined by $T_i = T$ and $T_{2i} = S$ for each $i \in \mathbb{N}$ satisfies (2.1) with

$$A_1 = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}, A_2 = A_3 = A_4 = B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and all the other conditions of Theorem 2.2 also hold. Thus, we have $x^* \in X$ such that $x^* \in T_i x^*$ for sufficiently large values of i. Hence we conclude that $x^* \in Tx^*$ and $x^* \in Sx^*$.

Now, we state and prove our second result, in this result we denote the closed ball with centered at x_0 and radius r by $B(x_0, r)$.

Theorem 2.4. *Let (X,d) be a complete generalized metric space, $x_0 \in X$ and $r = (r_{11}, r_{21}, \cdots, r_{m1})^T$ be any nonzero matrix in $\mathbb{R}_m$ with $r_{i1} \ge 0$ for each $i \in \{1,2,\cdots,m\}$. Let $\{T_i\colon B(x_0,r) \to CL(X)\}$ be a sequence of mappings such that for each $x, y \in B(x_0,r)$ and $u \in T_i x$, we have $v \in T_{i+1} y$ satisfying the following inequality:*

$$\begin{aligned} d(u,v) \ \le \ & A_1 d(x,y) + A_2 d(x,u) + A_3 d(y,v) \\ & + A_4 d(x,v) + B d(y,u), \end{aligned} \tag{2.4}$$

where $A_1, A_2, A_3, A_4, B \in M_{m,m}(\mathbb{R}_+)$ and $(I - A_3 - A_4)^{-1}$ exists. Further, assume that the following conditions hold:

(i) *the matrix $A = (I - A_3 - A_4)^{-1}(A_1 + A_2 + A_4)$ converges to zero;*

(ii) *there exist $x_0 \in X$ and $x_1 \in T_1 x_0$ such that $(I - A)^{-1} d(x_0, x_1) \le r$;*

Then there exists an $\omega \in X$ such that $\omega \in T_i \omega$ for sufficiently large i.

Proof. By hypothesis (ii), we have $x_0 \in X$ and $x_1 \in T_1 x_0$ such that $(I - A)^{-1} d(x_0, x_1) \le r$. Thus we have

$$(I - A)^{-1} d(x_0, x_1) \le r \le (I - A)^{-1} r.$$

This implies that $d(x_0, x_1) \leq r$. Hence $x_1 \in B(x_0, r)$. From (2.4), for $x_0, x_1 \in B(x_0, r)$ and $x_1 \in T_1x_0$, we have $x_2 \in T_2x_0$ such that

$$\begin{aligned} d(x_1, x_2) &\leq A_1d(x_0, x_1) + A_2d(x_0, x_1) + A_3d(x_1, x_2) \\ &\quad +A_4d(x_0, x_2) + Bd(x_1, x_1) \\ &\leq A_1d(x_0, x_1) + A_2d(x_0, x_1) + A_3d(x_1, x_2) \\ &\quad +A_4[d(x_0, x_1) + d(x_1, x_2)] + B0. \end{aligned}$$

After simplifying this inequality we get

$$\begin{aligned} d(x_1, x_2) &\leq (I - A_3 - A_4)^{-1}(A_1 + A_2 + A_4)d(x_0, x_1) \\ &= Ad(x_0, x_1). \end{aligned} \tag{2.5}$$

Further, we get

$$\begin{aligned} (I - A)^{-1}d(x_1, x_2) &\leq (I - A)^{-1}Ad(x_0, x_1) \\ &= A(I - A)^{-1}d(x_0, x_1) \\ &\leq Ar. \end{aligned} \tag{2.6}$$

By using the triangle inequality and (2.6), we have

$$\begin{aligned} (I - A)^{-1}d(x_0, x_2) &\leq (I - A)^{-1}d(x_0, x_1) + (I - A)^{-1}d(x_1, x_2) \\ &\leq Ir + Ar \\ &\leq (I + A + A^2 + A^3 + \cdots)r \\ &= (I - A)^{-1}r. \end{aligned}$$

This implies that $d(x_0, x_2) \leq r$, that is, $x_2 \in B(x_0, r)$. From (2.4), for $x_1, x_2 \in B(x_0, r)$ and $x_2 \in T_2x_1$, we have $x_3 \in T_3x_2$ such that

$$\begin{aligned} d(x_2, x_3) &\leq A_1d(x_1, x_2) + A_2d(x_1, x_2) + A_3d(x_2, x_3) \\ &\quad +A_4d(x_1, x_3) + Bd(x_2, x_2) \\ &\leq A_1d(x_1, x_2) + A_2d(x_1, x_2) + A_3d(x_2, x_3) \\ &\quad +A_4[d(x_1, x_2) + d(x_2, x_3)] + B0. \end{aligned}$$

This implies that

$$\begin{aligned} d(x_2, x_3) &\leq (I - A_3 - A_4)^{-1}(A_1 + A_2 + A_4)d(x_1, x_2) \\ &= Ad(x_1, x_2). \end{aligned} \tag{2.7}$$

From (2.5) and (2.7), we have

$$d(x_2, x_3) \leq A^2d(x_0, x_1).$$

Further, from the above inequality, we get

$$\begin{aligned}(I-A)^{-1}d(x_2,x_3) &\leq (I-A)^{-1}A^2d(x_0,x_1)\\ &= A^2(I-A)^{-1}d(x_0,x_1)\\ &\leq A^2r. \end{aligned} \tag{2.8}$$

Again by using the triangle inequality, we get

$$\begin{aligned}(I-A)^{-1}d(x_0,x_3) &\leq (I-A)^{-1}d(x_0,x_1)+(I-A)^{-1}d(x_1,x_2)\\ &\quad +(I-A)^{-1}d(x_2,x_3)\\ &\leq Ir+Ar+A^2r\\ &= (I+A+A^2+A^3+\cdots)r\\ &= (I-A)^{-1}r.\end{aligned}$$

Thus $d(x_0,x_3) \leq r$, hence $x_3 \in B(x_0,r)$. Continuing in this process, we construct a sequence $\{x_n\} \subseteq B(x_0,r)$ such that for each $n \in \mathbb{N}$,

(i) $x_n \in T_nx_{n-1}$;
(ii) $d(x_n,x_{n+1}) \leq A^nd(x_0,x_1)$;
(iii) $(I-A)^{-1}d(x_0,x_n) \leq (I-A)^{-1}r$.

From the proof of Theorem 2.2, we conclude that $\{x_n\}$ is a Cauchy sequence in $B(x_0,r)$. Since $B(x_0,r)$ is closed in complete space X, we have $x^* \in B(x_0,r)$ such that $x_n \to x^*$. From (2.4), for $x_n, x^* \in B(x_0,r)$ and $x_{n+1} \in T_{n+1}x_n$ we have $v_{n+2} \in T_{n+2}x^*$ such that

$$\begin{aligned}d(x_{n+1},v_{n+2}) &\leq A_1d(x_n,x^*)+A_2d(x_n,x_{n+1})+A_3d(x^*,v_{n+2})\\ &\quad +A_4d(x_n,v_{n+2})+Bd(x^*,x_{n+1}).\end{aligned}$$

By using the triangle inequality, we have

$$\begin{aligned}d(x^*,v_{n+2}) &\leq d(x^*,x_{n+1})+d(x_{n+1},v_{n+2})\\ &\leq d(x^*,x_{n+1})+A_1d(x_n,x^*)\\ &\quad +A_2d(x_n,x_{n+1})+A_3d(x^*,v_{n+2})\\ &\quad +A_4[d(x_n,x^*)+d(x^*,v_{n+2})]+Bd(x^*,x_{n+1}).\end{aligned}$$

This implies that

$$(I-(A_3+A_4))d(x^*,v_{n+2}) \leq 0, \text{ as } n \to \infty.$$

Since the inverse of the matrix $I-(A_3+A_4)$ exists, we have $d(x^*,v_{n+2}) \to 0$ as $n \to \infty$. Hence $v_{n+2} \to x^*$. This means that $x^* \in T_nx^*$ for sufficiently large n. □

3. Consequences

As consequences, we mention only fixed point results which can be obtained from our results.

The following theorems are special cases of our main results by considering the sequence of mappings $\{T_i\}$, as $T_i = T$ for each $i \in \mathbb{N}$.

Theorem 3.1. *Let (X, d) be a complete generalized metric space and $\Lambda : X \times X \to M_{m,m}(\mathbb{R}_+)$. Let $T \colon X \to CL(X)$ be a mapping such that for each $x, y \in X$ with $\Lambda(x, y) \geq I$ and $u \in Tx$, we have $v \in Ty$ satisfying the following inequality:*

$$\begin{aligned} d(u, v) \quad &\leq \quad A_1 d(x, y) + A_2 d(x, u) \\ &\quad + A_3 d(y, v) + A_4 d(x, v) + B d(y, u), \end{aligned}$$

where $A_1, A_2, A_3, A_4, B \in M_{m,m}(\mathbb{R}_+)$ and $(I - A_3 - A_4)^{-1}$ exists. Further, assume that the following conditions hold:

(i) *the matrix $A = (I - A_3 - A_4)^{-1}(A_1 + A_2 + A_4)$ converges to zero;*
(ii) *there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\Lambda(x_0, x_1) \geq I$;*
(iii) *T is Λ_*-admissible, that is, for $x \in X$ and $y \in Tx$ with $\Lambda(x, y) \geq I$, we have $\Lambda(y, v) \geq I$ for each $v \in Ty$;*
(iv) *for each sequence $\{x_n\} \subseteq X$ such that $x_n \to x$ and $\Lambda(x_n, x_{n+1}) \geq I$ for all $n \in \mathbb{N}$, we have $\Lambda(x_n, x) \geq I$.*

Then T has a fixed point.

Theorem 3.2. *Let (X, d) be a complete generalized metric space, $x_0 \in X$ and $r = (r_{11}, r_{21}, \cdots, r_{m1})^T$ be any nonzero matrix in $\mathbb{R}_m$ with $r_{i1} \geq 0$ for each $i \in \{1, 2, \cdots, m\}$. Let $T \colon B(x_0, r) \to CL(X)$ be a mapping such that for each $x, y \in B(x_0, r)$ and $u \in Tx$, we have $v \in Ty$ satisfying the following inequality;*

$$\begin{aligned} d(u, v) \quad &\leq \quad A_1 d(x, y) + A_2 d(x, u) + A_3 d(y, v) \\ &\quad + A_4 d(x, v) + B d(y, u), \end{aligned} \tag{3.1}$$

where $A_1, A_2, A_3, A_4, B \in M_{m,m}(\mathbb{R}_+)$ and $(I - A_3 - A_4)^{-1}$ exists. Further, assume that the following conditions hold:

(i) *the matrix $A = (I - A_3 - A_4)^{-1}(A_1 + A_2 + A_4)$ converges to zero;*
(ii) *there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $(I - A)^{-1} d(x_0, x_1) \leq r$.*

Then T has a fixed point.

Example 3.3. Let $X = \mathbb{R}^2$ be equipped with a generalized metric defined by

$$d(x, y) = \begin{pmatrix} |x_1 - y_1| \\ |x_2 - y_2| \end{pmatrix} \text{ for each } x = (x_1, x_2), y = (y_1, y_2) \in X.$$

Take $x_0 = (2,0)$ and $r = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$. Then we have $B(x_0, r) = \{(x, y) : 0 \leq x \leq 4 \text{ and } y = 0\}$. Define a mapping

$$T\colon B(x_0, r) \to CL(X), \quad T(x_1, x_2) = \left\{(5,0), \left(\frac{x_1}{2}, x_2\right)\right\}.$$

Then, it is easy to see that for each $x, y \in B(x_0, r)$ and $u \in Tx$, we have $v \in Ty$ satisfying (3.1) with

$$A_1 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix}, A_2 = A_3 = A_4 = B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Further for $(2,0)$, we have $(1,0) \in T(2,0)$ satisfying $(I - A)^{-1} d(x_0, x_1) \leq r$. Thus, by Theorem 3.2, T has a fixed point.

The following results can be obtain from Theorem 3.1. By considering $B_1 = A_1$, $B_2 = A_2 = A_3$ and $B_3 = A_4 = B$.

Corollary 3.4. *Let (X, d) be a complete generalized metric space and $\Lambda : X \times X \to M_{m,m}(\mathbb{R}_+)$. Let $T\colon X \to CL(X)$ be a mapping such that for each $x, y \in X$ with $\Lambda(x, y) \geq I$ and $u \in Tx$, we have $v \in Ty$ satisfying the following inequality:*

$$\begin{aligned} d(u, v) \quad \leq \quad & B_1 d(x, y) + B_2[d(x, u) + d(y, v)] \\ & + B_3[d(x, v) + d(y, u)], \end{aligned}$$

where $B_1, B_2, B_3 \in M_{m,m}(\mathbb{R}_+)$ and $(I - B_2 - B_3)^{-1}$ exists. Further, assume that the following conditions hold:

(i) *the matrix $A = (I - B_2 - B_3)^{-1}(B_1 + B_2 + B_3)$ converges to zero;*
(ii) *there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\Lambda(x_0, x_1) \geq I$;*
(iii) *T is Λ_*-admissible;*
(iv) *for each sequence $\{x_n\} \subseteq X$ such that $x_n \to x$ and $\Lambda(x_n, x_{n+1}) \geq I$ for all $n \in \mathbb{N}$, we have $\Lambda(x_n, x) \geq I$.*

Then T has a fixed point.

Subsequently, (X, d) is a generalized metric space and $G = (V, E)$ is a directed graph such that the set V of its vertices coincides with X and the set E of its edges contains loops; that is, $E \supseteq \triangle$, where $\triangle$ is the diagonal of the Cartesian product $X \times X$. The following theorem can be obtained from Theorem 3.1, by defining the function $\Lambda : X \times X \to M_{m,m}(\mathbb{R}_+)$ as

$$\Lambda(x, y) = \begin{cases} I_{m\times m} \text{ if } (x, y) \in E, \\ 0_{m\times m} \text{ otherwise,} \end{cases}$$

where $I_{m\times m}$ is an $m \times m$ identity matrix and $0_{m\times m}$ is $m \times m$ zero matrix.

Theorem 3.5. *Let (X, d) be a complete generalized metric space with the graph G, let $T\colon X \to CL(X)$ be a mapping such that for each $x, y \in X$ with $(x, y) \in E$ and $u \in Tx$, we have $v \in Ty$ satisfying the following inequaity:*

$$\begin{aligned} d(u,v) \quad &\leq \quad A_1 d(x,y) + A_2 d(x,u) + A_3 d(y,v) \\ &\quad +A_4 d(x,v) + B d(y,u), \end{aligned}$$

where $A_1, A_2, A_3, A_4, B \in M_{m,m}(\mathbb{R}_+)$ and $(I - A_3 - A_4)^{-1}$ exists. Further, assume that the following conditions hold:

(i) *the matrix $A = (I - A_3 - A_4)^{-1}(A_1 + A_2 + A_4)$ converges to zero;*
(ii) *there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0, x_1) \in E$;*
(iii) *for $x \in X$ and $y \in Tx$ with $(x, y) \in E$, we have $(y, v) \in E$ for each $v \in Ty$;*
(iv) *for each sequence $\{x_n\} \subseteq X$ such that $x_n \to x$ and $(x_n, x_{n+1}) \in E$ for all $n \in \mathbb{N}$, we have $(x_n, x) \in E$.*

Then T has a fixed point.

The following corollary can be obtained from Theorem 3.5. By considering the graph $G = (V, E)$ as defined below

$$V = X,\ E = \{(x, y) \in X \times X : x \preceq y\},$$

where $\preceq$ is a partial ordering on X.

Corollary 3.6. *Let (X, d) be a complete generalized metric space with partial ordering $\preceq$, let $T\colon X \to CL(X)$ be a mapping such that for each $x, y \in X$ with $x \preceq y$ and $u \in Tx$, we have $v \in Ty$ satisfying the following inequality:*

$$\begin{aligned} d(u,v) \quad &\leq \quad A_1 d(x,y) + A_2 d(x,u) + A_3 d(y,v) \\ &\quad +A_4 d(x,v) + B d(y,u), \end{aligned}$$

where $A_1, A_2, A_3, A_4, B \in M_{m,m}(\mathbb{R}_+)$ and $(I - A_3 - A_4)^{-1}$ exists. Further, assume that the following conditions hold:

(i) *the matrix $A = (I - A_3 - A_4)^{-1}(A_1 + A_2 + A_4)$ converges to zero;*
(ii) *there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $x_0 \preceq x_1$;*
(iii) *for $x \in X$ and $y \in Tx$ with $x \preceq y$, we have $y \preceq v$ for each $v \in Ty$;*
(iv) *for each sequence $\{x_n\} \subseteq X$ such that $x_n \to x$ and $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$, we have $x_n \preceq x$.*

Then T has a fixed point.

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